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A Robust ABC-Gibbs Method for Bathtub Failure Rate Modeling: Theory, Simulation, and Reliability Assessment

Mustapha Muhammad ¹, Mohammed K. Shakhathreh ^{2,*} , Badamasi Abba ³, Isyaku Muhammad ⁴ 
and Hassan S. Bakouch ^{5,6} 

¹ Department of Mathematics, Guangdong University of Petrochemical Technology, Maoming 525000, China; mmmahmoud12@sci.just.edu.jo or mmuhammad.mth@buk.edu.ng

² Department of Mathematics and Statistics, Faculty of Science, Jordan University of Science and Technology, P.O. Box 3030, Irbid 22110, Jordan

³ School of Business Administration, Hunan University, Changsha 410082, China; badamasiabba@gmail.com

⁴ School of Automotive Intelligent Manufacturing, Hubei University of Automotive Technology, Shiyan 442002, China; isyakuedu@yahoo.com

⁵ Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia; h.bakouch@qu.edu.sa

⁶ Department of Mathematics, Faculty of Science, Tanta University, Tanta 31527, Egypt

* Correspondence: mkshakhathreh6@just.edu.jo

Abstract

Reliability data in engineering and biomedical studies often exhibit non-monotone failure mechanisms, including bathtub-shaped failure rates (BTFRs). Existing flexible lifetime models capture such behavior, but statistical inference is challenging due to intractable or unstable likelihoods. We propose a likelihood-free Bayesian framework for BTFR models using approximate Bayesian computation Gibbs (ABC-Gibbs) methodology. To support this, a new competing-risk lifetime model is introduced by combining an extended Weibull and a Chen distribution in series, capable of modeling diverse non-monotone patterns, including flat bathtub regions. Within ABC-Gibbs, parameters are estimated without explicit likelihood evaluation using the Kolmogorov–Smirnov statistic. Key reliability characteristics—failure rate, mean residual life, and their change points—are investigated, providing insights for burn-in optimization and system aging. Model features such as cumulative residual entropy and Kullback–Leibler divergence are analyzed. Simulations and real data sets demonstrate that the ABC-Gibbs method is a robust and effective alternative to existing BTFR models for complex reliability analysis.

Keywords: approximate Bayesian computation; Kolmogorov–Smirnov measure; bathtub failure rate; simulation; mean residual life



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1. Introduction

Reliability analysis plays a critical role in the design, maintenance, and operation of systems across various engineering and biomedical applications. A central challenge in these studies is the estimation of failure rates and the subsequent prediction of system longevity under uncertain conditions. While traditional methods, such as maximum likelihood estimation (MLE), have been widely adopted for parameter estimation, these approaches often fall short when dealing with complex systems, intricate failure behaviors, big data or the absence of explicit likelihood functions. This limitation is particularly evident

in situations where data cannot be easily modeled using standard statistical methods in various applications (see, [1,2]).

In response to these challenges, approximate Bayesian computation (ABC) has emerged as a powerful alternative for parameter estimation, especially in models where likelihood functions are intractable, involve big data, or are computationally expensive to obtain. The ABC-Gibbs algorithm, which enhances the standard ABC approach by integrating Gibbs sampling, offers a robust framework for parameter estimation in such complex systems. This study introduces a novel ABC-Gibbs methodology that employs the Kolmogorov–Smirnov (KS) distance function to efficiently estimate parameters of a new bathtub-shaped failure rate model. This model combines elements of the extended Weibull and Chen distributions, offering a flexible and effective tool for analyzing complex systems with non-monotone failure rates. Our approach provides a significant advancement in modeling systems that exhibit bathtub failure rates, a common pattern observed in reliability data, particularly in electronics and mechanical systems.

Although the use of ABC in statistics and applied studies remains relatively limited compared with likelihood-based methods, several previous and recent studies have demonstrated its usefulness across different fields. In reliability and degradation modeling, ABC has been increasingly explored as a likelihood-free tool for estimating parameters of complex failure and degradation models. For example, Ref. [3] applied ABC to parameter estimation in failure models, while Ref. [4] used ABC to estimate the parameters of a Wiener-process-based model with transmuted normal random effects. Similarly, Ref. [5] examined the performance of likelihood-free inference for a Chen-distribution-based Wiener process. Moreover, ABC has also been applied to stochastic-process models when the likelihood function is difficult to derive or computationally demanding. For instance, Ref. [6] developed an ABC-MCMC approach for estimating the parameters of a gamma process in the presence of noisy data. Furthermore, beyond reliability applications, ABC has been used in biological and experimental systems where complex mechanisms and limited analytical likelihoods make conventional inference challenging. In this regard, Ref. [7] employed ABC for model parameter estimation in the analysis of experimental data in systems biology. In addition, ABC has been considered for model comparison, model discrimination, and parametric inference problems, as reported in [8–10].

Correspondingly, the failure rate (FR) function plays a central role in the analysis of lifetime data. The mathematical characterization of a lifetime distribution associated with a given life event is inherently linked to its FR structure. In engineering practice, many real-world datasets exhibit a BTFR or other nonlinear FR patterns. The final phase of the bathtub curve typically arises from cumulative damage, aging, or material degradation, leading to an increasing trend in the FR function. Nevertheless, contrary to the traditional assumption that the FR necessarily increases rapidly with time, several external factors such as design characteristics, operating environment, applied stress, and power quality can substantially influence the failure behavior of a component (see [11]).

A particularly important feature of the BTFR is the duration of its flat (or nearly constant) region, which spans the interval between the end of the burn-in period and the onset of wear-out. This region is of critical interest in reliability studies. Burn-in refers to a pre-use process in which products or systems are subjected to controlled electrical, thermal, or mechanical stress for a specified duration in order to eliminate early-life failures, commonly referred to as infant mortality. Components that fail during this period are discarded, ensuring that only items meeting acceptable quality standards are deployed. Although extending the burn-in period can reduce warranty-related costs by screening out defective units, it simultaneously increases manufacturing expenses. Consequently, determining an optimal burn-in time is essential. For bathtub-shaped lifetime models, this

optimal burn-in duration is often defined as the time at which the mean residual life (MRL) function reaches its maximum [12].

In recent years, there has been a notable surge in technological progress and complexity, especially in vital domains like contemporary engines, computerization, automation, and artificial intelligence (see [13], Chapter 2). The potential far-reaching consequences of failure in these areas have sparked considerable attention. Responding to this challenge has become a focal point, with a strong emphasis on advancing robust FR models and employing powerful estimation methods. This concerted effort aims to elevate the standards of reliability studies, quality control, system management, and the analysis of system survival. Extensive efforts have been devoted to developing lifetime distributions capable of characterizing bathtub-shaped FR patterns. These contributions can be broadly organized according to different improvement directions. Several studies have focused on Weibull-based extensions to enhance distributional flexibility and capture non-monotone FR behavior. Examples include the new modified Weibull distribution proposed by [14], the log-normal modified Weibull distribution in [15], the Sine-Weibull Geometric distribution introduced by [16], and the new additive Weibull extension developed by [17]. Moreover, additive FR models have been introduced by combining different baseline distributions or components to provide a reasonable flexibility in modeling complex lifetime patterns. Representative examples include the additive Chen–Gompertz model [18], the three-component additive Weibull model [19], the flexible additive Weibull model [20], the additive Teissier–Weibull model [21], and the additive Chen distribution [22]. Furthermore, competing-risk and hybrid formulations have been proposed to describe systems affected by two or more failure mechanisms, such as the flexible Dhillon–Weibull competing-risk model [23] and the flexible exponential power–Gompertz distribution [24]. In addition, discrete lifetime extensions, such as the discrete additive Perks–Weibull distribution [25], have been developed for situations where lifetime data are recorded in discrete form.

Despite the growing literature on ABC methods and BTFR models, existing studies typically address these topics separately. ABC methodologies have rarely been explored in the context of non-monotone failure rate models, while most BTFR models rely on likelihood-based estimation techniques. Motivated by this gap, the present paper integrates ABC–Gibbs methodology with the modeling and reliability analysis of bathtub-shaped failure rates, offering a likelihood-free inferential framework together with a new flexible competing-risk model.

The novelty of the proposed model lies in its specific competing-risk structure, which combines a Chen-type exponential aging component with an extended Weibull-type component modified by an inverse-time damping factor. This interaction provides flexible control over early-life failures, the flat useful-life period, and late-life wear-out behavior, enabling the model to capture prolonged bathtub-shaped regions and other non-monotone failure-rate patterns more effectively than many existing BTFR models.

The Main Contributions and Organization of the Paper

The main contributions of this paper are summarized as follows:

1. This paper develops an ABC–Gibbs framework specifically designed for lifetime models exhibiting BTFR. To the best of our knowledge, this represents one of the first systematic applications of ABC–Gibbs methodology to reliability models with non-monotone failure rate structures.
2. We demonstrate that ABC–Gibbs provides a robust likelihood-free alternative for Bayesian inference in complex multi-parameter BTFR models, where classical likelihood-based methods may be numerically unstable on complicated datasets.

3. A novel and flexible BTFR model is introduced by combining extended Weibull and Chen lifetime distributions in a competing-risk framework. The proposed model is capable of capturing long flat bathtub regions and diverse non-monotone failure patterns, which are difficult to model using many existing distributions.
4. Beyond parameter estimation, this paper investigates key reliability characteristics of the proposed model, including the FR, the MRL, and their associated change points. The relationship between FR and MRL is studied, and providing insights into burn-in optimization and system reliability assessment. Moreover, characteristics such as the cumulative residual entropy (CRE) and Kullback–Leibler divergence (KLD) of the proposed model are explored.
5. The performance of the proposed ABC-Gibbs methodology is examined through extensive simulation studies and real engineering datasets exhibiting bathtub-shaped failure rates. Comparisons with likelihood-based estimation and existing BTFR models demonstrate the practical effectiveness of the proposed approach.

The paper is arranged as follows. In Section 2, ABC-Gibbs is discussed. In Section 3, the proposed bathtub FR model and its characteristics are discussed for the purpose of testing the ABC-Gibbs scheme. In Section 4, model parameter estimation by ABC-Gibbs and maximum likelihood are presented. Simulation studies based on ABC-Gibbs are conducted to assess the performance of the ABC-Gibbs using KS as the distance function under various thresholds. In Section 5, two real-data studies are illustrated. In Section 6, conclusions and further studies are presented.

2. The Approximate Bayesian Computation Method

Let a probability distribution $f(\Theta)$ have parameter vector Θ . Let a sample observation be defined by D_{obs} and prior distribution $\pi(\Theta)$, and let the posterior distribution of Θ be derived by the product $\mathcal{L}(\Theta) \pi(\Theta)$, where $\mathcal{L}$ is the likelihood function. This is the usual Bayes estimation:

$$\pi(\Theta|D_{obs}) \propto \mathcal{L}(\Theta|D_{obs})\pi(\Theta).$$

The first basic ABC technique was introduced by [26] for solving an application in population genetics. The basic ABC algorithm operates by involving the use of summary statistics $s(\cdot)$ and generating samples from the joint distribution of the parameters vector Θ , and then simulating data using an available method—usually, the accept–rejection method. This technique avoids the requirement of likelihood evaluation; instead, the algorithm first generates a set of parameter vectors $\Theta^{(k)}$ sampled from the given prior density $\pi(\Theta)$, say, $\Theta^{(1)}, \Theta^{(2)}, \Theta^{(3)}, \dots, \Theta^{(n)}$, $k = 1, 2, \dots, n$. Next, the algorithm will simulate a data set with the same size as $s(D)$ using a generative model $\mathcal{M}(s(D)|\Theta)$ or a given distribution, say, $D^{(1)}, D^{(2)}, D^{(3)}, \dots, D^{(n)}$. Finally, a distance function is set $\rho(s(D), s(D_{obs})) \leq \epsilon$ as a condition to accept the set of parameters $\Theta^{(k)}$ satisfying $\rho(s(D), s(D_{obs})) \leq \epsilon$, where $s(\cdot)$ is sufficient statistics or summary statistics, and ϵ is a tolerance threshold. The accepted parameters $\Theta^{(k)}$ are considered the set of ABC estimates. This method is also called the likelihood-free Bayes method. We demonstrate the ABC algorithm in Algorithm A1 in Appendix A, and the flowchart in Figure 1.

