

Proper Total Domination in Generalized Fans and Wheels

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Abstract

A set S of vertices in a graph G is a total dominating set if every vertex of G is adjacent to (totally dominated by) some vertex of S . The number of vertices in S that totally dominate a vertex v of G is denoted by $\sigma_S(v)$. A total dominating set S is a pt -dominating set (or pt -dominating set) if $\sigma_S(u) \neq \sigma_S(v)$ for every two adjacent vertices u and v of G . The pt -domination number $\gamma_{pt}(G)$ of a graph G is the minimum cardinality of a pt -dominating set in G . Two well-known classes of highly symmetric graphs are investigated here, namely fans and wheels. For $n \geq 2$, the fan F_n is the join $P_n \vee K_1$ of the path P_n of order n and the complete graph K_1 of order 1, while for $n \geq 3$, the wheel W_n is the join $C_n \vee K_1$ of the cycle C_n of order n and K_1 . For a positive integer t , the general fan $F_{n,t}$ is the graph $P_n \vee \overline{K}_t$ and the general wheel $W_{n,t}$ is the graph $C_n \vee \overline{K}_t$ where $\overline{K}_t$ is the empty graph of order t . All fans, wheels, general fans and general wheels are determined that possess pt -dominating sets. Furthermore, pt -domination numbers of all these graphs are determined as well.

Keywords: total domination; proper total domination; proper total domination number; fan; wheel; general fan and general wheel

MSC: 05C69; 05C38; 05C76; 05C15

1. Introduction

A vertex u in a graph G is said to *totally dominate* a vertex v if u is adjacent to v . A set S of vertices in G is a *total dominating set* if every vertex of G is totally dominated by at least one vertex of S . Consequently, a graph G has a total dominating set if and only if G has no isolated vertices. The minimum cardinality of a total dominating set in G is the *total domination number* $\gamma_t(G)$ of G . A total dominating set S with $|S| = \gamma_t(G)$ is called a γ_t -set of G . This concept was introduced by Cockayne, Dawes and Hedetniemi [1] in 1980. More information on total domination in graphs (as well as some special classes of graphs) can be found in [2–8], for example.

For a total dominating set S of a graph G , the number of vertices of S that totally dominate a vertex v of G is denoted by $\sigma_S(v)$. While there exists no graph G with a total dominating set S such that $\sigma_S(u) \neq \sigma_S(v)$ for every two distinct vertices u and v of G , such sets exist for every two adjacent vertices u and v of G . The concept of total domination has been studied extensively with many interesting results and applications (see [5,6], for example). A 2013 book by Henning and Yeo [6] deals exclusively with total domination in graphs with many real life applications. Another area of research in graph theory is the study of graph coloring; its origin is the famous four-color problem. Today, graph coloring is one of the most popular areas of graph theory because of its numerous fascinating problems and applications to modern society. In a proper coloring of a graph, adjacent



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vertices are required to be colored differently. The minimum number of colors in a proper coloring of a graph G is its chromatic number $\chi(G)$. Combining total domination and proper graph coloring gave rise to proper total domination in graphs.

A total dominating set S in a graph G is a *proper total dominating set* (or a *pt-dominating set*) if $\sigma_S(u) \neq \sigma_S(v)$ for every two adjacent vertices u and v of G . This concept was introduced by Chartrand and studied in [9–12]. This concept has a close connection to the topics studied in [13–16] and the methodological and applied themes in proper total dominations align closely with many recent research topics in other areas of science (see [17,18], for example).

In general, it is challenging to determine whether a given graph possesses a *pt-dominating set*. While several well-known classes of graphs have been shown to possess a *pt-dominating set*, for many classes this remains unknown (see [9,11,12]). Hence, the following observations are useful. A graph G is *locally irregular* if $\deg u \neq \deg v$ for every two adjacent vertices u and v of G (see [19]). The vertex set $S = V(G)$ of a locally irregular graph G is a total dominating set of G since $\sigma_S(v) = \deg v$ for every vertex v of G . In fact, locally irregular graphs are the only graphs G for which $V(G)$ is a *pt-dominating set*.

Observation 1 ([11]). *If S is a pt-dominating set of a connected graph G , then the subgraph $G[S]$ induced by S is locally irregular without isolated vertices. Thus, $|S| \geq 3$ while $|S| = 3$ if and only if $G[S] = P_3$. Furthermore, the vertex set of a graph G is a pt-dominating set of G if and only if G is locally irregular.*

The set $N(v)$ of vertices adjacent to a vertex v in a graph G is the *neighborhood* of v and the set $N[v] = \{v\} \cup N(v)$ is the *closed neighborhood* of v . Two vertices u and v in a graph G are called *twins* if $N(u) = N(v)$. If u and v are adjacent vertices in a graph G such that $N[u] = N[v]$, then u and v are *adjacent twins*.

Observation 2 ([11]). *If u and v are adjacent twins of a connected graph G with pt-dominating set S , then exactly one of u and v belongs to S . Consequently, if G contains a complete subgraph K_t of order $t \geq 3$ such that every two vertices of K_t are twins of G , then G has no pt-dominating set. In particular, every complete graph of order 3 or more has no pt-dominating set.*

The minimum cardinality of a *pt-dominating set* in a graph G is the *proper total domination number* (*pt-domination number*) $\gamma_{pt}(G)$ of G . A *pt-dominating set* S with $|S| = \gamma_{pt}(G)$ is called a γ_{pt} -set of G . The *pt-domination numbers* have been determined for several well-known classes of graphs possessing a *pt-dominating set*, including paths, cycles, trees, complete multipartite graphs, and Cartesian products $G \square K_2$ of a graph G and K_2 (see [9,11,12]). Since proper total domination is more restrictive than total domination, it follows that $\gamma_t(G) \leq \gamma_{pt}(G)$ for every graph G with a *pt-dominating set*. Relationships between the classic total domination number and proper total domination number were investigated in [9]. All pairs a, b of positive integers were determined for which there is a graph G with a *pt-dominating set* such that $\gamma_t(G) = a$ and $\gamma_{pt}(G) = b$. For each graph G with a *pt-dominating set* S , the numbers in the set $\Sigma_S = \{\sigma_S(v) : v \in V(G)\}$ give rise to a proper coloring of G . The *proper total chromatic number* $\chi_{pt}(G)$ of G is the minimum value of $|\Sigma_S|$ among all *pt-dominating sets* S of G . Hence, $\chi(G) \leq \chi_{pt}(G)$ for every graph G with a *pt-dominating set*. Furthermore, it was shown in [10] that for every pair a, b of positive integers with $2 \leq a \leq b$, there is a graph G with $\chi(G) = a$ and $\chi_{pt}(G) = b$. In general, it is challenging to determine the exact value of the *pt-domination number* of a given graph having a *pt-dominating set*. Among those graphs whose *pt-domination numbers* have been determined are paths and cycles.

Theorem 1 ([9]). For an integer $n \geq 3$, let P_n be a path of order n and C_n a cycle of order n .

- (a) The path P_n has a pt -dominating set if and only if $n \equiv 3 \pmod{4}$. Furthermore, P_n has a unique pt -dominating set and $\gamma_{pt}(P_n) = \frac{3(n+1)}{4}$ when $n \equiv 3 \pmod{4}$.
- (b) The cycle C_n has a pt -dominating set if and only if $n \equiv 0 \pmod{4}$. Furthermore, C_n has a unique pt -dominating set and $\gamma_{pt}(C_n) = \frac{3n}{4}$ when $n \geq 4$ and $n \equiv 0 \pmod{4}$.

Results on paths and cycles have led to a study of classes of highly symmetric connected graphs related to paths and cycles. One such class consists of Cartesian products $G \square H$ where G and H are either paths or cycles in [9,12]. In particular, the grid $P_n \square P_m$ has a pt -dominating set for all integers n, m with $n \geq m \geq 2$ (see [9]). On the other hand, the prism $C_n \square K_2$ possesses a pt -dominating set if and only if $n \geq 4$ is even. While the cylindrical graph $C_n \square P_m$ possesses a pt -dominating set for every even integer $n \geq 4$ and every integer $m \geq 3$, many cylindrical graphs $C_n \square P_m$ have no pt -dominating set for odd integers $n \geq 3$. For example, $C_3 \square P_m$ possesses a pt -dominating set if and only if $m = 3$ (see [12]). For odd integers $n \geq 5$, it is not known whether there is a characterization of cylindrical graphs $C_n \square P_m$ possessing a pt -dominating set. While a cyclical graph $C_n \square C_m$ has a pt -dominating set if at least one of n and m is even, the graph $C_3 \square C_m$ does not possess a pt -dominating set for each odd integer $m \geq 3$. It is not known which cyclical graphs $C_n \square C_m$ possess a pt -dominating set for odd integers $n, m \geq 5$. Furthermore, for those grids, prisms, cylindrical graphs, and cyclical graphs possessing a pt -dominating set, the exact values of their pt -domination numbers are not known in general. All of these lead us to consider two other well-known classes of highly symmetric graphs closely related to paths and cycles, namely, fans, wheels, and generalized fans and wheels.

For two vertex-disjoint graphs F and H , the graph $F \vee H$ is the *join* of F and H , that is, $V(F \vee H) = V(F) \cup V(H)$ and $E(F \vee H) = E(F) \cup E(H) \cup \{uv : u \in V(F), v \in V(H)\}$. The graph $F_n = P_n \vee K_1$ of order $n + 1 \geq 3$ is referred to as a *fan* and the graph $W_n = C_n \vee K_1$ of order $n + 1 \geq 4$ is a *wheel*. More generally, for each positive integer t , the graph $F_{n,t} = P_n \vee \overline{K}_t$ is referred to as a *generalized fan* and the graph $W_{n,t} = C_n \vee \overline{K}_t$ is a *generalized wheel*, where $\overline{K}_t$ is the empty graph of order t . Here, we determine all generalized fans and all generalized wheels that possess a pt -dominating set as well as their pt -domination numbers. In Section 2, all fans possessing a pt -dominating set and their pt -domination numbers are determined. In Section 3, all wheels possessing a pt -dominating set and their pt -domination numbers are determined. We then extend results on fans and wheels to generalized fans and generalized wheels in Section 4. We refer to the books [5,20] for notation and terminology not presented here.

