

Article

On the Total π Chromatic Number of Petal, Wing, and Tentacle Graph Families

Arati Salunke ¹ , Archana Bhangre ^{2,*}  and H. R. Bhapkar ³ 
¹ Department of Mathematics, School of Engineering and Sciences, MIT Art, Design and Technology University, Pune 412201, India; aratimaths@gmail.com

² Maharashtra Academy of Naval Education and Training, MIT Art, Design and Technology University, Pune 412201, India

³ Department of Mathematics, Central University of Kashmir, Ganderbal 191201, India; hrbhapkar@gmail.com

* Correspondence: archana.bhangre17@gmail.com

Abstract

The total π chromatic number $T\pi CN(G)$ of a graph G requires at least three colors to create a total coloring which provides every vertex with its distinct color combination between its own color and the colors of all neighboring vertices and attached edges. We determine $T\pi CN(G)$ for eight graph families of geometric and biological character: the Jempiring, Butterfly, Lotus, Sunflower, Jellyfish, Snail, Octopus, and Lily graphs. The lower bound for each family comes from degree arguments which apply at the hub while the upper bound applies through a modular arithmetic coloring system that creates distinct color sets through explicit verification. Seven families satisfy $T\pi CN(G) = \Delta(G) + 1$ which demonstrates Type 1 behavior while the Jellyfish graph satisfies $T\pi CN(J(n, n)) = n + 4$ for all $n \geq 2$, a consequence of its diamond body which creates inter-hub constraints that force one extra color beyond the Type-1 bound. These findings extend the existing collection of graph families which have established total π chromatic numbers.

Keywords: total π coloring; total π chromatic number; Jempiring graph; Snail graph; Octopus graph; Lily graph; graph coloring

1. Introduction

Graph coloring theory has developed from its original vertex-coloring method into a complex system of parameters which represent advanced structural characteristics of graphs. The concept of total coloring functions as a fundamental generalization which Behzad [1] and Vizing [2] discovered through their independent research. The Total Coloring Conjecture asserts that total chromatic number $\chi''(G)$ reaches a maximum value of $\Delta(G) + 2$ for all simple graphs G where the maximum degree of graph G is represented as $\Delta(G)$. Despite extensive verification for specific families [3], this conjecture remains open in general, as surveyed by Yap [4] and Geetha, Narayanan, and Somasundaram [5]. Graphs attaining the lower bound $\chi''(G) = \Delta(G) + 1$ are termed Type 1, and determining which structural features force or prevent Type 1 behavior is a central theme in the field.

Modern coloring systems extend local constraints to neighboring vertices and incident edges, enabling unique identification of every vertex through its chromatic environment. The *color set* of a vertex v under a total coloring c is defined as

$$S(v) = \{c(v)\} \cup \{c(u) : uv \in E(G)\} \cup \{c(e) : e \text{ incident with } v\},$$



Academic Editor: Michel Planat

Received: 28 March 2026

Revised: 21 June 2026

Accepted: 24 June 2026

Published: 27 June 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and

conditions of the Creative Commons

Attribution (CC BY) license.

which collects the color of v together with the colors of all its neighbors and all edges incident to v . A *total pi coloring* is a total coloring in which every pair of distinct vertices receives a different color set. The *total pi chromatic number* $T\pi CN(G)$ is the minimum number of colors required for such a coloring. Since every total pi coloring is a total coloring, we have $T\pi CN(G) \geq \Delta(G) + 1$. The color set of a vertex encodes its local structural environment: its size and composition reflect the degree of v , the degrees of its neighbors, and the surrounding edge structure, making total pi coloring highly sensitive to local graph topology. Vertices residing in structurally similar neighborhoods tend to receive identical color sets under economical colorings, thereby requiring additional colors to achieve separation. Neighbor-distinguishing edge colorings [6], adjacent-vertex-distinguishing total colorings [7], and set-distinguishing frameworks [8] represent related approaches to chromatic separability, each imposing identification constraints of varying strength.

The current research establishes the precise value of $T\pi CN(G)$ for eight graph families which were developed through geometric and natural design principles. The eight graph families include the *Jempiring graph* Je_n , *Butterfly graph* $BF(n, n)$, *Lotus graph* $L(n)$, *Sunflower graph* SF_n , *Jellyfish graph* $J(n, n)$, *Snail graph* Sn_n , *Octopus graph* Op_n , and *Lily graph* Li_n . Each family construct begins with a central hub which connects to structured appendages that take the form of petals, wings, tentacles or spiral arms through a specific regular pattern design which uses positive integer n as a parameter. Geometrically structured families of this kind have attracted sustained attention in total coloring [5] and distinguishing coloring [9] yet the total pi chromatic number has not previously been determined for any of the families studied here.

A lower bound for each family is derived from a color-set cardinality argument applied at the maximum-degree vertex, and an explicit constructive coloring using closed-form modular arithmetic functions establishes the upper bound via a global color-set distinctness verification. Seven of the eight families satisfy $T\pi CN(G) = \Delta(G) + 1$, confirming Type 1 total pi chromatic behavior; specifically, $T\pi CN(Je_n) = n + 1$, $T\pi CN(BF(n, n)) = 2n + 3$, $T\pi CN(L(n)) = n + 1$, $T\pi CN(SF_n) = n + 1$, $T\pi CN(Sn_n) = n + 1$, $T\pi CN(Op_n) = n + 1$, and $T\pi CN(Li_n) = n + 1$. The Jellyfish graph is the sole exception: $T\pi CN(J(n, n)) = n + 4$ for all $n \geq 2$, a consequence of the inter-hub constraints imposed by its diamond body structure. Hub-and-spoke architectures with structured peripheral appendages systematically achieve optimal chromatic bounds, whereas graph families whose bodies contain cycles connecting multiple high-degree vertices may require additional colors. The paper is organized as follows. Section 2 surveys related work on total coloring and distinguishing chromatic parameters, followed by formal definitions and notation in Section 3. Sections 4–11 present structural descriptions, main theorems, complete proofs, and color-set verifications for each of the eight families. Section 12 summarizes the findings and discusses directions for future research.

2. Literature Review

Behzad [1] and Vizing [2] introduced total coloring of graphs, which now serves as the foundation for the Total Coloring Conjecture stating that $\chi''(G) \leq \Delta(G) + 2$ for every simple graph G . Vijayaditya [10] first verified the conjecture for graphs with $\Delta(G) = 3$, and Kostochka [3] extended it to all graphs with $\Delta(G) \leq 5$. McDiarmid and Sánchez-Arroyo [11] established the NP-hardness of determining the total chromatic number for regular bipartite graphs, confirming that exact results for specific families require dedicated structural arguments. For graphs with large maximum degree, Molloy and Reed [12] showed that the total chromatic number lies within the range $[\Delta(G) + 1, \Delta(G) + O(\ln \Delta(G))]$. Yap [4] provides a comprehensive treatment of the conjecture and its partial resolutions.

The scientists established total chromatic numbers for many different graph families. The bipartite graph conjecture received its solution from Kronk and Mitchem [13] while Geetha and Somasundaram [14] proved it for generalized Sierpiński graphs. Geetha, Somasundaram, and Fu [15] studied circulant graphs and Vignesh, Geetha, and Somasundaram [9] investigated product graph families which included corona and comb products while Somasundaram, Geetha, and Vignesh [16] established additional results about product graphs. Geetha, Narayanan, and Somasundaram [5] presented a comprehensive review of the current status of the conjecture among different graph classes. The combined findings show that the conjecture remains unsolved but the developed construction techniques for particular graph families enable researchers to create new methods which they can apply to different graph structures.

The goal of distinguishing vertices through their chromatic environments has motivated several identification-based coloring parameters. Zhang, Liu, and Wang [6] introduced adjacent vertex distinguishing edge colorings, in which the multisets of edge colors at adjacent vertices must differ, and determined the corresponding index for paths, cycles, and complete graphs. Zhang, Chen, Li, Yao, Lu, and Wang [7] extended this paradigm to total colorings, requiring that the color sets of adjacent vertices differ under a total coloring. Both parameters achieve vertex separation only for adjacent pairs, a weaker requirement than total pi coloring, which demands globally distinct color sets. Karoński, Łuczak, and Thomason [8] introduced the neighbor sum distinguishing framework, in which integer edge weights render adjacent vertices distinguishable by their weighted degree sums, and conjectured that weights from $\{1, 2, 3\}$ always suffice. Chartrand, Okamoto, and Zhang [17] defined the chromatic distinguishing number as the minimum number of colors for which each vertex receives a unique chromatic neighborhood code. Pilsniak and Wozniak [18] determined precise bounds on the total neighbor distinguishing index for regular graphs and graphs with large chromatic number. Harary [19] remains the standard reference for the structural graph theory foundations underlying these developments.

Bhange and Bhapkar [20] developed their framework for pi coloring through their introduction of perfect coloring and its variants which they applied to standard graph families. Thakare Bhange and Bhapkar [21] determined the incident vertex pi coloring number for fan and book and gear and windmill and Dutch windmill and crown graphs. The total coloring methods for structured graph families are explained in detail by Sanders and Zhao [22] and Campos and de Mello [23] which provides the necessary background for our proof strategies. The present paper extends the total pi coloring programme to eight new graph families of geometric and biological character, none of which has had its total pi chromatic number determined previously.

3. Mathematical Preliminaries

$G = (V, E)$ denotes a finite, simple, connected, undirected graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, the open neighborhood $N(v) = \{u \in V(G) : uv \in E(G)\}$ consists of all vertices adjacent to v , and the degree of v is $\deg(v) = |N(v)|$. The maximum degree of G is $\Delta(G) = \max_{v \in V(G)} \deg(v)$. Vertices are u_i and edges e_j with precise indexing conventions specified for each family.

Definition 1 (Total Coloring [1]). *A total coloring of G is a mapping $c: V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$ satisfying:*

1. $c(u) \neq c(v)$ for every edge $uv \in E(G)$;
2. $c(e_1) \neq c(e_2)$ for every pair of edges e_1, e_2 sharing a common endpoint;
3. $c(v) \neq c(e)$ for every vertex v and every edge e incident with v .

The total chromatic number $\chi''(G)$ is the minimum k for which a total coloring exists.

The Total Coloring Conjecture asserts that $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$ for every simple graph G [2]. A graph with $\chi''(G) = \Delta(G) + 1$ is called *Type 1*, and one requiring $\chi''(G) = \Delta(G) + 2$ is called *Type 2*.

Definition 2 (Color Set). Let c be a total coloring of G . The color set of a vertex $v \in V(G)$ is

$$S(v) = \{c(v)\} \cup \{c(u) : u \in N(v)\} \cup \{c(e) : e \text{ incident with } v\}.$$

Definition 3 (Total Pi Coloring). A total coloring c of G is a total pi coloring if $S(u) \neq S(v)$ for every pair of distinct vertices $u, v \in V(G)$. The total pi chromatic number $T\pi CN(G)$ is the minimum number of colors over all total pi colorings of G .

Since every total pi coloring is a total coloring, we have the following bound.

For any graph G ,

$$T\pi CN(G) \geq \chi''(G) \geq \Delta(G) + 1.$$

The following method is used in lower-bound arguments.

Under any total pi coloring c of G , if a vertex v has degree $\deg(v) = d$, then $S(v)$ must contain the color of v itself, the colors of its d neighbors, and the colors of its d incident edges. While colors may coincide across these three roles, the total coloring constraints imply that the color of v is distinct from the colors of all incident edges and all neighboring vertices. Consequently $|S(v)| \geq \deg(v) + 1$, and for the maximum-degree vertex we obtain $|S(v)| \geq \Delta(G) + 1$. Any coloring which achieves $T\pi CN(G) = \Delta(G) + 1$ must use exactly $\Delta(G) + 1$ colors which should be distributed to vertices according to their minimum color requirement for unique color assignment.

Graph Families Under Study

The eight families examined in this study are each parametrised by a positive integer n . Formal definitions, vertex and edge counts, and exact parameter ranges are given in the respective sections. Table 1 provides a structural overview for reference.

Table 1. Structural parameters and total π chromatic numbers of the graph families investigated in this paper.

Graph Family	Notation	$ V(G) $	$ E(G) $	$\Delta(G)$	$T\pi CN(G)$
Jempiring graph	Je_n	$\frac{3n}{2} + 1$	$3n$	n	$n + 1$
Butterfly graph	$BF(n, n)$	$2n + 3$	$4n$	$2n + 2$	$2n + 3$
Lotus graph	$L(n)$	$2n - 1$	$3n - 4$	n	$n + 1$
Sunflower graph	SF_n	$2n + 1$	$4n$	n	$n + 1$
Jellyfish graph	$J(n, n)$	$2n + 4$	$2n + 5$	$n + 2$	$n + 4$ ($n \geq 2$)
Snail graph	Sn_n	$2n + 1$	$3n - 2$	n	$n + 1$
Octopus graph	Op_n	$n + 1$	$\frac{3n}{2} - 1$	n	$n + 1$
Lily graph	Li_n	$\frac{2n}{3} + 1$	n	n	$n + 1$

4. Jempiring Graph

4.1. Structure of the Jempiring Graph

The Jempiring graph Je_n defined for even integers n starting from 4 creates a three-layer graph whose petal structure centers around a main hub. The graph has a total of $\frac{3n}{2} + 1$ vertices and $3n$ edges.

Vertex layers. A single central vertex v_0 constitutes the hub. The remaining $\frac{3n}{2}$ vertices are indexed from v_1 to $v_{\frac{3n}{2}}$ and are distributed across two concentric rings. The *inner ring* consists of n vertices; for each petal with even index $i = 2, 4, \dots, n$ (representing the cumulative degree of v_0 as successive petals are added), the two inner ring vertices are $v_{\frac{3i}{2}-2}$ and $v_{\frac{3i}{2}-1}$. The *outer ring* consists of $\frac{n}{2}$ vertices $v_{\frac{3i}{2}}$, one per petal.