Thus, the accepted parameter vector and simulated data in this algorithm are samples from a distribution of the form $\pi(\Theta, D|\rho(s(D), s(D_{obs})) \leq \epsilon)$. Hence, the actual posterior distribution obtained from the ABC algorithm can approximately be written as

$$\pi(\Theta|D_{obs}) \approx \pi_{\epsilon}(\Theta|D_{obs}) \propto \int_D \pi(\Theta, D|\rho(s(D), s(D_{obs})) \leq \epsilon) dD, \quad (1)$$

where $\pi_\epsilon(\Theta|D_{obs})$ is the marginal ABC posterior of parameter vector Θ ; one can see [6] for more details and the proof of (1). This algorithm can yield a reliable approximation of the posterior distribution by carefully selecting appropriate summary statistics, denoted as $s(\cdot)$, and choosing a sufficiently small value for ϵ . However, the determination of the tolerance threshold ϵ and the selection of set of summary statistics $s(\cdot)$ involves a trade-off between computational cost and algorithm accuracy. Consequently, a smaller ϵ results in higher accuracy but also entails increased computational costs. In alignment with the underlying concept of ABC, a notable drawback of the ABC rejection sampler is its low acceptance rate when the prior is ill-suited or has a significantly broader support. In response to this limitation, practitioners have developed advanced hybrid ABC algorithms to enhance the method, one of which is the ABC-MCMC introduced by [27]. This method employs a Markov Chain Monte Carlo (MCMC) sampler within the likelihood-free ABC framework. Notably, the ABC-MCMC sampler directly draws samples from the proposal distribution. Unlike traditional methods, the acceptance probability in ABC-MCMC does not involve the likelihood function or the normalizing constant, thereby improving the efficiency of the sampler. It is essential to highlight that, with a symmetric proposal distribution, the acceptance probability relies solely on the prior distribution. The evolution of the ABC algorithm has led to further advancements, with the introduction of ABC-Gibbs by [28].

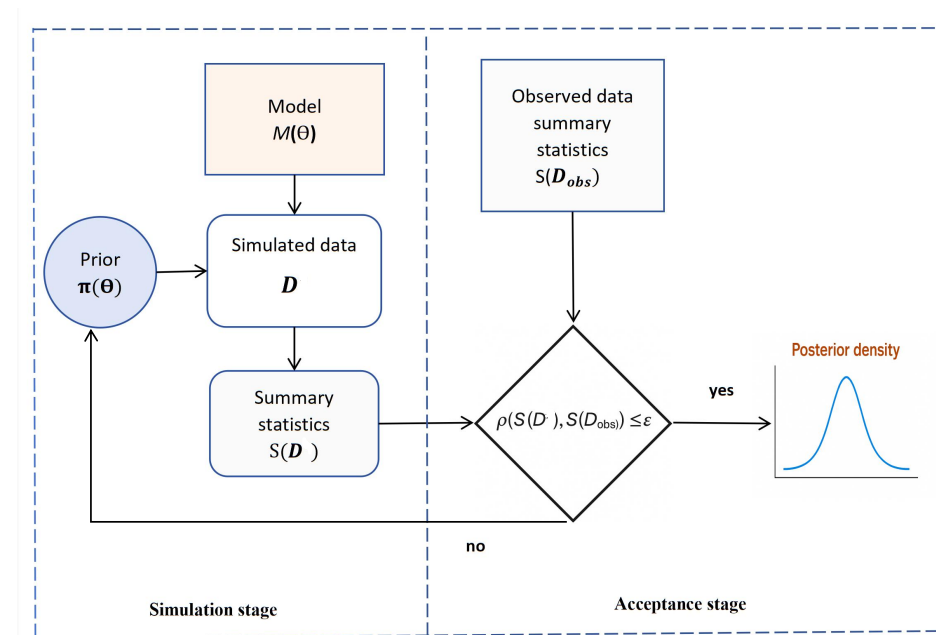


Figure 1. Flowchart of basic ABC algorithm.

2.1. Distance Measure, Summary Statistics, and Tolerance Threshold

To execute ABC, the selection of an appropriate distance function, denoted as $\rho(s(D), s(D_{obs}))$, is crucial and depends on the data's nature. It is evident that ρ is contingent on the choice of $s(\cdot)$. The use of sufficient statistics, particularly for large datasets, has been proposed for $s(\cdot)$ since sufficient statistics typically possess fewer dimensions compared to the complete dataset. Employing lower-dimensional statistics can enhance approximation accuracy while maintaining equivalent computational efforts. However, deriving sufficient statistics from high-dimensional complex or intractable models may be impractical, making this approach non-viable in practice. Consequently, it is feasible to consider a set of summary statistics $s_i(\cdot)$, such as a set of important quartiles, the mean, the mode, and so on. The effectiveness of ABC hinges on the judicious selection of these statistics (see [29]). The primary motivation behind transforming raw data into a set of summary statistics is

to improve the efficiency of comparisons between observed and simulated data. For large datasets, an important advantage of ABC is that inferences are based on informative summary statistics, which preserve key distributional features rather than a direct comparison of the full observed and simulated data, thus reducing the computational burden.

To improve scalability, the present study uses low-dimensional but informative summary statistics together with the Kolmogorov–Smirnov distance, thereby reducing computational complexity while maintaining the essential characteristics of the observed lifetime data.

In practical research applications, distance measures $\rho(\cdot, \cdot) = \|\cdot\|$, such as the Euclidean norm, correlation coefficient, or root mean square error, among others, are commonly used. However, it is crucial to retain only relevant summary statistics or, more generally, a distance function when constructing $\rho(\cdot, \cdot)$ to enable a reasonable inference of the desired target quantity.

The choice of the tolerance threshold ϵ is of utmost importance as it determines the acceptance condition. However, the distribution of $\rho(s(D), s(D_{obs}))$ is often unknown. Therefore, the algorithm can be initialized with a chosen initial value of ϵ for a trial. Subsequently, a check can be performed to analyze whether the selected ϵ meets certain criteria. For instance, [6] utilized the Euclidean norm as the distance function, while [30] suggested the use of some percentile of the distances as ϵ .

2.2. ABC-Gibbs Algorithm

The ABC-Gibbs sampling approach stands out as a distinct and notably efficient algorithmic strategy. Unlike directly computing the marginal posterior, this method involves sampling each parameter from its respective approximate marginal posterior distribution. ABC-Gibbs operates like the likelihood-free ABC sampler: here, it does not require the use of an explicit likelihood; rather, it indirectly targets the high-dimensional posterior by approximating its complete conditional distributions. This is accomplished through simulated parameter values and, potentially, the set of summary statistics, facilitating sampling and enabling approximate Gibbs update steps. Furthermore, adopting the Gibbs perspective allows for the consideration of the present values of other parameter components. This approach alleviates the need for simulating from an inefficient prior, as it utilizes a proposal distribution. This characteristic aligns this approach with earlier solutions found in the literature, such as the Metropolis version introduced by [27] and various sequential Monte Carlo schemes proposed by [31,32]. Notably, a Gibbs-based iteration of the ABC method presents a plethora of potential and substantial enhancements compared to earlier variants like basic ABC and ABC-MCMC. The ABC-Gibbs algorithm is detailed in Algorithm A2 in Appendix B.

3. Case Study Model: FR and MRL Characteristics

For the sake of testing the performance of the proposed ABC-Gibbs algorithm, we intend to propose a model, not to adopt the other existing ones. The proposed models' important reliability characteristics are discussed, such as reliability function (R), MRL, and the data-generating process.

Model Derivation

Consider a system composed of two independent subsystems arranged in series. Let X_1 and X_2 denote the respective lifetimes of the first and second subsystems, with cumulative distribution functions (CDFs) $F_{X_1}(\cdot)$ and $F_{X_2}(\cdot)$. The system lifetime T is

defined as $T = \min\{X_1, X_2\}$, which corresponds to the time to failure for the entire system. The reliability function of the system is therefore given by

$$R(t) = P(T > t) = \prod_{i=1}^2 [1 - F_{X_i}(t)], \quad t > 0. \quad (2)$$

To derive the proposed model, we refer to the $1 - F_{X_1}(x_1) = e^{-\lambda x_1^\theta e^{-\frac{\gamma}{x_1}}}$, an extended Weibull by [33]; the model characteristics were studied recently by [34]. $1 - F_{X_2}(x_2) = e^{\alpha(1-e^{t^\beta})}$ follows a form of Chen distribution. The R of the proposed model is given by (3) as

$$R(t) = e^{\alpha(1-e^{t^\beta}) - \lambda t^\theta e^{-\frac{\gamma}{t}}}, \quad t > 0, \quad (3)$$

with five-parameters $\alpha, \beta, \theta, \lambda > 0$ and $\gamma \geq 0$. The corresponding CDF is given by (4):

$$F(t) = 1 - e^{\alpha(1-e^{t^\beta}) - \lambda t^\theta e^{-\frac{\gamma}{t}}}. \quad (4)$$

While the probability density function (PDF) is given by

$$f(t) = \left[\alpha \beta t^{\beta-1} e^{t^\beta} + \lambda t^{\theta-1} e^{-\frac{\gamma}{t}} (\theta + \gamma t^{-1}) \right] e^{\alpha(1-e^{t^\beta}) - \lambda t^\theta e^{-\frac{\gamma}{t}}}. \quad (5)$$

The quantile function of the proposed distribution for a random data simulation process can be the real solution of the equation below:

$$\alpha e^{t^\beta} + \lambda t^\theta e^{-\frac{\gamma}{t}} = \alpha - \log(1 - p), \quad 0 \leq p \leq 1.$$

Since the equation above does not have an analytical closed-form solution, the values t_i must be determined numerically. This can be achieved through a root-finding method, which allows for efficient sampling and is particularly well-suited for simulation studies and Monte Carlo-based inference. In practice, the solution may be obtained using the `uniroot` function from the `HDInterval` package [35] in R (version 4.3.2). The corresponding algorithmic steps are summarized in Algorithm 1.

Algorithm 1 Generating random data

- Step 1: Specify $\alpha, \beta, \lambda, \theta$, and γ .
 Step 2: From uniform distribution, get samples of $p_i \sim \mathcal{U}(0, 1)$ for $i = 1, 2, \dots, n$.
 Step 3: Solve for t_i numerically for each p_i :

$$\alpha e^{t_i^\beta} + \lambda t_i^\theta e^{-\frac{\gamma}{t_i}} = \alpha - \log(1 - p_i).$$
-

3.1. FR Function of the Model

The FR function serves as a crucial component in a probability distribution, illustrating the growth in failure probability over time or aging. Its importance is particularly notable in various applications, with a significant role in the examination of failure time data in the realm of reliability studies. The FR function specific to the proposed model is expressed as

$$h(t) = \alpha \beta t^{\beta-1} e^{t^\beta} + \lambda t^{\theta-1} e^{-\frac{\gamma}{t}} [\theta + \gamma t^{-1}], \quad (6)$$

where, $\alpha, \beta, \lambda > 0$ and $\gamma \geq 0$. The influence of prior knowledge about the FR shape empowers us to make more informed decisions when selecting an appropriate model. In cases where the factors influencing the lifespan of a system or component change over time, it becomes crucial to address the problem through an analysis involving the FR.

We can establish the features of the new models's FR, which greatly depends on the shape parameter β and θ . From Equation (7), we can see that, for $\beta > 1$ and $\theta > 2$, the function $h'(t) > 0$; hence, the FR is an increasing failure rate (IFR). In the case when $\beta < 1$ and $\theta < 1$, the shape characteristics can be a decreasing failure rate (DFR) if $h'(t) < 0$. Further, if $\beta < 1$ and $\theta < 2$, the shape characteristics can be determined if $h'(t) = 0$, which can imply a BTFR and other non-monotone shapes; this can be displayed graphically as below.

$$h'(t) = \alpha\beta t^{\beta-2}e^{t^\beta}[\beta - 1 + \beta t^{\beta-1}] + \lambda\theta t^{\theta-2}e^{-\frac{\gamma}{t}}[\theta - 1 + \gamma t^{-1}] + \lambda\gamma t^{\theta-3}e^{-\frac{\gamma}{t}}[\theta - 2 + \gamma t^{-1}]. \quad (7)$$

We have illustrated various shapes of the proposed model's FR in Figures 2 and 3 for specific parameter values. Figure 2 distinctly reveals that the FR of the proposed model displays a bathtub-shaped profile with a prolonged-flat useful time region, successfully achieving the intended FR model. Furthermore, the cumulative failure rate (CFR) or cumulative hazard, denoted as $H(t)$, quantifies the accumulated overall risk associated with experiencing a specific event up to time t . The expression for the CFR pertaining to the proposed model is

$$H(t) = \int_0^t h(x)dx = -\ln[1 - F(t)] = \lambda t^\theta e^{-\frac{\gamma}{t}} - \alpha(1 - e^{t^\beta}).$$

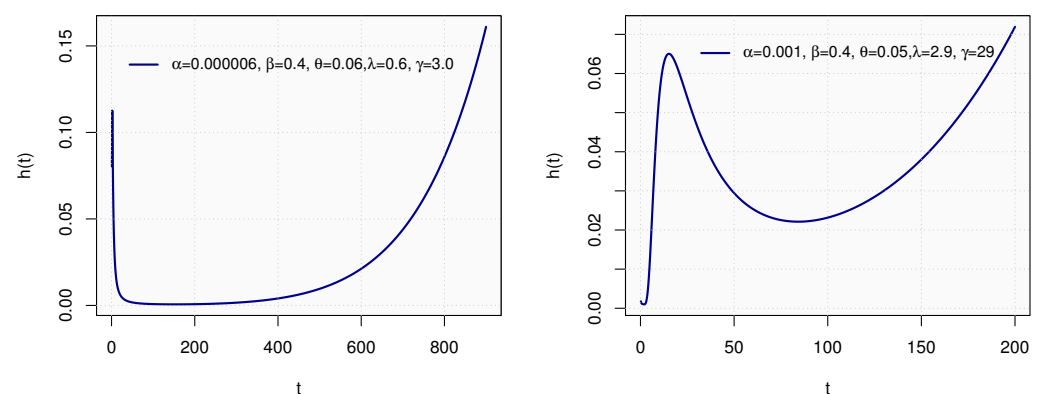


Figure 2. Plots of the FR function of the new model for some values of parameters: case A.

