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A Covariant Wave-Tensor Framework for Bohmian Mechanics on Classical Curved Spacetime: Lagrangian Structure and Post-Newtonian Predictions

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Abstract

We propose an exploratory framework for a Bohmian model of quantum matter propagating on a classical curved spacetime background. The gravitational sector is governed by classical Einstein field equations throughout; no quantisation of spacetime is attempted. The wave function emerges as the scalar contraction $\Psi = \psi_\nu \psi^\nu \in \mathbb{C}$ of a complex-valued tensorial field ψ^μ, encoding quantum dynamics in a geometric object. The wave tensor interacts with spacetime via the stress–energy tensor $T_{\mu\nu}$, mediated by a real scalar field a of dimension volume, so that $aT_{\mu\nu}\psi^\mu\psi^\nu$ yields the correct potential energy. We derive a covariant Adapted Schrödinger Equation as the unique minimal covariant lift of the standard equation, justify it from four guiding principles, and verify three internal consistency checks. Under seven explicit approximations the framework reproduces the Schrödinger equation with Coulomb potential for the hydrogen atom. We also derive a dynamical equation for ψ^μ that entails the Adapted Schrödinger Equation by contraction. Two open problems are then resolved. First, a complete Lagrangian formulation is provided: a real-valued action for Ψ yields the Adapted Schrödinger Equation via the Euler–Lagrange equations; a separate action for ψ^μ, extended by a non-polynomial term, yields the full dynamical equation variationally. Second, two experimental predictions are derived. Expanding to first post-Newtonian order, the perturbation Hamiltonian has coefficients $(3, 1)$ on the kinetic and potential operators; via the virial theorem these produce a coordinate-time blueshift, which after photon propagation yields the universal Einstein gravitational redshift $\delta\nu/\nu = \Phi/c^2$, confirming consistency with the equivalence principle. The same kinetic coefficient independently predicts that free quantum wave packets spread more slowly by the fractional amount $3|\Phi|/c^2$, a correction absent in standard non-relativistic quantum mechanics.

Keywords: Bohmian mechanics; general relativity; classical curved spacetime; wave tensor; pilot wave; stress–energy tensor; covariant Schrödinger equation; Lagrangian formulation; gravitational redshift; Einstein equivalence principle

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1. Introduction

The unification of General Relativity (GR) and quantum mechanics (QM) into a single overarching theory is the main enigma of today's physics. On the one hand, GR is classical in several aspects, including the fact that particles have concrete trajectories; on the other hand, QM has either non-classical properties or non-locality, depending on the interpretation chosen—GR and QM are, apparently, inherently different and perhaps irreconcilable.

Several efforts exist in the literature to address this problem, including String Theory [1,2], Loop Quantum Gravity [3], and Quantum Field Theory in Curved Spacetime [4]. Most of these approaches adapt GR to QM rather than the other way around. Moreover, in approaches that do take GR into consideration [5], non-deterministic reasoning plays a major role.

It is important to clarify at the outset the precise objective and scope of this work. Broadly speaking, one may attempt either (i) Bohmian Quantum Gravity, in which spacetime geometry is subject to quantum dynamics and back-reaction, or (ii) a Bohmian model on a classical curved spacetime, in which quantum matter propagates on a background satisfying Einstein's classical field equations. The present paper belongs firmly to the second category: we work on a spacetime manifold (M, g) whose metric satisfies Einstein's field equations with a classical stress–energy tensor, seeking a covariant, deterministic equation of motion for the quantum wave function on that background. The gravitational sector is treated classically throughout; no quantisation of spacetime is attempted, no full back-reaction programme is pursued, and no resolution of the quantum gravity problem is claimed. This work is an exploratory reformulation framework, not a unification theory in the sense of (i). For a review of Bohmian approaches to quantum gravity in the first sense, including many-body and back-reaction problems, the reader is referred to [6–8]; our goals and methods differ substantially from those works.

We propose a fresh approach by adhering, as much as possible, to GR and its tensorial and deterministic philosophy. We firstly consider a deterministic interpretation of QM (hence non-local by Bell's Theorem [9]), namely Bohmian Mechanics (BM) (also known as de Broglie–Bohm Theory or the Pilot Wave Theory) [10,11]; secondly, we consider a complex-valued tensor ψ^ν giving rise to the usual complex wave function via $\Psi = \psi_\nu \psi^\nu \in \mathbb{C}$. Our approach is a first step in developing a Bohmian model on classical curved spacetime in the sense of category (ii), where more theoretical and experimental study is needed, in particular to further characterise the parameter a.

The tensor ψ^ν not only represents the wave function tensorially, but also interacts with spacetime via the stress–energy tensor. These two aspects—the wave function arising from a complex tensor, and that tensor interacting with spacetime—are the fundamental novelties of the approach. In this paper we analyse the equations for one single particle without accounting for spin.

Our approach differs from existing work relating BM to relativity, for instance [12,13] by H. Nikolić, which develops Dirac-like equations for Minkowski-like metrics, and developments thereof [8,14]. We consider the full GR framework accounting for Einstein's tensor and the stress–energy tensor, and encode the wave function in a complex tensor that interacts with spacetime via $T_{\mu\nu}$. While the Dirac equation on curved spacetime provides a well-established relativistic framework for spin-$\frac{1}{2}$ particles [4], it does not incorporate the deterministic pilot wave structure of BM nor encode the wave function as a geometric tensor object interacting with the full stress–energy tensor.

1.1. General Relativity

We assume the reader is familiar with GR and its mathematical framework [15,16]. We work on a spacetime manifold (M, g) with signature $(-, +, +, +)$, using Greek letters $\mu, \nu, \ldots$ for spacetime coordinates and Latin letters $j, k, \ldots$ for spatial coordinates. As usual, $g_{\mu\nu}$ denotes the metric, $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ the Einstein tensor, and $T_{\mu\nu}$ the stress–energy tensor. The fundamental equation of GR is

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$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{1}$$

We use the metric to raise and lower indices in the standard way.

1.2. Bohmian Mechanics

Bohmian Mechanics (BM), also known as de Broglie–Bohm Theory or the Pilot Wave Theory, was introduced by Bohm [10]. Unlike canonical QM, BM assigns well-defined positions and velocities to particles at all times; it is non-local and free from the measurement problem. Particles are *guided* by a *pilot wave* that determines their trajectories deterministically. The complex wave function $\psi(\mathbf{r}, t) \in \mathbb{C}$ obeys Schrödinger's equation:

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}, t) + V(\mathbf{r}, t)\psi(\mathbf{r}, t), \tag{2}$$

where m is the particle mass, $\mathbf{r} = (x^1, x^2, x^3)$ its position, and $V(\mathbf{r}, t)$ the potential energy. The fundamental equation of BM is the *guiding equation*:

$$\mathbf{v}(t) = \frac{\hbar}{m} \operatorname{Im}\left(\frac{\nabla \psi(\mathbf{r}, t)}{\psi(\mathbf{r}, t)} \right). \tag{3}$$

One fundamental feature of BM is the *equivariance property*: if particles are initially distributed according to $|\psi|^2$, this distribution is preserved in time, so the probability of finding a particle in a region Ω is

$$\mathbb{P}(\mathbf{r} \in \Omega, t) = \int_\Omega |\psi(\mathbf{r}, t)|^2 \, d^3\mathbf{r}. \tag{4}$$

We recommend [11] for further details on BM and [17] for an overview.

1.3. Scope and Status of the Framework

Because the present proposal sits between several established programmes, we state its status precisely at the outset.

A modified dynamics, not merely an interpretation. It is essential to distinguish two layers. *Standard* Bohmian mechanics is an *interpretation*: laid over the ordinary Schrödinger (or Klein–Gordon) dynamics, it reproduces, by construction, every empirical prediction of orthodox quantum theory [11,18]. Were the present framework only the Bohmian reading of a standard wave equation, it could make no new predictions, and we would claim none. This is *not* what we propose. The Adapted Schrödinger Equation (12) is a *distinct dynamical law*: its kinetic structure on a curved background—the covariant $3 + 1$ Laplacian and the resulting post-Newtonian coefficients $(3, 1)$—is *not* identical to the minimal coupling of the standard Schrödinger or Klein–Gordon equation to gravity. The framework is therefore an *overarching dynamical theory*: it reduces to known physics where this has been tested (it reproduces the hydrogen spectrum, Section 4, and the universal Einstein redshift, Section 7), but is permitted to *deviate* from the standard non-relativistic reduction in regimes not yet probed. The novel empirical content of the framework (Section 7) originates entirely from this *dynamical* deviation, and not from the Bohmian interpretation. A theory that does not agree with established physics in every regime is precisely what is needed to generate genuinely new, falsifiable predictions; we make this deviation explicit and bound its present size in Section 7.

What is specifically Bohmian. The Bohmian content of the construction is the *ontology and the guidance structure*: a single-valued actual configuration (beable) that always possesses a definite value, the guiding equation (Section 2.9) that determines its trajectory from Ψ, and the Born-rule/equivariance account of measurement (Section 2.8). This ontology is what makes the proposal a *complete* theory rather than an isolated equation; in particular, as shown in Section 2.3, it furnishes a single, definite source $T_{\mu\nu}[Q]$ for the classical Einstein equation even when Ψ is in superposition. In short: the *dynamics* is carried by (12); the *interpretation* is carried by the guiding equation and the beable.

Why a Schrödinger-type equation rather than Klein–Gordon. One might ask why we do not simply solve the Klein–Gordon equation on the curved background. The Klein–Gordon equation is second-order in time and does not admit a positive-definite single-particle probability density, which obstructs both the Born rule and the pilot wave guidance for a single particle; these are exactly the difficulties the relativistic Bohmian literature must navigate [12,13,18]. We therefore seek a *Schrödinger-type* equation—first-order in the observer's time and equipped with a positive-definite $|\Psi|^2$—that is nonetheless covariant through the observer-relative $3+1$ split. The Adapted Schrödinger Equation is this object: not a competitor to Klein–Gordon as a relativistic field equation, but a covariant single-particle equation engineered to support a definite Bohmian configuration and the Born rule on a curved background.

Relation to prior work. Bohmian dynamics on relativistic or curved backgrounds has been studied before, by routes different from ours: Nikolić's foliation-based Bohmian quantum field theory [13,19]; the geometrisation of relativistic Bohmian Mechanics from a scalar theory of spacetime [20]; Bohm-like trajectories on singular backgrounds [21]; operational weak-value trajectories that coincide with the Klein–Gordon current and are explicitly interpretive [14,18]; and, most closely, a recent Lagrangian-based covariant Bohmian extension generating "hidden curvature" [22]. These works either remain interpretations of standard dynamics or geometrise the existing Klein–Gordon dynamics. The present framework differs in proposing a *new dynamical coupling* between the wave function—encoded as the tensor contraction $\Psi = \psi_\nu \psi^\nu$—and the stress–energy tensor $T_{\mu\nu}$, yielding the modified, testable dynamics described above; how its predictions differ from a minimally coupled curved-spacetime reduction is made explicit in Section 7.

On "derivation". Finally, the Adapted Schrödinger Equation is not merely postulated. It is obtained in two independent ways: as the unique minimal covariant lift of the standard Schrödinger equation under four covariance principles (Section 2), and from a variational principle, as the Euler–Lagrange equation of a real action (Section 6).

2. Equations for a Single Particle Without Spin

We now develop equations governing a single spinless particle in our framework.

2.1. The Wave Tensor and Its Complex Structure

We consider a relativistic *wave tensor* ψ^ν and define

$$\Psi := \psi_\nu \psi^\nu. \tag{5}$$

This scalar is to be interpreted as the usual wave function: it depends on t, x^1, x^2, x^3 (dependence omitted for brevity) and has the same units as the wave function. Although Ψ is a tensor contraction, it depends on spacetime coordinates just as the Ricci scalar does; the same holds for a.

Complex-valuedness. Both the wave tensor ψ^μ and the resulting scalar Ψ are *complex-valued*. Writing each component as $\psi^\mu = \psi^\mu_{\mathrm{R}} + i\psi^\mu_{\mathrm{I}}$ with $\psi^\mu_{\mathrm{R}}, \psi^\mu_{\mathrm{I}} \in \mathbb{R}$, the contraction gives

$$\Psi = g_{\mu\nu}\psi^\mu\psi^\nu = \underbrace{g_{\mu\nu}\Big(\psi^\mu_{\mathrm{R}}\psi^\nu_{\mathrm{R}} - \psi^\mu_{\mathrm{I}}\psi^\nu_{\mathrm{I}}\Big)}_{\mathrm{Re}(\Psi)} + i\,\underbrace{g_{\mu\nu}\Big(\psi^\mu_{\mathrm{R}}\psi^\nu_{\mathrm{I}} + \psi^\mu_{\mathrm{I}}\psi^\nu_{\mathrm{R}}\Big)}_{\mathrm{Im}(\Psi)}, \tag{6}$$

which is generically complex. The complex phase information of the standard wave function is encoded in the interplay between the real and imaginary parts of the tensor components. Note that $T_{\mu\nu}$ in Einstein's equation is real-valued (as usual in GR), whereas ψ^μ and Ψ are complex.