Petal structure and edge labeling. For each even index $i = 2, 4, \dots, n$, the corresponding petal contains five edges:

- e_{3i-5} : left spoke from v_0 to $v_{\frac{3i}{2}-2}$,
- e_{3i-4} : right spoke from v_0 to $v_{\frac{3i}{2}-1}$,
- e_{3i-3} : inner chord from $v_{\frac{3i}{2}-2}$ to $v_{\frac{3i}{2}-1}$,
- e_{3i-2} : left petal arm from $v_{\frac{3i}{2}-2}$ to $v_{\frac{3i}{2}}$,
- e_{3i-1} : right petal arm from $v_{\frac{3i}{2}-1}$ to $v_{\frac{3i}{2}}$.

Adjacent petals are connected along the outer ring by the edge e_{3i-6} , which joins $v_{\frac{3i}{2}-3}$ (the outer vertex of the preceding petal) to $v_{\frac{3i}{2}}$ (the outer vertex of the current petal), for $i = 4, 6, \dots, n$. The outer ring is closed by the wrap-around edge e_{3n} , joining $v_{\frac{3n}{2}}$ back to v_3 . The total edge count is computed as follows. There are $n/2$ petals, each contributing 5 intra-petal edges, giving $5 \times (n/2) = 5n/2$ intra-petal edges. There are also $n/2 - 1$ inter-petal outer cycle edges (one joining each consecutive pair of outer ring vertices, for $i = 4, 6, \dots, n$) plus the single wrap-around edge e_{3n} , totalling $(n/2 - 1) + 1 = n/2$ outer cycle edges. Hence

$$|E(Je_n)| = \underbrace{5 \cdot \frac{n}{2}}_{\text{intra-petal}} + \underbrace{\frac{n}{2}}_{\text{outer cycle}} = \frac{5n}{2} + \frac{n}{2} = 3n.$$

Degree sequence. The central vertex v_0 is incident with the n spoke edges, so $\deg(v_0) = n$. Each inner ring vertex is incident with one spoke, one inner chord, and one petal arm, giving degree 3. Each outer ring vertex is adjacent to the two inner ring vertices of its own petal and to the two neighboring outer ring vertices via outer cycle edges, giving degree 4. Consequently, for $n \geq 4$,

$$|V(Je_n)| = \frac{3n}{2} + 1, \quad |E(Je_n)| = 3n, \quad \Delta(Je_n) = n.$$

Figure 1 illustrates the total π coloring scheme for Je_n .

4.2. Neighborhood Sets

We derive each closed neighborhood W_v directly from the petal structure defined above, making explicit which vertices and edges are adjacent or incident to v .

The central hub v_0 is adjacent only to inner ring vertices via spoke edges; it has no direct connection to any outer ring vertex. For petal k (corresponding to even index $i = 2k$, $k = 1, 2, \dots, \frac{n}{2}$), the two spokes incident to v_0 are e_{6k-5} (left spoke to v_{3k-2}) and e_{6k-4} (right spoke to v_{3k-1}). The sequence $6k - 5$ for $k = 1, \dots, \frac{n}{2}$ gives edge indices $1, 7, 13, \dots, 3n - 5$, which are precisely those $\equiv 1 \pmod{6}$ in the range $[1, 3n - 5]$. The sequence $6k - 4$ gives $2, 8, 14, \dots, 3n - 4$, precisely those $\equiv 2 \pmod{6}$ in the range $[2, 3n - 4]$. The inner ring vertices adjacent to v_0 are v_{3k-2} and v_{3k-1} for $k = 1, \dots, \frac{n}{2}$, which are exactly the vertices v_i with $i \not\equiv 0 \pmod{3}$ and $1 \leq i \leq \frac{3n}{2}$ (outer ring vertices $v_3, v_6, \dots, v_{\frac{3n}{2}}$ have indices divisible by 3 and are not adjacent to v_0). Therefore:

$$W_{v_0} = \{v_0\} \cup \{v_i : i \not\equiv 0 \pmod{3}, 1 \leq i \leq \frac{3n}{2}\} \\ \cup \{e_k : k \equiv 1 \pmod{6}, 1 \leq k \leq 3n - 5\} \cup \{e_l : l \equiv 2 \pmod{6}, 2 \leq l \leq 3n - 4\}.$$

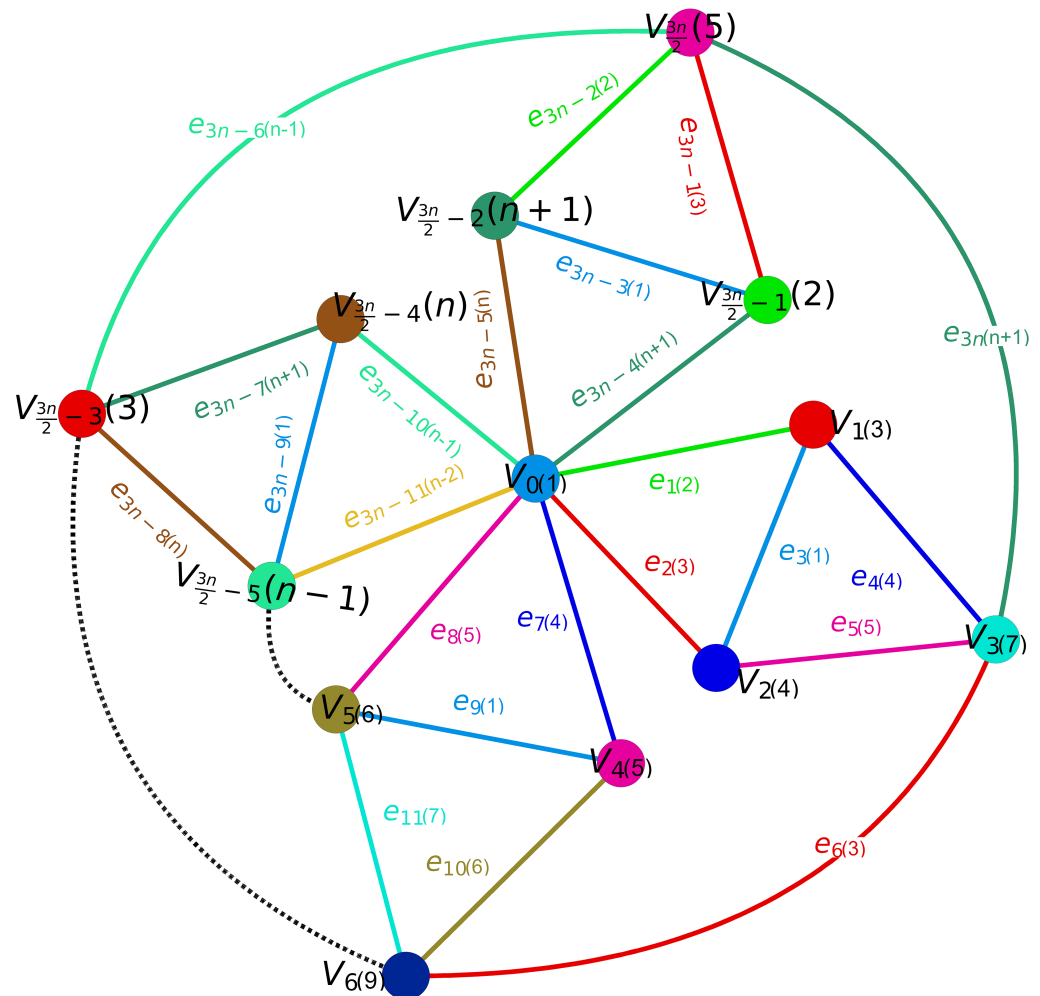


Figure 1. Total π coloring scheme for Jempiring Graph J_{e_n} .

Verification for $n = 4$: the two petals give spokes e_1, e_2 (petal $k = 1$) and e_7, e_8 (petal $k = 2$), and inner ring vertices v_1, v_2, v_4, v_5 , giving $|W_{v_0}| = 1 + 4 + 4 = 9$ and $\deg(v_0) = 4 = n$, consistent with the degree sequence.

For an inner ring vertex v_t with $t \equiv 1 \pmod{3}, t = 1, 4, \dots, n$:

$$W_{v_t} = \{v_0, v_t, v_{t+1}, v_{t+2}, e_{2t-1}, e_{2t+1}, e_{2t+2}\}.$$

For an inner ring vertex v_t with $t \equiv 2 \pmod{3}, t = 2, 5, \dots, n$:

$$W_{v_t} = \{v_0, v_{t-1}, v_t, v_{t+1}, e_{2t-2}, e_{2t-1}, e_{2t+1}\}.$$

For a general outer ring vertex v_t with $t \equiv 0 \pmod{3}, t = 6, 9, \dots, \frac{3n}{2} - 3$:

$$W_{v_t} = \{v_{t-3}, v_{t-2}, v_{t-1}, v_t, v_{t+3}, e_{2t-6}, e_{2t-2}, e_{2t-1}, e_{2t}\}.$$

The boundary outer ring vertices have

$$W_{v_3} = \{v_1, v_2, v_3, v_6, v_{\frac{3n}{2}}, e_4, e_5, e_6, e_{3n}\},$$

$$W_{v_{\frac{3n}{2}}} = \{v_{\frac{3n}{2}-3}, v_{\frac{3n}{2}-2}, v_{\frac{3n}{2}-1}, v_{\frac{3n}{2}}, e_{3n-6}, e_{3n-2}, e_{3n-1}, e_{3n}\}.$$

For every vertex v , the color set satisfies $S(v) = \{c(w) : w \in W_v\}$ under any total coloring c . Because the hub v_0 , the inner ring vertices (of two residue classes), and the outer

ring vertices (including boundary cases v_3 and $v_{\frac{3n}{2}}$) have structurally distinct neighborhoods, their W_v sets differ in size and composition, and the proof below exploits this to confirm that the coloring assigns a distinct $S(v)$ to each vertex type.

4.3. Jempiring Graph Result

Theorem 1. For every even integer $n \geq 4$, $T\pi CN(Je_n) = n + 1$.

Proof. Lower bound. The central vertex v_0 is adjacent to n distinct inner ring vertices and is incident with n spoke edges. A valid total coloring requires $c(v_0)$ to differ from the colors of all n neighbors and all n incident spokes. These $2n$ elements, together with v_0 itself, force the color set $S(v_0)$ to have size at least $n + 1$. Hence $T\pi CN(Je_n) \geq n + 1$.

Upper bound. We exhibit an explicit total pi coloring of Je_n using exactly $n + 1$ colors.

Vertex coloring function $c(v_t)$.

$$c(v_0) = 1. \quad (1)$$

$$c(v_t) = \frac{2t+7}{3}, \quad t \equiv 1 \pmod{3}, \quad t = 1, 4, \dots, \frac{3n}{2} - 5. \quad (2)$$

$$c(v_t) = \frac{2t+8}{3}, \quad t \equiv 2 \pmod{3}, \quad t = 2, 5, \dots, \frac{3n}{2} - 4. \quad (3)$$

$$c(v_t) = \frac{2t+15}{3}, \quad t \equiv 0 \pmod{3}, \quad t = 3, 6, \dots, \frac{3n}{2} - 6. \quad (4)$$

The wrap-around boundary vertices receive special assignments:

$$c(v_{\frac{3n}{2}-3}) = 3, \quad c(v_{\frac{3n}{2}-1}) = 2, \quad c(v_{\frac{3n}{2}}) = 5. \quad (5)$$

Edge coloring function $c(e_t)$.

$$c(e_t) = \frac{t+5}{3}, \quad t \equiv 1 \pmod{6}, \quad t = 1, 7, \dots, 3n. \quad (6)$$

$$c(e_t) = \frac{t+7}{3}, \quad t \equiv 2 \pmod{6}, \quad t = 2, 8, \dots, 3n - 4. \quad (7)$$

$$c(e_t) = 1, \quad t \equiv 3 \pmod{6}, \quad t = 3, 9, \dots, 3n - 3. \quad (8)$$

$$c(e_t) = \frac{t+8}{3}, \quad t \equiv 4 \pmod{6}, \quad t = 4, 10, \dots, 3n - 8. \quad (9)$$

$$c(e_t) = \frac{t+10}{3}, \quad t \equiv 5 \pmod{6}, \quad t = 5, 11, \dots, 3n - 7. \quad (10)$$

$$c(e_t) = \frac{t}{3} + 1, \quad t \equiv 0 \pmod{6}, \quad t = 6, 12, \dots, 3n - 6. \quad (11)$$

The last two edges of the outer cycle receive special assignments:

$$c(e_{3n-2}) = 2, \quad c(e_{3n-1}) = 3. \quad (12)$$

Verification of the total coloring conditions. Adjacent vertices are properly colored: the central vertex has color 1, while every inner ring vertex has color at least 3 under the general formulas, and the special boundary vertices receive colors 2, 3, and 5, none of which equal 1. Within each petal, the two inner ring vertices $v_{\frac{3i}{2}-2}$ and $v_{\frac{3i}{2}-1}$ receive colors whose numerators $2t + 7$ and $2t + 8$ differ by 1, so they are assigned distinct integer values since t and $t + 1$ belong to different residue classes modulo 3. Adjacent edges incident at v_0 belong to residue classes 1 and 2 modulo 6, so their colors $\frac{t+5}{3}$ and $\frac{t+7}{3}$ are distinct integers. Inner chord edges (residue 3 mod 6) all receive color 1, which is also the color of v_0 ; however, these edges are not incident at v_0 , so no incidence conflict arises. All other vertex-edge

incidence pairs are proper by the separation between vertex and edge color indices built into the respective residue-class formulas.