2. Proper Total Domination in Fans

Since the fan $F_2 = P_2 \vee K_1 = K_3$ has no pt -dominating set by Observation 2, we are only concerned with fans F_n with $n \geq 3$. The following observation is useful.

Observation 3. Let G and H be connected graphs such that $G \subseteq H$. If G does not have a pt -dominating set and H has a pt -dominating set, then every pt -dominating set of H contains at least one vertex not belonging to G .

Proof. Assume, to the contrary, that H has a pt -dominating set T such that $T \subseteq V(G)$. Let u and v be adjacent vertices of G . Since T is a pt -dominating set of H , it follows that $\sigma_T(u) \neq \sigma_T(v)$, so T is a pt -dominating set of G , which is impossible. $\square$

The following result describes those fans possessing a pt -dominating set.

Proposition 1. For an integer $n \geq 3$, the fan F_n has a pt -dominating set if and only if $n \neq 4$.

Proof. In the fan $F_n = P_n \vee K_1$, let $P_n = (v_1, v_2, \dots, v_n)$ and $V(K_1) = \{v\}$. For $n = 3, 5, 6$, the fan F_n has a pt -dominating set. For example, $V(F_3) - \{v_2\}$ is a pt -dominating set of F_3 , $V(F_5) - \{v_4\}$ is a pt -dominating set of F_5 , and $V(F_6) - \{v_4, v_6\}$ is a pt -dominating set of F_6 . (This is shown in Figure 1).

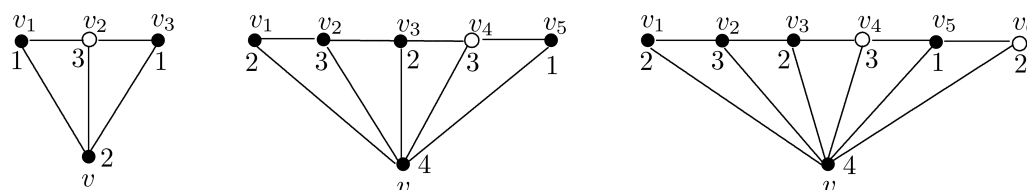


Figure 1. pt -dominating sets in F_n for $n = 3, 5, 6$.

Next, we show that F_4 does not have a pt -dominating set. Assume, to the contrary, that F_4 has a pt -dominating set S . Since F_4 is not locally irregular, it follows that $V(F_4)$ is not a pt -dominating set by Observation 1 and thus $|S| \in \{3, 4\}$. Furthermore, $F_4[S]$ is locally irregular by Observation 1 again. Since P_4 has no pt -dominating set, it follows that $v \in S$ by Observation 3.

- ★ First, suppose that $|S| = 4$. Since $v \in S$ and the star $K_{1,3}$ of order 4 is the only connected locally irregular graph of order 4, it follows that $F_4[S] = K_{1,3}$, whose center is v . However, F_4 does not contain such an induced subgraph $K_{1,3}$, which is a contradiction.
- ★ Next, suppose that $|S| = 3$. Since $v \in S$ and P_3 is the only connected locally irregular graph of order 3, it follows that $F_4[S] = P_3 = (v_i, v, v_j)$ where $j \geq i + 2$. We may assume that either (1) $F_4[S] = (v_2, v, v_4)$ or (2) $F_4[S] = (v_1, v, v_4)$. If (1) occurs, then $\sigma_S(v_1) = \sigma_S(v) = 2$, while if (2) occurs, then $\sigma_S(v_2) = \sigma_S(v_3) = 2$. A contradiction is produced in either case.

For $n \geq 7$, let $S = \{v\} \cup \{v_i : i \text{ is odd and } 1 \leq i \leq n\}$. If $n \geq 7$ is odd, then $\sigma_S(v_i) = 1$ for odd integers i with $1 \leq i \leq n$, $\sigma_S(v_i) = 3$ for even integers i with $2 \leq i \leq n - 1$, and $\sigma_S(v) \geq (n + 1)/2 \geq 4$. If $n \geq 8$ is even, then $\sigma_S(v_i) = 1$ for odd integers i with $1 \leq i \leq n - 1$, $\sigma_S(v_i) = 3$ for even integers i with $2 \leq i \leq n - 2$, $\sigma_S(v_n) = 2$, and $\sigma_S(v) \geq n/2 \geq 4$. In either case, S is a pt -dominating set of F_n . $\square$

In order to determine the exact value of $\gamma_{pt}(F_n)$ for all those integers n for which F_n has a pt -dominating set, we first present some preliminary information. In the fan $F_n = P_n \vee K_1$, let $P_n = (v_1, v_2, \dots, v_n)$ and $V(K_1) = \{v\}$. For a pt -dominating set S of F_n , a block B of P_n with respect to S is a maximal set of consecutive vertices of P_n (or a maximal subpath of P_n) such that either all vertices of B belong to S or no vertices of B belong to S . The path $P_n = (v_1, v_2, \dots, v_n)$ can therefore be expressed as one of the following four sequences, depending on the locations of the initial block and terminal block in the sequence:

$$(B_1, B'_1, B_2, B'_2, \dots, B_k, B'_k), (B_1, B'_1, B_2, B'_2, \dots, B_k, B'_k, B_{k+1}), \\ (B'_1, B_1, B'_2, B_2, \dots, B'_k, B_k), \text{ and } (B'_1, B_1, B'_2, B_2, \dots, B'_k, B_k, B'_{k+1})$$

such that $B_i \subseteq S$ and $B'_i \cap S = \emptyset$ for $1 \leq i \leq k$ or $1 \leq i \leq k + 1$. The first block in a sequence is the *initial block* of the sequence and the last block in a sequence is the *terminal block*. First, we verify the following claims.

Claim 1: If B is a block with $B \subseteq S$, then $|B| \in \{1, 3\}$.

Claim 2: Let B' be a block with $B' \cap S = \emptyset$. If B' is the initial or terminal block, then $|B'| \in \{1, 2\}$ and if B' is an interior block, then $|B'| \in \{1, 3\}$.

Claim 3: If B'_1 is the initial block, then $|B_1| = 1$. Similarly, if B'_i is the terminal block where $i \in \{k, k + 1\}$, then $|B_i| = 1$.

- ★ If Claim 1 is not true, then there is a block B with $B \subseteq S$ such that $|B| = 2$ or $|B| \geq 4$. First, suppose that $|B| = 2$ and $B = (v_i, v_{i+1})$ where $1 \leq i \leq n-1$. If $v \in S$, then v_i is dominated only by v_{i+1} and v in S and v_{i+1} is dominated only by v_i and v in S . Thus, $\sigma_S(v_i) = \sigma_S(v_{i+1}) = 2$, which is impossible. If $v \notin S$, then v_i is dominated only by v_{i+1} in S and v_{i+1} is dominated only by v_i in S . Thus, $\sigma_S(v_i) = \sigma_S(v_{i+1}) = 1$, which is impossible. Next, suppose that $|B| = 4$ and $B = (v_i, v_{i+1}, v_{i+2}, \dots, v_{i+\ell})$ for some integer $\ell \geq 3$. If $v \in S$, then v_{i+1} is dominated only by v_i, v_{i+2} , and v in S and v_{i+2} is dominated only by v_{i+1}, v_{i+3} and v in S . Thus, $\sigma_S(v_{i+1}) = \sigma_S(v_{i+2}) = 3$, which is impossible. If $v \notin S$, then v_{i+1} is dominated only by v_i and v_{i+2} in S and v_{i+2} is dominated only by v_{i+1} and v_{i+3} in S . Hence, $\sigma_S(v_{i+1}) = \sigma_S(v_{i+2}) = 2$, which is impossible. This verifies Claim 1.
- ★ Assume, to the contrary, that Claim 2 is not true. First, suppose that B' is the initial block or the terminal block with $|B'| \geq 3$. We may assume that $B' = (v_1, v_2, \dots, v_\ell)$ for some integer $\ell \geq 3$. Since $v_1, v_2, v_3 \notin S$, it follows that v_1 and v_2 are dominated only by the vertex v . This implies that $v \in S$ and $\sigma_S(v_1) = \sigma_S(v_2) = 1$, which is impossible. Next, suppose that B' is an interior block with $|B'| \notin \{1, 3\}$. Then either $|B'| = 2$ or $|B'| \geq 4$. For $|B'| = 2$, let $B' = (v_i, v_{i+1})$ where $2 \leq i \leq n-2$ and $v_{i-1}, v_{i+2} \in S$. If $v \in S$, then v_i is dominated only by v_{i-1} and v in S and v_{i+1} is dominated only by v_{i+2} and v in S . Thus, $\sigma_S(v_i) = \sigma_S(v_{i+1}) = 2$, which is impossible. If $v \notin S$, then v_i is dominated only by v_{i-1} in S and v_{i+1} is dominated only by v_{i+2} in S . Hence, $\sigma_S(v_i) = \sigma_S(v_{i+1}) = 1$, which is impossible. For $|B'| \geq 4$, let $B' = (v_{i+1}, v_{i+2}, \dots, v_{i+\ell})$ where $i \geq 2, \ell \geq 4$, and $i + \ell \leq n-1$. Then v_{i+2} and v_{i+3} are dominated only by v . This implies that $v \in S$ and $\sigma_S(v_{i+2}) = \sigma_S(v_{i+3}) = 1$, which is also impossible. This verifies Claim 2.
- ★ If Claim 3 is not true, then $|B_1| \neq 1$ and thus $|B_1| = 3$ by Claim 1. Since $|B'_1| \in \{1, 2\}$ by Claim 2, it follows that $B_1 = (v_j, v_{j+1}, v_{j+2})$ where $j \in \{2, 3\}$ and $v_{j-1} \in B'_1$. This implies that $\sigma_S(v_{j-1}) = \sigma_S(v_j) = 2$ if $v \in S$ and $\sigma_S(v_{j-1}) = \sigma_S(v_j) = 1$ if $v \notin S$, which is impossible. Similarly, if B'_i is the terminal block where $i \in \{k, k+1\}$, then $|B_i| = 1$. Thus, Claim 3 is true.