Bohmian mapping. The complex scalar Ψ plays exactly the role of the standard pilot wave in BM:

- The Born-rule probability density is $|\Psi|^2 = \Psi\bar{\Psi} \geq 0$, where $\bar{\Psi}$ is the complex conjugate.
- The guidance equation (Section 2.9) uses $\mathrm{Im}(\nabla\Psi/\Psi)$, exactly as in standard BM, requiring Ψ to be complex-valued.
- The polar decomposition $\Psi = Re^{iS/\hbar}$ (with R, S real) recovers the standard Bohmian amplitude and phase.

We also consider a real scalar field $a \in \mathbb{R}$ responsible for mediating the relation between the wave function and spacetime.

Non-uniqueness of ψ^μ and gauge freedom. The map $\psi^\mu \mapsto \Psi = \psi_\nu\psi^\nu$ is many-to-one: any transformation $\psi^\mu \to R^\mu{}_\nu\psi^\nu$ preserving the bilinear form (e.g. complex rotations satisfying $R^\mu{}_\rho R^\nu{}_\sigma g_{\mu\nu} = g_{\rho\sigma}$) leaves Ψ unchanged. This non-uniqueness is a genuine ambiguity of the framework at the present stage and can be interpreted as a gauge freedom in ψ^μ. Fixing it requires either an additional dynamical equation for ψ^μ (provided by (31)) or a gauge condition. Note that different choices of ψ^μ satisfying (31) with the same initial data all yield the same Ψ via the proof in Section 2, so the physical observable Ψ and all predictions derived from it are unaffected by this gauge freedom. In the hydrogen atom calculation the alignment approximation (Assumption 7) implicitly fixes this freedom by aligning ψ^i with the radial direction. Establishing a systematic gauge principle for ψ^μ is an open problem identified for future work.

To be explicit about the status: the transformations $R^\mu{}_\nu$ that preserve the bilinear form $\psi_\nu\psi^\nu$ constitute a well-defined local symmetry group (a complexified rotation group acting on the tensorial index of ψ^μ), and the physical content of the theory — the scalar Ψ, the dynamics (12), the guidance, and every prediction—is invariant under it, exactly as the electromagnetic potential A_μ carries a gauge freedom that leaves the field strength and all observables invariant. What is *not* yet supplied is a canonical gauge-fixing condition (or a first-principles reason to prefer one) for ψ^μ itself; the dynamical equation, Equation (31), constrains, but does not uniquely fix, the representative. We therefore do not claim a complete gauge principle: we claim only that the residual freedom is a symmetry of the observable sector, and we record the construction of a canonical gauge condition as Limitation (L2).

2.2. Einstein's Equation

The equation is

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}; \tag{7}$$

as usual, only real-valued tensors appear here. Given a generic matter field ϕ, the stress–energy tensor is defined as [15]

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}, \qquad S_{\text{matter}} = \int \mathcal{L}_{\text{matter}}(\phi, \nabla_\mu\phi, g_{\mu\nu})\sqrt{-g}\, d^4x, \qquad g = \det(g_{\mu\nu}). \tag{8}$$

Discussion. Equation (7) is symbolically identical to (1), but in our framework it must be read in conjunction with all subsequent equations, whose components interact with it.

Role of $T_{\mu\nu}$. Throughout this paper, $T_{\mu\nu}$ is treated as a *prescribed background field*: it is determined by the classical matter content and is *not* sourced by the quantum wave tensor ψ^μ at this order. This avoids conceptual double-counting: $T_{\mu\nu}$ sources the Einstein equations (determining the metric) and *independently* enters the quantum evolution equation as the carrier of potential-energy information. The quantum matter described by ψ^μ is therefore a test field propagating on the background (M, g) already determined by the classical $T_{\mu\nu}$, and does not itself necessarily contribute to $T_{\mu\nu}$ in the present framework. A treatment including back-reaction is left for future work.

2.3. Consistency of the Matter–Gravity Coupling

A foundational objection to any coupling of quantum matter to classical gravity is that the Einstein equation (7) requires a single, definite stress–energy tensor on its right-hand side, whereas quantum matter generically exists in a superposition of distinct energy–momentum configurations. The standard *mean-field* response—sourcing curvature with the expectation value $\langle\psi|\hat{T}_{\mu\nu}|\psi\rangle$—is known to be problematic: the gravitational field would then track the mean of a (possibly macroscopic) superposition rather than the definite configuration that is in fact observed, in tension with the experiment of Page and Geilker [23].

Our framework is Bohmian, and this is precisely what dissolves the objection. In the de Broglie–Bohm theory the quantum system is described not only by the wave function but by an actual configuration Q—the primitive ontology, or *beable*—which is single-valued at all times and evolves deterministically under the guidance of Ψ [10,11,17]. The source of spacetime curvature is then the *actual* stress–energy tensor $T_{\mu\nu}[Q]$ built from this definite configuration, and not the expectation value $\langle\hat{T}_{\mu\nu}\rangle$ nor the wave function itself. Because Q is definite at every instant, the right-hand side of (7) is always a single, well-defined tensor even when Ψ is in a superposition. This is exactly the route to a consistent semi-classical coupling proposed by Struyve [24,25], and it is one of the principal reasons we adopt a Bohmian rather than an orthodox formulation.

We emphasise the scope of this resolution and its relation to the role of $T_{\mu\nu}$ stated above. In the present paper $T_{\mu\nu}$ enters as a prescribed background field and plays the role of the carrier of potential-energy information in the Adapted Schrödinger Equation (12); it is conceptually distinct from, and should not be conflated with, the matter's own stress–energy that would source curvature through back-reaction. The metrics used in the hydrogen atom and post-Newtonian computations (Sections 4 and 7) are external classical backgrounds, and the quantum particle does not source curvature at this order; the superposition-of-sources problem therefore does *not* arise in any computation performed here. The argument above establishes the complementary conceptual point: when the framework is extended to include back-reaction (Limitation (L6) below), the Bohmian ontology supplies a definite source $T_{\mu\nu}[Q]$ by construction, so the coupling to the classical Einstein equation remains well-posed.

2.4. Adapted Schrödinger's Equation

We consider

$$i\hbar\nabla_0\Psi = -\frac{\hbar^2}{2m}\nabla_j\nabla^j\Psi + aT_{\mu\nu}\psi^\mu\psi^\nu, \tag{9}$$

where m is the rest mass, ∇_0 is the time covariant derivative, and ∇_j the spatial covariant derivative. As noted above, $T_{\mu\nu}$ is real-valued throughout. The scalar a mediates the coupling between $T_{\mu\nu}$ and the wave function. The covariant Laplacian is

$$\nabla_j\nabla^j\Psi = \frac{1}{\sqrt{h}}\partial_j\Big(\sqrt{h}\,g^{jk}\partial_k\Psi\Big), \qquad h = \det\Big(g_{jk}\Big). \tag{10}$$

For complex-valued fields we extend covariant derivatives linearly: if $\Psi = \Psi_\mathrm{R} + i\Psi_\mathrm{I}$, then $\nabla_\mu\Psi := (\nabla_\mu\Psi_\mathrm{R}) + i(\nabla_\mu\Psi_\mathrm{I})$.

To make (9) fully covariant, we introduce u^μ, the unit future-directed tangent to the world-line of a reference observer (τ is proper time):

$$u^\mu = \frac{dx^\mu}{d\tau}, \qquad g_{\mu\nu}u^\mu u^\nu = -1. \tag{11}$$

The metric decomposes as $g^{\mu\nu} = -u^\mu u^\nu + h^{\mu\nu}$, where $h^{\mu\nu}$ is the spatial projector orthogonal to u^μ. Replacing $\nabla_0 \to u^\mu \nabla_\mu$ and $\nabla_j \nabla^j \to h^{\mu\nu}\nabla_\mu\nabla_\nu$, we obtain the fully covariant **Adapted Schrödinger Equation**:

$$\boxed{i\hbar u^\mu \nabla_\mu \Psi = -\frac{\hbar^2}{2m} h^{\mu\nu}\nabla_\mu\nabla_\nu\Psi + aT_{\mu\nu}\psi^\mu\psi^\nu.} \tag{12}$$

2.4.1. Dimensional Analysis of the Coupling Term

We verify dimensional consistency of $aT_{\mu\nu}\psi^\mu\psi^\nu$. In Equation (2), the potential term $V\psi$ has dimensions [energy] $\times$ [wave function], so the ratio $aT_{\mu\nu}\psi^\mu\psi^\nu/\Psi$ must have dimensions of energy.

The stress–energy tensor carries dimensions of energy per unit volume

$$[T_{\mu\nu}] = \frac{\mathrm{J}}{\mathrm{m}^3} \quad \text{(SI units)}, \tag{13}$$

as evident from T_{00} being energy density and from $E = \int_\Omega T_{00}\sqrt{-g}\, d^3x$. Therefore,

$$[a] \times \left[\frac{\mathrm{J}}{\mathrm{m}^3}\right] = \mathrm{J} \qquad \Longrightarrow \qquad [a] = \mathrm{m}^3 \quad \text{(volume)}. \tag{14}$$

The scalar field a must carry *dimensions of volume*: it converts the energy-density information in $T_{\mu\nu}$ into a potential energy by integrating over a characteristic volume associated with the particle. The choice $a = 4\pi r^3$ in Section 4 is dimensionally natural, with $4\pi r^3$ being of the same order as the volume of a ball of radius r.

2.4.2. Guiding Principles and Justification

Equation (12) is not an ad hoc postulate but the *unique minimal covariant lift* of Schrödinger's equation in our setting. It must reduce to (2) in the flat, non-relativistic limit and satisfy four requirements:

(P1) **Full spacetime covariance.** The equation must be tensorial and hold in any coordinate system. Partial derivatives must be replaced by covariant derivatives throughout.

(P2) **Covariant $3+1$ splitting via a preferred observer.** The Schrödinger equation distinguishes time from space. In a covariant setting this is encoded by the future-directed unit timelike vector field u^μ, inducing $g^{\mu\nu} = -u^\mu u^\nu + h^{\mu\nu}$. The operators $u^\mu\nabla_\mu$ and $h^{\mu\nu}\nabla_\mu\nabla_\nu$ are the *unique* covariant objects reproducing ∂_t and ∇^2 in the flat limit.

(P3) **Potential energy from the stress–energy tensor.** In GR all information about matter and energy is encoded in $T_{\mu\nu}$. Our framework demands that $V\psi$ be replaced by a coupling to $T_{\mu\nu}$ mediated by the real scalar a: the natural and minimal way to import gravitational and matter-field information into the quantum evolution.

(P4) **Full contraction of $T_{\mu\nu}$ by the wave tensor.** Since $\Psi = \psi_\nu\psi^\nu$ and $T_{\mu\nu}$ carries two free indices, a consistent coupling requires *both* indices to be contracted by the wave tensor, uniquely selecting $aT_{\mu\nu}\psi^\mu\psi^\nu$. Alternatives such as $T^\mu{}_\mu\Psi$ or $T_{\mu\nu}u^\mu\psi^\nu$ either fail to exploit the full tensorial structure of $T_{\mu\nu}$ or break the symmetry between its indices.

Given (P1)–(P4), the form of (12) is fully determined up to the specification of a: one must take $u^\mu \nabla_\mu$ as the time-evolution operator, $h^{\mu\nu}\nabla_\mu\nabla_\nu$ as the kinetic operator, and $aT_{\mu\nu}\psi^\mu\psi^\nu$ as the potential coupling. Equation (12) is therefore the unique equation of this structure that is covariant, reduces to (2) in the flat non-relativistic limit, and couples the wave tensor to the full stress–energy tensor via both its free indices.

2.5. The Scalar Field a: Phenomenological Status and Limitations

Phenomenological status. The scalar a is at present a *phenomenological* mediator: its value is not derived from a first-principles action or symmetry argument. Instead, a is fixed by matching to known physics in each regime. Deriving a from a variational or symmetry principle is identified as a central open problem. The present results should be understood accordingly: the framework is an exploratory proposal whose coupling function must be independently calibrated for each physical system, rather than a complete predictive theory.

The choice $a = 4\pi r^3$. For the hydrogen atom (Section 4), the choice $a = 4\pi r^3$ reflects the natural volumetric geometry of the spherical Coulomb field: $4\pi r^3$ is proportional to the volume of a ball of radius r (with the exact prefactor absorbed into the approximations of Section 4), and the dimensional analysis shows that a must carry units of volume. The choice is therefore the *simplest dimensionally correct scalar* built from the local spatial radius r consistent with spherical symmetry.

Stated constraints. Constraints established in this paper: (1) a must be real-valued with dimensions of volume; (2) for a spherically symmetric, non-relativistic, single-particle system, $a = 4\pi r^3$ recovers the correct Coulomb potential; (3) a is observer-relative in the same way as other quantities in the $3+1$ formalism. The value of a in multi-particle, strong-field settings, or those that are not spherically symmetric, is not established here and requires further study.