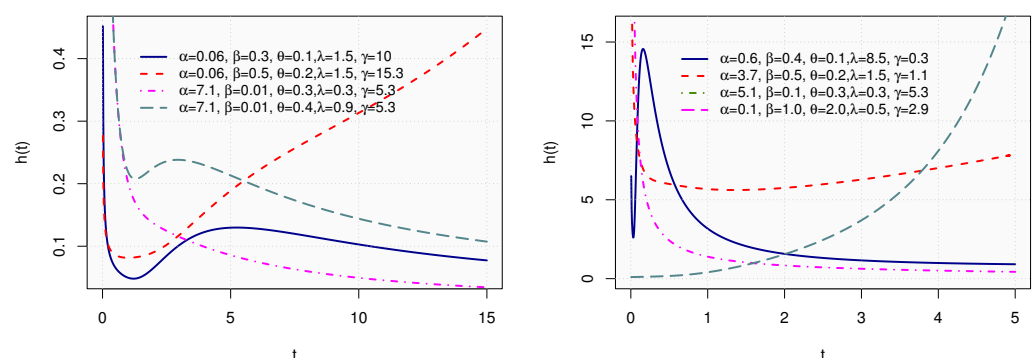


Figure 3. Plots of the FR function of the new model for various parameter values: case B.

3.2. MRL and Mean Time to Failure (MTTF)

In reliability theory and lifetime analysis, the MRL function provides an important way to describe the expected future lifetime of a unit that has already survived up to a given time. Rather than focusing only on the probability of failure, MRL measures the average remaining operating time, conditional on survival beyond a specified age. This makes it useful for evaluating aging behavior, maintenance planning, replacement decisions,

and survival patterns in engineering systems, software reliability, and biomedical lifetime studies. The uniqueness of the MRL characterizes the corresponding lifetime distribution distinctly, as discussed in [36]. The residual lifetime of T at time x is defined as the conditional random variable $T - x | T > x$. The MRL can be expressed mathematically as $\mu_T(x) = E(T - x | T > x)$. To compute the MRL of the proposed model, we require exponential series representation and Lemma A1, given in Appendix C. The MRL is derived as

$$\mu_T(x) = \frac{1}{R(x)} \sum_{i,j,k=0}^{\infty} \phi_{i,j,k}(\alpha, \lambda, \theta, \gamma) \gamma(-(\theta j + \beta k + 1), \frac{j\gamma}{x}), \quad (8)$$

where $\phi_{i,j,k}(\alpha, \lambda, \theta, \gamma) = \frac{(-1)^{i+j} \alpha^i \lambda^j i^k (j\gamma)^{\theta j + \beta k + 1} e^{\alpha}}{i! j! k!}$.

Clearly, the computation of the MRL is complicated. Interestingly, we derive an approximation value for large values of t that admits a closed-form expression, as stated in the following theorem.

Theorem 1. For sufficiently large t (i.e., as $t \rightarrow \infty$), the MRL of the proposed model admits the following asymptotic form.

$$\mu_T(t) \sim \begin{cases} \frac{t^{1-\beta}}{\alpha\beta} \exp(-t^\beta), & 0 < \theta < 1, \\ \frac{1}{\alpha\beta t^{\beta-1} \exp(t^\beta) + \lambda}, & \theta = 1, \\ 0, & \theta > 1. \end{cases} \quad (9)$$

Proof. Appendix C. $\square$

Table 1 reports selected numerical values of the MRL function computed using (8), together with the asymptotic approximation given in (9), for various choices of α , β , θ , λ , and γ , and for different time points. It is evident that the two approaches yield consistently similar results as the time point increases.

Table 1. MRL for some parameter values.

$(\alpha, \beta, \theta, \lambda, \gamma)$	t	Equation (8)	Equation (9)
(1, 1, 0.5, 0.3, 0.5)	1	0.2731	0.3679
	1.5	0.1831	0.2231
	3	0.0473	0.0498
	5	0.0067	0.0067
(1, 1, 1, 1, 1)	1	0.2385	0.2689
	1.5	0.1636	0.1824
	3	0.0455	0.0474
	5	0.0067	0.0067
(2, 0.5, 1, 1.5, 0.5)	1	0.2714	0.2689
	1.5	0.2625	0.2646
	3	0.2301	0.2346
	5	0.1895	0.1929

The MTTF is a measure used to estimate the average time that a non-repairable system or component operates before failing. This metric is typically applied to systems that cannot be repaired, such as specific electronic components or disposable items. The MTTF for the proposed model is expressed as

$$\mu_{TF}(x) = \sum_{i,j,k=0}^{\infty} \phi_{i,j,k}(\alpha, \lambda, \theta, \gamma) \Omega_{i,j,k}(\beta, \theta, \gamma; u),$$

where $\Omega_{i,j,k}(\beta, \theta, \gamma; u) = \int_0^{\infty} u^{-(\theta j + \beta k + 1) - 1} e^{-u} du$, which is easily computable numerically using mathematical packages such as R version 4.4.1 and MATLAB R2020b.

3.2.1. Relationship Between MRL and FR

The reliability function is linked to the FR through the well-known identity $R(t) = \exp\left(-\int_0^t h(x) dx\right)$, while the MRL is defined as $\mu_T(t) = \frac{1}{R(t)} \int_t^{\infty} R(x) dx$. While the failure rate can be represented in a straightforward manner through the MRL and its derivative, obtaining an explicit expression for the MRL in terms of the FR generally requires evaluating an integral. In particular,

$$h(t) = \frac{\mu'_T(t) + 1}{\mu_T(t)}, \quad (10)$$

with the restriction $\mu'_T(t) \geq -1$ [37]. Moreover, the reliability function can be equivalently expressed in terms of the MRL as $R(t) = \frac{\mu_T(0)}{\mu_T(t)} \exp\left(-\int_0^t \frac{1}{\mu_T(x)} dx\right)$. The interplay between MRL and FR has been studied extensively in the literature. For instance, Ref. [38] showed that, when the FR exhibits a BTFR pattern, the associated MRL follows an upside-down bathtub shape. They also established the inequality $\frac{1}{\mu_T(0)} \geq h(0)$, where the reciprocal of the mean time to failure exceeds the initial hazard rate. Refs. [39,40] examined the so-called “roller-coaster” FR, consisting of alternating decreasing and increasing phases, and established that the associated MRL inherits the opposite roller-coaster structure. These findings emphasize the deep structural connections between the MRL and the FR; not only can one function characterize the lifetime distribution, but the qualitative shape of one often implies a corresponding shape in the other. Figure 4 presents graphical evidence showing how the proposed model’s FR shape links with shape of the MRL.

3.2.2. Change Points of MRL and FR

A turning point of the MRL function reflects a shift in the expected remaining life, while a turning point of the FR signals a change in failure tendency from early-life failures to wear-out failures. Identifying these change points is essential in reliability analysis, maintenance planning, and risk assessment. In particular, for BTFR distributions, the location of these turning points provides critical insights into the duration of the flat (useful life) region. In this context, we investigate the interplay between the change points of BTFR and MRL. Although BTFR and MRL have been studied in the literature, relatively few works explicitly connect their change points. For instance, Ref. [39] established that, if a differentiable BTFR satisfies $\mu_T(t)h(t) \leq 1$ and admits a unique change point $c_t^0 \in (0, \infty)$, then the associated MRL is upside-down bathtub-shaped with a unique change point $c_t^* \in (0, c_t^0)$. Thus, the turning point of the MRL always precedes that of the FR. To quantify this difference, one may define the absolute and relative gaps between the FR and MRL change points as

$$\text{Difference: } \Delta_D = c_t^0 - c_t^*, \quad \text{Relative: } \nabla_R = \frac{\Delta_D}{c_t^*}.$$

These measures provide a useful index of the flatness of the bathtub curve: larger differences imply a more gradual transition in the FR, since the trend of the MRL does not change in the vicinity of the FR turning point. Recent contributions on the change point of MRL include [15], which examined MRL change points in log-normal modified Weibull models. Next, we investigate the critical change points for the selected BTFR and corresponding

MRL of the proposed model. It is observed that the optimal burn-in time that maximizes the residual life may occur considerably earlier than the point at which the FR attains its minimum [39]. For the proposed model, the change points of the FR can be determined by solving $\frac{dh(t)}{dt} = 0$ in (7), while those of the MRL, using the relation in $\mu'_T(x) = h(t)\mu_T(x) - 1$, involve solving for $\mu'_T(x) = 0$, which implies optimizing $h(t)\mu_T(x) = 1$, specifically by solving (11) or (12). Although closed-form solutions are not available, these quantities can be computed numerically.

$$\frac{\alpha\beta t^{\beta-1}e^{t^\beta} + \lambda t^{\theta-1}e^{-\frac{\gamma}{t}}[\theta + \gamma t^{-1}]}{e^{\alpha(1-e^{(t+x)^\beta}) - \lambda(t+x)^\theta}e^{-\frac{\gamma}{(t+x)}}} \int_0^\infty e^{\alpha(1-e^{(t+x)^\beta}) - \lambda(t+x)^\theta}e^{-\frac{\gamma}{(t+x)}} dt = 1, \quad (11)$$

or

$$\frac{\alpha\beta t^{\beta-1}e^{t^\beta} + \lambda t^{\theta-1}e^{-\frac{\gamma}{t}}[\theta + \gamma t^{-1}]}{e^{\alpha(1-e^{(t+x)^\beta}) - \lambda(t+x)^\theta}e^{-\frac{\gamma}{(t+x)}}} \sum_{i,j,k=0}^\infty \phi_{i,j,k}(\alpha, \lambda, \theta, \gamma) \gamma(-(\theta j + \beta k + 1), \frac{j\gamma}{x}) = 1. \quad (12)$$

Table 2 presents some numerical values of the change points of the proposed model's BTFR and MRL together with absolute and relative gaps between the FR and MRL change points. From Table 2, it is clear that, as α increases, the BTFR change point c_t^0 shifts slightly to the right, while both the difference Δ_D and the ratio ∇_R decrease. Secondly, increases in β and θ tend to move the change points of both FR and MRL to earlier times, while often yielding larger ratios ∇_R , reflecting a stronger relative difference between the two criteria. Thirdly, higher values of γ raise the magnitude of the BTFR but reduce Δ_D and ∇_R . Overall, the results confirm that the optimal burn-in time maximizing the MRL generally occurs earlier than the FR minimum, and that the gap between the two narrows as the parameter values increase, based on the selected set of parameters.