Color set analysis. For the central vertex:

$$S(v_0) = \{1, 2, 3, \dots, n+1\}. \quad (13)$$

For inner ring vertices with $t \equiv 1 \pmod{3}$, $t = 4$ to $\frac{3n}{2} - 11$:

$$S(v_t) = \left\{1, \frac{2t+4}{3}, \frac{2t+7}{3}, \frac{2t+10}{3}, \frac{2t+19}{3}\right\}. \quad (14)$$

For inner ring vertices with $t \equiv 2 \pmod{3}$, $t = 5$ to $\frac{3n}{2} - 10$:

$$S(v_t) = \left\{1, \frac{2t+5}{3}, \frac{2t+8}{3}, \frac{2t+11}{3}, \frac{2t+17}{3}\right\}. \quad (15)$$

For general outer ring vertices with $t \equiv 0 \pmod{3}$, $t = 6$ to $\frac{3n}{2} - 9$:

$$S(v_t) = \left\{\frac{2t-3}{3}, \frac{2t+3}{3}, \frac{2t+6}{3}, \frac{2t+9}{3}, \frac{2t+15}{3}, \frac{2t+21}{3}\right\}. \quad (16)$$

The boundary and wrap-around vertices have the following special color sets:

$$S(v_3) = \{3, 4, 5, 7, 9, n+1\}, \quad (17)$$

$$S\left(v_{\frac{3n}{2}-6}\right) = \{n-5, n-3, n-2, n-1, n+1, 3\}, \quad (18)$$

$$S\left(v_{\frac{3n}{2}-5}\right) = \{1, 3, n-2, n-1, n\}, \quad (19)$$

$$S\left(v_{\frac{3n}{2}-4}\right) = \{1, 3, n-1, n, n+1\}, \quad (20)$$

$$S\left(v_{\frac{3n}{2}-3}\right) = \{3, 5, n-3, n-1, n, n+1\}, \quad (21)$$

$$S\left(v_{\frac{3n}{2}-2}\right) = \{1, 2, 5, n, n+1\}, \quad (22)$$

$$S\left(v_{\frac{3n}{2}-1}\right) = \{1, 2, 3, 5, n+1\}, \quad (23)$$

$$S\left(v_{\frac{3n}{2}}\right) = \{2, 3, 5, 7, n-1, n+1\}. \quad (24)$$

Derivation of the general color set formulas. For an inner ring vertex v_t with $t = 3k - 2$ ($t \equiv 1 \pmod{3}$, $k \geq 2$), the four elements of $S(v_t)$ arise as follows. The vertex color is $c(v_t) = 2k + 1$. The hub neighbor contributes $c(v_0) = 1$. The right inner ring partner $v_{t+1} = v_{3k-1}$ contributes $c(v_{3k-1}) = 2k + 2$. The outer ring vertex $v_{t+2} = v_{3k}$ of the same petal contributes $c(v_{3k}) = 2k + 5$. The left spoke e_{6k-5} contributes $c(e_{6k-5}) = 2k$. The inner chord e_{6k-3} contributes $c(e_{6k-3}) = 1$ (which merges with $c(v_0)$). The left petal arm e_{6k-2} contributes $c(e_{6k-2}) = 2k + 2$ (which merges with $c(v_{3k-1})$). Hence $S(v_t) = \{1, 2k, 2k+1, 2k+2, 2k+5\}$, which in terms of t gives $\left\{1, \frac{2t+4}{3}, \frac{2t+7}{3}, \frac{2t+10}{3}, \frac{2t+19}{3}\right\}$.

Similarly, for v_t with $t = 3k - 1$ ($t \equiv 2 \pmod{3}$, $k \geq 2$): vertex color $2k + 2$; hub 1; left partner v_{3k-2} gives $2k + 1$; outer vertex v_{3k} gives $2k + 5$; right spoke e_{6k-4} gives $2k + 1$ (merges); inner chord e_{6k-3} gives 1 (merges); right petal arm e_{6k-1} gives $2k + 3$. Hence $S(v_t) = \{1, 2k+1, 2k+2, 2k+3, 2k+5\} = \left\{1, \frac{2t+5}{3}, \frac{2t+8}{3}, \frac{2t+11}{3}, \frac{2t+17}{3}\right\}$.

For an outer ring vertex v_t with $t = 3k$ ($t \equiv 0 \pmod{3}$, $k = 2$ to $\frac{n}{2} - 3$): vertex color $2k + 5$; left inner v_{3k-2} gives $2k + 1$; right inner v_{3k-1} gives $2k + 2$; previous outer v_{3k-3} gives $2k + 3$; next outer v_{3k+3} gives $2k + 7$; left petal arm e_{6k-2} gives $2k + 2$ (merges); right petal arm e_{6k-1} gives $2k + 3$ (merges); incoming outer cycle e_{6k-6} gives $2k - 1$; outgoing outer cycle e_{6k} gives $2k + 1$ (merges). Hence $S(v_t) = \{2k - 1, 2k + 1, 2k + 2, 2k + 3, 2k + 5, 2k + 7\} = \left\{\frac{2t-3}{3}, \frac{2t+3}{3}, \frac{2t+6}{3}, \frac{2t+9}{3}, \frac{2t+15}{3}, \frac{2t+21}{3}\right\}$.

These formulas were verified computationally for $n = 12$, where all 19 color sets are distinct.

Global distinctness. We show the map $v \mapsto S(v)$ is injective by structural group. The hub v_0 uniquely carries the full palette $\{1, \dots, n+1\}$ of cardinality $n+1$, so no other vertex can share its color set. For inner ring vertices with $t \equiv 1 \pmod{3}$: $S(v_t) = \{1, 2k, 2k+1, 2k+2, 2k+5\}$ where $k = (t+2)/3$; the minimum non-unity element $2k$ determines k , hence $t = 3k - 2$ uniquely. For $t \equiv 2 \pmod{3}$: the second element $2k+1$ determines $k = (t+1)/3$, hence $t = 3k - 1$ uniquely. For outer ring vertices with $t \equiv 0 \pmod{3}$: $S(v_t)$ does *not* contain the color 1 (since $c(v_0) = 1$ and v_t is not adjacent to v_0), so no inner ring set can equal any outer ring set. Within the outer ring, the minimum element $2k-1$ where $k = t/3$ determines t uniquely. The eight boundary sets are verified individually above and each differs from all general-range sets by at least one element. Hence $\text{T}\pi\text{CN}(Je_n) \leq n+1$.

Combining both bounds, $\text{T}\pi\text{CN}(Je_n) = n+1$. $\square$

5. Butterfly Graph

5.1. Structure of the Butterfly Graph

The butterfly graph $BF(n, n)$ uses a positive integer parameter n which must be at least 2 to define the order of its shell structure. The graph consists of a central hub vertex which serves as its base and two symmetrical wing structures which extend from it. The graph has a total of $2n$ plus 3 vertices and $4n$ edges. The butterfly graph builds upon the Jempiring graph construction method shown in Figure 1 by creating a hub-and-spoke system which connects two linear chains to the central vertex through symmetrical attachments.

Vertex set. A single central vertex O (also denoted x_0) forms the hub. The *left wing* comprises n chain vertices $x_1, x_2, \dots, x_n$ together with one pendant vertex x . The *right wing* comprises n chain vertices $y_1, y_2, \dots, y_n$ together with one pendant vertex y . Thus

$$V(BF(n, n)) = \{O\} \cup \{x_i : 1 \leq i \leq n\} \cup \{x\} \cup \{y_i : 1 \leq i \leq n\} \cup \{y\}, \quad |V| = 2n + 3.$$

Edge set. The edges are organized into six groups:

- r : pendant spoke connecting O to x ,
- r_i ($i = 1, \dots, n$): left spokes connecting O to x_i ,
- d_i ($i = 1, \dots, n-1$): left chain edges connecting x_i to x_{i+1} ,
- g : pendant spoke connecting O to y ,
- g_i ($i = 1, \dots, n$): right spokes connecting O to y_i ,
- f_i ($i = 1, \dots, n-1$): right chain edges connecting y_i to y_{i+1} .

The total number of edges is $1 + n + (n-1) + 1 + n + (n-1) = 4n$.

Degree sequence. The central vertex O is incident with all $2n+2$ spoke edges, so $\deg(O) = 2n+2 = \Delta(BF(n, n))$. Each pendant vertex x and y has degree 1. The terminal chain vertices x_1, x_n, y_1 , and y_n each have degree 2, while every interior chain vertex x_i and y_i for $2 \leq i \leq n-1$ has degree 3. Consequently,

$$|V(BF(n, n))| = 2n + 3, \quad |E(BF(n, n))| = 4n, \quad \Delta(BF(n, n)) = 2n + 2.$$

Figure 2 illustrates the total π coloring scheme for $BF(n, n)$.

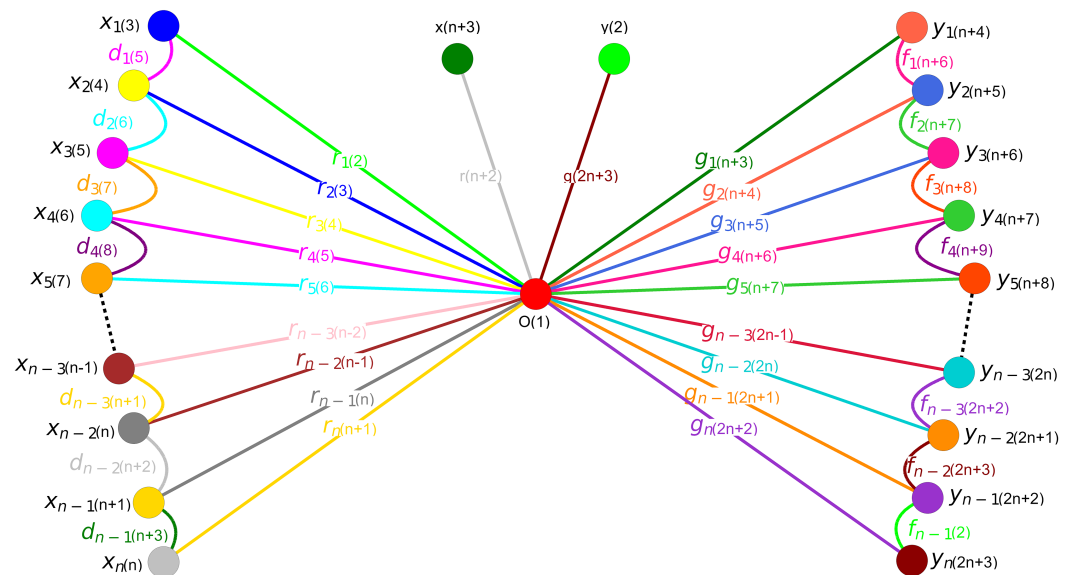


Figure 2. Total π coloring scheme for Butterfly Graph $BF(n, n)$.

5.2. Neighborhood Sets

The closed neighborhood W_v of a vertex v collects v itself together with all adjacent vertices and all incident edges.

For the central vertex:

$$W_O = \{O\} \cup \{x_i, y_i, r_i, g_i : 1 \leq i \leq n\} \cup \{x, y, r, g\}.$$

For the pendant vertices:

$$W_x = \{O, x, r\}, \quad W_y = \{O, y, g\}.$$

For the first left chain vertex:

$$W_{x_1} = \{O, x_1, x_2, r_1, d_1\}.$$

For a general interior left chain vertex ($i = 2$ to $n - 1$):

$$W_{x_i} = \{O, x_{i-1}, x_i, x_{i+1}, d_{i-1}, d_i, r_i\}.$$

For the last left chain vertex:

$$W_{x_n} = \{O, x_{n-1}, x_n, r_n, d_{n-1}\}.$$

For the first right chain vertex:

$$W_{y_1} = \{O, y_1, y_2, g_1, f_1\}.$$

For a general interior right chain vertex ($i = 2$ to $n - 1$):

$$W_{y_i} = \{O, y_{i-1}, y_i, y_{i+1}, f_{i-1}, f_i, g_i\}.$$

For the last right chain vertex:

$$W_{y_n} = \{O, y_{n-1}, y_n, g_n, f_{n-1}\}.$$

For every vertex v , the color set satisfies $S(v) = \{c(w) : w \in W_v\}$ under any total coloring c . The central vertex O , the two wing pendants x and y , and the chain vertices x_i and y_i each occupy structurally distinct positions in $BF(n, n)$ and therefore have W_v sets that differ in both size and composition; the proof below uses this to confirm global color-set distinctness.

5.3. Butterfly Graph Result

Theorem 2. For every integer $n \geq 2$, $T\pi CN(BF(n, n)) = 2n + 3$.

Proof. Lower bound. The central vertex O has degree $\Delta(BF(n, n)) = 2n + 2$, being adjacent to n left chain vertices, n right chain vertices, and the two pendant vertices x and y . It is simultaneously incident with $2n + 2$ spoke edges. A valid total coloring requires $c(O)$ to differ from the colors of all $2n + 2$ neighbors and all $2n + 2$ incident edges, forcing $|S(O)| \geq 2n + 3$. Hence $T\pi CN(BF(n, n)) \geq 2n + 3$.

Upper bound. We construct an explicit total pi coloring of $BF(n, n)$ using exactly $2n + 3$ colors from the palette $\{1, 2, \dots, 2n + 3\}$.