2.6. Local Equation for a Concrete Scalar Field a

We construct a covariant version of $a = 4\pi r^3$ for a one-particle system, with $r = \sqrt{h_{\mu\nu}x^\mu x^\nu}$—see Section 4 where the hydrogen atom is analysed for the motivation for this choice; the general idea is that a plays the role of a volume element, consistent with the dimensional analysis of the previous section. Fix a generic point p_0 in spacetime, and let $\gamma(\tau)$ be a time-like reference world-line with $p_0 = \gamma(\tau_0)$. Let $u^\mu(p_0) = \frac{dx^\mu}{d\tau}\big|_{\tau_0}$ be the unit four-velocity at p_0, satisfying $g_{\mu\nu}(p_0)u^\mu(p_0)u^\nu(p_0) = -1$. The exponential map at p_0 is

$$\exp_{p_0} : T_{p_0}M \to M, \qquad \exp_{p_0}(\zeta^\mu) = \gamma_{\zeta^\mu}(1), \tag{15}$$

where $\gamma_\zeta(\lambda)$ is the geodesic with $\gamma_\zeta(0) = p_0$, $\dot{\gamma}_\zeta(0) = \zeta \in T_{p_0}M$. For any p in the normal neighbourhood there is a unique *geodesic separation vector* $\xi(p) \in T_{p_0}M$ with $p = \exp_{p_0}(\xi(p))$ and $\|\xi\| = s_{\mathrm{geo}}(p_0, p)$, so $[\xi^\mu] =$ length. The spatial projector at p_0 is

$$h_{\mu\nu}(p_0) = g_{\mu\nu}(p_0) + u_\mu(p_0)u_\nu(p_0), \tag{16}$$

satisfying $h_{\mu\nu}u^\nu = 0$ and $h_\mu{}^\rho h_{\rho\nu} = h_{\mu\nu}$. The covariant scalar radius is

$$r(p) = \sqrt{h_{\mu\nu}(p_0)\,\xi^\mu(p)\,\xi^\nu(p)}. \tag{17}$$

Since $h_{\mu\nu}$ is a $(0,2)$-tensor and ξ^μ a $(1,0)$-tensor, both at p_0, their contraction is a scalar. One verifies directly that r is invariant under smooth coordinate changes: if $\tilde{x}^\alpha$ is another chart, $\tilde{h}_{\alpha\beta}\tilde{\xi}^\alpha\tilde{\xi}^\beta = h_{\rho\sigma}\xi^\rho\xi^\sigma$, so $\tilde{r}(p) = r(p)$.

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In the normal neighbourhood of p_0, the **Local Adapted Schrödinger Equation** is

$$\boxed{i\hbar u^\mu \nabla_\mu \Psi = -\frac{\hbar^2}{2m} h^{\mu\nu} \nabla_\mu \nabla_\nu \Psi + 4\pi r^3 T_{\mu\nu} \psi^\mu \psi^\nu.} \tag{18}$$

Discussion and Observer Dependence

Equation (12) is one of the most important equations of our paper. Its left-hand side involves the observer four-velocity u^μ, defining a $3+1$ foliation of spacetime entirely analogous to the ADM decomposition of GR [15]. Throughout this paper we work with a *fixed* smooth timelike unit vector field u^μ on an open region $\mathcal{U} \subseteq M$, selecting a preferred foliation $\{\Sigma_t\}$ into spacelike hypersurfaces of constant time. All equations involving u^μ, $h^{\mu\nu}$, and t are relative to this foliation.

Limitations. The scalars Ψ and a depend on the choice of observer as energy depends on the reference frame in special relativity. In particular, r defined in (17) is constructed relative to a fixed observer at p_0; a different observer would define a different r and hence a different a. The Local Adapted Schrödinger Equation (18) holds only in the normal neighbourhood of p_0 and should not be extrapolated beyond it. These are inherent limitations of embedding a non-relativistic quantum equation into a covariant framework, to be addressed in future work by specifying the observer field concretely (e.g. as the fluid four-velocity or as a Killing vector field).

Physical interpretation. The potential term $V\psi$ of (2) is replaced by $aT_{\mu\nu}\psi^\mu\psi^\nu$ in (12). The stress–energy tensor represents the density and flow of energy and momentum; the total energy in a spatial region Ω is $E = \int_\Omega T_{00}\sqrt{-g}\, d^3x$. The field *a extracts the potential energy* from $T_{\mu\nu}$ by contracting it with the wave tensor. Equation (12) is not necessarily linear in general, but becomes approximately linear with appropriate assumptions on a, as demonstrated in Section 4.

2.7. Consistency Checks

We provide three internal consistency checks within the stated scope.

2.7.1. Flat-Spacetime Limit

Set $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and $u^\mu = (1, 0, 0, 0)$. Covariant derivatives reduce to partial derivatives. Then,

$$i\hbar u^\mu \nabla_\mu \Psi = i\hbar \frac{\partial \Psi}{\partial t}, \tag{19}$$

$$-\frac{\hbar^2}{2m} h^{\mu\nu} \nabla_\mu \nabla_\nu \Psi = -\frac{\hbar^2}{2m} \nabla^2 \Psi. \tag{20}$$

The coupling term $aT_{\mu\nu}\psi^\mu\psi^\nu$ reduces (under the hydrogen atom approximations of Section 4) to $V\Psi$ with $V = -e^2/(4\pi\epsilon_0 r)$. Equation (12) thus reduces *exactly* to (2), confirming that the covariant framework is a genuine generalisation of standard BM.

2.7.2. Conservation Structure

Using $T_{\mu\nu}$ in both the Einstein equations and the quantum evolution equation creates no inconsistency with GR's conservation laws. The Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ together with (7) implies $\nabla_\mu T^{\mu\nu} = 0$; this conservation law holds for the background $T_{\mu\nu}$ independently of the quantum wave equation. Since ψ^μ is a test field that does not contribute to $T_{\mu\nu}$ at this order, the two uses of $T_{\mu\nu}$ are logically independent: one determines the geometry, the other governs quantum dynamics on that geometry.

2.7.3. Probability Normalisation and Equivariance

In the flat, non-relativistic limit, Equation (12) reduces to the standard Schrödinger equation, for which equivariance of $|\Psi|^2$ is a standard BM result [11]. The continuity equation

$$\frac{\partial|\Psi|^2}{\partial t} + \nabla\cdot\mathbf{J} = 0, \qquad \mathbf{J} = \frac{\hbar}{m}\,\mathrm{Im}(\bar{\Psi}\nabla\Psi), \tag{21}$$

ensures that $|\Psi|^2$ is preserved as the probability distribution. The normalisation condition (23) on each leaf Σ_t is therefore preserved in this limit by the standard BM argument. In the fully curved case, verifying preservation of (23) requires establishing a covariant continuity equation; this is an open problem flagged for future work.

To describe the point precisely, what is required in the curved case is a covariant current J^μ with $\nabla_\mu J^\mu = 0$, reducing to $J^\mu = (\hbar/m)\,\mathrm{Im}(\bar{\Psi}\,\nabla^\mu\Psi)$ in the flat limit, whose flux through the leaves Σ_t of the foliation defined by u^μ reproduces (23) on every leaf. Equivariance of the Born distribution $|\Psi|^2$ then follows provided this current is generated by the guidance velocity. We have established this only in the flat, non-relativistic limit, where (12) reduces to the standard Schrödinger equation and the result is the textbook one; on a general curved background, constructing such a conserved J^μ from (12) and proving equivariance remains open. We state this plainly rather than assume it (Limitation (L3)): the Born rule on curved spacetime is, at this stage, a well-motivated assumption supported by the flat-limit result, not a theorem of the framework.

2.8. Measurement

For a fixed $j \in \{1,2,3\}$ and a given observer,

$$\mathbb{P}_{c\le x^j\le d}(t) := \int_{\Sigma_t^{j;[c,d]}} |\Psi|^2\sqrt{h}\;d^3x, \tag{22}$$

where $\Sigma_t^{j;[c,d]}$ is the hypersurface of constant t with $c \le x^j \le d$. The normalisation condition is

$$\int_{\Sigma_t} |\Psi|^2\sqrt{h}\;d^3x = 1, \tag{23}$$

to hold on each leaf Σ_t of the fixed foliation. Different choices of observer field yield different foliations and hence different normalisation conditions; this is an inherent feature of embedding a non-relativistic probability interpretation into a covariant framework [15]. Equations (22) and (23) are the covariant translation of (4).

2.9. Adapted Guiding Equation

For a fixed $j \in \{1,2,3\}$,

$$\frac{dx^j}{d\tau} = \frac{\hbar}{m}\,g^{j\nu}\,\mathrm{Im}\left(\frac{\nabla_\nu\Psi}{\Psi}\right), \tag{24}$$

where τ is proper time, $d\tau = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu}$. We use proper time because it is the natural parameter for velocities in GR. The imaginary part $\mathrm{Im}(\nabla_\nu\Psi/\Psi)$ is well-defined because Ψ is complex-valued (Equation (6)), in direct analogy with the standard Bohmian guidance equation.

Remark on covariance. Equation (24) is written only for the spatial components $j \in \{1,2,3\}$, following the standard BM strategy of singling out space within the $3+1$ foliation defined by u^μ. A fully covariant four-velocity guidance equation of the form $u^\mu_{particle} = f^\mu(\Psi,\nabla\Psi)$ can in principle be constructed from the full four-gradient of Ψ; however, the timelike component would then reproduce the evolution equation, Equation (12), rather than provide independent information, as is familiar from the covariant formulation of BM in flat spacetime [12,13]. Developing a fully covariant guidance equation consistent with (12) is an open problem left for future work.

3. Spacetime and the Wave Function

In our formulation, spacetime curvature affects the evolution of Ψ via the covariant Laplacian and also enters directly through the interaction term $aT_{\mu\nu}\psi^{\mu}\psi^{\nu}$, linking the gravitational field with the quantum potential.

The geometric structure underlying the whole construction is the observer-relative $3+1$ split of spacetime, summarised in Figure 1: a timelike unit vector field u^{μ} foliates spacetime into spacelike slices Σ_t, and the metric decomposes as $g^{\mu\nu} = -u^{\mu}u^{\nu} + h^{\mu\nu}$. The Adapted Schrödinger Equation (12) is assembled from the time derivative $u^{\mu}\nabla_{\mu}$ and the spatial Laplacian $h^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$ defined by this split, so that the equation is fully covariant while retaining the observer-relative notion of time required by the Schrödinger structure. In Figure 1 you have a graphical representation.

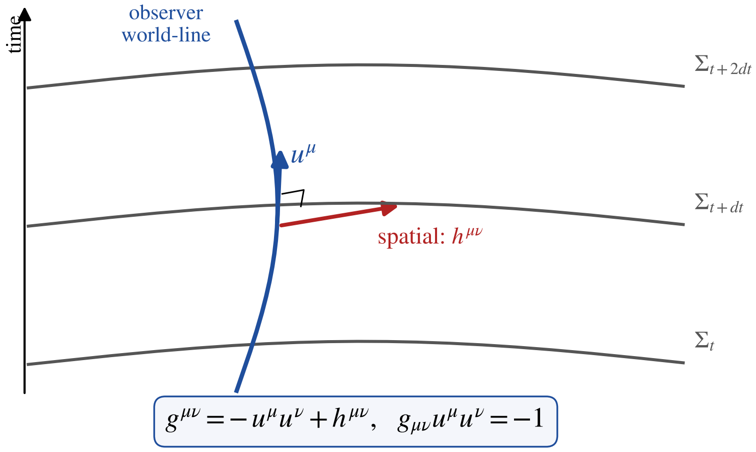

Figure 1. The observer-relative $3+1$ foliation underlying the framework. A future-directed timelike unit vector field u^{μ} is everywhere orthogonal to the spacelike slices Σ_t; the inverse metric splits as $g^{\mu\nu} = -u^{\mu}u^{\nu} + h^{\mu\nu}$, with $h^{\mu\nu}$ the spatial projector. The covariant time derivative $u^{\mu}\nabla_{\mu}$ and the spatial Laplacian $h^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$ entering (12) are defined relative to this split.

Combining (7) and (9) yields

$$\frac{ac^4}{8\pi G}\left(G_{\mu\nu}\psi^{\mu}\psi^{\nu} + \Lambda g_{\mu\nu}\psi^{\mu}\psi^{\nu}\right) = i\hbar\nabla_0\Psi + \frac{\hbar^2}{2m}\nabla_j\nabla^j\Psi. \tag{25}$$

Equation (25) expresses how the wave function interacts with Einstein's tensor; in a sense, the curvature of spacetime affects the wave function, but the latter may also produce effects on the former (this is an aspect that should be explored in the future).

Very naïvely, (9) and (25) tell the wave function how to *vibrate* in accordance with the deformation of spacetime. In the context of Equation (18), Equation (25) becomes the covariant equation

$$\frac{ac^4}{8\pi G}\left(G_{\mu\nu}\psi^{\mu}\psi^{\nu} + \Lambda g_{\mu\nu}\psi^{\mu}\psi^{\nu}\right) = i\hbar u^{\mu}\nabla_{\mu}\Psi + \frac{\hbar^2}{2m}h^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\Psi. \tag{26}$$

4. The Hydrogen Atom

We now show that Equation (18) approximates the known hydrogen atom. Since we focus on one-particle systems, we ignore the proton's internal structure and account only for its electromagnetic influence on the electron. Because we use the flat metric, the results are approximate.