We are now prepared to determine the value of $\gamma_{pt}(F_n)$ for all those integers n for which F_n has a pt -dominating set.

Theorem 2. For an integer $n \geq 3$ with $n \neq 4$, $\gamma_{pt}(F_3) = 3$, $\gamma_{pt}(F_n) = 5$ for $5 \leq n \leq 8$, $\gamma_{pt}(F_9) = 4$, and for $n \geq 10$,

$$\gamma_{pt}(F_n) = \begin{cases} p+1 & \text{if } n \in \{4p, 4p+1\} \text{ for an integer } p \geq 3 \\ p+2 & \text{if } n \in \{4p+2, 4p+3\} \text{ for an integer } p \geq 2. \end{cases}$$

Proof. In the fan $F_n = P_n \vee K_1$, let $P_n = (v_1, v_2, \dots, v_n)$ and $V(K_1) = \{v\}$. First, suppose that $3 \leq n \leq 8$ and $n \neq 4$. For $n = 3$, let $S = V(F_3) - \{v_2\}$; for $n = 5$, let $S = V(F_n) - \{v_4\}$; for $n = 6$, let $S = V(F_n) - \{v_4, v_6\}$; and for $n = 7, 8$, let $S = \{v\} \cup \{v_i : i \text{ is odd and } 1 \leq i \leq n\}$. In each case, S is a pt -dominating set of F_n . Since $\gamma_{pt}(F_3) = 3$, it remains to show that $\gamma_{pt}(F_n) \geq 5$ for $5 \leq n \leq 8$. Assume, to the contrary, that $\gamma_{pt}(F_n) \leq 4$ for some integer n with $5 \leq n \leq 8$. Let S be a γ_{pt} -set of F_n . Then $|S| \in \{3, 4\}$ and $F_n[S]$ is a locally irregular graph (without isolated vertices) in F_n . There are two graphs of order 3 without isolated vertices, namely P_3 and K_3 , while K_3 is not locally irregular. Thus, P_3 is the only locally irregular graph without isolated vertices of order 3. Hence, if $|S| = 3$, then $F_n[S] = P_3$. There are seven graphs of order 4 without isolated vertices, namely, $K_{1,3}, K_{1,3} + e, P_4, C_4, C_4 + e, K_4, 2K_2$, while only $K_{1,3}$ is locally irregular. Hence, if $|S| = 4$, then $F_n[S] = K_{1,3}$.

- ★ First, suppose that $|S| = 3$ and $F_n[S] = P_3 = (x, y, z)$. If $v \notin S$, then either there exists $x' \in V(P_n) - S$ that is adjacent to x such that $\sigma_S(x) = \sigma_S(x') = 1$ or there exists $z' \in V(P_n) - S$ that is adjacent to z such that $\sigma_S(z) = \sigma_S(z') = 1$, which is a contradiction. Thus, we may assume that $v \in S$. Then $P_3 = (v_i, v, v_j)$ where $i < j$ and $j \geq i + 2$ (since v is adjacent to every vertex of P_n). If $i \geq 2$, then $\sigma_S(v_{i-1}) = \sigma_S(v) = 2$; if $j \leq n - 1$, then $\sigma_S(v_{j+1}) = \sigma_S(v) = 2$; and if $i = 1$ and $j = n$, then $\sigma_S(v_{i+1}) = \sigma_S(v) = 2$. In each case, a contradiction is produced.
- ★ Next, suppose that $|S| = 4$ and $F_n[S] = K_{1,3}$. Since v is adjacent to every vertex of P_n , it is impossible to have an induced star of order 4 or more in F_n whose center is not v . Thus, v is the center of $K_{1,3}$. Let $S = \{v, v_i, v_j, v_k\}$ where $1 \leq i < j < k$. Then $j \geq i + 2$ and $k \geq j + 2$. If $j = i + 2$, then $\sigma_S(v_{i+1}) = \sigma_S(v) = 3$. Thus, $j \geq i + 4 \geq 5$ by Claim 2. Similarly, $k \geq j + 4 \geq 9$. However then, $n \geq 9$, which is impossible.

Thus, $\gamma_{pt}(F_n) = 5$ for $5 \leq n \leq 8$.

Next, suppose that $n = 9$. Since $S = \{v, v_1, v_5, v_9\}$ is a pt -dominating set of F_9 , it follows that $\gamma_{pt}(F_9) \leq 4$ (see Figure 2). Next, we show that $\gamma_{pt}(F_9) \neq 3$. Assume, to the contrary, that $\gamma_{pt}(F_9) = 3$. Let S be a γ_{pt} -set of F_9 . Since $F_9[S]$ is locally irregular, $F_9[S] = P_3$. Thus, either (i) $F_9[S] = (v_i, v_{i+1}, v_{i+2})$ for some integer i with $1 \leq i \leq 7$ or (ii) $F_9[S] = (v_i, v, v_j)$ where $1 \leq i < j \leq 9$ and $j \geq i + 2$. In either case, we may assume that $1 \leq i \leq 4$. If (i) occurs, then v_9 is not dominated by any vertex of S , a contradiction. If (ii) occurs, then $\sigma_S(v_{j+1}) = \sigma_S(v) = 2$ if $j \neq 9$ and $\sigma_S(v_{j-1}) = \sigma_S(v) = 2$ if $j = 9$, also a contradiction. Thus, $\gamma_{pt}(F_9) = 4$.

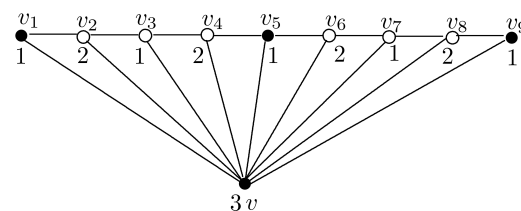


Figure 2. A pt -dominating set in F_9 .

For an integer $n \geq 10$, let $n = 4p + r$ where $r \in \{0, 1, 2, 3\}$. We consider two cases.

Case 1. $n = 4p$ or $n = 4p + 1$ for some integer $p \geq 3$. In this case, we show that $\gamma_{pt}(F_n) = p + 1$. Let $S_1 = \{v\} \cup \{v_i : i \equiv 3 \pmod{4} \text{ for } 1 \leq i \leq n\}$. Then $|S_1| = p + 1$. Since $\sigma_{S_1}(v_i) = 1$ if i is odd, $\sigma_{S_1}(v_i) = 2$ if i is even, and $\sigma_{S_1}(v) = p \geq 3$, it follows that S_1 is a pt -dominating set of F_n , so $\gamma_{pt}(F_n) \leq p + 1$. (See Figure 3 for $n = 13$ and a pt -dominating set in F_{12} can be obtained by deleting the vertex v_{13} .) It remains to show that $\gamma_{pt}(F_n) \geq p + 1$. Assume, to the contrary, that there is an integer $n \in \{4p, 4p + 1\}$ for some integer $p \geq 3$ such that F_n has a pt -dominating set S for which $|S| \leq p$. By Claim 2, if B' is a block of consecutive vertices on P_n such that $|B' \cap S| = \emptyset$, then $|B'| \leq 3$. This implies that for every four consecutive vertices on P_n , at least one of these vertices belongs to S . Thus, $|S| = p$ and $S \subseteq V(P_n)$. Further, there is a block B' on P_n with $B' \cap S = \emptyset$ that is a 3-path (v_j, v_{j+1}, v_{j+2}) for some integer j with $2 \leq j \leq n - 2$. However, in this case, v_{j+1} is not dominated by any vertex in S , which is a contradiction. Notice that the set S_1 is not a pt -dominating set when $n = 4p + 1$ where $p = 2$, since if $n = 9$, then S_1 contains two vertices of P_9 and thus $\sigma_{S_1}(v) = \sigma_{S_1}(v_2) = 2$. Therefore, $p \geq 3$ when $n = 4p + 1 \geq 13$ is necessary, as illustrated in Figure 3.

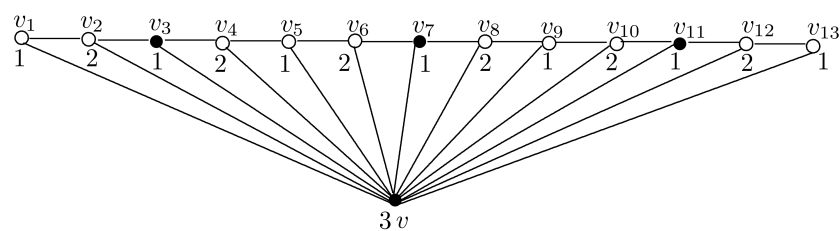


Figure 3. A pt -dominating set in F_{13} .