Vertex coloring.

$$c(O) = 1. \quad (25)$$

$$c(x_i) = i + 2, \quad i = 1, 2, \dots, n. \quad (26)$$

$$c(y_i) = n + i + 3, \quad i = 1, 2, \dots, n. \quad (27)$$

$$c(x) = n + 3. \quad (28)$$

$$c(y) = 2. \quad (29)$$

Edge coloring.

$$c(r_i) = i + 1, \quad i = 1, 2, \dots, n. \quad (30)$$

$$c(g_i) = n + i + 2, \quad i = 1, 2, \dots, n. \quad (31)$$

$$c(r) = n + 2. \quad (32)$$

$$c(g) = 2n + 3. \quad (33)$$

$$c(d_i) = i + 4, \quad i = 1, 2, \dots, n - 1. \quad (34)$$

$$c(f_i) = n + i + 5, \quad i = 1, 2, \dots, n - 2. \quad (35)$$

$$c(f_{n-1}) = 2. \quad (36)$$

Verification that c is a valid total coloring. At the central vertex O : the neighbor colors are $\{c(x_1), \dots, c(x_n)\} \cup \{c(y_1), \dots, c(y_n)\} \cup \{c(x), c(y)\} = \{3, \dots, n + 2\} \cup \{n + 4, \dots, 2n + 3\} \cup \{n + 3, 2\} = \{2, 3, \dots, 2n + 3\}$, all distinct from $c(O) = 1$. The incident edge colors at O are $\{c(r_1), \dots, c(r_n)\} \cup \{c(g_1), \dots, c(g_n)\} \cup \{c(r), c(g)\} = \{2, \dots, n + 1\} \cup \{n + 3, \dots, 2n + 2\} \cup \{n + 2, 2n + 3\} = \{2, 3, \dots, 2n + 3\}$, likewise all distinct from $c(O) = 1$. For an interior left chain vertex x_i ($2 \leq i \leq n - 1$): $c(x_i) = i + 2$ differs from $c(r_i) = i + 1$, $c(d_{i-1}) = i + 3$, $c(d_i) = i + 4$, $c(x_{i-1}) = i + 1$, and $c(x_{i+1}) = i + 3$, so all incidence and adjacency constraints hold. The same verification applies symmetrically to the right wing. The special assignment $c(f_{n-1}) = 2$ satisfies $c(f_{n-1}) \neq c(y_{n-1}) = 2n + 2$ and $c(f_{n-1}) \neq c(y_n) = 2n + 3$, and differs from the adjacent edge colors $c(f_{n-2}) = 2n + 3$ and $c(g_{n-1}) = 2n + 1$. Note that $c(f_{n-1}) = 2 = c(y)$, where y is the right pendant vertex. This repetition of color 2 is permitted because f_{n-1} and y are neither adjacent nor incident: f_{n-1} connects y_{n-1} to y_n on the right chain, while y is a pendant vertex attached only to O via spoke g , so the two elements share no endpoint and do not touch anywhere in the graph. A total coloring only

forbids equal colors on adjacent vertices, adjacent edges, or a vertex–edge incident pair; no such relationship exists between f_{n-1} and y .

Color set computation.

For the central vertex:

$$S(O) = \{1, 2, 3, \dots, 2n + 3\}. \quad (37)$$

For the pendant vertices:

$$S(x) = \{1, n + 2, n + 3\}. \quad (38)$$

$$S(y) = \{1, 2, 2n + 3\}. \quad (39)$$

For the first left chain vertex:

$$S(x_1) = \{1, 2, 3, 4, 5\}. \quad (40)$$

For a general interior left chain vertex ($i = 2$ to $n - 1$):

$$S(x_i) = \{1, i + 1, i + 2, i + 3, i + 4\}. \quad (41)$$

Note that $c(x_{i-1}) = i + 1 = c(r_i)$ and $c(x_{i+1}) = i + 3 = c(d_{i-1})$, so these pairs merge within $S(x_i)$, giving a set of cardinality 5. Verification for $n = 4$: the interior vertices x_2 and x_3 give $S(x_2) = \{c(x_2), c(O), c(x_1), c(x_3), c(r_2), c(d_1), c(d_2)\} = \{4, 1, 3, 5, 3, 5, 6\} = \{1, 3, 4, 5, 6\}$ (which matches the formula $\{1, 3, 4, 5, 6\}$), and $S(x_3) = \{5, 1, 4, 6, 4, 6, 7\} = \{1, 4, 5, 6, 7\}$ (which matches the formula $\{1, 4, 5, 6, 7\}$).

For the last left chain vertex:

$$S(x_n) = \{1, n + 1, n + 2, n + 3\}. \quad (42)$$

For right chain vertices ($i = 1$ to $n - 2$):

$$S(y_i) = \{1, n + i + 2, n + i + 3, n + i + 4, n + i + 5\}. \quad (43)$$

For the penultimate right chain vertex:

$$S(y_{n-1}) = \{1, 2, 2n + 1, 2n + 2, 2n + 3\}. \quad (44)$$

For the last right chain vertex:

$$S(y_n) = \{1, 2, 2n + 2, 2n + 3\}. \quad (45)$$

Global distinctness. We show $v \mapsto S(v)$ is injective by structural group. The hub O uniquely carries the full palette of cardinality $2n + 3$. The pendant sets $S(x) = \{1, n + 2, n + 3\}$ and $S(y) = \{1, 2, 2n + 3\}$ are the only sets of cardinality 3 and are distinguished by their maximum elements ($n + 3$ vs $2n + 3$). For the general left chain ($i = 2$ to $n - 1$): $S(x_i) = \{1, i + 1, i + 2, i + 3, i + 4\}$; if $S(x_i) = S(x_j)$ then their second elements satisfy $i + 1 = j + 1$, so $i = j$. For the general right chain ($i = 1$ to $n - 2$): $S(y_i) = \{1, n + i + 2, n + i + 3, n + i + 4, n + i + 5\}$; if $S(y_i) = S(y_j)$ then $n + i + 2 = n + j + 2$, so $i = j$. Left chain sets have maximum element at most $n + 3$ (for $i = n - 1$); right chain sets have minimum non-unity element at least $n + 3$ (for $i = 1$). They can share the value $n + 3$ in principle, but $S(x_{n-1}) = \{1, n, n + 1, n + 2, n + 3\}$ while $S(y_1) = \{1, n + 3, n + 4, n + 5, n + 6\}$: these are

distinct sets. The boundary sets $S(x_1)$, $S(x_n)$, $S(y_{n-1})$, and $S(y_n)$ are verified individually above. Hence $T\pi CN(BF(n, n)) \leq 2n + 3$.

Combining both bounds, $T\pi CN(BF(n, n)) = 2n + 3$. $\square$

6. Lotus Graph

6.1. Structure of the Lotus Graph

The lotus graph $L(n)$, parametrized by an integer $n \geq 4$, is a three-component graph whose architecture reflects the botanical structure of a lotus flower: a central hub, a downward stem, a fan of stamens, and curved petals arcing between adjacent stamen tips. The graph comprises $2n - 1$ vertices and $3n - 4$ edges.

Vertex set. A single central vertex x_0 serves as the hub. The *stamen vertices* $x_1, x_2, \dots, x_n$ are all directly connected to x_0 ; the vertex x_1 is a pendant (the stem) while $x_2, \dots, x_n$ fan outward symmetrically. The *petal vertices* $y_1, y_2, \dots, y_{n-2}$ sit between adjacent stamen tips and are not adjacent to x_0 . Thus

$$V(L(n)) = \{x_0\} \cup \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n-2\}, \quad |V| = 2n - 1.$$

Edge set. The edges fall into two groups:

- d_i ($i = 1, \dots, n$): stamen spokes connecting x_0 to x_i ,
- g_{2i-1} and g_{2i} ($i = 1, \dots, n-2$): petal arc edges, where g_{2i-1} connects x_{i+1} to y_i and g_{2i} connects y_i to x_{i+2} .

The total number of edges is $n + 2(n-2) = 3n - 4$.

Degree sequence. The central vertex x_0 is incident with the n stamen spokes, giving $\deg(x_0) = n = \Delta(L(n))$. The stem vertex x_1 is incident only with d_1 , so $\deg(x_1) = 1$. The second stamen vertex x_2 is incident with d_2 and the arc g_1 (from the first petal), giving $\deg(x_2) = 2$. A general interior stamen vertex x_i for $3 \leq i \leq n-1$ is incident with d_i , the incoming arc $g_{2(i-2)}$ (from petal y_{i-2}), and the outgoing arc g_{2i-3} (to petal y_{i-1}), giving $\deg(x_i) = 3$. The last stamen vertex x_n is incident with d_n and the arc g_{2n-4} , giving $\deg(x_n) = 2$. Each petal vertex y_i is incident with g_{2i-1} and g_{2i} , so $\deg(y_i) = 2$. Consequently,

$$|V(L(n))| = 2n - 1, \quad |E(L(n))| = 3n - 4, \quad \Delta(L(n)) = n.$$

Figure 3 illustrates the total π coloring scheme for $L(n)$.

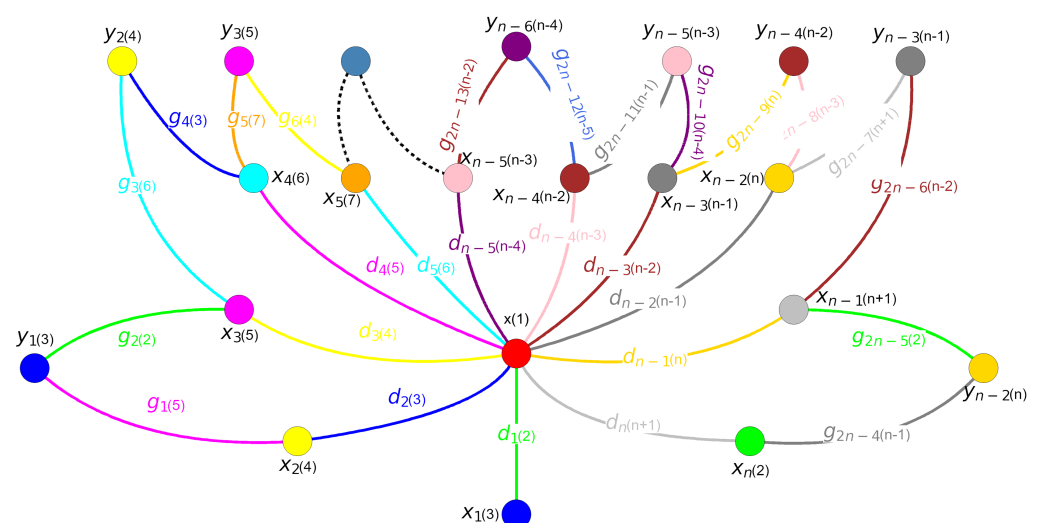


Figure 3. Total π coloring scheme for Lotus Graph $L(n)$.

6.2. Neighborhood Sets

The closed neighborhood W_v of a vertex v collects v itself together with all adjacent vertices and all incident edges.

For the central vertex:

$$W_{x_0} = \{x_0, x_1, x_2, \dots, x_n, d_1, d_2, \dots, d_n\}.$$

For the stem vertex:

$$W_{x_1} = \{x_0, x_1, d_1\}.$$

For the second stamen vertex:

$$W_{x_2} = \{x_0, x_2, y_1, d_2, g_1\}.$$

For the last stamen vertex:

$$W_{x_n} = \{x_0, x_n, y_{n-2}, d_n, g_{2n-4}\}.$$

For a general interior stamen vertex ($i = 3$ to $n - 1$):

$$W_{x_i} = \{x_0, x_i, y_{i-2}, y_{i-1}, d_i, g_{2i-4}, g_{2i-3}\}.$$

For a general petal vertex ($i = 1$ to $n - 2$):

$$W_{y_i} = \{y_i, x_{i+1}, x_{i+2}, g_{2i-1}, g_{2i}\}.$$

For every vertex v , the color set satisfies $S(v) = \{c(w) : w \in W_v\}$ under any total coloring c . The central vertex x_0 , the stem x_1 , the stamen vertices x_i (with boundary cases x_2 and x_n treated separately), and the petal vertices y_i each have neighborhoods of different sizes; the proof below uses these distinctions to verify that no two vertices share a color set.

6.3. Lotus Graph Result

Theorem 3. For every integer $n \geq 4$, $T\pi CN(L(n)) = n + 1$.

Proof. Lower bound. The central vertex x_0 is adjacent to all n stamen vertices $x_1, \dots, x_n$ and is incident with n stamen spokes $d_1, \dots, d_n$. A valid total coloring requires $c(x_0)$ to differ from the colors of all these neighbors and incident edges, forcing $|S(x_0)| \geq n + 1$. Hence $T\pi CN(L(n)) \geq n + 1$.

Upper bound. We construct an explicit total pi coloring of $L(n)$ using exactly $n + 1$ colors from the palette $\{1, 2, \dots, n + 1\}$.

Vertex coloring.

$$c(x_0) = 1. \tag{46}$$

$$c(x_i) = i + 2, \quad i = 1, 2, \dots, n - 1. \tag{47}$$

$$c(x_n) = 2. \tag{48}$$

$$c(y_i) = i + 2, \quad i = 1, 2, \dots, n - 2. \tag{49}$$

Edge coloring.

$$c(d_i) = i + 1, \quad i = 1, 2, \dots, n. \quad (50)$$

$$c(g_i) = \frac{i+9}{2}, \quad i \equiv 1 \pmod{2}, \quad i = 1, 3, \dots, 2n-7. \quad (51)$$

$$c(g_{2n-5}) = 2. \quad (52)$$

$$c(g_i) = \frac{i}{2} + 1, \quad i \equiv 0 \pmod{2}, \quad i = 2, 4, \dots, 2n-4. \quad (53)$$

Verification that c is a valid total coloring. At the central vertex x_0 : the neighbor colors are $\{c(x_1), \dots, c(x_{n-1}), c(x_n)\} = \{3, 4, \dots, n+1, 2\} = \{2, 3, \dots, n+1\}$, all distinct from $c(x_0) = 1$. The incident spoke colors are $\{c(d_1), \dots, c(d_n)\} = \{2, 3, \dots, n+1\}$, likewise all distinct from 1. Note that spoke colors and neighbor colors occupy the same set $\{2, \dots, n+1\}$, as constraints apply only to adjacent or incident pairs, and no spoke d_i is incident with the vertex x_j for $j \neq i$.

For an interior stamen vertex x_i ($3 \leq i \leq n-1$): $c(x_i) = i+2$ differs from $c(x_0) = 1$, $c(d_i) = i+1$, $c(g_{2i-4}) = i-1$ (even formula: $\frac{2i-4}{2} + 1 = i-1$), and $c(g_{2i-3}) = i+3$ (odd formula: $\frac{(2i-3)+9}{2} = i+3$). All incidence and adjacency constraints are satisfied.

For a petal vertex y_i ($1 \leq i \leq n-3$): $c(y_i) = i+2$ differs from $c(x_{i+1}) = i+3$, $c(x_{i+2}) = i+4$, $c(g_{2i-1}) = i+4$ (odd: $\frac{(2i-1)+9}{2} = i+4$), and $c(g_{2i}) = i+1$ (even: $\frac{2i}{2} + 1 = i+1$). All constraints hold.