Remark: Purely leptonic atoms. The neglect of nuclear structure is an approximation for hydrogen, where the proton has finite size and internal (hadronic) structure. It becomes *exact*, however, for purely leptonic hydrogen-like atoms such as muonium ($M = \mu^+ e^-$) and positronium ($e^+ e^-$), whose constituents are structureless point leptons with no finite-size or nuclear corrections; this is precisely why such atoms are regarded as clean laboratories for bound-state QED [26,27]. The framework applies to muonium unchanged, with the proton replaced by the antimuon μ^+ and the energy levels rescaled by the reduced mass $\mu_{\text{red}} = m_e m_\mu / (m_e + m_\mu) \approx 0.9952\, m_e$ (using $m_\mu \approx 206.77\, m_e$). The gravitational predictions of Section 7 carry over directly: the redshift $\delta\nu/\nu = \Phi/c^2$ is universal and system-independent, so it is identical for muonium, which—being free of the nuclear-structure uncertainties that limit hydrogen—is in fact an especially clean system in which to confront the spectral prediction. For the wave-packet-spreading prediction the *fractional* effect $3|\Phi|/c^2$ is likewise mass-independent, while the absolute lag $\delta\tau \propto m$ is smaller for the lighter muonium than for the heavy alkali atoms of Table 4; muonium therefore favours the spectroscopic test and heavy atoms the interferometric one. Current muonium 1*S*–2*S* and fine-structure spectroscopy already reach the precision frontier of bound-state QED [27], making purely leptonic atoms a natural target for future tests of the framework.

We separate clearly what is *assumed* from what is *derived*.

4.1. Assumptions for the Approximation

Assumption 1 (Flat spacetime)**.** *$g_{\mu\nu} \approx \eta_{\mu\nu}$, so $u^\mu = (1, 0, 0, 0)$ and covariant derivatives reduce to partial derivatives. This is valid when gravitational effects on the electron are negligible relative to electromagnetic ones.*

Assumption 2 (Maxwell stress–energy tensor)**.** *The electromagnetic field of the proton is $T_{\mu\nu} = \frac{1}{\mu_0}(F_{\mu\lambda} F_\nu{}^\lambda - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta})$.*

Assumption 3 (No magnetic field)**.** *We set $\mathbf{B} = 0$, retaining only the electrostatic Coulomb field, so $F_{0i} = -F_{i0} = E_i$ with $E_i = \frac{e x_i}{4\pi\epsilon_0 r^3}$.*

Assumption 4 (Radial approximation)**.** *Following standard simplifications [28,29], we approximate by a purely radial Coulomb field ($x_i \approx r$) and neglect the isotropic pressure term:*

$$T_{\mu\nu} \approx -\epsilon_0 E_i E_j. \tag{27}$$

Retaining the pressure term would add $O(\epsilon_0 |\mathbf{E}|^2 \delta_{ij})$ corrections to T_{ij}, subleading in the non-relativistic limit.

Assumption 5 (Dominantly spatial wave tensor)**.** *We set $\psi^0 \approx 0$. The temporal component contributes corrections of order $(v/c)^2$.*

Assumption 6 (Coupling $a = 4\pi r^3$)**.** *As motivated by the dimensional analysis of Section 2.5, with $r = \sqrt{x_i x^i}$ in flat Cartesian coordinates.*

Assumption 7 (Alignment approximation)**.** *We use $x_i x_j \psi^i \psi^j \approx r^2 \Psi$, valid when the wave tensor is approximately radially aligned. Off-diagonal terms produce angular corrections of order $(1 - \cos^2\theta)$, where θ is the angle between ψ^i and x^i.*

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4.2. Derivation

The standard hydrogen potential is $V(r) = -e^2/(4\pi\epsilon_0 r)$, where e is the electron charge and ϵ_0 the permittivity of free space. Under Assumptions 1 and 2, Equation (18) becomes

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + 4\pi r^3 T_{\mu\nu}\psi^\mu\psi^\nu. \tag{28}$$

In flat space with Cartesian coordinates and rest observer $u^\alpha = (1,0,0,0)$, the geodesic separation vector is $\xi^\alpha(p) = \Delta x^\alpha$, and for equal-time separations the covariant radius reduces to $r = \sqrt{(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2}$. For simplicity we set $r = \sqrt{x_i x^i}$. Under Assumptions 3–7:

$$\begin{aligned} aT_{\mu\nu}\psi^\mu\psi^\nu &\approx -4\pi r^3\epsilon_0 E_i E_j \psi^i\psi^j \quad \text{[Assumption 4]} \\ &= -\frac{4\pi r^3\epsilon_0 e^2 x_i x_j}{(4\pi\epsilon_0)^2 r^6}\psi^i\psi^j \\ &\approx -\frac{e^2 x_i x_j}{4\pi\epsilon_0 r^3}\psi^i\psi^j \\ &\approx -\frac{e^2 r^2}{4\pi\epsilon_0 r^3}\Psi = -\frac{e^2}{4\pi\epsilon_0 r}\Psi. \quad \text{[Assumption 7]} \end{aligned} \tag{29}$$

Combining all terms gives the standard hydrogen atom equation:

$$\boxed{i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi - \frac{e^2}{4\pi\epsilon_0 r}\Psi.} \tag{30}$$

This derivation is not circular: $a = 4\pi r^3$ is chosen by dimensional reasoning (volume units), and the Coulomb potential emerges as a consequence.

5. Dynamical Equation for ψ^μ

We have proposed Equation (12) for Ψ. We now propose the following **dynamical equation** for the complex field $\psi^\mu \in \mathbb{C}^4$ and show that it entails (12). This equation is motivated mathematically rather than physically; a Lagrangian for Ψ is provided in Section 6, and a direct Lagrangian for ψ^μ yielding this equation is provided in Section 6.3 (this derivation requires $\psi^\mu \neq 0$):

$$i\hbar u^\alpha\nabla_\alpha\psi^\mu = -\frac{\hbar^2}{2m}h^{\rho\sigma}\nabla_\rho\nabla_\sigma\psi^\mu + \frac{a}{2}T^\mu{}_\nu\psi^\nu - \frac{\hbar^2}{2m}\frac{\mathcal{G}[\psi]}{\|\psi^\mu\|^2}\psi^\mu, \tag{31}$$

where $\mathcal{G}[\psi] := h^{\rho\sigma}(\nabla_\rho\psi_\nu)(\nabla_\sigma\psi^\nu)$. Equation (31) is tensorial; $\|\psi^\mu\|^2 := \psi_\mu\psi^\mu = \Psi$ denotes the complex bilinear norm (not the Hermitian norm $\psi_\mu\bar{\psi}^\mu$), so that the contraction $\psi_\nu \cdot (\mathcal{G}[\psi]/\|\psi^\mu\|^2)\psi^\nu = \mathcal{G}[\psi]\cdot\Psi/\Psi = \mathcal{G}[\psi]$ cancels exactly in the derivation below. We exclude $\psi^\mu = 0$.

5.1. Useful Preliminaries

Since $\psi_\mu = g_{\mu\nu}\psi^\nu$ and $\nabla_\alpha g_{\mu\nu} = 0$, we have $\nabla_\alpha\psi_\mu = g_{\mu\nu}\nabla_\alpha\psi^\nu$. This gives the identity

$$\nabla_\alpha\Psi = \nabla_\alpha(\psi_\mu\psi^\mu) = \psi^\mu g_{\mu\nu}\nabla_\alpha\psi^\nu + \psi_\mu\nabla_\alpha\psi^\mu = 2\psi_\mu\nabla_\alpha\psi^\mu, \tag{32}$$

from which we derive

$$h^{\rho\sigma}\nabla_\rho\nabla_\sigma\Psi = 2h^{\rho\sigma}(\nabla_\rho\psi_\mu)(\nabla_\sigma\psi^\mu) + 2\psi_\mu h^{\rho\sigma}\nabla_\rho\nabla_\sigma\psi^\mu. \tag{33}$$

5.2. Derivation

Starting from (32): $i\hbar u^\alpha \nabla_\alpha \Psi = 2\psi_\mu (i\hbar u^\alpha \nabla_\alpha \psi^\mu)$. Substituting (31),

$$\begin{aligned} i\hbar u^\alpha \nabla_\alpha \Psi &= 2\psi_\mu \left(-\frac{\hbar^2}{2m} h^{\rho\sigma} \nabla_\rho \nabla_\sigma \psi^\mu + \frac{a}{2} T^\mu{}_\nu \psi^\nu - \frac{\hbar^2}{2m} \frac{\mathcal{G}[\psi]}{\|\psi^\mu\|^2} \psi^\mu \right) \\ &= -\frac{\hbar^2}{m} \psi_\mu h^{\rho\sigma} \nabla_\rho \nabla_\sigma \psi^\mu + a T_{\mu\nu} \psi^\mu \psi^\nu - \frac{\hbar^2}{m} \mathcal{G}[\psi]. \end{aligned} \tag{34}$$

Using (33) to replace $\psi_\mu h^{\rho\sigma} \nabla_\rho \nabla_\sigma \psi^\mu = \frac{1}{2} h^{\rho\sigma} \nabla_\rho \nabla_\sigma \Psi - h^{\rho\sigma} (\nabla_\rho \psi_\mu)(\nabla_\sigma \psi^\mu)$, and noting that the $\frac{\hbar^2}{m} \mathcal{G}[\psi]$ terms cancel,

$$i\hbar u^\alpha \nabla_\alpha \Psi = -\frac{\hbar^2}{2m} h^{\rho\sigma} \nabla_\rho \nabla_\sigma \Psi + a T_{\mu\nu} \psi^\mu \psi^\nu, \tag{35}$$

which is exactly Equation (12).

6. Lagrangian Formulation: Direct Derivation of the Adapted Schrödinger Equation

The central equation of the paper is the Adapted Schrödinger Equation (12) for the complex scalar Ψ. The dynamical equation, Equation (31), for ψ^μ was constructed *mathematically* as a tool from which (12) can be derived by contraction. A more direct and physically transparent route is to build the action principle around Ψ itself, treating ψ^μ as a prescribed background field—exactly as the Coulomb potential $V(r)$ is a prescribed background in the standard Schrödinger Lagrangian. This section shows that (12) follows from a natural, *real-valued* Lagrangian density for Ψ via the Euler–Lagrange equations, with no nonlinear correction terms required.

6.1. The Action for the Scalar Wave Function

Fix a smooth future-directed timelike unit vector field u^μ on an open region $\mathcal{U} \subseteq M$ with $\nabla_\mu u^\mu = 0$, and let $h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ be the corresponding spatial projector. We propose the action

$$S[\Psi, \bar{\Psi}] = \int_{\mathcal{U}} \mathcal{L}_\Psi \sqrt{-g}\, d^4x, \tag{36}$$

with Lagrangian density

$$\boxed{\mathcal{L}_\Psi = \underbrace{\frac{i\hbar}{2}(\bar{\Psi}\, u^\alpha \nabla_\alpha \Psi - (u^\alpha \nabla_\alpha \bar{\Psi})\, \Psi)}_{\text{(I) time-kinetic}} - \underbrace{\frac{\hbar^2}{2m} h^{\mu\nu} (\nabla_\mu \bar{\Psi})(\nabla_\nu \Psi)}_{\text{(II) spatial-kinetic}} - \underbrace{a\, T_{\mu\nu}(\psi^\mu \psi^\nu \bar{\Psi} + \bar{\psi}^\mu \bar{\psi}^\nu \Psi)}_{\text{(III) potential coupling}}.} \tag{37}$$

Here, the overline denotes complex conjugation; $a, T_{\mu\nu}$ are the scalar field and stress–energy tensor of the paper; and $\psi^\mu, \bar{\psi}^\mu$ enter term (III) as a *prescribed background* (analogous to $V(r)$ in the standard Schrödinger Lagrangian).

Reality. $\mathcal{L}_\Psi$ is real-valued. Term (II) is manifestly real since $h^{\mu\nu}$ is real and symmetric. Term (III) is real because $T_{\mu\nu}, a$ are real and $\overline{a T_{\mu\nu} \psi^\mu \psi^\nu \bar{\Psi}} = a T_{\mu\nu} \bar{\psi}^\mu \bar{\psi}^\nu \Psi$, so the two summands in (III) are complex conjugates of each other. Term (I) is directly real: its complex conjugate is

$$\overline{\frac{i\hbar}{2}(\bar{\Psi} u^\alpha \nabla_\alpha \Psi - (u^\alpha \nabla_\alpha \bar{\Psi})\Psi)} = \frac{-i\hbar}{2}(\Psi u^\alpha \nabla_\alpha \bar{\Psi} - (u^\alpha \nabla_\alpha \Psi)\bar{\Psi}) = \frac{i\hbar}{2}(\bar{\Psi} u^\alpha \nabla_\alpha \Psi - (u^\alpha \nabla_\alpha \bar{\Psi})\Psi), \tag{38}$$

which equals term (I) itself. Hence, $\mathcal{L}_\Psi \in \mathbb{R}$.

Remark on the symmetrised potential. The most natural analogue of the standard potential term $-V|\Psi|^2$ in the Lagrangian would be $-a T_{\mu\nu} \psi^\mu \psi^\nu |\Psi|^2 / \Psi$, but this is non-polynomial in Ψ. The symmetrised form $-aT_{\mu\nu}(\psi^\mu\psi^\nu\bar{\Psi}+\bar{\psi}^\mu\bar{\psi}^\nu\Psi)$ is the simplest polynomial (bilinear in $(\Psi,\bar{\Psi})$, with ψ^μ as background) that is real and yields $-aT_{\mu\nu}\psi^\mu\psi^\nu$ under variation with respect to $\bar{\Psi}$. It differs from the standard potential by the additional term $-aT_{\mu\nu}\bar{\psi}^\mu\bar{\psi}^\nu\Psi$, which is a prescribed real function of spacetime and does not contribute to the equation of motion for Ψ (it does not depend on $\bar{\Psi}$). Its sole role is to ensure the reality of the action.