Case 2. $n = 4p + 2$ or $n = 4p + 3$ for an integer $p \geq 2$. In this case, we show that $\gamma_{pt}(F_n) = p + 2$. Let $S_1 = \{v\} \cup \{v_i : i \equiv 1 \pmod{4} \text{ for } 1 \leq i \leq n\}$. Then $|S_1| = p + 2$. Since $\sigma_{S_1}(v_i) = 1$ if i is odd, $\sigma_{S_1}(v_i) = 2$ if i is even, and $\sigma_{S_1}(v) = p \geq 3$, it follows that S_1 is a pt -dominating set of F_n and thus $\gamma_{pt}(F_n) \leq p + 2$. (See Figure 4 for $n = 11$ and a pt -dominating set in $P_{10} \vee K_1$ can be obtained by deleting the vertex v_{11} .) It remains to show that $\gamma_{pt}(F_n) \geq p + 2$. Assume, to the contrary, that there is an integer $n \in \{4p + 2, 4p + 3\}$ for some integer $p \geq 2$ such that F_n has a pt -dominating set S for which $|S| \leq p + 1$. By Claims 1 and 2, for every four consecutive vertices in P_n , at least one of these vertices belongs to S . Thus, $|S| = p + 1$ and $S \subseteq V(P_n)$. Further, there is a block B' in P_n with $B' \cap S = \emptyset$ that is a 3-path (v_j, v_{j+1}, v_{j+2}) for some integer j with $2 \leq j \leq n - 2$. However then, v_{j+1} is not dominated by any vertex in S , which is a contradiction. $\square$

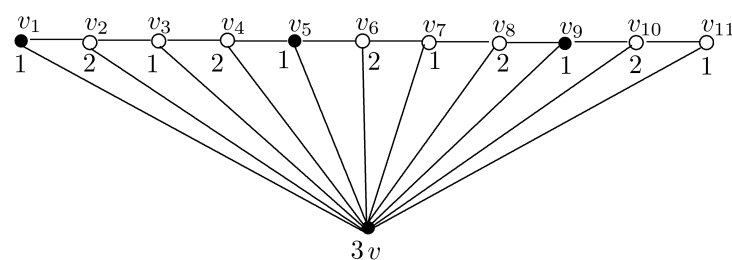


Figure 4. A pt -dominating set in F_{11} .

3. Proper Total Domination in Wheels

Since the wheel $W_3 = K_4$ has no pt -dominating set, we assume that $n \geq 4$. First, we present some preliminary information. For the wheel $W_n = C_n \vee K_1$, let $C_n = (v_1, v_2, \dots, v_n, v_1)$ and $V(K_1) = \{v\}$. For a pt -dominating set S of W_n , a *block* B of C_n with respect to S is a maximal set of consecutive vertices (or a maximal subpath) of C_n if either all vertices of B belong to S or no vertices of B belong to S . The cycle C_n can therefore be expressed as a sequence $(B_1, B'_1, B_2, B'_2, \dots, B_k, B'_k)$ of $2k$ blocks for some positive integer k such that $B_i \subseteq S$ and $B'_i \cap S = \emptyset$ for $1 \leq i \leq k$. We verify the following claim.

Claim: For each integer i with $1 \leq i \leq k$, $|B_i| \in \{1, 3\}$ and $|B'_i| \in \{1, 3\}$.

Suppose that this claim is false. We may assume, without loss of generality, that either $|B_1| \notin \{1, 3\}$ or $|B'_1| \notin \{1, 3\}$. Let $B_1 = (v_1, v_2, v_3, \dots, v_p)$ and $B'_1 = (v_{p+1}, v_{p+2}, \dots, v_{p+q})$ where either $p \notin \{1, 3\}$ or $q \notin \{1, 3\}$. If $p \geq 4$, then $\sigma_S(v_2) = \sigma_S(v_3)$. If $p = 2$, then $\sigma_S(v_1) = \sigma_S(v_2)$. Thus, $p = 2$ and $p \geq 4$ are impossible. Therefore, $|B_i| \in \{1, 3\}$ for $1 \leq i \leq k$. If $q \geq 4$, then $\sigma_S(v_{p+2}) = \sigma_S(v_{p+3})$; while if $q = 2$, then $\sigma_S(v_{p+1}) = \sigma_S(v_{p+2})$. Thus, $q = 2$ and $q \geq 4$ are impossible. Consequently, $|B'_i| \in \{1, 3\}$ for $1 \leq i \leq k$. This verifies the claim.

Proposition 2. For an integer $n \geq 4$, the wheel W_n has a pt -dominating set if and only if n is even.

Proof. For the wheel W_n , let $C_n = (v_1, v_2, \dots, v_n, v_1)$ and $V(K_1) = \{v\}$. If $n = 4$, then $\{v_1, v_2, v_3\}$ is a pt -dominating set of W_4 , while if $n = 6$, then $\{v_1, v_2, v_3, v_5\}$ is a pt -

dominating set of W_6 . If $n \geq 8$, let $S = \{v\} \cup \{v_i : i \text{ is odd and } 1 \leq i \leq n-1\}$. See Figure 5. Since $\sigma_S(v_i) = 1$ for odd integers i with $1 \leq i \leq n-1$, $\sigma_S(v_i) = 3$ for even integers i with $2 \leq i \leq n$, and $\sigma_S(v) \geq n/2 \geq 4$, it follows that S is a pt -dominating set of W_n .

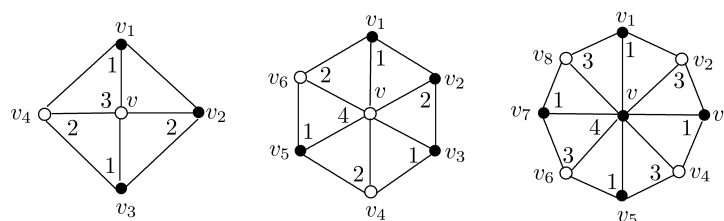


Figure 5. pt -dominating sets in W_n for $n \in \{4, 6, 8\}$.

There remains a need to verify the converse. Assume, to the contrary, that there is an odd integer $n \geq 5$ such that W_n has a pt -dominating set S . Then the cycle C_n in W_n is then expressed as a sequence $(B_1, B'_1, B_2, B'_2, \dots, B_k, B'_k)$ of $2k$ blocks for some positive integer k such that $B_i \subseteq S$ and $B'_i \cap S = \emptyset$ for $1 \leq i \leq k$. By the claim, $n = \sum_{i=1}^k (|B_i| + |B'_i|)$ is even, which is a contradiction. $\square$

We now determine the value of $\gamma_{pt}(W_n)$ for every even integer $n \geq 4$.

Theorem 3. For an even integer $n \geq 4$, $\gamma_{pt}(W_4) = 3$, $\gamma_{pt}(W_6) = 4$, $\gamma_{pt}(W_8) = \gamma_{pt}(W_{10}) = 5$, and for $n \geq 12$,

$$\gamma_{pt}(W_n) = \begin{cases} p+1 & \text{if } n = 4p \text{ for some integer } p \geq 3 \\ p+2 & \text{if } n = 4p+2 \text{ for some integer } p \geq 3. \end{cases}$$

Proof. For the wheel W_n where $n \geq 4$ is an even integer, let $C_n = (v_1, v_2, \dots, v_n, v_1)$ and $V(K_1) = \{v\}$. Since $\gamma_{pt}(W_4) = 3$ where $\{v_1, v_2, v_3\}$ is a γ_{pt} -set of W_4 , we may assume that $n \geq 6$. First, suppose that $n \in \{6, 8, 10\}$.

- ★ For $n = 6$, the set $\{v_1, v_2, v_3, v_5\}$ is a pt -dominating set of W_6 and thus $\gamma_{pt}(W_6) \leq 4$. Assume, to the contrary, that W_6 has a γ_{pt} -set S with $|S| = 3$. Then $W_n[S] = P_3 = (x, y, z)$ is locally irregular by Observation 1. If $v \notin S$, then S is not a total dominating set of W_6 , a contradiction. If $v \in S$, then we may assume that $P_3 = (v_1, v, v_j)$ where $j \in \{3, 4\}$. However then, $\sigma_S(v) = \sigma_S(v_6) = 2$, a contradiction. Thus, $\gamma_{pt}(W_6) \geq 4$, so $\gamma_{pt}(W_6) = 4$.
- ★ For $n = 8$, the set $\{v, v_1, v_3, v_5, v_7\}$ is a pt -dominating set of W_8 and thus $\gamma_{pt}(W_8) \leq 5$. Assume, to the contrary, that W_8 has a γ_{pt} -set S with $|S| \leq 4$. Then $|S| \in \{3, 4\}$ and $W_n[S]$ is locally irregular by Observation 1. If $|S| = 3$, then $W_n[S] = P_3 = (x, y, z)$. If $v \notin S$, then S is not a total dominating set of W_8 , a contradiction. If $v \in S$, then we may assume that $P_3 = (v_1, v, v_j)$ where $3 \leq j \leq 5$. However then, $\sigma_S(v) = \sigma_S(v_8) = 2$. If $|S| = 4$, then $W_n[S] = K_{1,3}$ is a star centered at v . We may assume that $S = \{v, v_1, v_i, v_j\}$ where $i \geq 3$ and $j - i \geq 2$. If $i = 3$, then $\sigma_S(v) = \sigma_S(v_2) = 3$, a contradiction. By the claim, $i = 5$. Similarly, $j - i = 4$ and thus $j = i + 4 = 9$, which is impossible since $n = 8$. Thus, $\gamma_{pt}(W_8) \geq 5$ and thus $\gamma_{pt}(W_8) = 5$.
- ★ For $n = 10$, the set $\{v, v_1, v_3, v_5, v_7, v_9\}$ is a pt -dominating set of W_{10} and thus $\gamma_{pt}(W_{10}) \leq 5$. Assume, to the contrary, that W_{10} has a γ_{pt} -set S with $|S| \leq 4$. Then $|S| \in \{3, 4\}$ and $W_n[S]$ is locally irregular by Observation 1. If $|S| = 3$, then $W_n[S] = P_3 = (x, y, z)$. If $v \notin S$, then S is not a total dominating set of W_{10} , a contradiction. If $v \in S$, then we may assume that $P_3 = (v_1, v, v_j)$ where $3 \leq j \leq 6$. However, $\sigma_S(v) = \sigma_S(v_{10}) = 2$. If $|S| = 4$, then $W_n[S] = K_{1,3}$ is a star centered at v .