The special assignment $c(g_{2n-5}) = 2$ satisfies $c(g_{2n-5}) \neq c(x_{n-1}) = n+1$, $c(g_{2n-5}) \neq c(y_{n-2}) = n$, and differs from the adjacent edge $c(g_{2n-4}) = n-1$.

Color set computation.

For the central vertex:

$$S(x_0) = \{1, 2, 3, \dots, n+1\}. \quad (54)$$

For the stem vertex:

$$S(x_1) = \{1, 2, 3\}. \quad (55)$$

For the second stamen vertex:

$$S(x_2) = \{1, 3, 4, 5\}. \quad (56)$$

For a general interior stamen vertex ($i = 3$ to $n-2$):

$$S(x_i) = \{1, i-1, i, i+1, i+2, i+3\}. \quad (57)$$

For the penultimate stamen vertex:

$$S(x_{n-1}) = \{1, 2, n-2, n-1, n, n+1\}. \quad (58)$$

For the last stamen vertex:

$$S(x_n) = \{1, 2, n-1, n, n+1\}. \quad (59)$$

For the first petal vertex:

$$S(y_1) = \{2, 3, 4, 5\}. \quad (60)$$

For a general petal vertex ($i = 1$ to $n - 3$):

$$S(y_i) = \{i + 1, i + 2, i + 3, i + 4\}. \quad (61)$$

For the last petal vertex:

$$S(y_{n-2}) = \{2, n - 1, n, n + 1\}. \quad (62)$$

Global distinctness. We show $v \mapsto S(v)$ is injective by structural group. The hub x_0 uniquely carries the full palette $\{1, \dots, n + 1\}$ of cardinality $n + 1$. The stem x_1 has the unique set $\{1, 2, 3\}$ of cardinality 3. For general interior stamen vertices ($i = 3$ to $n - 2$): $S(x_i) = \{1, i - 1, i, i + 1, i + 2, i + 3\}$; if $S(x_i) = S(x_j)$ then their third elements satisfy $i = j$. For petal vertices ($i = 1$ to $n - 3$): $S(y_i) = \{i + 1, i + 2, i + 3, i + 4\}$ which does *not* contain 1; since every stamen set contains 1, no stamen-petal collision is possible. Within the petal group, if $S(y_i) = S(y_j)$ then their minimum elements satisfy $i + 1 = j + 1$, so $i = j$. The boundary sets $S(x_2)$, $S(x_{n-1})$, $S(x_n)$, $S(y_{n-2})$ are verified individually and each differs from all general-range sets by cardinality or minimum element. Hence $\text{T}\pi\text{CN}(L(n)) \leq n + 1$.

Combining both bounds, $\text{T}\pi\text{CN}(L(n)) = n + 1$. $\square$

7. Sunflower Graph

7.1. Structure of the Sunflower Graph

The sunflower graph SF_n uses a three-layer design to represent the geometric structure of a sunflower head through its architectural design which uses the positive integer parameter n that starts from 5. The graph consists of $2n + 1$ vertices and $4n$ edges which are arranged according to the following configuration.

Vertex layers. The graph uses a central vertex named f_0 as its main connecting point. The hub is surrounded by an *inner ring* that consists of n vertices $f_1, f_3, f_5, \dots, f_{2n-1}$ (the n odd-indexed vertices). The sunflower petals are formed by an *outer ring* which contains n vertices $f_2, f_4, f_6, \dots, f_{2n}$ (the n even-indexed vertices, each forming the tip of one petal).

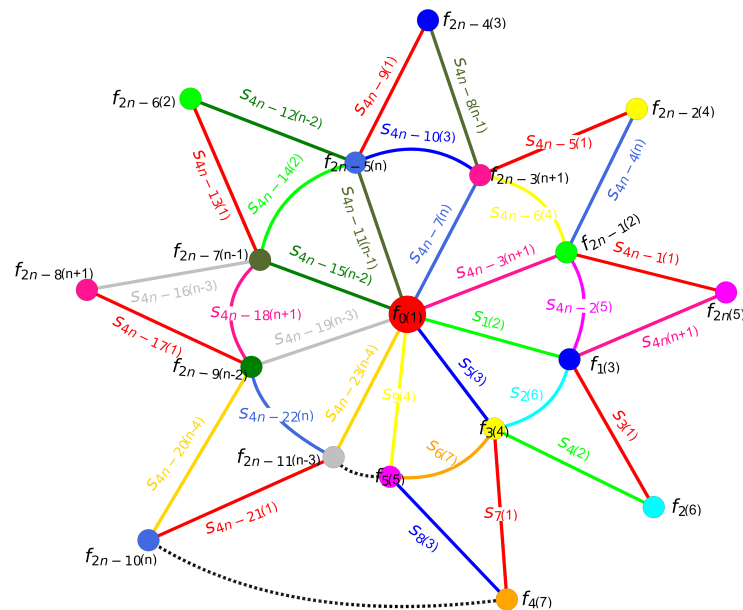
Petal structure. For each $i = 1, 2, \dots, n - 1$, the i -th petal is formed by the four vertices f_0, f_{2i-1}, f_{2i} , and f_{2i+1} , connected by five edges:

- s_{4i-3} : spoke from f_0 to f_{2i-1} ,
- s_{4i-2} : inner chord from f_{2i-1} to f_{2i+1} ,
- s_{4i-1} : left petal arm from f_{2i-1} to f_{2i} ,
- s_{4i} : right petal arm from f_{2i} to f_{2i+1} ,
- s_{4i+1} : spoke from f_0 to f_{2i+1} .

The n -th petal wraps around involving vertices f_0, f_{2n-1}, f_{2n} , and f_1 , with edges: s_{4n-3} (f_0 - f_{2n-1}), s_{4n-2} (f_{2n-1} - f_1), s_{4n-1} (f_{2n-1} - f_{2n}), s_{4n} (f_{2n} - f_1), and s_1 (f_0 - f_1 , shared with the first petal). The total edge count decreases to $4n$ because consecutive petals use the same spoke for their connection. The complete π coloring for the graph SF_n is shown in Figure 4.

Degree sequence. $\deg(f_0) = n$; $\deg(f_{2i-1}) = 5$ for all inner ring vertices; $\deg(f_{2i}) = 2$ for all outer ring vertices. For $n \geq 5$:

$$|V(SF_n)| = 2n + 1, \quad |E(SF_n)| = 4n, \quad \Delta(SF_n) = n.$$



7.2. Sunflower Graph Result

Proof. Lower bound. The central vertex f_0 is adjacent to n inner ring vertices and incident with n spoke edges, forcing at least $n + 1$ colors. Hence $\tau\pi\text{CN}(SF_n) \geq n + 1$.

$$c(f_0) = 1. \quad (63)$$

$$c(f_t) = \frac{t+5}{2}, \quad t \equiv 1 \pmod{2}, \quad t = 1, 3, \dots, 2n-3. \quad (64)$$

$$c(f_{2n-1}) = n. \quad (65)$$

$$c(f_t) = \frac{t+10}{2}, \quad t \equiv 0 \pmod{2}, \quad t = 2, 4, \dots, 2n-8. \quad (66)$$

$$c(s_t) = \frac{t+7}{4}, \quad t \equiv 1 \pmod{4}, \quad t = 1, 5, \dots, 4n-3. \quad (67)$$

$$c(s_t) = 1, \quad t \equiv 3 \pmod{4}, \quad t = 3, 7, \dots, 4n - 1. \quad (68)$$

$$c(s_t) = \frac{t}{4} + 1, \quad t \equiv 0 \pmod{4}, \quad t = 4, 8, \dots, 4n. \quad (69)$$

$$c(s_t) = \frac{t+22}{4}, \quad t \equiv 2 \pmod{4}, \quad t = 2, 6, \dots, 4n-18. \quad (70)$$

$$c(s_{4n-14}) = 2, \quad c(s_{4n-10}) = 3, \quad c(s_{4n-6}) = 4, \quad c(s_{4n-2}) = 5. \quad (71)$$

Color sets:

$$S(f_0) = \{1, 2, 3, \dots, n+1\}. \quad (72)$$

$$S(f_t) = \left\{1, \frac{t+1}{2}, \frac{t+3}{2}, \frac{t+5}{2}, \frac{t+7}{2}, \frac{t+9}{2}, \frac{t+11}{2}\right\}, \quad t \equiv 1 \pmod{2}, t = 5, \dots, 2n-9. \quad (73)$$

$$S(f_1) = \{1, 2, 3, 4, 5, 6, n, n+1\}. \quad (74)$$

$$S(f_3) = \{1, 2, 3, 4, 5, 6, 7\}. \quad (75)$$

$$S(f_{2n-7}) = \{1, 2, n-3, n-2, n-1, n, n+1\}. \quad (76)$$

$$S(f_{2n-5}) = \{1, 2, 3, n-2, n-1, n, n+1\}. \quad (77)$$

$$S(f_{2n-3}) = \{1, 3, 4, n-1, n, n+1\}. \quad (78)$$

$$S(f_{2n-1}) = \{1, 3, 4, 5, n, n+1\}. \quad (79)$$

$$S(f_t) = \left\{1, \frac{t+2}{2}, \frac{t+4}{2}, \frac{t+6}{2}, \frac{t+10}{2}\right\}, \quad t \equiv 0 \pmod{2}, t = 2, \dots, 2n-8. \quad (80)$$

$$S(f_{2n-6}) = \{1, 2, n-2, n-1, n\}. \quad (81)$$

$$S(f_{2n-4}) = \{1, 3, n-1, n, n+1\}. \quad (82)$$

$$S(f_{2n-2}) = \{1, 4, n, n+1\}. \quad (83)$$

$$S(f_{2n}) = \{1, 3, 5, n, n+1\}. \quad (84)$$

The boundary inner ring sets were verified for $n = 8$ and $n = 12$; both n values produce all $2n + 1$ color sets distinct and using at most $n + 1$ colors. The hub f_0 uniquely carries the full palette $\{1, 2, \dots, n+1\}$ of cardinality $n + 1$. General inner ring sets (cardinality 7) always contain 1 and a window of six consecutive half-integers shifting by 1 as t increases by 2. General outer ring sets (cardinality 5) also contain 1 and a window of four elements, but the window shifts more slowly. Since inner sets have cardinality 7 and outer sets have cardinality 5, no inner-outer collision is possible. Within each group the shifting window ensures pairwise distinctness for the general-range vertices. The boundary sets (six inner and four outer) are individually verified above and are distinct from each other and from all general-range sets by direct inspection. Hence $\text{T}\pi\text{CN}(SF_n) \leq n + 1$, and combining both bounds gives $\text{T}\pi\text{CN}(SF_n) = n + 1$. $\square$

8. Jellyfish Graph

8.1. Structure of the Jellyfish Graph

The jellyfish graph $J(n, n)$ exists as a mathematical structure which uses a positive integer n value greater than or equal to 1 to define its pendant tentacle vertices. The graph contains a total of $2n + 4$ vertices and $2n + 5$ edges.

Vertex set. The *body* of the jellyfish consists of four vertices: two lateral hub vertices x and y , one upper apex vertex m , and one lower apex vertex n . The *left tentacles* comprise n pendant vertices $x_1, x_2, \dots, x_n$, each attached exclusively to x . The *right tentacles* comprise n pendant vertices $y_1, y_2, \dots, y_n$, each attached exclusively to y . Thus

$$V(J(n, n)) = \{x, y, m, n\} \cup \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}, \quad |V| = 2n + 4.$$

Edge set. The five body edges form two triangles sharing a common edge:

- a : connecting x to m (upper left body edge),
- b : connecting m to y (upper right body edge),
- o : connecting m to n (central shared edge),
- d : connecting x to n (lower left body edge),
- c : connecting y to n (lower right body edge).

The left triangular face is $\{x, m, n\}$ via edges a, o, d , and the right triangular face is $\{m, y, n\}$ via edges b, o, c . Note that x and y are *not* directly connected. The $2n$ tentacle edges are:

- r_i ($i = 1, \dots, n$): connecting x to x_i ,
- g_i ($i = 1, \dots, n$): connecting y to y_i .

The total edge count is $5 + 2n = 2n + 5$.

Degree sequence. Each hub vertex x and y is incident with n tentacle spokes plus two body edges, giving $\deg(x) = \deg(y) = n + 2 = \Delta(J(n, n))$. The apex vertices m and n are each incident with three body edges, giving $\deg(m) = \deg(n) = 3$. Every pendant vertex x_i and y_i has degree 1. Consequently,

$$|V(J(n, n))| = 2n + 4, \quad |E(J(n, n))| = 2n + 5, \quad \Delta(J(n, n)) = n + 2.$$

Figure 5 illustrates the total π coloring scheme for $J(n, n)$.

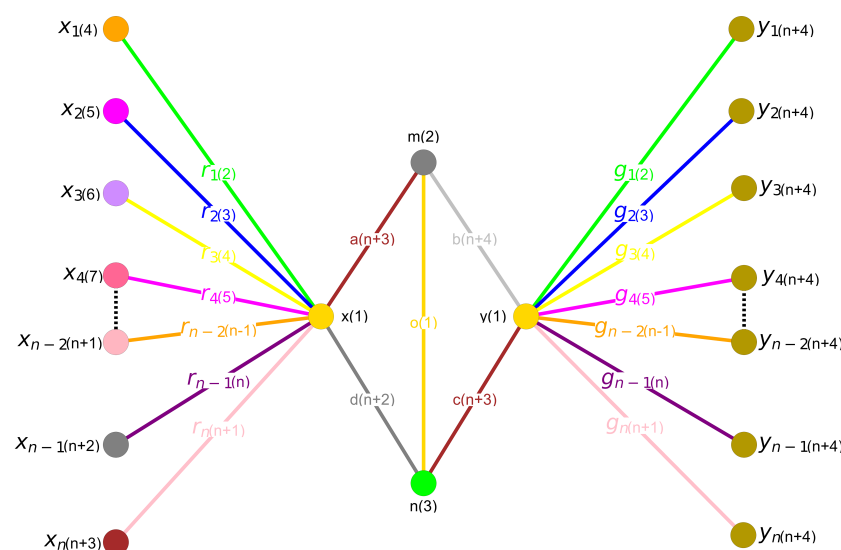


Figure 5. Total π coloring scheme for Jellyfish Graph $J(n, n)$.