Physical interpretation. Term (I) is the covariant lift of the standard Schrödinger time-derivative, with ∂_t replaced by the observer-covariant derivative $u^\alpha\nabla_\alpha$. Term (II) is the covariant spatial kinetic energy density, with the flat Laplacian replaced by $h^{\mu\nu}\nabla_\mu\nabla_\nu$. Term (III) is the potential-energy coupling: the symmetrised combination ensures reality of the Lagrangian while producing $aT_{\mu\nu}\psi^\mu\psi^\nu$ in the equation of motion, exactly as required by (12).

6.2. Euler–Lagrange Equations and Recovery of the Adapted Schrödinger Equation

We vary the action (36) with respect to $\bar{\Psi}$ (treating Ψ and $\bar{\Psi}$ as independent). The Euler–Lagrange equation on the curved background reads

$$\frac{\partial\mathcal{L}_\Psi}{\partial\bar{\Psi}}-\nabla_\alpha\frac{\partial\mathcal{L}_\Psi}{\partial(\nabla_\alpha\bar{\Psi})}=0. \tag{39}$$

We compute each term.

6.2.1. Time-Kinetic Term (I)

Split term (I) into $\mathcal{L}_a=\frac{i\hbar}{2}\bar{\Psi}u^\alpha\nabla_\alpha\Psi$ (containing undifferentiated $\bar{\Psi}$) and $\mathcal{L}_b=-\frac{i\hbar}{2}(u^\alpha\nabla_\alpha\bar{\Psi})\Psi$ (containing $\nabla_\alpha\bar{\Psi}$):

$$\frac{\partial\mathcal{L}_a}{\partial\bar{\Psi}}=\frac{i\hbar}{2}u^\alpha\nabla_\alpha\Psi, \tag{40}$$

$$\frac{\partial\mathcal{L}_b}{\partial(\nabla_\alpha\bar{\Psi})}=-\frac{i\hbar}{2}u^\alpha\Psi,\quad \nabla_\alpha\left(-\frac{i\hbar}{2}u^\alpha\Psi\right)=-\frac{i\hbar}{2}(\nabla_\alpha u^\alpha)\Psi-\frac{i\hbar}{2}u^\alpha\nabla_\alpha\Psi. \tag{41}$$

The combined contribution of term (I) to (39) is

$$\frac{i\hbar}{2}u^\alpha\nabla_\alpha\Psi-\left(-\frac{i\hbar}{2}(\nabla_\alpha u^\alpha)\Psi-\frac{i\hbar}{2}u^\alpha\nabla_\alpha\Psi\right)=i\hbar u^\alpha\nabla_\alpha\Psi+\frac{i\hbar}{2}(\nabla_\alpha u^\alpha)\Psi. \tag{42}$$

6.2.2. Spatial-Kinetic Term (II)

No undifferentiated $\bar{\Psi}$ appears, so $\frac{\partial\mathcal{L}_{\rm II}}{\partial\bar{\Psi}}=0$. For the derivative part,

$$\frac{\partial\mathcal{L}_{\rm II}}{\partial(\nabla_\alpha\bar{\Psi})}=-\frac{\hbar^2}{2m}h^{\alpha\nu}\nabla_\nu\Psi. \tag{43}$$

From $h^{\mu\nu}=g^{\mu\nu}+u^\mu u^\nu$ and $\nabla_\alpha g^{\alpha\nu}=0$,

$$\nabla_\alpha h^{\alpha\nu}=(\nabla_\alpha u^\alpha)u^\nu+u^\alpha\nabla_\alpha u^\nu. \tag{44}$$

The second term $u^\alpha\nabla_\alpha u^\nu$ is the four-acceleration of the observer. For a *divergence-free observer* ($\nabla_\alpha u^\alpha=0$), equation (44) gives $\nabla_\alpha h^{\alpha\nu}=u^\alpha\nabla_\alpha u^\nu$. For an observer that is also *geodesic* ($u^\alpha\nabla_\alpha u^\nu=0$), this vanishes exactly. For a static observer in a weak gravitational field (non-geodesic), the term $u^\alpha(\nabla_\alpha u^\nu)\nabla_\nu\Psi$ is of relative order $ga_0/c^2\approx 6\times 10^{-27}$ compared to the Laplacian, where g is the surface gravity and a_0 the Bohr radius; it is negligible for all atomic-physics applications. We therefore impose

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Assumption about four-acceleration: The observer field u^μ satisfies $\nabla_\mu u^\mu = 0$, and the four-acceleration correction $u^\alpha(\nabla_\alpha u^\nu)\nabla_\nu\Psi$ is negligible relative to $h^{\mu\nu}\nabla_\mu\nabla_\nu\Psi$.

Under Assumption about four-acceleration, $\nabla_\alpha(h^{\alpha\nu}\nabla_\nu\Psi) \approx h^{\mu\nu}\nabla_\mu\nabla_\nu\Psi$ (the four-acceleration correction is negligible), and the contribution of term (II) to (39) is

$$0 - \nabla_\alpha\left(-\frac{\hbar^2}{2m}h^{\alpha\nu}\nabla_\nu\Psi\right) \approx \frac{\hbar^2}{2m}h^{\mu\nu}\nabla_\mu\nabla_\nu\Psi. \tag{45}$$

6.2.3. Potential Term (III)

Write (III) as $-aT_{\mu\nu}\psi^\mu\psi^\nu\bar{\Psi} - aT_{\mu\nu}\bar{\psi}^\mu\bar{\psi}^\nu\Psi$. The first part depends linearly on $\bar{\Psi}$ (giving $-aT_{\mu\nu}\psi^\mu\psi^\nu$ upon differentiation); the second part depends on Ψ only (contributing zero to $\partial\mathcal{L}/\partial\bar{\Psi}$). Hence,

$$\frac{\partial\mathcal{L}_{\mathrm{III}}}{\partial\bar{\Psi}} = -aT_{\mu\nu}\psi^\mu\psi^\nu. \tag{46}$$

No derivative of $\bar{\Psi}$ appears in (III), so the second EL term is zero.

6.2.4. Full Equation

Collecting (42), (45), (46) and setting the sum to zero, using $\nabla_\alpha u^\alpha = 0$,

$$i\hbar u^\alpha\nabla_\alpha\Psi + \frac{\hbar^2}{2m}h^{\mu\nu}\nabla_\mu\nabla_\nu\Psi - aT_{\mu\nu}\psi^\mu\psi^\nu \approx 0, \tag{47}$$

where the $\approx$ reflects Assumption 6.2.2 (the four-acceleration correction is of relative order $ga_0/c^2 \approx 6\times10^{-27}$ and is negligible for all atomic-physics applications). Rearranging,

$$\boxed{i\hbar u^\alpha\nabla_\alpha\Psi = -\frac{\hbar^2}{2m}h^{\mu\nu}\nabla_\mu\nabla_\nu\Psi + aT_{\mu\nu}\psi^\mu\psi^\nu.} \tag{48}$$

This is the Adapted Schrödinger Equation (12), recovered to within the stated approximation (exact for geodesic observers; negligible correction for static observers in weak gravity).

6.3. Lagrangian for the Wave Tensor ψ^μ

For completeness, we now construct an action directly for the complex field ψ^μ, resolving the open problem flagged at the end of Section 2. We treat ψ^μ and $\bar{\psi}^\mu$ as independent complex fields and propose

$$\boxed{\mathcal{L}_\psi = \underbrace{\frac{i\hbar}{2}\left(\bar{\psi}_\mu u^\alpha\nabla_\alpha\psi^\mu - (u^\alpha\nabla_\alpha\bar{\psi}_\mu)\psi^\mu\right)}_{\text{(I) time-kinetic}} - \underbrace{\frac{\hbar^2}{2m}h^{\rho\sigma}(\nabla_\rho\bar{\psi}_\mu)(\nabla_\sigma\psi^\mu)}_{\text{(II) spatial-kinetic}} - \underbrace{\frac{a}{2}T_{\mu\nu}\psi^\mu\bar{\psi}^\nu}_{\text{(III) potential}}.} \tag{49}$$

The structure mirrors (37), with Ψ replaced by the vector components ψ^μ, lowered-index contractions providing the scalar norm, and the potential carrying a factor $\frac{1}{2}$ (which compensates for the factor of 2 that arises upon contraction with ψ_ν). Reality of $\mathcal{L}_\psi$ follows by the same argument as for $\mathcal{L}_\Psi$: term (I) equals its own complex conjugate; term (II) is manifestly real; term (III) satisfies $\overline{T_{\mu\nu}\psi^\mu\bar{\psi}^\nu} = T_{\mu\nu}\bar{\psi}^\mu\psi^\nu = T_{\mu\nu}\psi^\nu\bar{\psi}^\mu$ (by symmetry of $T_{\mu\nu}$), which equals term (III) itself.

6.3.1. Euler–Lagrange Equations

Varying (49) with respect to $\bar{\psi}_\nu$ and applying the Euler–Lagrange formula (following the same steps as Section 6), one finds that under Assumption 6.2.2,

$$i\hbar u^\alpha \nabla_\alpha \psi^\nu \approx -\frac{\hbar^2}{2m} h^{\rho\sigma} \nabla_\rho \nabla_\sigma \psi^\nu + \frac{a}{2} T^\nu{}_\mu \psi^\mu. \tag{50}$$

This is Equation (31) *without* the nonlinear correction term $-\frac{\hbar^2}{2m}\frac{\mathcal{G}[\psi]}{\|\psi^\mu\|^2}\psi^\nu$. The key steps are identical to those in Section 6: term (I) contributes $i\hbar u^\alpha \nabla_\alpha \psi^\nu$ (plus a term proportional to $\nabla_\alpha u^\alpha = 0$); term (II) contributes $\frac{\hbar^2}{2m} h^{\rho\sigma} \nabla_\rho \nabla_\sigma \psi^\nu$ (with the four-acceleration correction negligible by Assumption 6.2.2); and term (III) contributes $-\frac{a}{2} T^\nu{}_\mu \psi^\mu$ (since $\frac{\partial}{\partial \bar{\psi}_\nu}(T_{\mu\lambda}\psi^\mu \bar{\psi}^\lambda) = T_\mu{}^\nu \psi^\mu$).

6.3.2. The Nonlinear Correction and the Extended Action

Contracting (50) with ψ_ν and using identities (32)–(33), one finds

$$i\hbar u^\alpha \nabla_\alpha \Psi = -\frac{\hbar^2}{2m} h^{\mu\nu} \nabla_\mu \nabla_\nu \Psi + a T_{\mu\nu} \psi^\mu \psi^\nu + \frac{\hbar^2}{m} \mathcal{G}[\psi], \tag{51}$$

which reproduces (12) if and only if $\mathcal{G}[\psi] = 0$. The residual $\frac{\hbar^2}{m}\mathcal{G}[\psi]$ is cancelled precisely by the nonlinear term of (31). To reproduce (31) in full from a variational principle, we define the **extended action**

$$S_{ext}[\psi^\mu, \bar{\psi}^\mu] = \int_{\mathcal{U}} (\mathcal{L}_\psi + \mathcal{L}_{nl}) \sqrt{-g}\, d^4x, \tag{52}$$

where

$$\mathcal{L}_{nl} = -\frac{\hbar^2}{2m} \frac{h^{\rho\sigma}(\nabla_\rho \psi_\lambda)(\nabla_\sigma \psi^\lambda)}{\psi_\mu \psi^\mu} \bar{\psi}_\nu \psi^\nu. \tag{53}$$

Since $h^{\rho\sigma}(\nabla_\rho \psi_\lambda)(\nabla_\sigma \psi^\lambda) = \mathcal{G}[\psi]$ and $\psi_\mu \psi^\mu = \Psi$ are independent of $\bar{\psi}^\nu$, varying $\mathcal{L}_{nl}$ with respect to $\bar{\psi}_\nu$ is immediate: $\frac{\partial \mathcal{L}_{nl}}{\partial \bar{\psi}_\nu} = -\frac{\hbar^2}{2m}\frac{\mathcal{G}[\psi]}{\Psi}\psi^\nu$, with no derivative term contributing. This is exactly the nonlinear term in (31). Therefore the Euler–Lagrange equations of S_{ext} reproduce (31) in full, and, by the derivation of Section 2, entail (12).

Remark on reality. Unlike $\mathcal{L}_\psi$, the term $\mathcal{L}_{nl}$ is *not* real-valued in general: since $\mathcal{G}[\psi]$ and Ψ are both complex (as ψ^μ is complex-valued), the ratio $\mathcal{G}[\psi]/\Psi$ is generically complex, and $\overline{\mathcal{L}_{nl}} \neq \mathcal{L}_{nl}$. The extended action S_{ext} is therefore best understood as a formal variational device that correctly reproduces (31) via the Euler–Lagrange equations, rather than as a physical action with all standard Lagrangian properties. For applications where a real action is required, the Ψ-based action $S[\Psi, \bar{\Psi}]$ of Section 6 is the appropriate primary object.