We may assume that $S = \{v, v_1, v_i, v_j\}$ where $i - 1 \geq 2$ and $j - i \geq 2$. If $i = 3$, then $\sigma_S(v) = \sigma_S(v_2) = 3$, a contradiction. By the claim, $i = 5$ and $j = i + 4 = 9$. However, $\sigma_S(v) = \sigma_S(v_{10}) = 3$, a contradiction. Thus, $\gamma_{pt}(W_{10}) \geq 5$, so $\gamma_{pt}(W_{10}) = 5$.

Next, suppose that $n \geq 12$. Thus, either $n = 4p$ or $n = 4p + 2$ for some integer $p \geq 3$. We consider these two cases.

Case 1. $n = 4p$ for some integer $p \geq 3$. We show that $\gamma_{pt}(W_n) = p + 1$. Let

$$S_1 = \{v\} \cup \{v_i : i \equiv 1 \pmod{4} \text{ for } 1 \leq i \leq n - 3\}.$$

Then $|S_1| = p + 1$. Since $\sigma_{S_1}(v_i) = 1$ if i is odd, $\sigma_{S_1}(v_i) = 2$ if i is even, and $\sigma_{S_1}(v) = p \geq 3$, it follows that S_1 is a pt -dominating set of W_n , so $\gamma_{pt}(W_n) \leq p + 1$.

It remains to be shown that $\gamma_{pt}(W_n) \geq p + 1$. Assume, to the contrary, that there is an integer $n = 4p$ for some integer $p \geq 3$ such that W_n has a pt -dominating set S for which $|S| \leq p$. By the claim, for every four consecutive vertices on C_n , at least one of these vertices belongs to S . Thus, $|S| = p$ and $S \subseteq V(C_n)$. Further, there is a block $B' = (x, y, z)$ on C_n such that $B' \cap S = \emptyset$. However, in this case, y is not dominated by any vertex in S , which is a contradiction. Therefore, $\gamma_{pt}(W_n) \geq p + 1$ and thus $\gamma_{pt}(W_n) = p + 1$.

Case 2. $n = 4p + 2$ for some integer $p \geq 3$. We show that $\gamma_{pt}(W_n) = p + 2$. Let

$$S_2 = \{v\} \cup \{v_i : i \equiv 1 \pmod{4} \text{ for } 1 \leq i \leq n - 1\}.$$

Then $|S_2| = p + 2$. Since $\sigma_{S_2}(v_i) = 1$ if i is odd, $\sigma_{S_2}(v_i) = 2$ if i is even and $i \neq n$, $\sigma_{S_2}(v_n) = 3$, and $\sigma_{S_2}(v) = p + 1 \geq 4$, it follows that S_2 is a pt -dominating set of F_n and thus $\gamma_{pt}(W_n) \leq p + 2$.

It remains to show that $\gamma_{pt}(W_n) \geq p + 2$. Assume, to the contrary, that there is an integer $n = 4p + 2$ for some integer $p \geq 3$ such that W_n has a pt -dominating set S for which $|S| \leq p + 1$. By the claim, for every four consecutive vertices on C_n , at least one of these vertices belongs to S . Thus, $|S| = p + 1$ and $S \subseteq V(C_n)$. Further, there is a block $B' = (x, y, z)$ such that $B' \cap S = \emptyset$. However then, y is not dominated by any vertex in S , which is a contradiction. Therefore, $\gamma_{pt}(W_n) \geq p + 2$, so $\gamma_{pt}(W_n) = p + 2$. $\square$

4. Proper Total Domination in Generalized Fans and Wheels

In this section, we extend the results obtained in Sections 2 and 3 to more generalized fans $F_{n,t} = P_n \vee \overline{K}_t$ and wheels $W_{n,t} = C_n \vee \overline{K}_t$ for every positive integer t , where $\overline{K}_t$ is the empty graph of order t . For a pt -dominating set S in a graph G , let $\Sigma_S(G) = \{\sigma_S(v) : v \in V(G)\}$. First, we establish a connection between a pt -dominating set in the graph $G \vee \overline{K}_1$ and a pt -dominating set in the graph $G \vee \overline{K}_t$ for an integer $t \geq 2$.

Theorem 4. For a graph G , let $H_t = G \vee \overline{K}_t$ for a positive integer t .

- If H_1 has a pt -dominating set S , then S is also a pt -dominating set of H_t for each integer $t \geq 2$ and thus $\gamma_{pt}(H_t) \leq \gamma_{pt}(H_1)$.
- If H_t has a pt -dominating set T for some integer $t \geq 2$ such that $|T| - 1 \notin \Sigma_T(G)$, then H_1 has a pt -dominating set.

Proof. Let $V(\overline{K}_t) = \{w_1, w_2, \dots, w_t\}$. If $t = 1$, then $V(\overline{K}_1) = \{w_1\}$ in $H_1 = G \vee K_1$.

First, we verify (a). Suppose that S is a pt -dominating set in H_1 . We show that S is a pt -dominating set of H_t for each integer $t \geq 2$. Since a vertex in $V(\overline{K}_t)$ is dominated only by a vertex in G , it follows that $p = |S \cap V(G)| \geq 1$ and $\sigma_S(w_i) = p$ for $1 \leq i \leq t$. Let u and v be adjacent vertices of H_t . Then at least one of u and v belongs to G , say $u \in V(G)$. If $v \in V(G)$, then $\sigma_S(u) \neq \sigma_S(v)$ since S is a pt -dominating set of H_1 . If $v \in V(\overline{K}_t)$, then $\sigma_S(u) \neq \sigma_S(w_1) = \sigma_S(v)$, again, since S is a pt -dominating set of H_1 . Thus, S is a pt -dominating set of H_t . This is illustrated in Figure 6 for $G = P_6$ and $t = 2$, so $H_1 = F_6$

and $H_2 = F_{6,2}$. Furthermore, if S is a γ_{pt} -set of H_1 , then S is a pt -dominating set of H_t for each integer $t \geq 2$ and thus $\gamma_{pt}(H_t) \leq |S| = \gamma_{pt}(H_1)$.

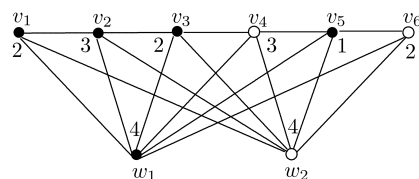


Figure 6. A pt -dominating set in $F_{6,2}$.

Next, we verify (b). Suppose that T is a pt -dominating set of $H_t = G \vee \bar{K}_t$ for some integer $t \geq 2$ such that $|T| - 1 \notin \Sigma_T(G)$. We show that H_1 has a pt -dominating set.

- ★ First, suppose that $T \subseteq V(G)$. Then $\sigma_S(w_i) = |T|$ for $1 \leq i \leq t$. Let u and v be adjacent vertices of H_1 . We may assume that $u \in V(G)$. If $v \in V(G)$, then $\sigma_T(u) \neq \sigma_T(v)$ since T is a pt -dominating set of H_t . If $v = w_1$, then $\sigma_T(u) \neq \sigma_T(w_1) = |T|$, again since T is a pt -dominating set of H_t . Thus, T is a pt -dominating set of H_1 .
- ★ Next, suppose that $T \not\subseteq V(G)$. Thus, T contains at least one vertex in $\bar{K}_t$. Let $T = S \cup S'$ where $S \subseteq V(G)$ and $S' \subseteq V(\bar{K}_t)$. Then $S \neq \emptyset$ and $S' \neq \emptyset$. We may assume that $w_1 \in S'$. Let $p = |S| \geq 1$, $q = |S'| \geq 1$, and $|T| = p + q$. Then $\sigma_T(w_i) = p$ for $1 \leq i \leq t$. Next, let $T' = S \cup \{w_1\}$. We show that T' is a pt -dominating set of H_1 . Let u and v be adjacent vertices of H_1 . We may assume that $u \in V(G)$. If $q = 1$, then $T' = T$ and thus $\sigma_{T'}(u) \neq \sigma_{T'}(v)$. Thus, we may assume that $q \geq 2$. If $v \in V(G)$, then $\sigma_{T'}(u) = \sigma_T(u) - (q - 1) \neq \sigma_T(v) - (q - 1) = \sigma_{T'}(v)$ since T is a pt -dominating set of H_t . If $v = w_1$, then $\sigma_{T'}(u) = \sigma_T(u) - (q - 1)$ and $\sigma_{T'}(v) = \sigma_T(v) = p$. Since $\sigma_T(u) \neq |T| - 1 = p + q - 1$, it follows that $\sigma_{T'}(u) \neq \sigma_{T'}(v)$. Thus, T' is a pt -dominating set of H_1 . This is illustrated in Figure 7 for $G = P_4$ and $t = 2$ and thus $H_1 = F_4$ and $H_2 = F_{4,2}$.

□

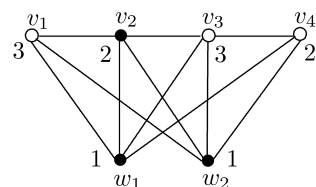


Figure 7. A pt -dominating set in $F_{4,2}$.