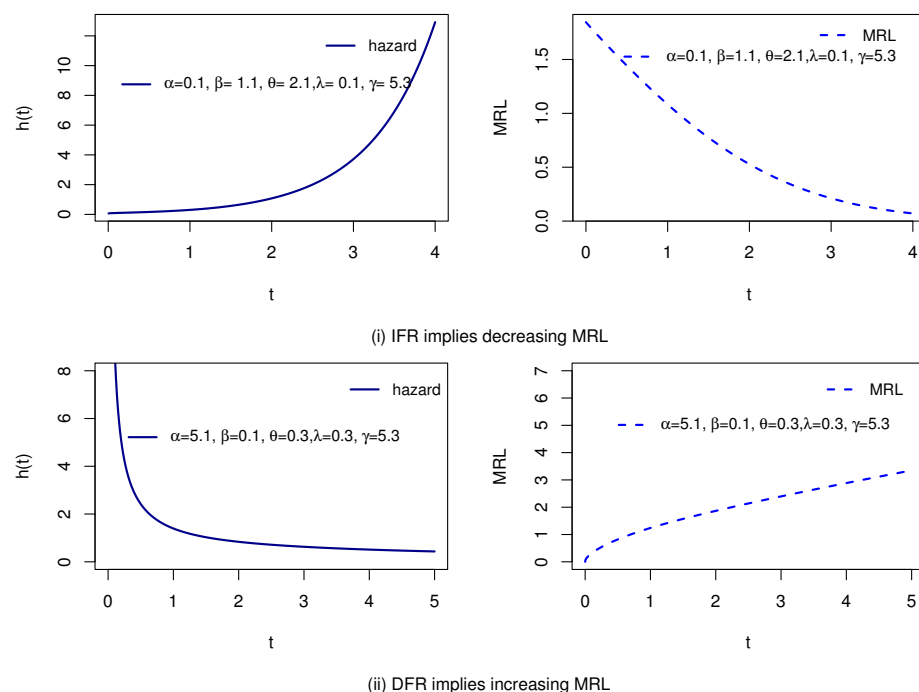


Figure 4. Cont.

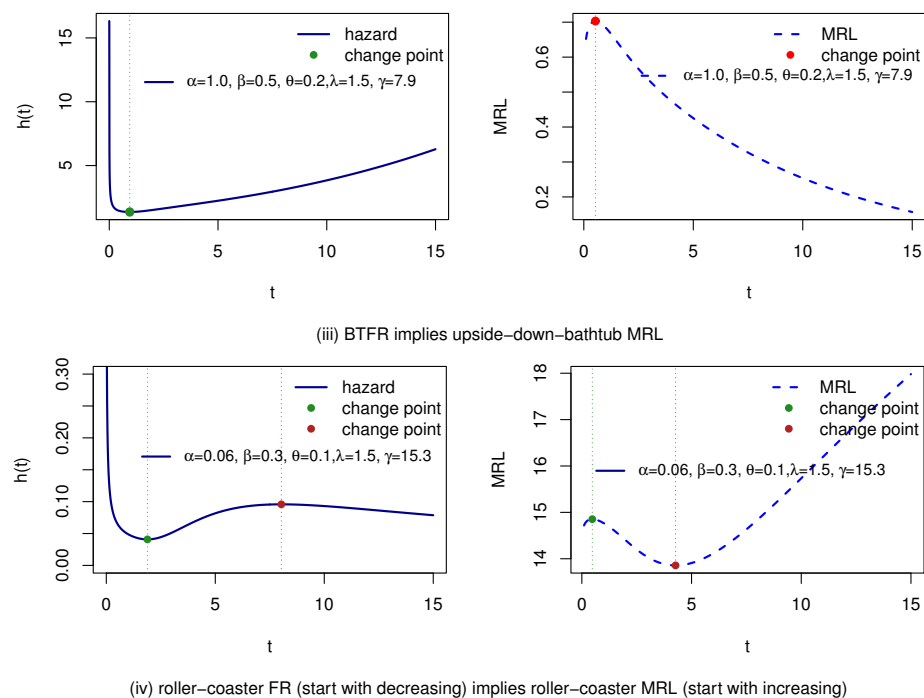


Figure 4. Graphical evidence of the relationship between the FR and MRL of the proposed model.

Table 2. Change points of MRL and BTFR for selected parameter sets of the proposed model.

α	β	θ	λ	γ	$(c_t^0, h(t))$	(c_t^*, MRL)	Δ_D	∇_R
1.0	0.5	0.2	1.5	7.9	(0.94126, 1.36282)	(0.53067, 0.70318)	0.41060	0.43622
1.1	0.4	0.1	1.2	6.5	(1.64326, 1.16564)	(1.26654, 0.85029)	0.37672	0.22925
1.2	0.5	0.2	1.5	7.9	(0.94882, 1.63476)	(0.57987, 0.59264)	0.36894	0.38885
1.5	0.5	0.2	1.5	7.9	(0.95710, 2.04262)	(0.63704, 0.47903)	0.32002	0.33438
1.8	0.5	0.2	1.5	7.9	(0.96299, 2.45044)	(0.68046, 0.40167)	0.28253	0.29339
2.0	0.5	0.2	1.5	7.9	(0.96611, 2.72231)	(0.70406, 0.36253)	0.26205	0.27124
3.0	0.5	0.2	1.5	7.9	(0.97613, 4.08159)	(0.78372, 0.24346)	0.19241	0.19712
3.1	0.6	0.9	1.9	1.2	(0.29093, 5.08416)	(0.19171, 0.19046)	0.09922	0.34103
3.7	0.5	0.2	1.5	1.1	(1.35527, 5.61880)	(1.19488, 0.17762)	0.16039	0.11834
3.7	0.5	0.9	1.6	1.1	(0.48053, 5.89434)	(0.38248, 0.16872)	0.09806	0.20406
3.8	0.5	0.9	1.6	1.2	(0.49622, 5.96289)	(0.39651, 0.16673)	0.09971	0.20094
3.8	0.6	1.0	1.7	1.3	(0.33467, 6.10381)	(0.23759, 0.16062)	0.09707	0.29005
6.0	0.5	0.2	1.5	7.9	(0.98733, 8.15917)	(0.87995, 0.12235)	0.10733	0.10871
6.0	0.6	0.2	1.5	7.9	(0.50874, 9.18812)	(0.42093, 0.10836)	0.08781	0.17269
6.8	0.6	0.5	2.5	8.0	(0.50875, 10.41320)	(0.42951, 0.09571)	0.07923	0.15574

3.3. Cumulative Residual Entropy

In information theory and related scientific fields, entropy is used to quantify the degree of uncertainty associated with an event when only partial knowledge of the underlying system is available. It is a fundamental tool in areas such as reliability theory and the life sciences. In this study, we consider the CRE, which is defined as $\text{CRE} = -\int_{-\infty}^{\infty} R(t) \log R(t) dt$. The CRE of the proposed model is given below, and the derivation of the CRE is given in Appendix D. Table 3 shows that CRE decreases sharply as the model parameters increase, indicating that higher parameter values lead to lower uncertainty in a system.

$$\begin{aligned}
CRE = & -\alpha e^{\alpha} \sum_{n,m,k=0}^{\infty} \frac{(-\alpha)^n (-\lambda)^m n^k}{n! m! k!} (m\gamma)^{m\theta+k\beta+1} \\
& \times \left[\gamma(-(m\theta+k\beta+1), \vartheta) + \varphi(m\theta+k\beta+2, \vartheta) \right] \\
& + \alpha e^{\alpha} \sum_{n,m,k=0}^{\infty} \frac{(-\alpha)^n (-\lambda)^m (n+1)^k}{n! m! k!} (m\gamma)^{m\theta+k\beta+1} \\
& \times \left[\gamma(-(m\theta+k\beta+1), \vartheta) + \varphi(m\theta+k\beta+2, \vartheta) \right] \\
& + \lambda e^{\alpha} \sum_{n,m,k=0}^{\infty} \frac{(-\alpha)^n (-\lambda)^m n^k}{n! m! k!} ((m+1)\gamma)^{m\theta+k\beta+1} \\
& \times \left[\gamma(-((m+1)\theta+k\beta+1), \vartheta) \right. \\
& \quad \left. + \varphi((m+1)\theta+k\beta+2, \vartheta) \right].
\end{aligned}$$

Table 3. Numerical values of CRE for some selected parameter values.

α	β	θ	λ	γ	CRE	α	β	θ	λ	γ	CRE
0.5	0.1	0.1	0.5	0.5	360.5684	1.5	0.2	0.2	1.5	1.5	0.51228
	0.15	0.15			23.5989		0.3	0.3			0.4164
	0.2	0.2			6.93805		0.5	0.5			0.3617
	0.3	0.2			2.44795		0.8	0.9			0.3116
	0.4	0.3			1.40549		1.5	1.5			0.2582
	0.4	0.4			1.3498	2.0	2.5	2.6	3.5	2.5	0.19174
	0.5	0.5			0.98655	2.8	3.5	4.6	5.5	4.5	0.15534
	0.6	0.8			0.7630	3.1	4.0	5.0	6.0	5.5	0.1423
	0.7	0.8			0.6553	7.1	4.6	5.3	6.8	5.9	0.1188
	0.8	0.9			0.5870	9.1	7.5	8.4	9.0	8.9	0.0877
	0.8	1.0			0.5794	10.5	9.5	9.5	9.4	10.0	0.0741
	1.2	1.2			0.4182	15.5	10.5	10.5	10.4	20.0	0.0675

3.4. Kullback–Leibler Divergence

Assessing the identifiability of a probabilistic model is a fundamental step in statistical inference, since it indicates whether the observed data contain enough information to uniquely determine the model parameters. The KLD is a commonly used measure for comparing two distributions and plays a key role in evaluating parameter identifiability. As pointed out in [41], the KLD provides a rigorous theoretical basis for examining whether a model's parameters can be uniquely identified. Accordingly, in this study, we adopt a KLD-based criterion, which is defined as follows.

Let $Z(y, \xi)$ denote a statistical model with parameter vector $\xi \in \mathbb{R}^d$. The parameter ξ is said to be *locally* (or *globally*) identifiable if it is the only solution of the equation $\text{KLD}(\xi, \zeta) = 0$ within an open neighborhood (or over the entire parameter space) of ξ in $\mathbb{R}^d$. The KLD between ξ and ζ is given by

$$\text{KLD}(\xi, \zeta) = \mathbb{E}_{\xi} \left[\log \left(\frac{Z(t, \xi)}{Z(t, \zeta)} \right) \right] = \int Z(t, \xi) \log \left(\frac{Z(t, \xi)}{Z(t, \zeta)} \right) dt. \quad (13)$$

It is well-known that $\text{KLD}(\xi, \zeta) \geq 0$ for all ζ , with equality if and only if $Z(t, \xi) = Z(t, \zeta)$ almost everywhere. Nevertheless, for many complex distributions, deriving a closed-form expression for the KLD in (13) is analytically intractable. Additional theoretical background

and illustrative examples can be found in [41]. In this study, $\text{KLD}(\xi, \zeta)$ for the proposed model is therefore evaluated numerically. Let $T_1 \sim f(t; \xi)$ and $T_2 \sim f(t; \zeta)$, where the parameter vectors are given by $\xi = (\alpha^{\oplus}, \beta^{\oplus}, \theta^{\oplus}, \lambda^{\oplus}, \gamma^{\oplus})'$ and $\zeta = (\alpha^{\ominus}, \beta^{\ominus}, \theta^{\ominus}, \lambda^{\ominus}, \gamma^{\ominus})'$. The KLD between these two parameter sets can be written as

$$\begin{aligned} \text{KLD}(\xi, \zeta) = & \int_0^{\infty} \left(\left[\alpha^{\oplus} \beta^{\oplus} t^{\beta^{\oplus}-1} e^{t^{\beta^{\oplus}}} + \lambda^{\oplus} t^{\theta^{\oplus}-1} e^{-\frac{\gamma^{\oplus}}{t}} (\theta^{\oplus} + \gamma^{\oplus} t^{-1}) \right] \right. \\ & \times \left. e^{\alpha^{\oplus}(1-e^{t^{\beta^{\oplus}}}) - \lambda^{\oplus} t^{\theta^{\oplus}} e^{-\frac{\gamma^{\oplus}}{t}}} \right) \\ & \times \log \left(\left[\alpha^{\oplus} \beta^{\oplus} t^{\beta^{\oplus}-1} e^{t^{\beta^{\oplus}}} + \lambda^{\oplus} t^{\theta^{\oplus}-1} e^{-\frac{\gamma^{\oplus}}{t}} (\theta^{\oplus} + \gamma^{\oplus} t^{-1}) \right] \right. \\ & \times \left. e^{\alpha^{\oplus}(1-e^{t^{\beta^{\oplus}}}) - \lambda^{\oplus} t^{\theta^{\oplus}} e^{-\frac{\gamma^{\oplus}}{t}}} \right) dt \\ & - \int_0^{\infty} \left(\left[\alpha^{\oplus} \beta^{\oplus} t^{\beta^{\oplus}-1} e^{t^{\beta^{\oplus}}} + \lambda^{\oplus} t^{\theta^{\oplus}-1} e^{-\frac{\gamma^{\oplus}}{t}} (\theta^{\oplus} + \gamma^{\oplus} t^{-1}) \right] \right. \\ & \times \left. e^{\alpha^{\oplus}(1-e^{t^{\beta^{\oplus}}}) - \lambda^{\oplus} t^{\theta^{\oplus}} e^{-\frac{\gamma^{\oplus}}{t}}} \right) \\ & \times \log \left(\left[\alpha^{\ominus} \beta^{\ominus} t^{\beta^{\ominus}-1} e^{t^{\beta^{\ominus}}} + \lambda^{\ominus} t^{\theta^{\ominus}-1} e^{-\frac{\gamma^{\ominus}}{t}} (\theta^{\ominus} + \gamma^{\ominus} t^{-1}) \right] \right. \\ & \times \left. e^{\alpha^{\ominus}(1-e^{t^{\beta^{\ominus}}}) - \lambda^{\ominus} t^{\theta^{\ominus}} e^{-\frac{\gamma^{\ominus}}{t}}} \right) dt. \end{aligned} \quad (14)$$

Due to the analytical intractability of (14), it is not straightforward to establish identifiability over the entire parameter space; thus, the KLD is evaluated numerically using R. To examine identifiability, we fix $\xi = (0.9, 0.8, 0.9, 0.6, 0.9)'$ and compare it with several alternative vectors ζ , including the case $\xi = \zeta$. The resulting KLD values in Table 4 show that $\text{KLD}(\xi, \zeta) > 0$ when $\xi \neq \zeta$ and equals zero only when $\xi = \zeta$, supporting identifiability for these parameter settings.