6.4. Comparison of the Two Lagrangian Approaches

Both actions—$S[\Psi, \bar{\Psi}]$ of Section 6 and $S_{ext}[\psi^\mu, \bar{\psi}^\mu]$ of Section 6.3—ultimately yield the Adapted Schrödinger Equation (12). They differ in three respects.

Directness. The Ψ-action yields (12) directly, in a single variational step with no auxiliary residual. The ψ^μ-action yields (31) first, and (12) only follows by a subsequent contraction. The Ψ-approach therefore provides the more economical derivation of the equation that governs observable probabilities.

Linearity. The Ψ-action is polynomial and bilinear in $(\Psi, \bar{\Psi})$; the extended ψ^μ-action requires the non-polynomial term $\mathcal{L}_{nl}$ (rational in $\psi^\mu/\sqrt{\Psi}$), which is well-defined only where $\Psi \neq 0$. Such non-polynomial Lagrangians appear in nonlinear sigma models [1], and their variational calculus is valid on the domain $\Psi \neq 0$ already required by the Bohmian guidance equation; but they introduce additional mathematical structure absent from the Ψ-approach.

Physical interpretation. In the Ψ-action, the coupling $aT_{\mu\nu}\psi^\mu\psi^\nu$ is a prescribed background potential, exactly as $V(r)$ is in the standard Schrödinger Lagrangian: it is fixed by the spacetime content and does not require ψ^μ to be a dynamical variable. In the ψ^μ-action, ψ^μ is the dynamical field, and the coupling involves $T_{\mu\nu}$ through the potential term. The Ψ-action therefore makes the connection to the familiar Schrödinger structure most transparent.

For these three reasons the Ψ-action of Section 6 is the primary variational formulation of the framework. The ψ^μ-action of Section 6.3 provides the complete variational foundation for the dynamical equation, Equation (31), resolving the open problem stated at the end of Section 2.

7. Experimental Prediction: Gravitational Modification of Hydrogen Spectral Lines

We now derive a concrete, quantitative experimental prediction of our framework. The Adapted Schrödinger Equation (12) is covariant and must reduce to a modified Schrödinger equation in the presence of a gravitational field. We work at first post-Newtonian order, compute the resulting perturbation to the hydrogen energy levels analytically, and obtain a precise prediction for the gravitational shift of spectral lines. The calculation is fully explicit and requires no free parameters beyond the metric.

What "prediction" means here. We stress that these predictions test the *dynamics* of (12), not the Bohmian interpretation. We agree that the Bohmian interpretation, on its own, makes no predictions beyond those of the wave equation it accompanies (Section 1.3); the content below follows from the modified kinetic structure of (12) and would equally be obtained by any interpretation of the same equation. Two distinct cases must be separated. The gravitational redshift of spectral lines (Section 7.4) is a *consistency check*: the result $\delta\nu/\nu = \Phi/c^2$ is the universal Einstein value common to any covariant quantum theory, and we present it to demonstrate that the framework respects the equivalence principle, not as a discriminating test. The slowing of free wave-packet spreading (Section 7.6) is, by contrast, a *genuinely novel* prediction: it is absent in standard non-relativistic quantum mechanics and depends on the framework-specific kinetic coefficient 3, so it would distinguish the present dynamics from a minimally coupled curved-spacetime reduction. All predicted deviations are of order $|\Phi|/c^2{\sim}10^{-9}$ at the Earth's surface, below present experimental precision; the framework is therefore consistent with all existing data and deviates only where measurements do not yet reach.

7.1. Setup: Post-Newtonian Metric and Decomposition

We place a hydrogen atom at a fixed height above a spherically symmetric gravitating body of mass M (e.g. the Earth). We work in units where $c = 1$ for the metric signature, so that the normalisation is $g_{\mu\nu}u^\mu u^\nu = -1$ and coordinate time t is measured in units of length (equivalently, $x^0 = ct$ in SI). In the weak-field, slow-motion (post-Newtonian) regime, the spacetime metric in isotropic coordinates takes the form

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(1+\frac{2\Phi}{c^2}\right)dt^2 + \left(1-\frac{2\Phi}{c^2}\right)\delta_{ij}dx^i dx^j + O\left(\frac{\Phi^2}{c^4}\right), \tag{54}$$

where $\Phi = \Phi(\mathbf{R}) < 0$ is the Newtonian gravitational potential at the atom's location $\mathbf{R}$ (e.g. $\Phi = -GM/R$ for a point mass) and $|\Phi/c^2| \ll 1$. The factor $1/c^2$ in Φ/c^2 is retained explicitly to mark the post-Newtonian small parameter, while the flat-space limit gives $g_{\mu\nu}^{flat} = \mathrm{diag}(-1,1,1,1)$ consistently with the normalisation above.

A static observer at $\mathbf{R}$ has four-velocity

$$u^\mu = \frac{1}{\sqrt{1+2\Phi/c^2}}(1,0,0,0) \approx \left(1-\frac{\Phi}{c^2},\,0,\,0,\,0\right) + O\!\left(\frac{\Phi^2}{c^4}\right), \tag{55}$$

satisfying $g_{\mu\nu}u^\mu u^\nu = -1$ and $\nabla_\mu u^\mu = 0$ exactly for any static observer in a static spacetime (as verified by $\nabla_\mu u^\mu = \frac{1}{\sqrt{-g}}\partial_0(\sqrt{-g}u^0) = 0$ since everything is time-independent and $u^i = 0$), consistently with Assumption 6.2.2. The spatial projector is

$$h^{ij} = \left(1-\frac{2\Phi}{c^2}\right)^{-1}\delta^{ij} \approx \left(1+\frac{2\Phi}{c^2}\right)\delta^{ij}. \tag{56}$$

7.2. Expanded Adapted Schrödinger Equation

We substitute the metric (54) into (12) and expand to first order in Φ/c^2. The time-evolution operator becomes

$$i\hbar\, u^\mu \nabla_\mu \Psi \approx i\hbar\left(1-\frac{\Phi}{c^2}\right)\frac{\partial\Psi}{\partial t} + O\!\left(\frac{\Phi^2}{c^4}\right), \tag{57}$$

where the Christoffel-symbol terms vanish identically in the time derivative of a scalar ($\nabla_\mu\Psi = \partial_\mu\Psi$). The spatial kinetic operator requires more care: the Christoffel correction to $h^{ij}\nabla_i\nabla_j\Psi$ is $h^{ij}\Gamma^k_{ij}\partial_k\Psi$, which has magnitude of order $|\nabla\Phi|/c^2 \cdot |\Psi|/a_0$ (the Christoffel symbols scale as $|\nabla\Phi|/c^2$ and the wave function gradient as $|\Psi|/a_0$). Comparing to $\nabla^2\Psi \sim |\Psi|/a_0^2$, the dimensionless relative magnitude is $a_0|\nabla\Phi|/c^2 \approx a_0 g/c^2 \approx 6\times 10^{-27}$, where $g = 9.807\ \mathrm{m\,s^{-2}}$; this is entirely negligible. The spatial kinetic term becomes

$$-\frac{\hbar^2}{2m}h^{ij}\nabla_i\nabla_j\Psi \approx -\frac{\hbar^2}{2m}\left(1+\frac{2\Phi}{c^2}\right)\nabla^2\Psi + O\!\left(\frac{\Phi^2}{c^4}\right). \tag{58}$$

Under the same approximations as in Section 4 (flat-space Coulomb field, Φ uniform over the atom), the coupling term remains $aT_{\mu\nu}\psi^\mu\psi^\nu \approx -\frac{e^2}{4\pi\epsilon_0 r}\Psi$. The gravitational dressing of $T_{\mu\nu}$ in the PN metric contributes a correction of order $(\Phi/c^2)\times aT_{\mu\nu}\psi^\mu\psi^\nu$, which is of order $(\Phi/c^2)^2$ relative to the leading term and is dropped consistently with our first-order expansion. Collecting all terms, Equation (12) becomes

$$i\hbar\left(1-\frac{\Phi}{c^2}\right)\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\left(1+\frac{2\Phi}{c^2}\right)\nabla^2\Psi - \frac{e^2}{4\pi\epsilon_0 r}\Psi. \tag{59}$$

Dividing both sides by $(1-\Phi/c^2)$ and using $(1-\Phi/c^2)^{-1} \approx 1+\Phi/c^2$:

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2m}\left(1+\frac{2\Phi}{c^2}\right)\left(1+\frac{\Phi}{c^2}\right)\nabla^2\Psi - \left(1+\frac{\Phi}{c^2}\right)\frac{e^2}{4\pi\epsilon_0 r}\Psi + O\!\left(\frac{\Phi^2}{c^4}\right) \\ &\approx \underbrace{\left(-\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\epsilon_0 r}\right)}_{\hat{H}_0}\Psi + \underbrace{\left(-\frac{3\Phi}{c^2}\frac{\hbar^2}{2m}\nabla^2 - \frac{\Phi}{c^2}\frac{e^2}{4\pi\epsilon_0 r}\right)}_{\hat{H}'}\Psi. \end{aligned} \tag{60}$$

7.3. Perturbation Hamiltonian and Energy Corrections

The unperturbed Hamiltonian is $\hat{H}_0 = \hat{T} + \hat{V}$ where $\hat{T} = -\frac{\hbar^2}{2m}\nabla^2$ and $\hat{V} = -\frac{e^2}{4\pi\epsilon_0 r}$. The gravitational perturbation reads

$$\hat{H}' = \frac{3\Phi}{c^2}\hat{T} + \frac{\Phi}{c^2}\hat{V} = \frac{\Phi}{c^2}(3\hat{T}+\hat{V}). \tag{61}$$

The coefficients 3 (on $\hat{T}$) and 1 (on $\hat{V}$) reflect the distinct origins of the corrections: the factor 3 arises from combining the spatial metric enhancement (factor 2) with the temporal metric correction (factor 1) after dividing, while the factor 1 on $\hat{V}$ comes solely from the time-dilation correction. This specific combination is a direct prediction of our covariant framework.

First-order perturbation theory gives the energy correction of the n-th hydrogen eigenstate $|n, l, m\rangle$:

$$E_n^{(1)} = \langle n, l, m|\hat{H}'|n, l, m\rangle = \frac{\Phi}{c^2}\big(3\langle\hat{T}\rangle_n + \langle\hat{V}\rangle_n\big). \tag{62}$$

We apply the quantum-mechanical virial theorem for the Coulomb potential ($\hat{V} \propto r^{-1}$) to the *unperturbed* eigenstates $|n, l, m\rangle$ of $\hat{H}_0$:

$$\langle\hat{T}\rangle_n = -E_n^{flat}, \qquad \langle\hat{V}\rangle_n = 2E_n^{flat}, \tag{63}$$

where $E_n^{flat} = -\frac{m_e e^4}{2(4\pi\epsilon_0)^2\hbar^2} \cdot \frac{1}{n^2} \equiv -\frac{E_1}{n^2}$ and $E_1 = 13.606$ eV. These expectation values are exact for the unperturbed hydrogen eigenstates and are used within first-order perturbation theory to evaluate $E_n^{(1)}$. Substituting,

$$E_n^{(1)} = \frac{\Phi}{c^2}\Big(3(-E_n^{flat}) + 2E_n^{flat}\Big) = \frac{\Phi}{c^2}\Big(-E_n^{flat}\Big) = \frac{\Phi}{c^2}\,\frac{E_1}{n^2}. \tag{64}$$

Since $\Phi < 0$ and $E_1 > 0$, we have $E_n^{(1)} < 0$: *the gravitational field makes every bound energy level more negative (more tightly bound)*. The corrected energy levels are

$$\boxed{E_n(\mathbf{R}) = E_n^{flat}\left(1 - \frac{\Phi(\mathbf{R})}{c^2}\right) = -\frac{E_1}{n^2}\left(1 - \frac{\Phi(\mathbf{R})}{c^2}\right).} \tag{65}$$

Table 1 lists the corrections for the lowest levels at Earth's surface ($\Phi_\oplus = -GM_\oplus/R_\oplus$, $\Phi_\oplus/c^2 = -6.954 \times 10^{-10}$). Please see what we described in Table 1.

Table 1. First-order gravitational corrections to hydrogen energy levels at Earth's surface ($\Phi_\oplus/c^2 = -6.954 \times 10^{-10}$, using $GM_\oplus = 3.986 \times 10^{14}$ m^{3}s^{-2}, $R_\oplus = 6.378 \times 10^6$ m, $c = 2.998 \times 10^8$ m s^{-1}). Positive $E_n^{(1)}$ denotes a downward shift (more tightly bound).

n	E_n^{flat} (eV)	$\lvert E_n^{(1)}\rvert$ (neV)	Relative Shift $\lvert E_n^{(1)}/E_n^{flat}\rvert$
1	−13.606	9.46	6.95×10^{-10}
2	−3.401	2.36	6.95×10^{-10}
3	−1.512	1.05	6.95×10^{-10}
4	−0.850	0.59	6.95×10^{-10}

7.4. Predicted Gravitational Shift of Spectral Lines

Before the detailed derivation, Figure 2 summarises the mechanism. A hydrogen transition emitted at coordinate-time frequency ν_0 deep in the gravitational potential is observed at a lower frequency ν_{obs} by a distant observer, the net fractional shift being the Einstein value $\delta\nu/\nu = \Phi/c^2$.