The condition that $\sigma_T(x) \neq |T| - 1$ for every vertex x of G in Theorem 4 is necessary. For example, consider the generalized fan $F_{4,2} = P_4 \vee \bar{K}_2$ where $P_4 = (v_1, v_2, v_3, v_4)$ and $V(\bar{K}_2) = \{w_1, w_2\}$ and let $T = \{v_2, w_1, w_2\}$. Since $\sigma_T(v_1) = \sigma_T(v_3) = 3$, $\sigma_T(v_2) = \sigma_S(v_4) = 2 = |T| - 1$, and $\sigma_T(w_1) = \sigma_T(w_2) = 1$, it follows that T is a pt -dominating set of $F_{4,2}$. By Proposition 1, the graph F_4 does not have a pt -dominating set.

The following are consequences of Propositions 1 and 2 and Theorem 4.

Corollary 1. For positive integers n and t where $n \geq 3$,

- (a) The generalized fan $F_{n,t}$ has a pt -dominating set if and only if $(n, t) \neq (4, 1)$;
- (b) The generalized wheel $W_{n,t}$ has a pt -dominating set if and only if n is even.

Proof. First, we verify (a). By Proposition 1, the graph $F_4 = F_{4,1}$ does not have a pt -dominating set. For the converse, assume that $(n, t) \neq (4, 1)$. For the graph $F_{4,t}$ where $t \geq 2$, let $P_4 = (v_1, v_2, v_3, v_4)$ and let w_1 and w_2 be two vertices in $\bar{K}_t$. Then $\{v_2, w_1, w_2\}$

is a pt -dominating set of $P_4 \vee \overline{K}_t$. By Proposition 1 and Theorem 4, the graph $F_{n,t}$ has a pt -dominating set for integers $n \geq 3$ and $t \geq 1$ such that $(n, t) \neq (4, 1)$.

Next, we verify (b). An argument similar to that in the proof of Proposition 2 shows that if $n \geq 3$ is odd and $t \geq 2$, then $W_{n,t}$ does not have a pt -dominating set. For the converse, if n is even and $t \geq 1$, then $W_{n,t}$ has a pt -dominating set by Proposition 2 and Theorem 4. $\square$

Next, we determine the pt -domination numbers of those generalized fans $F_{n,t}$ and generalized wheels $W_{n,t}$ possessing a pt -dominating set for each integer $t \geq 2$. In order to show this, we first present a lemma.

Lemma 1. For a graph G , let $H = G \vee \overline{K}_t$ for a positive integer t and let T be a pt -dominating set of H . If $T \cap V(G)$ is a total dominating set of G , then $T \cap V(G)$ is a pt -dominating set of G . In particular, if $T \subseteq V(G)$, then T is a pt -dominating set of G .

Proof. Let $S = T \cap V(G)$ and let $|T - S| = p \geq 0$. Since each vertex $x \in V(G)$ is adjacent to every vertex in $\overline{K}_t$, it follows that $\sigma_T(x) = \sigma_S(x) + p$. Furthermore, since S is a total dominating set of G , it follows that $\sigma_S(x) = \sigma_T(x) - p \geq 1$ for every vertex x of G . Let u and v be adjacent vertices of G . Since T is a pt -dominating set of H , it follows that $\sigma_T(u) \neq \sigma_T(v)$. Thus, $\sigma_S(u) = \sigma_T(u) - p \neq \sigma_T(v) - p = \sigma_S(v)$. Therefore, S is a pt -dominating set of G .

In particular, if $T \subseteq V(G)$, then $T \cap V(G) = T$. Since T is a pt -dominating set of H , every vertex of G is totally dominated by a vertex of T and thus T is a total dominating set of G . Consequently, T is a pt -dominating set of G . $\square$

The condition that $T \cap V(G)$ is a total dominating set of G in Lemma 1 is necessary. For example, consider the fan F_{10} where $V(K_1) = \{v\}$ and $P_{10} = (v_1, v_2, \dots, v_{10})$. Let $T = \{v, v_1, v_5, v_9\}$ in F_{10} (see Figure 8). Since $\sigma_T(v) = 3$, $\sigma_T(v_i) = 1$ if i is odd and $1 \leq i \leq 9$, while $\sigma_T(v_i) = 2$ if i is even and $2 \leq i \leq 10$, it follows that T is a pt -dominating set of F_{10} . However, $T \cap V(P_{10})$ is not a total dominating set of P_{10} , so it is not a pt -dominating set of P_{10} . In fact, P_{10} has no pt -dominating set by Theorem 1.

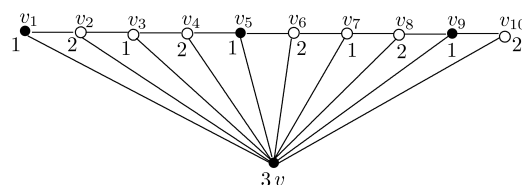


Figure 8. A pt -dominating set in F_{10} .

Theorem 5. For integers $n \geq 3$ and $t \geq 2$,

$$\gamma_{pt}(F_{n,t}) = \begin{cases} 3 & \text{if } 3 \leq n \leq 5 \\ \gamma_{pt}(F_n) & \text{if } n \geq 6. \end{cases}$$

Proof. For the generalized fan $F_{n,t} = P_n \vee \overline{K}_t$, let $P_n = (v_1, v_2, \dots, v_n)$ and $V(\overline{K}_t) = \{w_1, w_2, \dots, w_t\}$ where $n \geq 3$ and $t \geq 2$. First, we show that if $3 \leq n \leq 5$, then $\gamma_{pt}(F_{n,t}) = 3$. By Observation 1, it suffices to show that $F_{n,t}$ has a three-element pt -dominating set.

- ★ For $n = 3$, let $S = \{v_1, v_3, w_1\}$. Since $\sigma_S(v_1) = \sigma_S(v_3) = 1$, $\sigma_S(v_2) = 3$, and $\sigma_S(w_j) = 2$ for $1 \leq j \leq t$, it follows that S is a pt -dominating set of $F_{3,t}$.
- ★ For $n = 4$, let $S = \{v_2, w_1, w_2\}$. Since $\sigma_S(v_i) = 3$ for $i \in \{1, 3\}$, $\sigma_S(v_i) = 2$ for $i \in \{2, 4\}$, and $\sigma_S(w_j) = 1$ for $1 \leq j \leq t$, it follows that S is a pt -dominating set of $F_{4,t}$.

★ For $n = 5$, let $S = \{v_3, w_1, w_2\}$. Since $\sigma_S(v_i) = 2$ for $i \in \{1, 3, 5\}$, $\sigma_S(v_i) = 3$ for $i \in \{2, 4\}$, and $\sigma_S(w_j) = 1$ for $1 \leq j \leq t$, it follows that S is a pt -dominating set of $F_{5,t}$.

Thus, $\gamma_{pt}(F_{n,t}) = 3$ if $3 \leq n \leq 5$ and $t \geq 2$.

Next, we show that if $n \geq 6$, then $\gamma_{pt}(F_{n,t}) = \gamma_{pt}(F_n)$ for each integer $t \geq 2$. Since $\gamma_{pt}(F_{n,t}) \leq \gamma_{pt}(F_n)$ by Theorem 4, it remains to show that $\gamma_{pt}(F_{n,t}) \geq \gamma_{pt}(F_n)$. Assume, to the contrary, that there exist integers $n \geq 6$ and $t \geq 2$ such that $\gamma_{pt}(F_{n,t}) < \gamma_{pt}(F_n)$. Let T be a γ_{pt} -set of $F_{n,t}$. We consider two cases, according to whether $T \subseteq V(P_n)$ or $T \not\subseteq V(P_n)$.

Case 1. $T \subseteq V(P_n)$. It follows by Lemma 1 that T is a pt -dominating set of P_n . Since P_n has a pt -dominating set if and only if $n \equiv 3 \pmod{4}$ by Theorem 1, there is a contradiction if $n \not\equiv 3 \pmod{4}$. On the other hand, if $n \equiv 3 \pmod{4}$, then P_n has a unique pt -dominating set of cardinality $\frac{3(n+1)}{4}$, so $|T| = \frac{3(n+1)}{4} = \gamma_{pt}(P_n)$ by Theorem 1. However, then, $\frac{3(n+1)}{4} = |T| = \gamma_{pt}(F_{n,t}) < \gamma_{pt}(F_n) = \frac{n+5}{4}$ by Theorem 2, which is impossible. Therefore, Case 1 cannot occur.

Case 2. $T \not\subseteq V(P_n)$. Let $T = S \cup S'$ where $S \subseteq V(P_n)$ and $S' \subseteq V(\bar{K}_t)$. We may assume, without loss of generality, that $w_1 \in S'$. Then $S \neq \emptyset$ and $S' \neq \emptyset$. Thus, $p = |S| \geq 1$ and $q = |S'| \geq 1$. Notice that $\sigma_T(v_i) = \sigma_S(v_i) + q$ for $1 \leq i \leq n$ and $\sigma_T(w_j) = p$ for $1 \leq j \leq t$. Let $T' = S \cup \{w_1\}$. We show that T' is a pt -dominating set of F_n . Let x and y be adjacent vertices of F_n . Then at least one of x and y belongs to P_n . First, suppose that $x, y \in V(P_n)$, say $x = v_a$ and $y = v_b$ where $1 \leq a < b \leq n$. Since $\sigma_T(v_a) \neq \sigma_T(v_b)$ and $\sigma_T(v_i) = \sigma_S(v_i) + q$ for $1 \leq i \leq n$, it follows that $\sigma_S(v_a) \neq \sigma_S(v_b)$. Hence, $\sigma_{T'}(v_a) = \sigma_S(v_a) + 1 \neq \sigma_S(v_b) + 1 = \sigma_{T'}(v_b)$. Next, suppose that exactly one of x and y belongs to P_n , say, $x \in V(P_n)$ and $y = w_1$. Then $\sigma_{T'}(x) = \sigma_S(x) + 1$ and $\sigma_T(y) = p$. Since $\sigma_S(x) \in \{1, 2\}$, it follows that $\sigma_{T'}(x) \leq 3$. Therefore, if $p \geq 4$, then $\sigma_{T'}(x) \neq \sigma_{T'}(y)$. There remains for consideration the situation where $p \leq 3$. For this purpose, the block structure of the generalized fans needs to be investigated.