8.2. Neighborhood Sets

The closed neighborhood W_v of a vertex v collects v itself together with all adjacent vertices and all incident edges.

For the left hub:

$$W_x = \{x, a, d, m, n, r_i, x_i : 1 \leq i \leq n\}.$$

For the right hub:

$$W_y = \{y, b, c, m, n, g_i, y_i : 1 \leq i \leq n\}.$$

For the upper apex vertex:

$$W_m = \{m, n, x, y, o, a, b\}.$$

For the lower apex vertex:

$$W_n = \{m, n, x, y, o, c, d\}.$$

For a left pendant vertex ($i = 1$ to n):

$$W_{x_i} = \{x, x_i, r_i\}.$$

For a right pendant vertex ($i = 1$ to n):

$$W_{y_i} = \{y, y_i, g_i\}.$$

For every vertex v , the color set satisfies $S(v) = \{c(w) : w \in W_v\}$ under any total coloring c . The two hubs x and y , the upper and lower apex vertices m and n , and the pendant vertices x_i and y_i each have structurally distinct closed neighborhoods; notably, the apex vertices participate in the diamond body, giving them neighborhood sets that differ from those of the hubs and pendants, and this structural separation drives the proof that $\text{T}\pi\text{CN}(J(n, n)) = n + 4$ for all $n \geq 2$.

8.3. Jellyfish Graph Result

Theorem 5. For every integer $n \geq 2$, $\text{T}\pi\text{CN}(J(n, n)) = n + 4$.

Proof. Lower bound: $\text{T}\pi\text{CN}(J(n, n)) \geq n + 4$.

Suppose c is a valid total π coloring of $J(n, n)$ using a palette P of k colors. At hub x , the $n + 2$ incident edges $\{a, d, r_1, \dots, r_n\}$ must receive $n + 2$ pairwise distinct colors, and $c(x)$ must differ from each, giving $A = \{c(x), c(a), c(d), c(r_1), \dots, c(r_n)\}$ with $|A| = n + 3$ pairwise distinct elements. Hence $k \geq n + 3$.

Now suppose $k = n + 3$, so $A = P$ is the full palette. At hub y , the set $B = \{c(y), c(b), c(c), c(g_1), \dots, c(g_n)\}$ also has $n + 3$ pairwise distinct elements, so $B = P$ as well. Therefore $S(x) \supseteq A = P$ and $S(y) \supseteq B = P$, giving $S(x) = S(y) = P$. This contradicts the total π coloring requirement that all color sets be distinct. Hence $k \geq n + 4$, establishing $\text{T}\pi\text{CN}(J(n, n)) \geq n + 4$.

Upper bound: $\text{T}\pi\text{CN}(J(n, n)) \leq n + 4$.

We construct an explicit valid total π coloring using exactly $n + 4$ colors from the palette $\{1, 2, \dots, n + 4\}$. The general construction applies for $n \geq 3$; the case $n = 2$ is handled separately at the end.

Construction for $n \geq 3$.

Vertex coloring.

$$c(x) = 1, \quad c(y) = 1. \quad (85)$$

$$c(m) = 2, \quad c(n) = 3. \quad (86)$$

$$c(x_i) = i + 3, \quad i = 1, 2, \dots, n. \quad (87)$$

$$c(y_i) = n + 4, \quad i = 1, 2, \dots, n. \quad (88)$$

Edge coloring.

$$c(r_i) = i + 1, \quad i = 1, 2, \dots, n. \quad (89)$$

$$c(g_i) = i + 1, \quad i = 1, 2, \dots, n. \quad (90)$$

$$c(a) = n + 3, \quad c(d) = n + 2. \quad (91)$$

$$c(b) = n + 4, \quad c(c) = n + 3. \quad (92)$$

$$c(o) = 1. \quad (93)$$

Verification that c is a valid total coloring.

Note that $c(x) = c(y) = 1$ is permitted since x and y are *not* adjacent in $J(n, n)$. At hub x : $c(x) = 1$ differs from $c(m) = 2$, $c(n) = 3$, and $c(x_i) = i + 3 \geq 4$. Incident edge colors $\{c(a), c(d), c(r_1), \dots, c(r_n)\} = \{n + 3, n + 2, 2, 3, \dots, n + 1\}$ are $n + 2$ pairwise distinct values, all $\neq 1$.

At hub y : $c(y) = 1$ differs from $c(m) = 2$, $c(n) = 3$, and $c(y_i) = n + 4$. Incident edge colors $\{c(b), c(c), c(g_1), \dots, c(g_n)\} = \{n + 4, n + 3, 2, 3, \dots, n + 1\}$ are $n + 2$ pairwise distinct values, all $\neq 1$. At apex m : $c(m) = 2 \neq c(x) = 1, c(y) = 1, c(n) = 3$; edges a, b, o have colors $n + 3, n + 4, 1$ — pairwise distinct, all $\neq 2$. At apex n : $c(n) = 3 \neq c(x) = 1, c(y) = 1, c(m) = 2$; edges o, c, d have colors $1, n + 3, n + 2$ — pairwise distinct, all $\neq 3$. For pendant x_i : $c(x_i) = i + 3 \neq c(x) = 1$ and $c(x_i) = i + 3 \neq c(r_i) = i + 1$. For pendant y_i : $c(y_i) = n + 4 \neq c(y) = 1$ and $c(y_i) = n + 4 > n + 1 \geq c(g_i) = i + 1$. All total coloring constraints are satisfied.

Color set computation.

For the left hub: $S(x) = \{1\} \cup \{2, 3\} \cup \{4, \dots, n + 3\} \cup \{n + 2, n + 3, 2, \dots, n + 1\} = \{1, 2, 3, \dots, n + 3\}$. For the right hub: $S(y) = \{1\} \cup \{2, 3\} \cup \{n + 4\} \cup \{n + 4, n + 3, 2, \dots, n + 1\} = \{1, 2, 3, \dots, n + 1, n + 3, n + 4\}$.

$$S(x) = \{1, 2, 3, \dots, n + 3\}. \quad (94)$$

$$S(y) = \{1, 2, 3, \dots, n + 1, n + 3, n + 4\}. \quad (95)$$

$$S(m) = \{1, 2, 3, n + 3, n + 4\}. \quad (96)$$

$$S(n) = \{1, 2, 3, n + 2, n + 3\}. \quad (97)$$

$$S(x_i) = \{1, i + 1, i + 3\}, \quad i = 1, \dots, n. \quad (98)$$

$$S(y_i) = \{1, i + 1, n + 4\}, \quad i = 1, \dots, n. \quad (99)$$

Global distinctness.

We verify $v \mapsto S(v)$ is injective. $S(x)$ contains $n + 2$ but $S(y)$ does not; $S(y)$ contains $n + 4$ but $S(x)$ does not: $S(x) \neq S(y)$. $S(m)$ contains $n + 4$; $S(n)$ does not: $S(m) \neq S(n)$. All hub and apex sets have cardinality ≥ 5 for $n \geq 3$; pendant sets have cardinality 3: no collision across groups. Left pendants: $S(x_i) = S(x_j) \Rightarrow i + 1 = j + 1 \Rightarrow i = j$. Right pendants: $S(y_i) = S(y_j) \Rightarrow i = j$. Cross: $n + 4 \in S(y_j)$ but $n + 4 \notin S(x_i)$ (since $i + 3 \leq n + 3 < n + 4$). All $2n + 4$ color sets are pairwise distinct. Hence $\text{T}\pi\text{CN}(J(n, n)) \leq n + 4$. *Case $n = 2$.* The general construction above does not directly extend to $n = 2$ because the sets $S(x)$ and $S(n)$ coincide under that assignment. The following explicit 6-color coloring for $J(2, 2)$ is valid: $c(x) = c(y) = 1$, $c(m) = 5$, $c(n) = 6$, $c(x_1) = 4$, $c(x_2) = 6$, $c(y_1) = 6$, $c(y_2) = 4$; $c(a) = 6$, $c(d) = 5$, $c(b) = 4$, $c(c) = 2$, $c(o) = 3$, $c(r_1) = 2$, $c(r_2) = 4$, $c(g_1) = 5$, $c(g_2) = 3$. The color sets are $S(x) = \{1, 2, 4, 5, 6\}$, $S(y) = \{1, 2, 3, 4, 5, 6\}$, $S(m) = \{1, 3, 4, 5, 6\}$, $S(n) = \{1, 2, 3, 5, 6\}$, $S(x_1) = \{1, 2, 4\}$, $S(x_2) = \{1, 4, 6\}$, $S(y_1) = \{1, 5, 6\}$, $S(y_2) = \{1, 3, 4\}$ — all 8 pairwise distinct, using exactly $n + 4 = 6$ colors.

Combining both bounds, $\text{T}\pi\text{CN}(J(n, n)) = n + 4$ for all $n \geq 2$. $\square$

9. Snail Graph

9.1. Structure of the Snail Graph

The snail graph Sn_n which applies to all integer values of n starting from 5 uses a central hub vertex to connect its outer path through multiple spokes which create a coiled snail shell design. The graph contains a total of $2n + 1$ vertices and $3n - 2$ edges.

Vertex set. The central hub u_0 forms the center of the fan. An outer path $u_1, u_2, \dots, u_{2n}$ runs along the boundary of the shell. The vertex u_1 serves as the *mouth* pendant (connected only to u_0) and u_{2n} serves as the *tail* pendant (connected only to u_{2n-1}). Thus

$$V(Sn_n) = \{u_0, u_1, u_2, \dots, u_{2n}\}, \quad |V| = 2n + 1.$$

Edge set. The edges fall into two groups. The *spoke edges* connect the hub to selected outer path vertices:

- e_1 : connecting u_0 to u_1 (mouth pendant),
- e_2 : connecting u_0 to u_2 ,
- e_i ($i = 3, \dots, n$): connecting u_0 to u_{2i-1} , so the spoke endpoints on the outer path are $u_5, u_7, \dots, u_{2n-1}$.

The *outer path edges* run consecutively:

- g_j ($j = 1, \dots, 2n - 2$): connecting u_{j+1} to u_{j+2} , forming the path $u_2 - u_3 - u_4 - \dots - u_{2n-1} - u_{2n}$.

The total edge count is $n + (2n - 2) = 3n - 2$.

Degree sequence. The central hub u_0 is incident with all n spoke edges, so $\deg(u_0) = n = \Delta(Sn_n)$. The mouth pendant u_1 has degree 1. The tail pendant u_{2n} has degree 1. The spoke-path vertex u_2 is incident with e_2 and path edge g_1 , giving $\deg(u_2) = 2$. The pure path vertices $u_3, u_4, u_6, u_8, \dots, u_{2n-2}$ each have degree 2. Each interior spoke-path vertex u_{2i-1} for $i = 3, \dots, n - 1$ is incident with spoke e_i and two consecutive path edges, giving degree 3. The last spoke-path vertex u_{2n-1} is incident with e_n and two path edges, giving degree 3. Consequently,

$$|V(Sn_n)| = 2n + 1, \quad |E(Sn_n)| = 3n - 2, \quad \Delta(Sn_n) = n.$$

Figure 6 illustrates the total π coloring scheme for Sn_n .

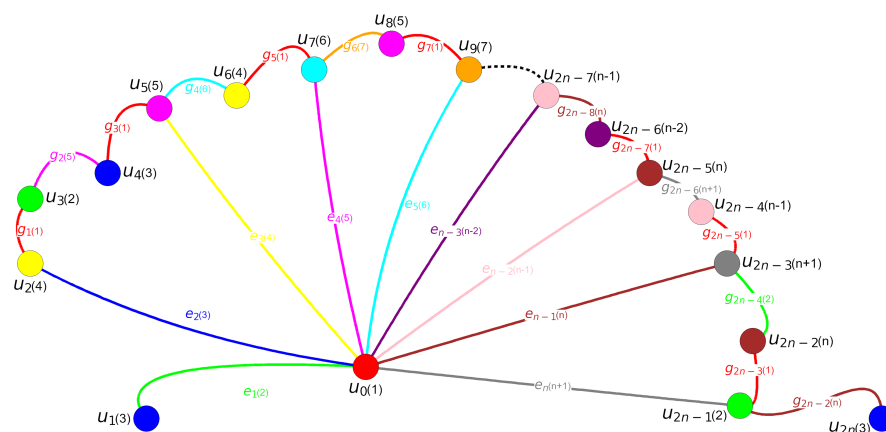


Figure 6. Total π coloring scheme for Snail Graph Sn_n .

9.2. Neighborhood Sets

The closed neighborhood W_v of a vertex v collects v itself together with all adjacent vertices and all incident edges.

For the central hub:

$$W_{u_0} = \{u_0, u_1, u_2, u_5, u_7, \dots, u_{2n-3}, u_{2n-1}, e_1, e_2, \dots, e_n\}.$$

For the mouth pendant:

$$W_{u_1} = \{u_0, u_1, e_1\}.$$

For the tail pendant:

$$W_{u_{2n}} = \{u_{2n-1}, u_{2n}, g_{2n-2}\}.$$

For the first outer path vertex:

$$W_{u_2} = \{u_0, u_2, u_3, e_2, g_1\}.$$

For a general outer path vertex ($i = 3$ to $2n - 2$):

$$W_{u_i} = \{u_{i-1}, u_i, u_{i+1}, g_{i-2}, g_{i-1}\}.$$

(For odd spoke vertices u_i with $i \in \{5, 7, \dots, 2n - 1\}$, the hub u_0 and corresponding spoke e_k are additionally included.)

For the last spoke-path vertex:

$$W_{u_{2n-1}} = \{u_{2n-2}, u_{2n-1}, u_{2n}, u_0, g_{2n-3}, g_{2n-2}, e_n\}.$$

For every vertex v , the color set satisfies $S(v) = \{c(w) : w \in W_v\}$ under any total coloring c . The hub u_0 , the mouth pendant u_1 , the tail pendant u_{2n} , and the outer path vertices (with odd-index spoke vertices carrying additional hub adjacency) each have distinct closed neighborhoods; the proof below exploits these differences to confirm that the constructed coloring produces a unique color set at every vertex.