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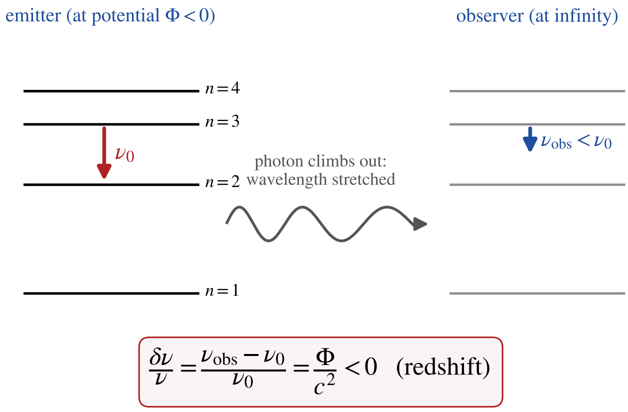

Figure 2. Schematic of the predicted gravitational shift of hydrogen spectral lines. A transition emitted at coordinate-time frequency ν_0 where the potential is $\Phi < 0$ is received at infinity with lower frequency $\nu_{obs} < \nu_0$; the net fractional shift is $\delta\nu/\nu = \Phi/c^2 < 0$ (redshift), recovering the universal Einstein value derived in Steps 1–3 below. Level spacing is schematic.

7.4.1. Step 1: Coordinate-Time Transition Frequency

The energy levels (65) give the coordinate-time transition energy for the transition $n \to m$ $(n > m)$:

$$\begin{aligned} h\nu^{coord}_{n\to m}(\mathbf{R}) &= E_n(\mathbf{R}) - E_m(\mathbf{R}) \\ &= E^{flat}_n(1 - \tfrac{\Phi}{c^2}) - E^{flat}_m(1 - \tfrac{\Phi}{c^2}) \\ &= \left(E^{flat}_n - E^{flat}_m\right)\left(1 - \frac{\Phi}{c^2}\right) \\ &= h\nu^{flat}_{n\to m}\left(1 - \frac{\Phi(\mathbf{R})}{c^2}\right). \end{aligned} \tag{66}$$

Since $\Phi < 0$, we have $1 - \Phi/c^2 > 1$, so $\nu^{coord} > \nu^{flat}$: atoms in a gravitational potential oscillate at a *higher* coordinate-time frequency than in flat space. This coordinate-time blueshift has fractional magnitude $-\Phi/c^2 = |\Phi|/c^2$.

7.4.2. Step 2: Photon Propagation to a Distant Observer

In a static spacetime the covariant photon energy $k_0 = g_{\mu\nu}k^\mu\xi^\nu$ (where $\xi^\mu = (1,0,0,0)$ is the timelike Killing vector) is conserved along every null geodesic. A photon emitted at $\mathbf{R}$ with coordinate-time frequency ν^{coord} has $k^0 = \nu^{coord}$ and hence conserved covariant energy

$$k_0 = g_{00}(\mathbf{R})\,k^0 \approx -\left(1 + \frac{2\Phi}{c^2}\right)\nu^{coord}. \tag{67}$$

A static observer at spatial infinity (where $g^\infty_{00} = -1$, $u^\mu_\infty = (1,0,0,0)$) measures the locally received frequency

$$\nu^{obs} = -k_\mu u^\mu_\infty = -k_0 = \left(1 + \frac{2\Phi(\mathbf{R})}{c^2}\right)\nu^{coord}. \tag{68}$$

Substituting (66),

$$\nu^{obs}_{n\to m}(\mathbf{R}) = \left(1 + \frac{2\Phi}{c^2}\right)\nu^{flat}\left(1 - \frac{\Phi}{c^2}\right) \approx \nu^{flat}\left(1 + \frac{\Phi(\mathbf{R})}{c^2}\right). \tag{69}$$

7.4.3. Step 3: Observed Fractional Shift

The fractional shift of the frequency received at infinity is

$$\boxed{\frac{\delta\nu^{obs}_{n\to m}}{\nu^{flat}_{n\to m}} = \frac{\Phi(\mathbf{R})}{c^2}.} \tag{70}$$

Since $\Phi < 0$, Equation (70) is a *redshift*: radiation emitted by atoms in a gravitational potential reaches a distant observer at lower frequency. The result is *independent of the quantum numbers n and m*: every hydrogen spectral line undergoes the same fractional gravitational redshift Φ/c^2. This is the universal Einstein gravitational redshift, recovered here as a two-step consequence: (i) the coordinate-time transition frequency is blueshifted by $|\Phi|/c^2$ (the gravitational field tightens the bound states), and (ii) the photon loses energy climbing out of the potential well, producing an exactly compensating (and then some) redshift of $2|\Phi|/c^2$, for a net observed redshift of $|\Phi|/c^2$. The specific coefficients $(3, 1)$ in perturbation (61) are what produce step (i) through the virial theorem; a different covariant formulation would give different coefficients and in general a different (non-universal) result.

Explicit predictions for principal lines. We give numerical predictions for an atom at Earth's surface ($\Phi_\oplus/c^2 = -6.954 \times 10^{-10}$). The absolute observed frequency shift is

$$\delta\nu^{obs}_{n\to m} = \frac{\Phi(\mathbf{R})}{c^2}\,\nu^{flat}_{n\to m}. \tag{71}$$

Table 2 lists the predicted shifts for the principal hydrogen series.

Table 2. Predicted gravitational frequency shifts of hydrogen spectral lines at Earth's surface ($\Phi_\oplus/c^2 = -6.954 \times 10^{-10}$). A negative $\delta\nu$ denotes a redshift. Flat-space frequencies are standard spectroscopic values [15,16]; all $|\delta\nu/\nu^{flat}| = 6.95 \times 10^{-10}$.

Series	Transition	ν^{flat} (THz)	$\delta\nu$ (kHz)	$\lvert\delta\nu/\nu^{flat}\rvert$
Lyman-α	$2 \to 1$	2466	−1713	6.95×10^{-10}
Lyman-β	$3 \to 1$	2923	−2031	6.95×10^{-10}
Balmer-α	$3 \to 2$	457	− 318	6.95×10^{-10}
Balmer-β	$4 \to 2$	617	− 429	6.95×10^{-10}
Paschen-α	$4 \to 3$	160	− 111	6.95×10^{-10}

7.5. Height-Dependent Differential Shift and Experimental Test

The prediction becomes most directly testable by comparing atomic transition frequencies at two different heights $h_1 < h_2$ above Earth's surface. For Earth's gravitational potential $\Phi(h) = -GM_\oplus/(R_\oplus + h)$, the difference in gravitational potential to first order is

$$\Delta\Phi = \Phi(h_2) - \Phi(h_1) \approx \frac{GM_\oplus}{R_\oplus^2}(h_2 - h_1) = g_\oplus\,\Delta h, \quad g_\oplus = 9.807\ \mathrm{m\,s^{-2}}. \tag{72}$$

Our framework predicts a fractional frequency difference between the atom at height h_2 (lower gravitational potential) and the atom at h_1 (higher potential, closer to Earth) of

$$\frac{\nu(h_2) - \nu(h_1)}{\nu^{flat}} = \frac{\Delta\Phi}{c^2} = \frac{g_\oplus\,\Delta h}{c^2} \approx 1.091 \times 10^{-16} \cdot \Delta h\ [\mathrm{m}]. \tag{73}$$

Table 3 shows the predicted differential frequency shift for the Lyman-α and Balmer-α transitions for several representative height differences Δh.

Table 3. Predicted differential gravitational frequency shift of hydrogen spectral lines between two heights separated by Δh in Earth's gravitational field ($g_\oplus = 9.807\ \mathrm{m\,s^{-2}}$, $c = 2.998 \times 10^8\ \mathrm{m\,s^{-1}}$, $g_\oplus/c^2 = 1.091 \times 10^{-16}\ \mathrm{m^{-1}}$). $\delta\nu = \nu(h_2) - \nu(h_1) > 0$ (upper atom blueshifted relative to lower).

Δh	$\Delta\Phi/c^2$	$\delta\nu_{\mathrm{Ly}\alpha}$ (kHz)	$\delta\nu_{\mathrm{Ba}\alpha}$ (kHz)	Measurable?
10 m	1.09×10^{-15}	2.69×10^{-3}	4.98×10^{-4}	Near threshold
1 km	1.09×10^{-13}	0.269	0.0499	Yes (optical clocks)
100 km	1.09×10^{-11}	26.9	4.98	Yes (H masers)
400 km	4.36×10^{-11}	107.6	19.9	Yes (ISS orbit)
36,000 km	3.93×10^{-9}	9 692	1 793	Yes (GPS orbit)

Physical mechanism. The prediction (73) emerges from the specific coefficient structure of the framework. The perturbing Hamiltonian (61) contains two distinct contributions: a kinetic correction $(3\Phi/c^2)\hat{T}$ from the modified spatial metric (factor $h^{ij} \approx 1 + 2\Phi/c^2$) and the time-dilation correction $(\Phi/c^2)\hat{V}$ from the modified u^μ. These two contributions combine via the virial theorem according to $3\langle\hat{T}\rangle_n + \langle\hat{V}\rangle_n = 3(-E_n^{flat}) + 2E_n^{flat} = -E_n^{flat}$, producing the universal ratio Φ/c^2 in (70). The result is not put in by hand: the coefficients 3 and 1 in (61) are determined entirely by the weak-field expansion of the PN metric, and the virial theorem for the Coulomb potential is what transforms them into the universal gravitational redshift.

Comparison with experiment. The Einstein gravitational redshift has been confirmed by numerous experiments, including the original Pound–Rebka nuclear resonance experiment (accuracy $\sim$1%) [30], the Gravity Probe A hydrogen-maser rocket experiment (accuracy $\sim 7 \times 10^{-5}$) [31], and optical atomic clock comparisons at the 10^{-16} level [32]. Our prediction (73) is consistent with all these observations. Conversely, any disagreement with the measured Φ/c^2 scaling would falsify the framework at the level of the discrepancy, providing a sharp experimental constraint on the post-Newtonian coefficients in the Adapted Schrödinger Equation.

Nature of this prediction. The gravitational redshift is primarily a *consistency check*: any covariant wave equation on a curved background that reduces to standard QM in flat spacetime reproduces it. The non-trivial content here is (i) that the specific coefficients $(3, 1)$ in (61), arising from the particular $3 + 1$ structure of (12), combine via the virial theorem to give *exactly* Φ/c^2 (rather than some other linear combination), which confirms the Einstein equivalence principle as an internal consequence of the framework; and (ii) that the prediction is falsifiable, giving specific numerical values for all hydrogen series.

7.6. Second Prediction: Gravitational Slowing of Free Wave-Packet Spreading

The gravitational redshift prediction above holds for bound-state transitions and is common to any covariant quantum theory. We now derive a *genuinely novel* prediction: a gravitationally induced modification of the *free* quantum wave-packet spreading rate, which is qualitatively absent in standard quantum mechanics.

7.6.1. Free Particle in a Gravitational Field

For a free quantum particle ($aT_{\mu\nu}\psi^\mu\psi^\nu = 0$), the Adapted Schrödinger Equation (12) in the post-Newtonian metric becomes, after dividing by $(1 - \Phi/c^2)$ as in Section 7,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\left(1 + 3\frac{\Phi}{c^2}\right)\nabla^2\Psi + O\left(\frac{\Phi^2}{c^4}\right). \tag{74}$$

This is the Schrödinger equation for a free particle with an *effective mass*

$$m_{\text{eff}}(\mathbf{R}) := \frac{m}{1 + 3\Phi(\mathbf{R})/c^2} \approx m\left(1 - \frac{3\Phi(\mathbf{R})}{c^2}\right). \tag{75}$$

Since $\Phi < 0$ at any gravitational source, $m_{\text{eff}} > m$: the gravitational field increases the effective inertial mass of the quantum wave function.

Standard QM prediction. In the standard non-relativistic treatment of a free particle in a gravitational field, gravity enters the Schrödinger equation as a Newtonian potential $m\Phi$:

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + m\Phi\right)\Psi. \tag{76}$$

Since Φ is spatially constant on atomic scales, the gravitational term adds only a global phase $e^{-im\Phi t/\hbar}$ and leaves the *kinetic* structure of the equation unchanged. Standard non-relativistic QM therefore predicts **zero correction** to the wave-packet spreading rate: the spreading time is τ_0 regardless of Φ. This is a consequence of Galilean symmetry and breaks down only when general-relativistic effects are included.

7.6.2. Quantum Spreading Time and Its Gravitational Correction

For a Gaussian wave packet of initial width σ_0, the width at time t in flat space is $\sigma(t)^2 = \sigma_0^2(1 + t^2/\tau_1^2)$ where $\tau_1 = 2m\sigma_0^2/\hbar$. The *characteristic spreading timescale* is

$$\tau_0 := \frac{m\sigma_0^2}{\hbar} = \tfrac{1}{2}\tau_1, \tag{77}$$

the time at which the quadratic spreading term equals the initial variance. (At $t = \tau_0$, $\sigma(\tau_0)^2 = \frac{5}{4}\sigma_0^2$, a $\sim$12% increase. The width doubles ($\sigma = 2\sigma_0$) at $t \approx 2\sqrt{3}\,\tau_0 \approx 3.46\,\tau_0$.) In our framework, Equation (74) shows that the effective diffusion parameter is $\hbar/(2m_{\text{eff}})$, giving

$$\tau(\mathbf{R}) = \frac{m_{\text{eff}}(\mathbf{R})\,\sigma_0^2}{\hbar} = \tau_0\left(1 - \frac{3\Phi(\mathbf{R})}{c^2}\right). \tag{78}$$

The fractional correction to the spreading time is

$$\boxed{\frac{\tau(\mathbf{R}) - \tau_0}{\tau_0} = -\frac{3\Phi(\mathbf{R})}{c^2} = \frac{3|\Phi(\mathbf{R})|}{c^2} > 0.} \tag{79}$$

Wave packets spread *more slowly* in a gravitational potential by a fractional amount $3|\Phi|/c^2$. The factor 3 arises from the same $(3, 1)$ coefficient structure in (61) that governs the spectral line shifts. This factor would be *different* in an alternative covariant formulation with different kinetic coefficients, making (79) a genuine discriminator between competing frameworks.