Table 4. Numerical values of KLD for some selected parameter values.

Hypothesis: $\xi \neq \zeta$			Hypothesis: $\xi \neq \zeta$ and $\xi = \zeta$		
	KLD	SD		KLD	SD
1. $\alpha^{\oplus} \neq \alpha^{\ominus}$	0.05339	4.5×10^{-6}	15. $\lambda^{\oplus} \neq \lambda^{\ominus}, \gamma^{\oplus} \neq \gamma^{\ominus}$	0.00538	6.6×10^{-5}
2. $\beta^{\oplus} \neq \beta^{\ominus}$	0.10387	1.9×10^{-5}	16. $\alpha^{\oplus} \neq \alpha^{\ominus}, \beta^{\oplus} \neq \beta^{\ominus}, \theta^{\oplus} \neq \theta^{\ominus}$	0.10718	5.4×10^{-5}
3. $\theta^{\oplus} \neq \theta^{\ominus}$	0.00114	1.2×10^{-5}	17. $\alpha^{\oplus} \neq \alpha^{\ominus}, \beta^{\oplus} \neq \beta^{\ominus}, \lambda^{\oplus} \neq \lambda^{\ominus}$	0.07277	9.7×10^{-5}
4. $\lambda^{\oplus} \neq \lambda^{\ominus}$	0.00097	6.8×10^{-6}	18. $\alpha^{\oplus} \neq \alpha^{\ominus}, \beta^{\oplus} \neq \beta^{\ominus}, \gamma^{\oplus} \neq \gamma^{\ominus}$	0.08494	4.6×10^{-5}
5. $\gamma^{\oplus} \neq \gamma^{\ominus}$	0.00011	2.1×10^{-5}	19. $\beta^{\oplus} \neq \beta^{\ominus}, \theta^{\oplus} \neq \theta^{\ominus}, \lambda^{\oplus} \neq \lambda^{\ominus}$	0.10516	1.8×10^{-5}
6. $\alpha^{\oplus} \neq \alpha^{\ominus}, \beta^{\oplus} \neq \beta^{\ominus}$	0.01235	2.6×10^{-5}	20. $\beta^{\oplus} \neq \beta^{\ominus}, \theta^{\oplus} \neq \theta^{\ominus}, \gamma^{\oplus} \neq \gamma^{\ominus}$	0.1052	8.6×10^{-5}
7. $\alpha^{\oplus} \neq \alpha^{\ominus}, \theta^{\oplus} \neq \theta^{\ominus}$	0.00545	7.7×10^{-5}	21. $\theta^{\oplus} \neq \theta^{\ominus}, \lambda^{\oplus} \neq \lambda^{\ominus}, \gamma^{\oplus} \neq \gamma^{\ominus}$	0.01303	2.2×10^{-6}
8. $\alpha^{\oplus} \neq \alpha^{\ominus}, \lambda^{\oplus} \neq \lambda^{\ominus}$	0.00253	1.3×10^{-5}	22. $\alpha^{\oplus} \neq \alpha^{\ominus}, \beta^{\oplus} \neq \beta^{\ominus}, \theta^{\oplus} \neq \theta^{\ominus}, \lambda^{\oplus} \neq \lambda^{\ominus}$	0.10390	7.5×10^{-6}
9. $\alpha^{\oplus} \neq \alpha^{\ominus}, \gamma^{\oplus} \neq \gamma^{\ominus}$	0.00405	0.000.2	23. $\alpha^{\oplus} \neq \alpha^{\ominus}, \beta^{\oplus} \neq \beta^{\ominus}, \theta^{\oplus} \neq \theta^{\ominus}, \gamma^{\oplus} \neq \gamma^{\ominus}$	0.08679	1.3×10^{-6}
10. $\beta^{\oplus} \neq \beta^{\ominus}, \theta^{\oplus} \neq \theta^{\ominus}$	0.00735	1.1×10^{-5}	24. $\alpha^{\oplus} \neq \alpha^{\ominus}, \theta^{\oplus} \neq \theta^{\ominus}, \lambda^{\oplus} \neq \lambda^{\ominus}, \gamma^{\oplus} \neq \gamma^{\ominus}$	0.05682	1.2×10^{-5}
11. $\beta^{\oplus} \neq \beta^{\ominus}, \lambda^{\oplus} \neq \lambda^{\ominus}$	0.01171	1.2×10^{-5}	25. $\alpha^{\oplus} \neq \alpha^{\ominus}, \beta^{\oplus} \neq \beta^{\ominus}, \lambda^{\oplus} \neq \lambda^{\ominus}, \gamma^{\oplus} \neq \gamma^{\ominus}$	0.13286	3.6×10^{-5}
12. $\beta^{\oplus} \neq \beta^{\ominus}, \gamma^{\oplus} \neq \gamma^{\ominus}$	0.00907	1.2×10^{-5}	26. $\beta^{\oplus} \neq \beta^{\ominus}, \theta^{\oplus} \neq \theta^{\ominus}, \lambda^{\oplus} \neq \lambda^{\ominus}, \gamma^{\oplus} \neq \gamma^{\ominus}$	0.21576	2.7×10^{-6}
13. $\theta^{\oplus} \neq \theta^{\ominus}, \lambda^{\oplus} \neq \lambda^{\ominus}$	0.00202	1.4×10^{-5}	27. $\alpha^{\oplus} \neq \alpha^{\ominus}, \beta^{\oplus} \neq \beta^{\ominus}, \theta^{\oplus} \neq \theta^{\ominus}, \lambda^{\oplus} \neq \lambda^{\ominus}, \gamma^{\oplus} \neq \gamma^{\ominus}$	0.16461	5.4×10^{-6}
14. $\theta^{\oplus} \neq \theta^{\ominus}, \gamma^{\oplus} \neq \gamma^{\ominus}$	0.00068	1.3×10^{-5}	28. $\alpha^{\oplus} = \alpha^{\ominus}, \beta^{\oplus} = \beta^{\ominus}, \theta^{\oplus} = \theta^{\ominus}, \lambda^{\oplus} = \lambda^{\ominus}, \gamma^{\oplus} = \gamma^{\ominus}$	0.0000	0.0000

4. Parameter Estimation and Simulation Studies

In this section, the ABC-Gibbs algorithm for the proposed model parameters estimation using KS as a distance function is developed. We focus on the performance of the ABC-Gibbs algorithm and the proposed distance function; hence, a simulation study is presented based on the ABC-Gibbs to assess the proposed methodology. For the sake of real-data studies, the MLE method is presented so as to include likelihood-based model comparison criteria.

4.1. ABC-Gibbs for the Proposed Model Parameters

We provided the formulation of the ABC-Gibbs for the proposed model. Another important factor here is the summary statistics, which are not involved in the ordinary Bayes method. Let D (or D_{obs}), containing $t_1, t_2, \dots, t_n$, be observed samples of size n from the proposed model. Let $\Theta = (\alpha, \beta, \theta, \lambda, \gamma)'$ be a vector of parameters, and let the ABC estimate be $\hat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\lambda}, \hat{\gamma})'$. Here, the choice of the prior is very sensitive because there is no prior information on the parameters. Thus, we assumed that Θ follows a gamma distribution since Θ is positive:

$$\pi(\omega_j) = \frac{\kappa_j^{\eta_j}}{\Gamma(\eta_j)} \omega_j^{\eta_j-1} e^{-\kappa_j \omega_j}, \quad \omega_j, \kappa_j, \eta_j > 0, \quad j = 1, \dots, 5.$$

However, we could not find any other systematic means for determining specific hyperparameters. Rather, we adopted the usual predictive checking to guess the prior knowledge and see if it aligned with our expectations; else, we adjusted the value of the hyperparameters. Therefore, $\alpha \sim \pi(\alpha|\kappa_1, \eta_1)$, $\beta \sim \pi(\beta|\kappa_2, \eta_2)$, $\theta \sim \pi(\theta|\kappa_3, \eta_3)$, $\lambda \sim \pi(\lambda|\kappa_4, \eta_4)$, $\gamma \sim \pi(\gamma|\kappa_5, \eta_5)$.

Let $s(D)$ be $y_1, \dots, y_m$, for some $m \ll n$. The approximate joint posterior density of $\Theta|D$ is given by (15).

$$\begin{aligned} \pi(\Theta|D) &\approx \pi(\Theta|s(D)) = \alpha^{\eta_1-1} \beta^{\eta_2-1} \theta^{\eta_3-1} \lambda^{\eta_4-1} \gamma^{\eta_5-1} \\ &\times \exp\left(-\kappa_1 \alpha - \kappa_2 \beta - \kappa_3 \theta - \kappa_4 \lambda - \kappa_5 \gamma + \sum_{i=1}^m \sigma_1(y_i)\right) \prod_{i=1}^m \sigma_2(y_i). \end{aligned} \quad (15)$$

For notational simplicity, we define $\sigma_1(y_i)$ and $\sigma_2(y_i)$ as the exponential and multiplicative components of the proposed PDF, respectively:

$$\sigma_1(y_i) = \alpha(1 - e^{y_i^\beta}) - \lambda y_i^\theta e^{-\frac{\gamma}{y_i}},$$

and

$$\sigma_2(y_i) = \alpha \beta y_i^{\beta-1} e^{y_i^\beta} + \lambda \theta y_i^{\theta-1} e^{-\frac{\gamma}{y_i}} + \lambda \gamma y_i^{\theta-2} e^{-\frac{\gamma}{y_i}}.$$

From the approximate posterior based on the summary statistics in (15), the proportional full conditional posterior densities used in the ABC-Gibbs updating scheme are obtained by retaining the terms involving the parameter being updated while treating the remaining parameters as fixed. They are given by

$$\begin{aligned}
\pi_1(\alpha|s(\mathbf{D})) &\propto \alpha^{\eta_1-1} \exp\left(-\kappa_1\alpha + \alpha \sum_{i=1}^m (1 - e^{y_i^\beta})\right) \sigma_2^*(y_i), \\
\pi_2(\beta|s(\mathbf{D})) &\propto \beta^{\eta_2-1} \exp\left(-\kappa_2\beta + \alpha \sum_{i=1}^m (1 - e^{y_i^\beta})\right) \sigma_2^*(y_i), \\
\pi_3(\theta|s(\mathbf{D})) &\propto \theta^{\eta_3-1} \exp\left(-\kappa_3\theta - \lambda \sum_{i=1}^m y_i^\theta e^{-\frac{\gamma}{y_i}}\right) \sigma_2^*(y_i), \\
\pi_4(\lambda|s(\mathbf{D})) &\propto \lambda^{\eta_4-1} \exp\left(-\kappa_4\lambda - \lambda \sum_{i=1}^m y_i^\theta e^{-\frac{\gamma}{y_i}}\right) \sigma_2^*(y_i), \\
\pi_5(\gamma|s(\mathbf{D})) &\propto \gamma^{\eta_5-1} \exp\left(-\kappa_5\gamma - \lambda \sum_{i=1}^m y_i^\theta e^{-\frac{\gamma}{y_i}}\right) \sigma_2^*(y_i),
\end{aligned} \tag{16}$$

where $\sigma_2^*(y_i) = \prod_{i=1}^m \sigma_2(y_i)$. Through the above approximate marginal posterior densities, we can determined the ABC-Gibbs estimates $\hat{\Theta}$ respectively as

$$\begin{aligned}
\hat{\alpha} &= \mathbb{E}[\alpha|\mathbf{D}] \approx \frac{1}{M} \sum_{j=1}^M \alpha^{(j)} \mathbb{1}_{\{\rho < \epsilon\}}, \\
\hat{\beta} &= \mathbb{E}[\beta|\mathbf{D}] \approx \frac{1}{M} \sum_{j=1}^M \beta^{(j)} \mathbb{1}_{\{\rho < \epsilon\}}, \\
\hat{\theta} &= \mathbb{E}[\theta|\mathbf{D}] \approx \frac{1}{M} \sum_{j=1}^M \theta^{(j)} \mathbb{1}_{\{\rho < \epsilon\}}, \\
\hat{\lambda} &= \mathbb{E}[\lambda|\mathbf{D}] \approx \frac{1}{M} \sum_{j=1}^M \lambda^{(j)} \mathbb{1}_{\{\rho < \epsilon\}}, \\
\hat{\gamma} &= \mathbb{E}[\gamma|\mathbf{D}] \approx \frac{1}{M} \sum_{j=1}^M \gamma^{(j)} \mathbb{1}_{\{\rho < \epsilon\}},
\end{aligned}$$

where $\mathbb{1}_{\rho(\cdot, \cdot) < \epsilon}$ is an indicator function. An approximate $100(1 - d)\%$ highest posterior density (HPD) credible interval of Θ can be computed as described in [42].