A block B of P_n with respect to S is a maximal set of consecutive vertices of P_n (or a maximal subpath of P_n) such that either all vertices of B belong to S or no vertices of B belong to S . The path $P_n = (v_1, v_2, \dots, v_n)$ can therefore be expressed as one of the following four sequences, depending on the locations of the initial block and terminal block in the sequence

$$(B_1, B'_1, B_2, B'_2, \dots, B_k, B'_k), (B_1, B'_1, B_2, B'_2, \dots, B_k, B'_k, B_{k+1}), \\ (B'_1, B_1, B'_2, B_2, \dots, B'_k, B_k), \text{ and } (B'_1, B_1, B'_2, B_2, \dots, B'_k, B_k, B'_{k+1})$$

such that $B_i \subseteq S$ and $B'_i \cap S = \emptyset$ for $1 \leq i \leq k$ or $1 \leq i \leq k+1$. The following claims can be verified by arguments similar to that used in the case of the fans.

Claim 1: If B is a block with $B \subseteq S$, then $|B| \in \{1, 3\}$.

Claim 2: Let B' be a block with $B' \cap S = \emptyset$. If B' is the initial or terminal block, then $|B'| \in \{1, 2\}$, while if B' is an interior block, then $|B'| \in \{1, 3\}$.

Claim 3: If B'_1 is the initial block, then $|B_1| = 1$. Similarly, if B'_i is the terminal block where $i \in \{k, k+1\}$, then $|B_i| = 1$.

First, we show that if $n \geq 14$, then $|S| = p \geq 4$. Assume, to the contrary, that $|S| \leq 3$. By Claims 1 and 2, for every four consecutive vertices on P_n , at least one vertex is in S , so $|S| \geq 3$. Thus, $|S| = 3$ and $|B_i| = 1$ for every block B_i in S . If $v_1 \in S$, then $|S| \geq 4$ since S contains at least three vertices in $\{v_2, v_3, \dots, v_{13}\}$, a contradiction. Thus, we may assume that $v_1 \notin S$. If $v_2 \in S$, then $|S| \geq 4$ since S contains at least three vertices in $\{v_3, v_4, \dots, v_{14}\}$, a contradiction. Hence, $v_2 \notin S$ and thus $v_3 \in S$ by Claim 3. This forces $v_7 \in S$, $v_{11} \in S$, and $v_i \notin S$ for $i \geq 12$. Since $n \geq 14$, the terminal block is not in S and contains at least three vertices, which contradicts Claim 2. Therefore, if $n \geq 14$, then $|S| = p \geq 4$.

Thus, we may assume that $6 \leq n \leq 13$. By Theorem 2, if $6 \leq n \leq 8$, then $\gamma_{pt}(F_n) = 5$, while if $9 \leq n \leq 13$, then $\gamma_{pt}(F_n) = 4$. We consider these two possibilities.

Subcase 2.1. $6 \leq n \leq 8$. We show that $\gamma_{pt}(F_{n,t}) = 5$ for each integer $t \geq 2$. Assume, to the contrary, that $\gamma_{pt}(F_{n,t}) \leq 4$ for some integer n with $6 \leq n \leq 8$ and an integer $t \geq 2$. Let T be a γ_{pt} -set of $F_{n,t}$. Then $|T| \in \{3, 4\}$ and $F_{n,t}[T]$ is a locally irregular graph (without isolated vertices) in $F_{n,t}$. If $|T| = 3$, then $F_{n,t}[T] = P_3$, while if $|T| = 4$, then $F_{n,t}[T] = K_{1,3}$.

- ★ First, suppose that $|T| = 3$ and $F_{n,t}[T] = P_3 = (x, y, z)$. If $T \cap V(\overline{K}_t) = \emptyset$, then since $n \geq 6$, there is a vertex of P_n that is not totally dominated by any vertex in T , which is a contradiction. If $T \cap V(\overline{K}_t) \neq \emptyset$, then we may assume that (i) $P_3 = (v_i, w_1, v_j)$ where $j \geq i + 2$ or (ii) $P_3 = (w_1, v_i, w_2)$. If (i) occurs, then $\sigma_T(w_1) = \sigma_T(v_{i-1}) = 2$ (when $i \geq 2$), $\sigma_T(w_1) = \sigma_T(v_{j+1}) = 2$ (when $j \leq n - 1$), and $\sigma_T(w_1) = \sigma_T(v_2) = 2$ (when $i = 1$ and $j = n$), which is impossible. If (ii) occurs, then we may assume that $P_3 = (w_1, v_i, w_2)$ and $1 \leq i \leq n/2$. Since $n \geq 6$, it follows that $\sigma_T(v_{n-1}) = \sigma_T(v_n) = 2$, which is impossible.
- ★ Next, suppose that $|T| = 4$ and $F_{n,t}[T] = K_{1,3}$. Thus, either (i) the center of $K_{1,3}$ is a vertex in $V(\overline{K}_t)$ or (ii) the center of $K_{1,3}$ is a vertex of P_n and $t \geq 3$. If (i) occurs, we may assume that w_1 is the center of $K_{1,3}$. Let $T = \{w_1, v_i, v_j, v_k\}$ where $1 \leq i < j < k$. Then $j \geq i + 2$ and $k \geq j + 2$. By Claim 2, it follows that $j - i \in \{2, 4\}$ and $k - j \in \{2, 4\}$. If $j - i = 2$, then $\sigma_T(v_{i+1}) = \sigma_T(w_1) = 3$, which is impossible. Thus, $j - i = 4$. Similarly, $k - j = 4$. This implies that $n \geq 9$, which is also impossible. If (ii) occurs, we may assume that $V(K_{1,3}) = \{v_i, w_1, w_2, w_3\}$ and $1 \leq i \leq n/2$. Since $n \geq 6$, it follows that $\sigma_T(v_{n-1}) = \sigma_T(v_n) = 3$, which too is impossible.

Thus, $\gamma_{pt}(F_{n,t}) = 5$ for $6 \leq n \leq 8$ and $t \geq 2$.

Subcase 2.2. $9 \leq n \leq 13$. We show that $\gamma_{pt}(F_{n,t}) = 4$ for each integer $t \geq 2$. Assume, to the contrary, that $\gamma_{pt}(F_{n,t}) = 3$ for some integer n with $9 \leq n \leq 13$ and an integer $t \geq 2$. Let T be a γ_{pt} -set of $F_{n,t}$. Then $F_{n,t}[T] = P_3 = (x, y, z)$, an argument similar to that in Subcase 2.1 when $|T| = 3$ shows that a contradiction is produced. Thus, $\gamma_{pt}(F_{n,t}) = 4$ for $9 \leq n \leq 13$ and $t \geq 2$. $\square$

We saw in Theorem 4 that $\gamma_{pt}(H_t) \leq \gamma_{pt}(H_1)$ for every integer $t \geq 2$. By Theorem 5, $\gamma_{pt}(F_{5,t}) = 3$ for every integer $t \geq 2$. Since $\gamma_{pt}(F_5) = 5$ by Theorem 2, it follows that $\gamma_{pt}(F_{5,t}) < \gamma_{pt}(F_5)$ and hence strict inequality is possible.

Theorem 6. For integers n and $t \geq 2$ where $n \geq 4$ is even,

$$\gamma_{pt}(W_{n,t}) = \gamma_{pt}(W_n).$$

Proof. For the generalized wheel $W_{n,t} = C_n \vee \overline{K}_t$ where $n \geq 4$ is even and $t \geq 2$, let $C_n = (v_1, v_2, \dots, v_n, v_1)$ and $V(\overline{K}_t) = \{w_1, w_2, \dots, w_t\}$. By Theorem 4, $\gamma_{pt}(W_{n,t}) \leq \gamma_{pt}(W_n)$ for every integer $t \geq 2$. It remains to show that $\gamma_{pt}(W_{n,t}) \geq \gamma_{pt}(W_n)$. Since $\gamma_{pt}(W_4) = 3$, it follows by Observation 1 that $\gamma_{pt}(W_{4,t}) = 3$ for every integer $t \geq 2$. Thus, we may assume that n is even and $n \geq 6$. Assume, to the contrary, that $\gamma_{pt}(W_{n,t}) < \gamma_{pt}(W_n)$ for some integers n and $t \geq 2$ where $n \geq 6$ is even. Let T be a γ_{pt} -set of $W_{n,t}$. We consider two cases, according to whether $T \subseteq V(C_n)$ or $T \not\subseteq V(C_n)$.

Case 1. $T \subseteq V(C_n)$. By Lemma 1, T is a pt -dominating set of C_n . Since C_n has a unique pt -dominating set of cardinality $\frac{3n}{4}$, it follows that $|T| = \frac{3n}{4} = \gamma_{pt}(C_n)$ by Theorem 1. However, in this case, $\frac{3n}{4} = |T| = \gamma_{pt}(W_{n,t}) < \gamma_{pt}(W_n) \leq \frac{n+10}{4}$ by Theorem 3. This is impossible since $n \geq 6$. Therefore, Case 1 cannot occur.