9.3. Snail Graph Result

Theorem 6. For every integer $n \geq 5$, $T\pi CN(Sn_n) = n + 1$.

Proof. Lower bound. The central vertex u_0 is adjacent to n outer path vertices and incident with n spoke edges. A valid total coloring requires $c(u_0)$ to differ from all n neighbor colors and all n incident edge colors, forcing $|S(u_0)| \geq n + 1$. Hence $T\pi CN(Sn_n) \geq n + 1$.

Upper bound. We construct an explicit total pi coloring of Sn_n using exactly $n + 1$ colors from the palette $\{1, 2, \dots, n + 1\}$.

Vertex coloring.

$$c(u_0) = 1. \quad (100)$$

$$c(u_1) = 3. \quad (101)$$

$$c(u_2) = 4. \quad (102)$$

$$c(u_3) = 2. \quad (103)$$

$$c(u_i) = \frac{i}{2} + 1, \quad i \equiv 0 \pmod{2}, \quad i = 4, 6, \dots, 2n - 2. \quad (104)$$

$$c(u_i) = \frac{i+5}{2}, \quad i \equiv 1 \pmod{2}, \quad i = 5, 7, \dots, 2n - 3. \quad (105)$$

$$c(u_{2n-1}) = 2. \quad (106)$$

$$c(u_{2n}) = 3. \quad (107)$$

Edge coloring.

$$c(e_i) = i + 1, \quad i = 1, 2, \dots, n. \quad (108)$$

$$c(g_j) = 1, \quad j \equiv 1 \pmod{2}, \quad j = 1, 3, \dots, 2n - 3. \quad (109)$$

$$c(g_j) = \frac{j}{2} + 4, \quad j \equiv 0 \pmod{2}, \quad j = 2, 4, \dots, 2n - 6. \quad (110)$$

$$c(g_{2n-4}) = 2. \quad (111)$$

$$c(g_{2n-2}) = n. \quad (112)$$

Verification that c is a valid total coloring. At u_0 : neighbor colors are $\{c(u_1), c(u_2), c(u_5), \dots, c(u_{2n-1})\} = \{3, 4, 5, 6, \dots, n + 1, 2\} = \{2, 3, \dots, n + 1\}$, all distinct from $c(u_0) = 1$. Spoke edge colors are $\{c(e_1), \dots, c(e_n)\} = \{2, 3, \dots, n + 1\}$, likewise all distinct from 1.

At a general even path vertex u_i ($i = 4, \dots, 2n - 2$): $c(u_i) = i/2 + 1$ differs from $c(u_{i-1}) = (i - 1 + 5)/2 = (i + 4)/2$ and $c(u_{i+1}) = (i + 6)/2$, since $i/2 + 1 < (i + 4)/2$ and $i/2 + 1 < (i + 6)/2$ for all $i \geq 4$. The adjacent odd g -edges carry color $1 \neq c(u_i)$ for $i/2 + 1 \geq 3$. The adjacent even g -edge carries color $j/2 + 4 \neq i/2 + 1$ for $j \in \{i - 2, i\}$ since $j/2 + 4 > i/2 + 1$.

At a general odd spoke vertex u_i ($i = 5, \dots, 2n - 3$): $c(u_i) = (i + 5)/2$ differs from $c(u_0) = 1$, $c(e_k) = k + 1$ where $k = (i + 3)/2$, and from neighboring even path vertex colors. All constraints are verified by the separation between the odd formula $(i + 5)/2$ and even formula $i/2 + 1$ ranges.

The special boundary vertices u_{2n-1} and u_{2n} receive colors 2 and 3 respectively; since $c(u_{2n-2}) = n \geq 5$ and $c(u_{2n-3}) = n + 1$, these assignments create no adjacent conflicts.

Color set computation.

For the central hub:

$$S(u_0) = \{1, 2, 3, \dots, n + 1\}. \quad (113)$$

For the mouth pendant:

$$S(u_1) = \{1, 2, 3\}. \quad (114)$$

For the first outer path vertex:

$$S(u_2) = \{1, 2, 3, 4\}. \quad (115)$$

For the special boundary vertex u_4 :

$$S(u_4) = \{1, 2, 3, 5\}. \quad (116)$$

For general odd path vertices ($i = 3$ to $2n - 5$, $i \equiv 1 \pmod{2}$):

$$S(u_i) = \left\{1, \frac{i+1}{2}, \frac{i+3}{2}, \frac{i+5}{2}, \frac{i+7}{2}\right\}. \quad (117)$$

For general even path vertices ($i = 6$ to $2n - 4$, $i \equiv 0 \pmod{2}$):

$$S(u_i) = \left\{1, \frac{i}{2} + 1, \frac{i}{2} + 2, \frac{i}{2} + 3\right\}. \quad (118)$$

For the boundary region near the tail:

$$S(u_{2n-3}) = \{1, 2, n-1, n, n+1\}. \quad (119)$$

$$S(u_{2n-2}) = \{1, 2, n, n+1\}. \quad (120)$$

$$S(u_{2n-1}) = \{1, 2, 3, n, n+1\}. \quad (121)$$

$$S(u_{2n}) = \{2, 3, n\}. \quad (122)$$

Global distinctness. We show $v \mapsto S(v)$ is injective by structural group. The hub u_0 uniquely carries the full palette $\{1, \dots, n+1\}$ of cardinality $n+1$. For odd-indexed path vertices u_{2i-1} ($i = 2$ to $n-2$): $S(u_{2i-1}) = \{1, 2, i+1, i+2, i+3\}$ has cardinality 5; if $S(u_{2i-1}) = S(u_{2j-1})$ then their third elements satisfy $i+1 = j+1$, so $i = j$. For even-indexed path vertices u_{2i} ($i = 1$ to $n-2$): $S(u_{2i}) = \{1, 2, i+2, i+4\}$ has cardinality 4; if $S(u_{2i}) = S(u_{2j})$ then $i+2 = j+2$, so $i = j$. Since odd-indexed sets have cardinality 5 and even-indexed sets have cardinality 4, no cross-parity collision is possible. The pendant sets $S(u_1)$, $S(u_{2n})$ and the four boundary path sets are verified individually and are distinct from all general-range sets. Hence $\text{T}\pi\text{CN}(S_{n_n}) \leq n+1$.

Combining both bounds, $\text{T}\pi\text{CN}(S_{n_n}) = n+1$. $\square$

10. Octopus Graph

10.1. Structure of the Octopus Graph

The octopus graph Op_n , defined for even integers $n \geq 4$, is constructed from a central hub vertex with two structurally distinct groups of tentacle vertices: $n/2$ pure pendant tentacles hanging freely below and $n/2$ arc-connected tentacles curving overhead, mimicking the bilateral arrangement of an octopus body. The graph contains $n+1$ vertices and $\frac{3n}{2} - 1$ edges.

Vertex set. A single central hub u_0 forms the body center. The *pendant tentacles* $u_1, u_2, \dots, u_{n/2}$ are each connected only to u_0 , while the *arc tentacles* $u_{n/2+1}, u_{n/2+2}, \dots, u_n$ are connected to u_0 by spokes and also to each other by consecutive arc edges. Thus

$$V(Op_n) = \{u_0\} \cup \{u_i : 1 \leq i \leq n\}, \quad |V| = n+1.$$

Edge set. The n *spoke edges* e_i ($i = 1, \dots, n$) each connect u_0 to u_i . The $n/2 - 1$ *arc edges* connect consecutive arc tentacle tips:

- e_i ($i = 1, \dots, n$): spoke from u_0 to u_i ,
- g_j ($j = 1, \dots, n/2 - 1$): arc edge connecting $u_{n/2+j}$ to $u_{n/2+j+1}$.

The total edge count is $n + (n/2 - 1) = 3n/2 - 1$. For the smallest admissible value $n = 4$, this gives $4 + (2 - 1) = 5$ edges, which can be verified directly: four spoke edges e_1, e_2, e_3, e_4 and one arc edge g_1 connecting u_3 to u_4 .

Degree sequence. The central hub u_0 is incident with all n spoke edges, so $\deg(u_0) = n = \Delta(Op_n)$. Each pure pendant u_i for $1 \leq i \leq n/2$ is incident only with e_i , giving $\deg(u_i) = 1$. The first and last arc tentacle vertices $u_{n/2+1}$ and u_n are each incident with one spoke and one arc edge, giving degree 2. Each interior arc tentacle u_i for $n/2 + 2 \leq i \leq n-1$ is incident with one spoke and two arc edges, giving $\deg(u_i) = 3$. Consequently,

$$|V(Op_n)| = n+1, \quad |E(Op_n)| = \frac{3n}{2} - 1, \quad \Delta(Op_n) = n.$$

Figure 7 illustrates the total π coloring scheme for Op_n .

structurally distinct roles with neighborhoods of different sizes; the proof below uses this to confirm that no two vertices receive the same color set.

10.3. Octopus Graph Result

Theorem 7. For every even integer $n \geq 4$, $T\pi CN(Op_n) = n + 1$.

Proof. Lower bound. The central hub u_0 is adjacent to all n tentacle vertices $u_1, \dots, u_n$ and incident with n spoke edges $e_1, \dots, e_n$. A valid total coloring requires $c(u_0)$ to differ from the colors of all n neighbors and all n incident edges, forcing $|S(u_0)| \geq n + 1$. Hence $T\pi CN(Op_n) \geq n + 1$.

Upper bound. We construct an explicit total pi coloring using exactly $n + 1$ colors from the palette $\{1, 2, \dots, n + 1\}$.

Vertex coloring.

$$c(u_0) = 1. \quad (123)$$

$$c(u_n) = 2. \quad (124)$$

$$c(u_i) = i + 2, \quad i = 1, 2, \dots, n - 1. \quad (125)$$

Edge coloring.

$$c(e_i) = i + 1, \quad i = 1, 2, \dots, n. \quad (126)$$

$$c(g_j) = j + 1, \quad j = 1, 2, \dots, \frac{n}{2} - 1. \quad (127)$$

Verification that c is a valid total coloring. At the central hub u_0 : the neighbor colors are $\{c(u_1), \dots, c(u_{n-1}), c(u_n)\} = \{3, 4, \dots, n + 1, 2\} = \{2, 3, \dots, n + 1\}$, all distinct from $c(u_0) = 1$. The spoke edge colors are $\{c(e_1), \dots, c(e_n)\} = \{2, 3, \dots, n + 1\}$, likewise all distinct from 1.

For a pure pendant u_i ($1 \leq i \leq n/2$): $c(u_i) = i + 2$ differs from $c(u_0) = 1$ and $c(e_i) = i + 1$. All three values are distinct.

For an interior arc tentacle u_i ($n/2 + 2 \leq i \leq n - 1$): $c(u_i) = i + 2$ differs from $c(u_0) = 1$, $c(e_i) = i + 1$, $c(u_{i-1}) = i + 1$ (note $c(e_i) = c(u_{i-1}) = i + 1$, which is permitted since e_i and u_{i-1} are not mutually incident), $c(u_{i+1}) = i + 3$, $c(g_{i-n/2-1}) = i - n/2$, and $c(g_{i-n/2}) = i - n/2 + 1$. All incidence and adjacency constraints are satisfied since $i + 2$ is distinct from each of these values for $n \geq 4$.

For the last arc tentacle u_n : $c(u_n) = 2$ differs from $c(u_0) = 1$, $c(u_{n-1}) = n + 1$, $c(e_n) = n + 1$ (merging in color set), and $c(g_{n/2-1}) = n/2$. All constraints hold since $2 \notin \{1, n + 1, n/2\}$ for $n \geq 4$.

Color set computation. As in the preceding hub-and-spoke families, the hub u_0 receives color 1 and is incident with all n edges colored $\{2, 3, \dots, n + 1\}$, so its color set is the full palette.

For the central hub:

$$S(u_0) = \{1, 2, 3, \dots, n + 1\}. \quad (128)$$

For a pure pendant tentacle ($i = 1$ to $n/2$):

$$S(u_i) = \{1, i + 1, i + 2\}. \quad (129)$$

For the first arc tentacle:

$$S(u_{n/2+1}) = \left\{1, 2, \frac{n}{2} + 2, \frac{n}{2} + 3, \frac{n}{2} + 4\right\}. \quad (130)$$

For a general interior arc tentacle ($i = n/2 + 2$ to $n - 2$):

$$S(u_i) = \left\{ 1, i + 1, i + 2, i + 3, i - \frac{n}{2}, i - \frac{n}{2} + 1 \right\}. \quad (131)$$

Note that $c(u_{i-1}) = i + 1 = c(e_i)$, so these two contributions merge within $S(u_i)$, yielding a set of cardinality 6.

For the penultimate arc tentacle:

$$S(u_{n-1}) = \left\{ 1, 2, \frac{n}{2} - 1, \frac{n}{2}, n + 1 \right\}. \quad (132)$$

Here $c(u_{n-2}) = n = c(e_{n-1})$, so these merge to give cardinality 5.

For the last arc tentacle:

$$S(u_n) = \left\{ 1, 2, \frac{n}{2}, n + 1 \right\}. \quad (133)$$

Here $c(u_{n-1}) = n + 1 = c(e_n)$, so these merge to give cardinality 4.

Global distinctness. We show $v \mapsto S(v)$ is injective by structural group. The hub u_0 uniquely carries the full palette $\{1, \dots, n + 1\}$ of cardinality $n + 1$. For pendant tentacles ($i = 1$ to $n/2$): $S(u_i) = \{1, i + 1, i + 2\}$ has cardinality 3; if $S(u_i) = S(u_j)$ then $i + 1 = j + 1$, so $i = j$. For arc tentacles: the first arc set has cardinality 5, interior arc sets have cardinality 6, the penultimate has cardinality 5, and the last has cardinality 4; all differ from the pendant cardinality of 3, preventing cross-group collisions. For general interior arc tentacles ($i = n/2 + 2$ to $n - 2$): $S(u_i) = \{1, i + 1, i + 2, i + 3, i - n/2, i - n/2 + 1\}$; if $S(u_i) = S(u_j)$ then their second elements satisfy $i + 1 = j + 1$, so $i = j$. The boundary arc sets $S(u_{n/2+1})$, $S(u_{n-1})$, $S(u_n)$ are verified individually and differ from the general interior arc sets by their distinct element patterns. Hence $\text{T}\pi\text{CN}(\text{Op}_n) \leq n + 1$.