Activating the Proposed ABC-Gibbs Algorithm

To initialize the ABC-Gibbs for the proposed model, we start with the selection of the distance function ρ as the Kolmogorov–Smirnov statistic and some set of tolerance threshold for trial, say, $\epsilon_1, \epsilon_2, \dots, \epsilon_n$. The Kolmogorov–Smirnov statistic quantifies a distance between the empirical distributions of two samples, defined as

$$\rho(s(\mathbf{D}^*), s(\mathbf{D}_{obs})) = \sup_{s(\cdot)} |F_1(s(\mathbf{D}^*)) - F_2(s(\mathbf{D}))|, \tag{17}$$

where $F_i(\cdot)$, $i = 1, 2$ are the empirical distribution functions of the two samples, and $\sup$ is the supremum function. $s_i(\cdot)$, $i = 1, \dots, m$, are the extreme order statistics: some quartiles, mean, median, etc. We use a normal proposal distribution $q(\cdot|\cdot)$ and restrict the generated candidates to the positive parameter space. The algorithm is run for a sufficiently large number of iterations N , with a selected distance measure ρ and tolerance threshold ϵ . Since the conditional posterior densities in (16) are not available in closed form, parameter candidates are generated using an MCMC-based Metropolis-within-Gibbs update. At each update, the simulated summary statistics are compared with the observed summary

statistics, and the candidate is accepted if the ABC condition is satisfied; otherwise, the previous value is retained. The proposed ABC-Gibbs algorithm for the proposed model is given in Algorithm 2.

Algorithm 2 ABC-Gibbs algorithm for the proposed model

1. Input: prior distribution $\pi(\Theta)$, ρ , tolerance threshold ϵ , model $f(\Theta)$, $s(D)$, and iteration number K
 2. Start: input $\Theta^{(0)} \sim \pi(\Theta)$, say, $\Theta^{(0)} = (\alpha^{(0)}, \beta^{(0)}, \theta^{(0)}, \lambda^{(0)}, \gamma^{(0)})^T$.
 3. **for** $k = 1$ to K **do**
 4. For each parameter in Θ_i in the Θ , $i = 1, 2, 3, 4, 5$. **do**
 5. Sample: $\Theta_i^* \sim \pi_i(\Theta_i | s(D), \Theta_{-i}^{k-1})$,
 6. Generate data set $s(D^*)$ using the algorithm in the Table 1.
 7. Compute $\rho = \sup_{s(\cdot)} |F_1(s(D)) - F_2(s(D^*))|$
 8. **if** $\rho \leq \epsilon$, **then**
 9. Acceptance Θ_i^* and set $\Theta_i^{(k)} = \Theta_i^*$
 10. **else**
 11. reject Θ_i^* and set $\Theta_i^{(k)} = \Theta_i^{(k-1)}$
 12. **end if**
 13. **end for**
 14. update $\Theta^{(k)} = (\Theta_1^{(k)}, \Theta_2^{(k)}, \dots, \Theta_n^{(k)})$
 15. **end for**
-

An approximate $100(1 - c)\%$ highest posterior density (HPD) credible interval for $\hat{\Theta}$ can be obtained using the procedure of [42], and can be achieved using HDInterval package [35] in R. This interval corresponds to the shortest credible range for each accepted parameter and can be written as

$$(\hat{\Theta}^{(1)}, \hat{\Theta}^{((1-c)k)}), (\hat{\Theta}^{(2)}, \hat{\Theta}^{((1-c)(k+1))}), \dots, (\hat{\Theta}^{(ck)}, \hat{\Theta}^{(k)}).$$

4.2. Simulation Experiment

In the subsequent section, we delve into the performance evaluation of the ABC-Gibbs estimates. Unlike conventional simulation studies that mainly assess the effect of increasing sample size through mean square error in replications, ABC focuses on accepted parameters based on a threshold and distance function because ABC-Gibbs does not require a complete likelihood, and ABC-Gibbs is less directly dependent on the full data dimension because it uses summary statistics. In addition, the tolerance threshold plays a role in controlling the bias of the estimates, so mean square error assessment with repeated simulation is not efficient because the acceptance rate varies. Therefore, in this study, the performance is assessed mainly through acceptance rate, tolerance levels, and acceptance behavior rather than relying on repeated-simulation mean square error. We selected specific values for parameters $\alpha, \beta, \theta, \lambda$, and γ . Due to the nature of ABC algorithms, we iterated the process 1,000,000 times for various moderate sample sizes, denoted as n . The summary statistics used are the mean, median, mode, minimum and maximum order statistics, and upper and lower quartiles. Four distinct simulation results of (i.e, set of parameters and sample size) $(\Theta; n)$ were presented, following the algorithm outlined in Algorithm 2. The simulations were carried out for $(0.5, 0.7, 1.5, 1.1, 1.5; 30)$, $(0.5, 0.7, 1.6, 0.8, 1.3; 50)$, $(0.7, 0.6, 1.5, 0.9, 1.4; 100)$, and $(0.9, 0.4, 1.3, 0.8, 1.3; 200)$. The gamma priors employed were $\kappa_1 = 6$, $\eta_1 = 5$, $\kappa_2 = 9$, $\eta_2 = 5$, $\kappa_3 = 6$, $\eta_3 = 5$, $\kappa_4 = 9$, $\eta_4 = 6$, and $\kappa_5 = 6$, $\eta_5 = 9$.

To assess the algorithm's performance, we applied three sets of thresholds, namely, $\epsilon_1 = 0.2$, $\epsilon_2 = 0.4$, and $\epsilon_3 = 0.6$, while utilizing Kolmogorov–Smirnov as distance measure. The resulting numerical outcomes of the simulations are detailed in Table 5. In addition, the simulation results, illustrated in Figures 5–8, indicate commendable performance. The estimated parameters, obtained under the three different thresholds, exhibit close agreement. Thus, the ABC-Gibbs algorithm performed effectively on the proposed model, demonstrating the importance of the judicious choice of the distance function. In certain instances, permissible thresholds may yield satisfactory results, but the advantage of a smaller distance in ABC lies in its ability to capture the best estimates with minimal bias. Evidently, in our simulation, the smallest distance provides the best estimate, which is undoubtedly the best fit considering our choice of distance measure.

Table 5. Numerical results of the simulation studies.

Sample Size	α	β	θ	λ	γ	Threshold (ϵ)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\gamma}$	Acceptance Rate (%)
30	0.5	0.7	1.5	1.1	1.5	0.2	0.4393	0.8260	1.6924	0.9249	1.5135	0.02617
						0.4	0.5350	0.6195	1.5249	0.8516	1.5347	0.48551
						0.6	0.6502	0.5733	1.5109	0.8597	1.5170	0.91016
50	0.8	0.7	1.6	0.8	1.3	0.2	0.7931	0.5931	1.5461	0.7916	1.4103	0.05131
						0.4	0.7101	0.6315	1.5813	0.7216	1.2356	0.32512
						0.6	0.9214	0.8135	1.7310	0.9314	1.3613	0.8135
100	0.7	0.6	1.5	0.9	1.4	0.2	0.7999	0.7463	1.5003	0.9481	1.4025	0.00741
						0.4	0.6987	0.5859	1.5006	0.8826	1.4885	0.00819
						0.6	0.6915	0.5636	1.5013	0.8696	1.5072	0.9837
200	0.9	0.4	1.3	0.8	1.3	0.2	0.9208	0.4546	1.5001	0.8432	1.4562	0.00235
						0.4	0.7512	0.5161	1.5029	0.8654	1.4992	0.01065
						0.6	0.6934	0.5571	1.5058	0.8612	1.5115	0.35648

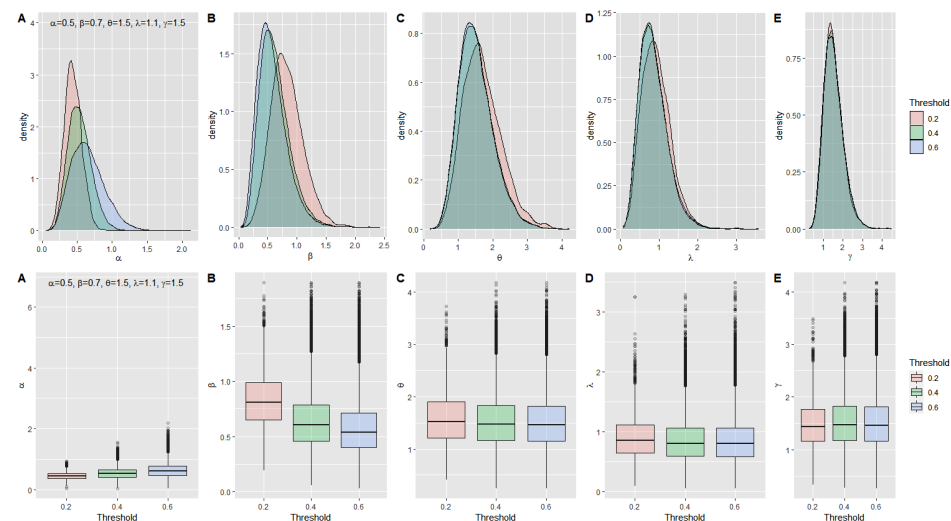


Figure 5. Simulation plots: density (top) and its corresponding box plot (bottom) for $(\Theta; n) = (0.5, 0.7, 1.5, 1.1, 1.5; 30)$.

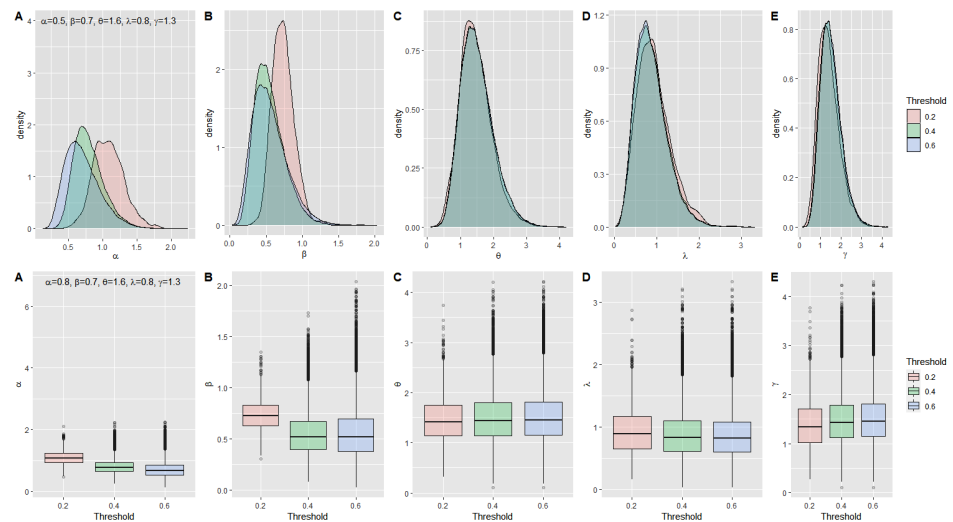


Figure 6. Simulation plots: density (top) and its corresponding box plot (bottom) for $(\Theta; n) = (0.5, 0.7, 1.6, 0.8, 1.3; 50)$.