Case 2. $T \not\subseteq V(C_n)$. Let $T = S \cup S'$ where $S \subseteq V(C_n)$ and $S' \subseteq V(\overline{K}_t)$. We may assume, without loss of generality, that $w_1 \in S'$. Then $S \neq \emptyset$ and $S' \neq \emptyset$. Thus, $p = |S| \geq 1$ and $q = |S'| \geq 1$. Furthermore, $\sigma_T(v_i) = \sigma_S(v_i) + q$ for $1 \leq i \leq n$ and $\sigma_T(w_j) = p$ for $1 \leq j \leq t$. Let $T' = S \cup \{w_1\}$. We show that T' is a pt -dominating set of W_n . Let x and y be adjacent vertices of W_n . Then at least one of x and y belongs to C_n . First, suppose

that $x, y \in V(C_n)$, say $x = v_a$ and $y = v_b$ where $1 \leq a < b \leq n$. Since $\sigma_T(v_a) \neq \sigma_T(v_b)$ and $\sigma_T(v_i) = \sigma_S(v_i) + q$ for $1 \leq i \leq n$, it follows that $\sigma_S(v_a) \neq \sigma_S(v_b)$. Hence, $\sigma_{T'}(v_a) = \sigma_S(v_a) + 1 \neq \sigma_S(v_b) + 1 = \sigma_{T'}(v_b)$. Next, suppose that exactly one of x and y belongs to P_n , say $x \in V(P_n)$ and $y = w_1$. Then $\sigma_{T'}(x) = \sigma_S(x) + 1$ and $\sigma_T(y) = p$. Since $\sigma_S(x) \in \{1, 2\}$, it follows that $\sigma_{T'}(x) \leq 3$. Therefore, if $p \geq 4$, then $\sigma_{T'}(x) \neq \sigma_{T'}(y)$. There remains a need to consider a situation when $p \leq 3$. Once again, the block structure of the generalized wheels needs to be investigated.

A block B of C_n with respect to S is a maximal set of consecutive vertices of C_n (or a maximal subpath of C_n) if either all vertices of B belong to S or no vertices of B belong to S . The cycle C_n can therefore be expressed as a sequence $(B_1, B'_1, B_2, B'_2, \dots, B_k, B'_k)$ of $2k$ blocks for some positive integer k such that $B_i \subseteq S$ and $B'_i \cap S = \emptyset$ where $1 \leq i \leq k$. The following claim can be verified by the argument used in the case of W_n .

Claim: For each integer i with $1 \leq i \leq k$, $|B_i| \in \{1, 3\}$ and $|B'_i| \in \{1, 3\}$.

First, we show that if n is even and $n \geq 14$, then $|S| = p \geq 4$. Assume, to the contrary, that $|S| \leq 3$. By the claim, for every four consecutive vertices on C_n , there is at least one vertex in S and thus $|S| \geq 3$. Thus, $|S| = 3$ and $|B_i| = 1$ for every block B_i in S . We may assume that $v_1 \in S$. Since S contains at least three vertices in $\{v_2, v_4, \dots, v_{13}\}$, it follows that $|S| \geq 4$, which is a contradiction. Therefore, if $n \geq 14$, then $|S| = p \geq 4$.

Thus, we may assume that n is even and $6 \leq n \leq 12$. By Theorem 3, if $n = 6, 12$, then $\gamma_{pt}(W_n) = 4$, while if $n = 8, 10$, then $\gamma_{pt}(W_n) = 5$. We consider these two subcases.

Subcase 2.1. $n = 6, 12$. We show that $\gamma_{pt}(W_{n,t}) = 4$ for every integer $t \geq 2$. Assume, to the contrary, that $\gamma_{pt}(W_{n,t}) = 3$ for some even integer $n \in \{6, 12\}$ and an integer $t \geq 2$. Let T be a γ_{pt} -set of $W_{n,t}$. Since $W_{n,t}[T]$ is locally irregular, $W_{n,t}[T] = P_3$. If $T \cap V(\bar{K}_t) = \emptyset$, then we may assume that $P_3 = (v_1, v_2, v_3)$. However, v_5 is not dominated by any vertex of T , which is impossible. If $T \cap V(\bar{K}_t) \neq \emptyset$, then we may assume that (i) $P_3 = (v_1, w_1, v_j)$ where $3 \leq j \leq 1 + n/2$ or (ii) $P_3 = (w_1, v_1, w_2)$. If (i) occurs, then $\sigma_T(w_1) = \sigma_T(v_n) = 2$ since $n \geq 6$, while if (ii) occurs, then $\sigma_T(v_3) = \sigma_T(v_4) = 2$. In either case, a contradiction is produced. Thus $\gamma_{pt}(W_{n,t}) = 4$ if $n \in \{6, 12\}$ and $t \geq 2$.

Subcase 2.2. $n = 8, 10$. We show that $\gamma_{pt}(W_{n,t}) = 5$ for every integer $t \geq 2$. Assume, to the contrary, that $\gamma_{pt}(W_{n,t}) \leq 4$ for some integer $n \in \{8, 10\}$ and an integer $t \geq 2$. Let T be a γ_{pt} -set of $W_{n,t}$. Then $|T| \in \{3, 4\}$. The argument used in Subcase 2.1 shows that $|T| \neq 3$. Hence, $|T| = 4$ and $W_{n,t}[T] = K_{1,3}$. Thus, either (i) the center of $K_{1,3}$ is a vertex in $V(\bar{K}_t)$ or (ii) the center of $K_{1,3}$ is a vertex of C_n and $t \geq 3$. If (i) occurs, we may assume that w_1 is the center of $K_{1,3}$ where $T = \{w_1, v_1, v_i, v_j\}$ for $3 \leq i < j$. By Claim 2, it follows that $i \in \{3, 5\}$ and $j - i \in \{2, 4\}$. If $i = 3$, then $\sigma_T(v_2) = \sigma_T(w_1) = 3$, which is impossible. Thus, $i = 5$ and $j - i = 4$ or $j = 9$, which is impossible if $n = 8$. If $n = 10$, then $\sigma_T(v_{10}) = \sigma_T(w_1) = 3$, which is impossible. If (ii) occurs, we may assume that $V(K_{1,3}) = \{v_i, w_1, w_2, w_3\}$ and $1 \leq i \leq n/2$. However, then $\sigma_T(v_{n-2}) = \sigma_T(v_{n-1}) = 3$, which too is impossible. $\square$

5. Conclusions

In conclusion, we summarize the results obtained and state some open problems for further study.

(1) Results on fans F_n and wheels W_n where $n \geq 3$.

- ★ The fan F_n has a pt -dominating set if and only if $n \neq 4$. Furthermore, if $n \geq 3$ is an integer with $n \neq 4$, then $\gamma_{pt}(F_3) = 3$, $\gamma_{pt}(F_n) = 5$ for $5 \leq n \leq 8$, $\gamma_{pt}(F_9) = 4$, and for $n \geq 10$,

$$\gamma_{pt}(F_n) = \begin{cases} p+1 & \text{if } n \in \{4p, 4p+1\} \text{ for an integer } p \geq 3 \\ p+2 & \text{if } n \in \{4p+2, 4p+3\} \text{ for an integer } p \geq 2. \end{cases}$$

- ★ The wheel W_n has a pt -dominating set if and only if n is even. Furthermore, if $n \geq 4$ is an even integer, then $\gamma_{pt}(W_4) = 3$, $\gamma_{pt}(W_6) = 4$, $\gamma_{pt}(W_8) = \gamma_{pt}(W_{10}) = 5$, and for $n \geq 12$,

$$\gamma_{pt}(W_n) = \begin{cases} p+1 & \text{if } n = 4p \text{ for some integer } p \geq 3 \\ p+2 & \text{if } n = 4p+2 \text{ for some integer } p \geq 3. \end{cases}$$

- (2) Results on generalized fans $F_{n,t}$ and wheels $W_{n,t}$ where $n \geq 3$ and $t \geq 2$.

- ★ The generalized fan $F_{n,t}$ has a pt -dominating set. Furthermore, if $3 \leq n \leq 5$, then $\gamma_{pt}(F_{n,t}) = 3$ and if $n \geq 6$, then $\gamma_{pt}(F_{n,t}) = \gamma_{pt}(F_n)$.
- ★ The generalized wheel $W_{n,t}$ has a pt -dominating set if and only if n is even. Furthermore, if $n \geq 4$ is even, then $\gamma_{pt}(W_{n,t}) = \gamma_{pt}(W_n)$.

A natural question here is to extend results obtained on generalized fans and wheels to more general classes of graphs. As mentioned earlier, it is challenging to determine whether a general graph possesses a pt -dominating set and to determine its pt -domination number. For example, while Theorem 1 provides a complete answer for all paths and cycles with a pt -dominating set as well as their pt -domination numbers, this is not the case for the Cartesian products of paths and cycles. On the other hand, with the aid of results on paths and cycles as well as the structure of pt -dominating sets in paths and cycles, a complete answer is obtained here for fans, wheels, and generalized fans and wheels. This observation suggests investigating the joins $T \vee \overline{K}_t$ of some well-known classes of trees T that are closely related to paths, and the joins $G \vee \overline{K}_t$ of some well-known classes of graphs G that are closely related to cycles. For example, for starlike trees (a subdivision of a star each of whose branches at the central vertex is a path), a characterization was established in [11] for every starlike tree possessing a pt -dominating set as well as the structures of pt -dominating sets. Another class of well-known trees is the caterpillars (a tree with the removal of all end-vertices results in a path), which have been investigated in [11]. A well-known class of graphs closely related to cycles is unicyclic graphs (connected graphs with exactly one cycle). Information on the join $G \vee K_t$ of a general graph G and $\overline{K}_t$ described in Theorem 4 and Lemma 1 might provide a potential approach to the investigation of pt -dominating sets in these graphs. Therefore, we close with the following problems.

Problem 1. For a positive integer t , determine which graphs $G \vee \overline{K}_t$ possess a pt -dominating set, where G is a starlike tree, a caterpillar, or a unicyclic graph. For those graphs $G \vee \overline{K}_t$ possessing a pt -dominating set, determine its pt -domination number.

Of course, there is the more general problem of investigating proper total domination in the joins of any two graphs.

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