7.6.3. Numerical Predictions

At Earth's surface ($|\Phi_\oplus|/c^2 = 6.954 \times 10^{-10}$):

$$\left.\frac{\delta\tau}{\tau_0}\right|_\oplus = 3 \times 6.954 \times 10^{-10} = 2.086 \times 10^{-9}. \tag{80}$$

Table 4 gives the absolute correction to the spreading time for several initial localisations, using rubidium-87 ($m = 1.443 \times 10^{-25}$ kg):

Table 4. Predicted gravitational correction to the free wave-packet spreading time for Rb-87 at Earth's surface. Standard QM predicts $\delta\tau = 0$ in all cases.

σ_0	τ_0	$\delta\tau = 3\lvert\Phi_\oplus\rvert/c^2 \cdot \tau_0$	Required Precision
1 µm	1.37 ms	2.86 ps	$\sim 10^{-9}$
10 µm	137 ms	0.286 ns	$\sim 10^{-9}$
100 µm	13.7 s	28.6 ns	$\sim 10^{-9}$
1 mm	22.9 min	2.86 µs	$\sim 10^{-9}$

The required relative precision of $\sim 10^{-9}$ in the spreading timescale is challenging but not unreasonable. Long-lived matter-wave packets of ultra-cold atoms in optical lattices and magnetic traps have been maintained with coherence times exceeding several seconds, and atom-interferometric phase measurements have reached sensitivities well below 10^{-9} in fractional quantities [4]. A dedicated experiment would require careful isolation from stray fields, vibrations, and thermal decoherence, and is therefore proposed as a future long-term test rather than an immediate measurement.Figure 3 summarizes what we have described.

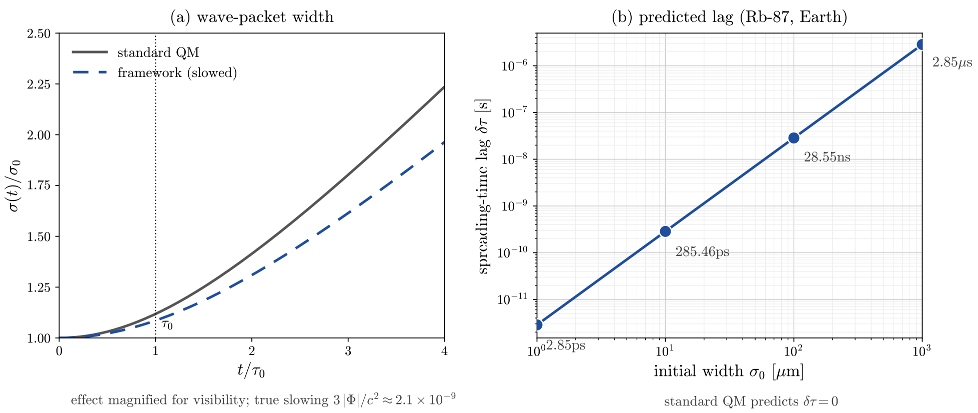

Figure 3. Predicted gravitational slowing of free wave-packet spreading. (**a**) Width $\sigma(t)/\sigma_0$ versus t/τ_0: Standard quantum mechanics (solid) versus the present framework (dashed), which spreads more slowly by the fractional amount $3|\Phi|/c^2$ [Equation (79)]. The effect is magnified for visibility; its true value at Earth's surface is $\approx 2.1 \times 10^{-9}$. (**b**) Absolute spreading-time lag $\delta\tau = 3|\Phi_\oplus|/c^2\, \tau_0$ versus initial width σ_0 for rubidium-87 (log–log; the data of Table 4). Standard quantum mechanics predicts $\delta\tau = 0$ in all cases.

7.6.4. Altitude Dependence and Experimental Strategy

Since $|\Phi| \propto 1/R$, the correction varies with altitude. The atom at the *lower* altitude (stronger gravity, larger m_{eff}) has the longer spreading time. For heights $h_1 < h_2$ with $\Delta h = h_2 - h_1 > 0$, the difference is

$$\frac{\tau(h_1) - \tau(h_2)}{\tau_0} = \frac{3\big(\Phi(h_2) - \Phi(h_1)\big)}{c^2} = \frac{3g_\oplus\,\Delta h}{c^2} \approx 3.273 \times 10^{-16} \cdot \Delta h\ [\text{m}]. \tag{81}$$

The right-hand side is positive: the lower atom spreads more slowly. For the ISS orbit ($\Delta h = 400$ km): $(\tau_\oplus - \tau_{ISS})/\tau_0 = 1.31 \times 10^{-10}$, i.e. the surface atom's spreading time exceeds that of the ISS atom by a fraction 1.31×10^{-10}. For a spreading time of $\tau_0 = 14$ s (corresponding to $\sigma_0 \approx 100$ µm for Rb-87), this gives $\tau_\oplus - \tau_{ISS} = 1.83$ ns—a difference accessible with precision atomic interferometry.

Distinction from standard quantum mechanics. This prediction is qualitatively different from what standard QM predicts in a gravitational field. In standard QM, a constant potential adds only a phase and leaves the spreading rate unchanged. Our framework predicts a *non-zero* slowing of the spreading rate, with the specific fractional correction $3|\Phi|/c^2$. The factor 3 (rather than 1 or 2 that might be expected from dimensional analysis) is a direct signature of the covariant $3+1$ structure of Equation (12). Measuring this factor would constitute a non-trivial test of the framework beyond merely confirming the equivalence principle.

Independence of the spreading prediction from a. Unlike the spectral line shift (which rests on the hydrogen energy levels and hence on the specific choice $a = 4\pi r^3$), the wave-packet spreading prediction (79) is derived for a *free* particle where $aT_{\mu\nu}\psi^\mu\psi^\nu = 0$. It therefore holds for any choice of a and any matter content: the coefficient 3 in $m_{\text{eff}} = m(1 - 3\Phi/c^2)$ comes solely from the post-Newtonian expansion of the kinetic operator $h^{\mu\nu}\nabla_\mu\nabla_\nu$ in (12) and is independent of the potential coupling. This makes (79) a genuine *a priori* prediction of the framework.

Comparison with minimal coupling to curved spacetime. In the standard approach of minimally coupling a non-relativistic particle to gravity [4], the leading-order curved-spacetime Schrödinger equation takes the form $i\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}(1 + c_1\Phi/c^2)\nabla^2\Psi + \ldots$ where the coefficient c_1 depends on the specific coupling scheme and conventions. Our framework gives $c_1 = 3$ through the covariant $3+1$ decomposition, which weights the spatial metric correction $(+2)$ and the temporal metric correction $(+1)$ additively. A different covariant equation with a different $3+1$ structure would in general give a different c_1. The measurement of this coefficient would therefore discriminate between alternative covariant extensions of quantum mechanics, going beyond the single number Φ/c^2 produced by the gravitational redshift.

8. Summary, Conclusions, and Outlook

We proposed an exploratory framework for a Bohmian model of quantum matter on a classical curved spacetime background. To be precise about scope: the gravitational sector is governed by the classical Einstein field equations throughout; *no quantisation of spacetime, no full back-reaction programme, and no resolution of the quantum gravity problem* are claimed.

The central contributions are as follows. We encode the complex quantum wave function as $\Psi = \psi_\nu\psi^\nu \in \mathbb{C}$, where ψ^μ is a complex-valued tensor, essential for Bohmian guidance dynamics and Born-rule probability. We derive the covariant Adapted Schrödinger Equation (12) as the unique minimal covariant lift of the standard equation satisfying four guiding principles (P1)–(P4). We show that the mediator a must carry dimensions of volume, with $a = 4\pi r^3$ the natural choice for spherically symmetric single-particle systems. We treat $T_{\mu\nu}$ as a prescribed background field, avoiding double-counting with the Einstein equation. We provide three consistency checks (flat-space limit, conservation structure, probability normalisation). Under seven explicit approximations the framework reproduces the hydrogen atom Schrödinger Equation with Coulomb potential. We conclude with the dynamical equation, Equation (31), for ψ^μ, from which (12) is derived.

Section 6 provides the primary variational foundation by constructing a real-valued action for Ψ directly, from which the Euler–Lagrange equations yield (12) in a single step under the geometric condition $\nabla_\mu u^\mu = 0$, with ψ^μ appearing as a prescribed background potential. Section 6.3 resolves the open problem from Section 2: the action with Lagrangian (49) yields the linear part of (31) via the Euler–Lagrange equations; the extended action with the additional non-polynomial term $\mathcal{L}_{nl}$ yields (31) in full, by a variational step whose simplicity follows from $\mathcal{G}[\psi]$ and Ψ being independent of $\bar{\psi}^\mu$. Section 7 derives two predictions. The first is a two-step consistency check: the perturbation Hamiltonian $\hat{H}' = \frac{\Phi}{c^2}(3\hat{T} + \hat{V})$, via the virial theorem, gives a coordinate-time blueshift of $|\Phi|/c^2$ in atomic transition frequencies; photon propagation (conservation of the covariant energy k_0 along null geodesics in the static metric) then yields the net Einstein gravitational redshift $\delta\nu^{obs}/\nu = \Phi/c^2$ at

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infinity. Quantitative predictions are given in Tables 1–3. The second prediction is novel: the free-particle Adapted Schrödinger Equation in the PN metric predicts that wave packets spread more slowly by the fractional amount $3|\Phi|/c^2$, absent in standard non-relativistic QM where a constant gravitational potential adds only a global phase. The factor 3 (rather than 1 that would arise from a minimal coupling) is a direct signature of the covariant $3+1$ kinetic structure. Numerical predictions are given in Table 4, with the ISS-to-ground comparison predicting a correction of order 1 ns for 100 µm-wide wave packets.

Future directions include: (1) understanding a and ψ^μ by studying further concrete systems and confronting the framework with experimental data; (2) generalising to multi-particle systems with spin, where the non-locality of BM plays a central role; (3) deriving a from symmetry or variational arguments, and clarifying conservation laws from the Lagrangian structure of Section 6; and (4) establishing a covariant continuity equation, verifying equivariance under the full curved dynamics of (12), and testing the wave-packet spreading prediction (79) with precision cold-atom experiments.

Open Problems and Limitations

We collect here the principal open problems and limitations of the present framework, to set clear expectations and guide future work.

(L1) Determination of a. The scalar field a is at present phenomenological, fixed by matching to known physics (Coulomb potential for hydrogen). Deriving a from a first-principles symmetry or variational principle is the central open problem. Without it, the framework is not fully predictive beyond calibrated systems.

(L2) Gauge freedom in ψ^μ. The decomposition $\Psi = \psi_\nu \psi^\nu$ is not unique: there are infinitely many ψ^μ giving the same Ψ. Equation (31) partially fixes this freedom (different choices of ψ^μ satisfying (dyn) must all give the same Ψ via the proof in Section 2), but a complete gauge principle for ψ^μ has not been established.

(L3) Born rule on curved spacetime. Equivariance of $|\Psi|^2$ (i.e., preservation of the Born-rule probability distribution by the joint dynamics of particles and wave function) has only been verified in the flat-space limit. Proving equivariance under (12) on a general curved background requires establishing a covariant continuity equation, which in turn requires controlling the flow of $|\Psi|^2$ in the foliation defined by u^μ. This is an essential open problem for the consistency of the Bohmian interpretation.

(L4) Covariant guidance equation. The guiding equation, Equation (24), gives only the spatial velocity components. A fully covariant four-vector guidance equation, required for a genuinely relativistic Bohmian formulation, remains to be constructed consistently with (12).

(L5) Multi-particle and field-theoretic extension. The present paper treats a single spinless particle. Extension to multi-particle systems (where the non-locality of BM is central and the wave function lives on configuration space rather than spacetime) and to field theory are major open directions.

(L6) Back-reaction and the source of curvature. Throughout, $T_{\mu\nu}$ is a prescribed background field and the quantum particle is a test field that does not source the metric. A full back-reaction programme, in which quantum matter sources the classical Einstein equation, is left for future work. As discussed in Section 2.3, the Bohmian ontology is well suited to this extension: the source would be the actual stress–energy $T_{\mu\nu}[Q]$ of the definite configuration Q, which is single-valued even when Ψ is in superposition [24,25], so the right-hand side of the Einstein equation remains well-defined. Working this out explicitly—including the consistency of the joint matter–metric dynamics—is a central open problem.

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Abbreviations

The following abbreviations are used in this manuscript:

GR	General Relativity
QM	Quantum Mechanics
BM	Bohmian Mechanics
ADM	Arnowitt–Deser–Misner

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