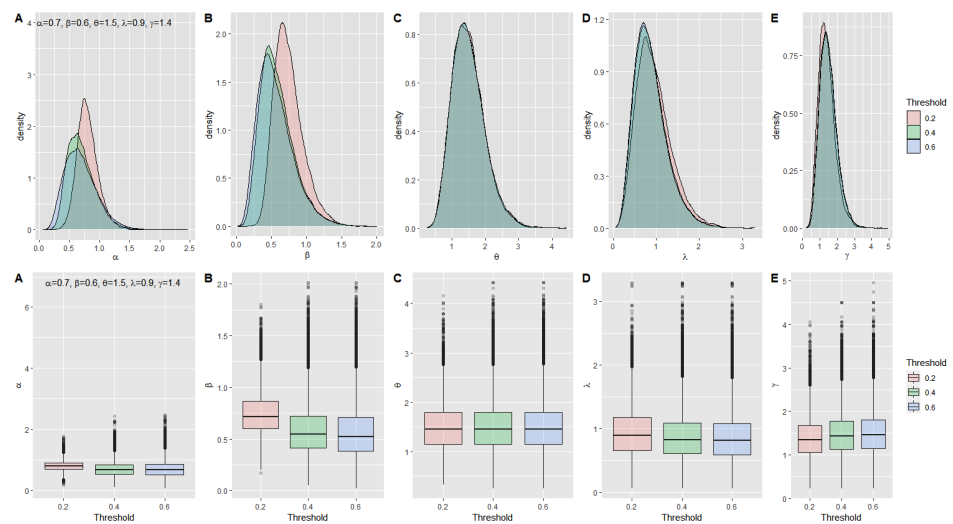


Figure 7. Simulation plots: density (top) and its corresponding box plot (bottom) for $(\Theta; n) = (0.7, 0.6, 1.5, 0.9, 1.4; 100)$.

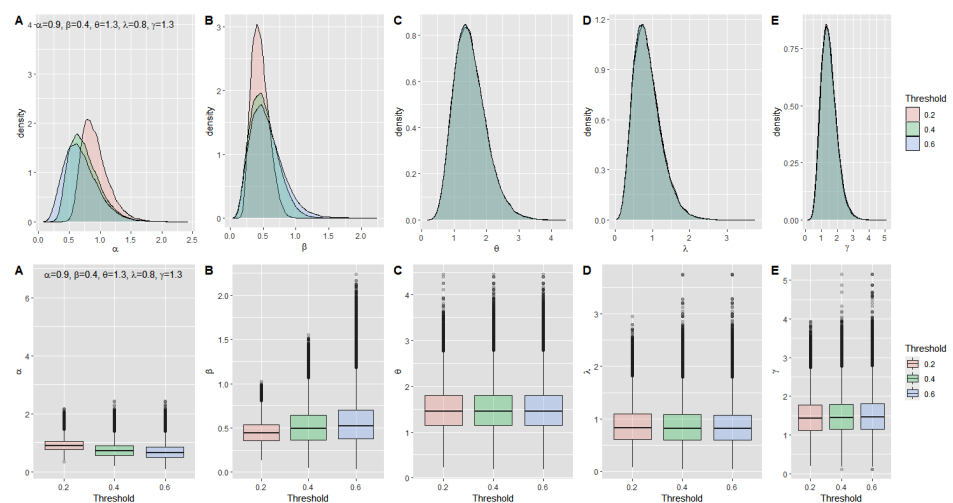


Figure 8. Simulation plots: density (top) and its corresponding box plot (bottom) for $(\Theta; n) = (0.9, 0.4, 1.3, 0.8, 1.3; 200)$.

4.3. MLE for the Model Parameters

Here, MLEs are provided for the sake of likelihood-based model selection criteria for model comparison. Let $t_1, t_2, t_3, \dots, t_n$ be observed values from a sample of size n from the new model. Let $\Theta = (\alpha, \beta, \theta, \lambda, \gamma)^T$; then, the MLEs of Θ , that is $\hat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\lambda}, \hat{\gamma})^T$ can be computed through the maximization of log-likelihood $L(\Theta)$, given as

$$L(\Theta) = \sum_{i=1}^n \log \left[\alpha \beta t_i^{\beta-1} e^{t_i^\beta} + \lambda \theta t_i^{\theta-1} e^{-\frac{\gamma}{t_i}} + \lambda \gamma t_i^{\theta-2} e^{-\frac{\gamma}{t_i}} \right] + \sum_{i=1}^n \left(\alpha (1 - e^{t_i^\beta}) - \lambda t_i^\theta e^{-\frac{\gamma}{t_i}} \right).$$

Alternatively, by the solutions of the nonlinear system,

$$\begin{aligned} \frac{\partial L}{\partial \alpha} &= \sum_{i=1}^n \frac{\beta t_i^{\beta-1} e^{t_i^\beta}}{v_1(t_i)} + \sum_{i=1}^n (1 - e^{t_i^\beta}), \\ \frac{\partial L}{\partial \beta} &= \sum_{i=1}^n \frac{\alpha t_i^{\beta-1} e^{t_i^\beta} + \alpha \beta v_2(t_i) [1 + t_i^\beta]}{v_1(t_i)} - \alpha \sum_{i=1}^n t_i^\beta \ln(t_i) e^{t_i^\beta}, \\ \frac{\partial L}{\partial \theta} &= \sum_{i=1}^n \frac{\lambda t_i^{\theta-1} e^{-\frac{\gamma}{t_i}} + \lambda v_3(t_i) [\theta + \gamma v_3(t_i) t_i^{-1}]}{v_1(t_i)} - \lambda \sum_{i=1}^n t_i^\theta \ln(t_i) e^{-\frac{\gamma}{t_i}}, \\ \frac{\partial L}{\partial \lambda} &= \sum_{i=1}^n \frac{t_i^{\theta-1} e^{-\frac{\gamma}{t_i}} [\theta + \gamma t_i^{-1}]}{v_1(t_i)} - \sum_{i=1}^n t_i^\theta e^{-\frac{\gamma}{t_i}}, \\ \frac{\partial L}{\partial \gamma} &= - \sum_{i=1}^n \frac{\lambda v_4(t_i) [\theta - 1 + \gamma t_i^{-1}]}{v_1(t_i)} + \lambda \sum_{i=1}^n t_i^{\theta-1} e^{-\frac{\gamma}{t_i}}, \end{aligned}$$

where $v_1(t_i) = \left[\alpha \beta t_i^{\beta-1} e^{t_i^\beta} + \lambda \theta t_i^{\theta-1} e^{-\frac{\gamma}{t_i}} + \lambda \gamma t_i^{\theta-2} e^{-\frac{\gamma}{t_i}} \right]$, $v_2(t_i) = t_i^{\beta-1} \ln t_i e^{t_i^\beta}$, $v_3(t_i) = t_i^{\theta-1} \ln t_i e^{-\frac{\gamma}{t_i}}$, and $v_4(t_i) = t_i^{\theta-2} e^{-\frac{\gamma}{t_i}}$. The above nonlinear equations can be solve numerically to obtain the MLEs; this can be achieved by using an iterative method for solving unconstrained nonlinear optimization problems such as the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm via package `maxLik` [43] in R software. The asymptotic distribution underlying $\hat{\Theta}$ is approximated as the multivariate normal $N_5(0, \hat{J}^{-1})$, under the usual parameters boundary conditions. The elements of the matrix J are $\partial^2 L(\Theta) / \partial \Theta \partial \Theta^T$. For $r = 1, \dots, 5$, let Θ_r be the r^{th} component of Θ . Then, a $100(1 - d)\%$ asymptotic confidence interval (ACI) for Θ_r is given by $ACI_r = \left(\Theta_r \pm w_{d/2} \sqrt{\hat{J}_{rr}} \right)$, where $\hat{J}_{rr}$ is the $(r, r)^{th}$ diagonal element of $\hat{J}^{-1}$ and $w_{d/2}$ is the quantile $1 - d/2$ of the standard normal distribution.

5. Real-Data Illustration

In this section, our objective is to demonstrate the efficiency of the proposed FR model through real-data applications, employing the proposed ABC-Gibbs methodology. All the provided datasets are characterized by bathtub-curve FR. The real data analysis helps in evaluating the performance of the proposed distribution in comparison to other well-known three-, four-, and five-parameter bathtub FR models outlined in Table 6.

Table 6. Competing models.

Competing Models	Reliability Function
Modified Weibull extension (MWE) [44]	$\exp(\alpha\lambda(1 - \exp((t/\alpha)^\beta))), t, \alpha, \beta, \lambda > 0$
Additive Chen–Weibull (ACW) [45]	$\exp(\lambda(1 - \exp(t^\gamma)) - (\alpha t)^\beta), t, \alpha, \beta, \gamma > 0, \lambda \geq 0$
Flexible exponential power Gomartz (FEPG) [24]	$\exp((1 - \exp((\alpha t)^\beta)) + \exp(-\lambda)(1 - \exp(\gamma t))), t, \beta, \lambda > 0, \alpha, \gamma \geq 0$
Exponentiated modified Weibull extension (EMWE) [46]	$1 - (1 - \exp(\alpha\lambda(1 - \exp((t/\alpha)^\beta))))^\gamma, t, \alpha, \beta, \lambda, \gamma > 0$
Additive modified Weibull (AMW) [47]	$\exp(-\alpha t^\theta \exp(\gamma t) + \exp(-\beta)(1 - \exp(\lambda t))), t, \alpha, \beta, \theta > 0, \gamma, \lambda \geq 0$
Exponentiated additive Weibull (EAddW) [48]	$1 - (1 - \exp(-\alpha t^\beta - \gamma t^\lambda))^\theta, t, \alpha, \beta, \lambda > 0, \gamma \geq 0$

To assess the fitting appropriateness of models, we employed established likelihood-based model selection metrics and goodness-of-fit metrics that do not involve likelihood: the log-likelihood ($\ell(\Theta)$), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and corrected Akaike Information Criterion (CAIC); and the Kolmogorov–Smirnov (KS), Anderson–Darling (AD), and Cramér–von Mises (CvM) metrics. A smaller value of these statistical measures indicates a better fit to the dataset. The AD and CvM metrics can be computed using the *nortest* package [49], while the KS metric uses the *GLDEX* package [50] in R. Furthermore, we performed a compatibility assessment by comparing the observed data with samples generated from a suitably specified model. A close agreement between the two indicates that the model is appropriate for the data analysis (see [51]). Accordingly, box plots are used to compare samples generated from the fitted models with the corresponding observed data. In addition, plots of the failure rate (FR), cumulative failure rate (CFR), reliability, and mean residual life (MRL) functions, together with box plots for the competing fitted models, are presented to facilitate comparative analysis. We also provide supporting results for both the ABC and MLE approaches, including the 95% HPD credible intervals and the 95% ACI for the parameters of the proposed model.

5.1. First Dataset

The dataset consists of failure times (in hours) for dielectric insulation used in generator armatures, reported by [52]. The box plot in Figure 9 (left) provides an initial summary of the data. The scaled TTT-transform shown in Figure 9 (right) exhibits a convex–concave pattern, indicating a bathtub-shaped FR. This behavior corresponds to a long flat or skewed unimodal bathtub curve, which poses challenges for many standard models. Consequently, the complexity of the observed FR motivates the use of a flexible modeling framework, for which the proposed model is particularly well-suited to capture the underlying failure dynamics.

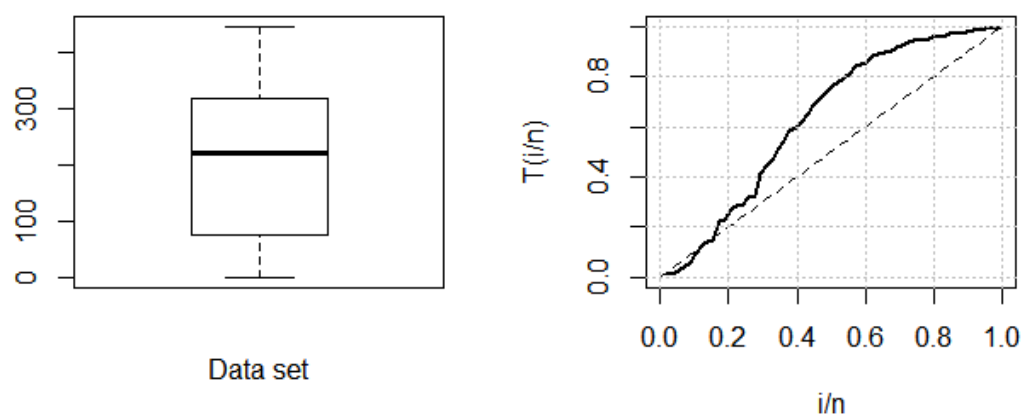


Figure 9. Empirical data box plot (left) and scaled TTT-transform plot (right) for the dielectric insulation data.