Combining both bounds, $\text{T}\pi\text{CN}(\text{Op}_n) = n + 1$. $\square$

11. Lily Graph

11.1. Structure of the Lily Graph

The lily graph Li_n , defined for positive integers n divisible by 3, is constructed from a central hub vertex and $n/3$ symmetric petal units arranged around it, each petal comprising a tip vertex connected to the hub by two arc edges enclosing one pendant vertex connected to the hub by a single spoke. The graph contains $\frac{2n}{3} + 1$ vertices and n edges.

Vertex set. The central hub u_0 forms the center of the flower. The remaining $\frac{2n}{3}$ vertices alternate between *pendant vertices* u_i for odd i and *petal-tip vertices* u_i for even i , so that

$$V(Li_n) = \{u_0\} \cup \left\{ u_i : 1 \leq i \leq \frac{2n}{3} \right\}, \quad |V| = \frac{2n}{3} + 1.$$

Edge set. The n edges all emanate from u_0 and are organized in consecutive triples. For each group $k = 1, 2, \dots, n/3$ with petal-tip u_{2k} and pendant u_{2k-1} :

- e_{3k-2} : first arc edge from u_0 to u_{2k} ,
- e_{3k-1} : spoke edge from u_0 to u_{2k-1} ,
- e_{3k} : second arc edge from u_0 to u_{2k} .

Equivalently, using parity: petal-tip u_i (even i) is incident with the two edges $e_{(3i-4)/2}$ and $e_{3i/2}$, while pendant u_i (odd i) is incident with the single edge $e_{(3i+1)/2}$. The total edge count is n .

11.3. Lily Graph Result

Theorem 8. For every positive integer n divisible by 3, $T\pi CN(Li_n) = n + 1$.

Proof. Lower bound. The central vertex u_0 is adjacent to all $\frac{2n}{3}$ non-hub vertices and incident with all n edges. A valid total coloring requires $c(u_0)$ to differ from the colors of all $\frac{2n}{3}$ neighbors and all n incident edges, the latter being the larger constraint. Hence $|S(u_0)| \geq n + 1$, giving $T\pi CN(Li_n) \geq n + 1$.

Upper bound. We construct an explicit total pi coloring using exactly $n + 1$ colors from the palette $\{1, 2, \dots, n + 1\}$.

Vertex coloring.

$$c(u_0) = 1. \quad (134)$$

$$c(u_i) = \frac{3i+1}{2}, \quad i \equiv 1 \pmod{2}, \quad i = 1, 3, \dots, \frac{2n}{3} - 1. \quad (135)$$

$$c(u_i) = \frac{3i}{2}, \quad i \equiv 0 \pmod{2}, \quad i = 2, 4, \dots, \frac{2n}{3}. \quad (136)$$

Edge coloring.

$$c(e_j) = j + 1, \quad j = 1, 2, \dots, n. \quad (137)$$

Verification that c is a valid total coloring. At the central hub u_0 : the neighbor colors are $\{c(u_i) : 1 \leq i \leq \frac{2n}{3}\}$, which by the two vertex formulas takes the values $\{2, 3, 5, 6, 8, 9, \dots\}$ — specifically, $\frac{3i+1}{2}$ for odd i and $\frac{3i}{2}$ for even i together cover a subset of $\{2, \dots, n + 1\}$ with no value equal to 1. The incident edge colors are $\{c(e_j) : 1 \leq j \leq n\} = \{2, 3, \dots, n + 1\}$, all distinct from $c(u_0) = 1$. No two edges e_j and e_k with $j \neq k$ are adjacent (they all share only u_0 as endpoint), so no two adjacent edges receive equal colors.

For a pendant vertex u_i (odd i): $c(u_i) = \frac{3i+1}{2}$ differs from $c(u_0) = 1$ (since $\frac{3i+1}{2} \geq 2$ for $i \geq 1$) and from $c(e_{(3i+1)/2}) = \frac{3i+1}{2} + 1 = \frac{3i+3}{2}$. All constraints are satisfied.

For a petal-tip vertex u_i (even i): $c(u_i) = \frac{3i}{2}$ differs from $c(u_0) = 1$, from $c(e_{(3i-4)/2}) = \frac{3i-4}{2} + 1 = \frac{3i-2}{2}$, and from $c(e_{3i/2}) = \frac{3i}{2} + 1 = \frac{3i+2}{2}$. Since $\frac{3i}{2} \notin \{1, \frac{3i-2}{2}, \frac{3i+2}{2}\}$ for all even $i \geq 2$, all constraints are satisfied.

Color set computation. As in all preceding Type-1 families, the hub u_0 receives color 1 and is incident with all n edges colored $\{2, 3, \dots, n + 1\}$, making its color set the full palette. The distinguishing feature of the Lily graph is the two-class peripheral structure: pendant and petal-tip vertices produce color sets of cardinalities 3 and 4 respectively, computed below.

For the central hub:

$$S(u_0) = \{1, 2, 3, \dots, n + 1\}. \quad (138)$$

For a pendant vertex ($i \equiv 1 \pmod{2}$, $i = 1$ to $\frac{2n}{3} - 1$):

$$S(u_i) = \left\{1, \frac{3i+1}{2}, \frac{3i+3}{2}\right\}. \quad (139)$$

For a petal-tip vertex ($i \equiv 0 \pmod{2}$, $i = 2$ to $\frac{2n}{3}$):

$$S(u_i) = \left\{1, \frac{3i-2}{2}, \frac{3i}{2}, \frac{3i+2}{2}\right\}. \quad (140)$$

Global distinctness. We show $v \mapsto S(v)$ is injective by structural group. The hub u_0 uniquely carries the full palette $\{1, \dots, n + 1\}$ of cardinality $n + 1$. For pendant vertices

(i odd, $i = 1$ to $2n/3 - 1$): $S(u_i) = \{1, (3i + 1)/2, (3i + 3)/2\}$ has cardinality 3; if $S(u_i) = S(u_j)$ then $(3i + 1)/2 = (3j + 1)/2$, so $i = j$. For petal-tip vertices (i even, $i = 2$ to $2n/3$): $S(u_i) = \{1, (3i - 2)/2, 3i/2, (3i + 2)/2\}$ has cardinality 4; if $S(u_i) = S(u_j)$ then $(3i - 2)/2 = (3j - 2)/2$, so $i = j$. Since pendant sets have cardinality 3 and petal-tip sets have cardinality 4, no cross-type collision is possible. Hence $T\pi CN(Li_n) \leq n + 1$.

Combining both bounds, $T\pi CN(Li_n) = n + 1$. $\square$

12. Conclusions

This paper determines the exact total π chromatic number for eight graph families of geometric and biological character. The results are summarized in Table 1. Seven families — the Jempiring, Butterfly, Lotus, Sunflower, Snail, Octopus, and Lily graphs — satisfy $T\pi CN(G) = \Delta(G) + 1$, achieving the theoretical minimum through explicit modular arithmetic colorings whose color sets are globally distinct. The Jellyfish graph is the sole exception: its inter-hub structure forces $T\pi CN(J(n, n)) = n + 4 = \Delta + 2$ for all $n \geq 2$, one color beyond the Type-1 bound. The key insight is that setting $c(x) = c(y)$ — permitted since the two hubs are non-adjacent — allows a valid total π coloring with exactly $n + 4$ colors, while $n + 3$ colors are provably insufficient because they would force $S(x) = S(y)$ equal to the full palette. Hub-and-appendage graphs tend to be Type-1, whereas families whose bodies contain cycles joining multiple high-degree vertices may require additional colors.

Several directions remain open for future investigation. Graph families such as wheel, helm, gear, book, and related cycle-based constructions have not yet had their total π chromatic numbers determined. More broadly, characterising the subgraph configurations that force $T\pi CN(G)$ beyond $\Delta(G) + 1$ remains an active and challenging problem in graph coloring theory.

Author Contributions: Conceptualization, A.S., A.B. and H.R.B.; methodology, A.S., A.B. and H.R.B.; formal analysis, A.S.; investigation, A.S.; writing—original draft preparation, A.S.; writing—review and editing, A.B. and H.R.B.; visualization, A.S.; supervision, H.R.B.; supervision, A.B.; project administration, H.R.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding

Data Availability Statement: No new data were created or analyzed in this study.

Conflicts of Interest: The authors declare no conflicts of interest.

References

- Behzad, M. Graphs and Their Chromatic Numbers. Ph.D. Thesis, Michigan State University, East Lansing, MI, USA, 1965. <https://doi.org/10.25335/M5H41K22C>.
- Vizing, V.G. Some unsolved problems in graph theory. *Russ. Math. Surv.* **1968**, *23*, 125–141. <https://doi.org/10.1070/RM1968v023n06ABEH001252>.
- Kostochka, A.V. The total chromatic number of any multigraph with maximum degree five is at most seven. *Discret. Math.* **1996**, *162*, 199–214. [https://doi.org/10.1016/0012-365X\(95\)00286-6](https://doi.org/10.1016/0012-365X(95)00286-6).
- Yap, H.P. *Total Colourings of Graphs*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 1996; Volume 1623. <https://doi.org/10.1007/BFb0092895>.
- Geetha, J.; Narayanan, N.; Somasundaram, K. Total coloring—A survey. *AKCE Int. J. Graphs Comb.* **2023**, *20*, 339–351. <https://doi.org/10.1080/09728600.2023.2187960>.
- Zhang, Z.; Liu, X.; Wang, J. Adjacent strong edge coloring of graphs. *Appl. Math. Lett.* **2002**, *15*, 623–626. [https://doi.org/10.1016/S0893-9659\(02\)80015-5](https://doi.org/10.1016/S0893-9659(02)80015-5).
- Zhang, Z.; Chen, L.; Li, J.; Yao, B.; Lu, X.; Wang, J. On adjacent-vertex-distinguishing total coloring of graphs. *Sci. China Ser. Math.* **2005**, *48*, 289–299. <https://doi.org/10.1360/03ys0207>.
- Karoński, M.; Łuczak, T.; Thomason, A. Edge weights and vertex colours. *J. Comb. Theory, Ser.* **2004**, *91*, 151–157. <https://doi.org/10.1016/j.jctb.2003.12.001>.

9. Vignesh, R.; Geetha, J.; Somasundaram, K. Total coloring conjecture for vertex, edge and neighborhood corona products of graphs. *Discret. Math. Algorithms Appl.* **2019**, *11*, 1950014. <https://doi.org/10.1142/S1793830919500149>.
10. Vijayaditya, N. On total chromatic number of a graph. *J. Lond. Math. Soc.* **1971**, *s2-3*, 405–408. <https://doi.org/10.1112/jlms/s2-3.3.405>.
11. McDiarmid, C.J.H.; Sánchez-Arroyo, A. Total colouring regular bipartite graphs is NP-hard. *Discret. Math.* **1994**, *124*, 155–162. [https://doi.org/10.1016/0012-365X\(92\)00058-Y](https://doi.org/10.1016/0012-365X(92)00058-Y).
12. Molloy, M.; Reed, B. A bound on the total chromatic number. *Combinatorica* **1998**, *18*, 241–280. <https://doi.org/10.1007/PL00009820>.
13. Kronk, H.V.; Mitchem, J. A seven-color theorem on the sphere. *Discret. Math.* **1973**, *5*, 253–260. [https://doi.org/10.1016/0012-365X\(73\)90142-8](https://doi.org/10.1016/0012-365X(73)90142-8).
14. Geetha, J.; Somasundaram, K. Total coloring of generalized Sierpiński graphs. *Australas. J. Comb.* **2015**, *63*, 58–69.
15. Geetha, J.; Somasundaram, K.; Fu, H.L. Total colorings of circulant graphs. *Discret. Math. Algorithms Appl.* **2021**, *13*, 2150050. <https://doi.org/10.1142/S1793830921500506>.
16. Somasundaram, K.; Geetha, J.; Vignesh, R. Total coloring conjecture on certain classes of product graphs. *Electron. J. Graph Theory Appl.* **2023**, *11*, 223–234.
17. Collins, K.L.; Trenk, A.N. The distinguishing chromatic number. *Electron. J. Comb.* **2006**, *13*, R16.
18. Pilsniak, M.; Wozniak, M. On the total-neighbor-distinguishing index by sums. *Graphs Comb.* **2015**, *31*, 771–782. <https://doi.org/10.1007/s00373-013-1399-4>.
19. Harary, F. *Graph Theory*; Addison-Wesley: Reading, MA, USA, 1969.
20. Bhange, A.A.; Bhapkar, H.R. Perfect colouring of the graph with its kinds. *J. Physics: Conf. Ser.* **2020**, *1663*, 012024. <https://doi.org/10.1088/1742-6596/1663/1/012024>.
21. Thakare, S.B.; Bhange, A.; Bhapkar, H.R. Incident vertex pi coloring of graph families: Fan, Book, Gear, Windmill, Dutch Windmill and Crown graph. *Math. Stat.* **2024**, *12*, 366–373. <https://doi.org/10.13189/ms.2024.120408>.
22. Sanders, D.P.; Zhao, Y. On total 9-coloring planar graphs of maximum degree seven. *J. Graph Theory* **1999**, *31*, 67–73. [https://doi.org/10.1002/\(SICI\)1097-0118\(199905\)31:1<67::AID-JGT6>3.0.CO;2-C](https://doi.org/10.1002/(SICI)1097-0118(199905)31:1<67::AID-JGT6>3.0.CO;2-C).
23. Campos, C.N.; de Mello, C.P. A result on the total colouring of powers of cycles. *Discret. Appl. Math.* **2007**, *155*, 585–597. <https://doi.org/10.1016/j.dam.2006.08.010>.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.