Table 7 reports the parameter estimates and goodness-of-fit measures for the proposed model and competing distributions fitted to the dielectric insulation data. Parameter estimation for the proposed model is carried out using both ABC-Gibbs and MLE, whereas the competing models are estimated via MLE. Model adequacy is evaluated using L , AIC, BIC, CAIC, KS, AD, and CVM statistics. The proposed model consistently provides the best fit, achieving the smallest values of AIC (708.71), BIC (719.02), CAIC (709.87), KS (0.038), AD (0.082), and CVM (0.010). In particular, the ABC estimates yield KS, AD, and CVM values nearly identical to those obtained under MLE, confirming the accuracy of the ABC approach. The compatibility analysis in Figure 10 further supports these findings, as samples generated from the proposed model closely match the observed data, while the competing models exhibit noticeably larger deviations.

Table 7. Parameter estimates and goodness-of-fit (p -value in parenthesis) measures of the competing models for the dielectric insulation data.

Model	Estimated Parameters	L	AIC	BIC	CAIC	KS	AD	CVM
New model _{ABC}	$\hat{\alpha} = 6.611 \times 10^{-6}$, $\hat{\beta} = 0.4254$ $\hat{\theta} = 0.6194$, $\hat{\lambda} = 0.0193$, $\hat{\gamma} = 2.4831$	-	-	-	-	0.038 (0.9999)	0.084 (0.9985)	0.010 (0.9993)
New model _{MLE}	$\hat{\alpha} = 5.422 \times 10^{-6}$, $\hat{\beta} = 0.4284$ $\hat{\theta} = 0.6195$, $\hat{\lambda} = 0.0194$, $\hat{\gamma} = 2.4931$	-349.36	708.71	719.02	709.87	0.038 (0.9999)	0.082 (0.9986)	0.010 (0.9994)
MWE	$\hat{\alpha} = 1.6329$, $\hat{\beta} = 0.3294$ $\hat{\lambda} = 0.0034$,	-357.78	721.55	727.73	721.99	0.120 (0.3422)	1.069 (0.0076)	0.179 (0.0094)
ACW	$\hat{\alpha} = 0.0032$, $\hat{\beta} = 2.7394$ $\hat{\lambda} = 0.0165$, $\hat{\gamma} = 0.2208$	-355.54	719.07	727.31	719.83	0.104 (0.5201)	0.723 (0.0561)	0.121 (0.0568)
EMWE	$\hat{\alpha} = 60$, $\hat{\beta} = 0.9281$ $\hat{\lambda} = 1.54e - 4$, $\hat{\gamma} = 0.4311$	-352.23	712.46	720.70	713.22	0.081 (0.8165)	0.295 (0.5857)	0.047 (0.5547)
AMW	$\hat{\alpha} = 0.0006$, $\hat{\beta} = 0.0191$ $\hat{\lambda} = 0.0023$, $\hat{\gamma} = 0.0194$, $\hat{\theta} = 0.02375$	-352.41	714.82	725.12	715.97	0.0960 (0.6236)	0.2805 (0.6304)	0.0467 (0.5549)
FEFG	$\hat{\alpha} = 0.00012$, $\hat{\beta} = 0.4802$ $\hat{\lambda} = 2.2407$, $\hat{\gamma} = 0.0082$	-353.47	714.94	723.18	715.69	0.089 (0.7024)	0.392 (0.3685)	0.066 (0.3111)
EAddW	$\hat{\alpha} = 0.0018$, $\hat{\beta} = 1.1905$ $\hat{\lambda} = 0.0908$, $\hat{\gamma} = 0.6835$, $\hat{\theta} = 4.3533$	-363.17	736.34	746.65	737.49	0.202 (0.0147)	1.406 (0.0011)	0.235 (0.0019)

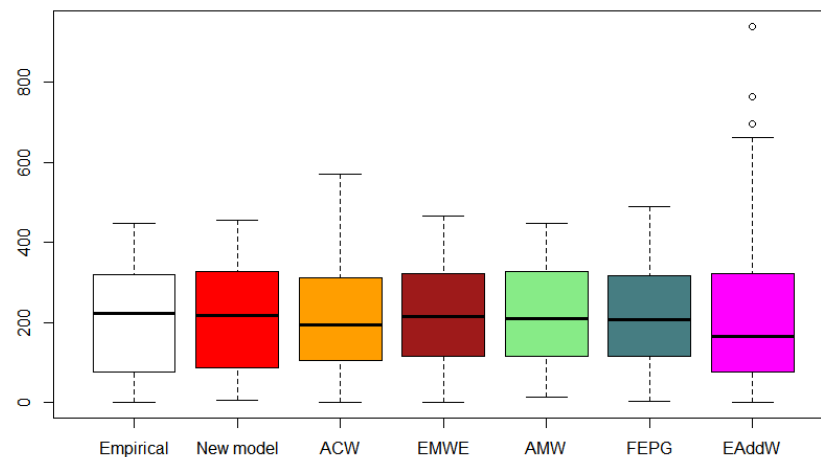


Figure 10. Empirical and fitted models' box plots for compatibility test regarding the dielectric insulation data.

Figure 11 compares the empirical and fitted FR functions, showing that the proposed model accurately captures the useful-life and wear-out phases. The Kaplan–Meier reliability curve in Figure 12 further demonstrates the close agreement between the empirical reliability function and that of the proposed model, outperforming competing alternatives. Similarly, Figure 13 shows that the proposed MRL function closely follows the empirical MRL and accurately identifies its turning point, highlighting the model's flexibility and robustness for reliability analysis and life-testing applications. ABC estima-

tion is implemented using the algorithm in Algorithm 2 with gamma priors specified by $\kappa_1 = 6$, $\eta_1 = 5$, $\kappa_2 = 9$, $\eta_2 = 5$, $\kappa_3 = 6$, $\eta_3 = 5$, $\kappa_4 = 9$, $\eta_4 = 6$, and $\kappa_5 = 6$, $\eta_5 = 9$, and a tolerance level $\epsilon = 0.05$. Figure 14 presents the scatter-plot matrix of the ABC posterior samples, indicating generally weak parameter correlations. The posterior distributions are further supported by the box plots in Figure 15. Finally, Table 8 reports the posterior summaries (mean, median, and 95% HPD credible intervals) for the ABC estimates, along with the 95% asymptotic confidence intervals for the MLEs, confirming the strong performance of both estimation approaches for the proposed model.

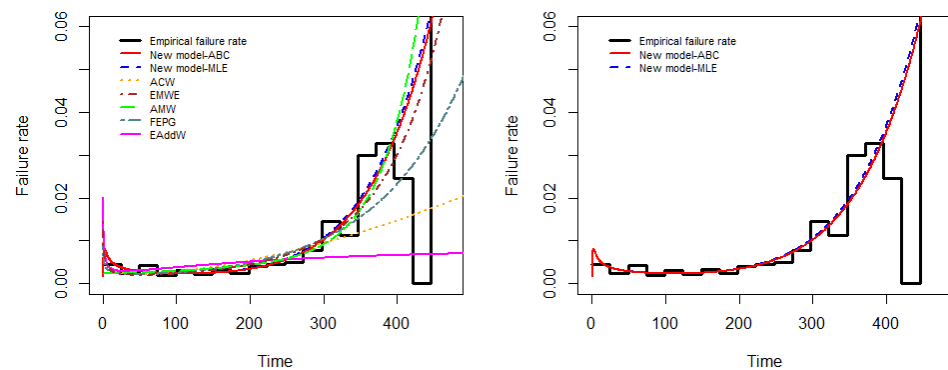


Figure 11. Plots of the empirical and fitted FR functions (**left**), and empirical and proposed model FR functions only (**right**) for the dielectric insulation data.

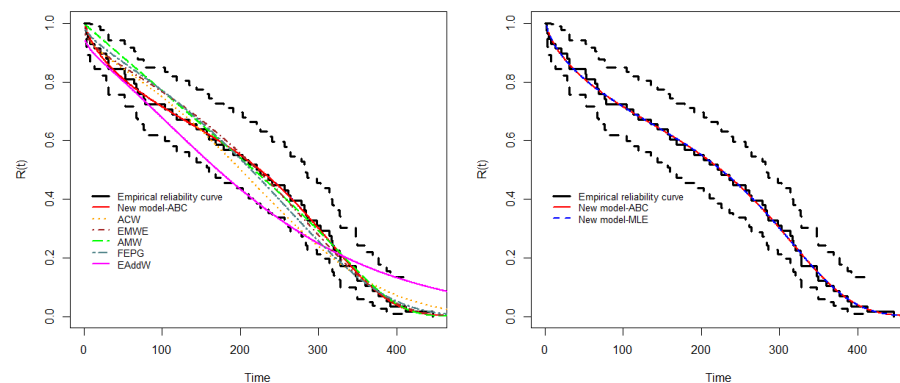


Figure 12. Plots of the empirical and fitted R functions (**left**), and empirical and proposed model R functions only (**right**) for the dielectric insulation data.

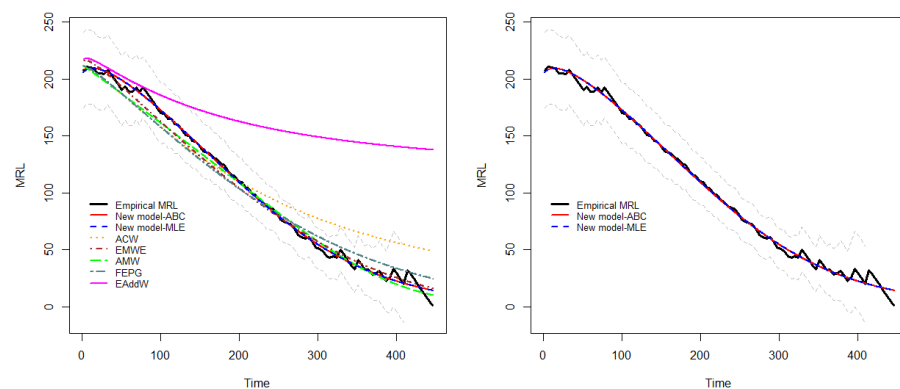
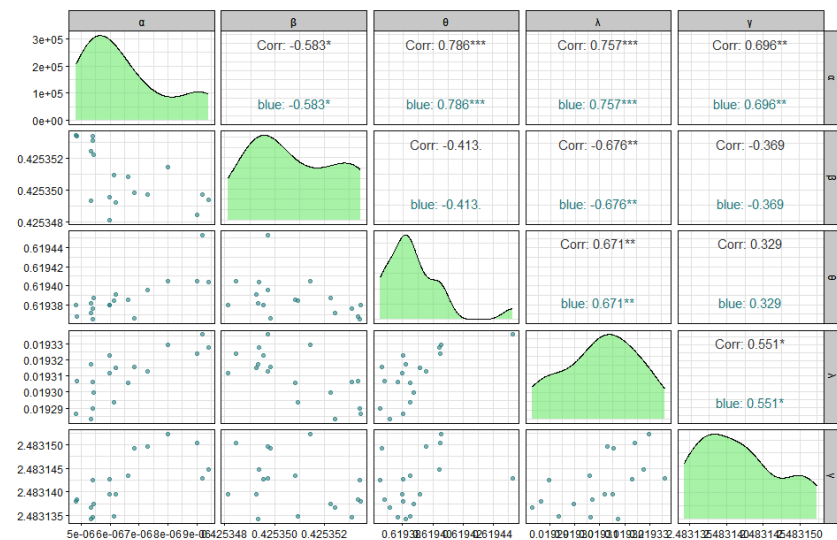
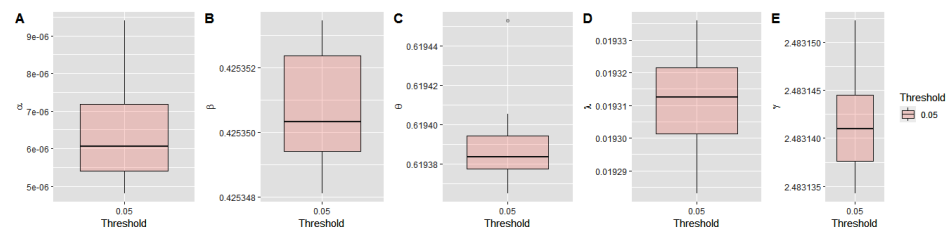


Figure 13. Plots of the empirical and fitted survival curve (**left**), and empirical and proposed model MRL functions only (**right**) for the dielectric insulation data.

Table 8. Confidence intervals of the estimates of the proposed model for the dielectric insulation data.

Parameters	Mean	Median	95% HPD	95% ACI
α	6.611×10^{-6}	6.062×10^{-6}	$[4.813 \times 10^{-6}, 9.401 \times 10^{-6}]$	$[0.0000, 0.0003]$
β	0.4254	4.258	$[0.4250, 0.4851]$	$[0.3509, 0.4361]$
θ	0.6194	0.6198	$[0.5136, 0.6445]$	$[0.2281, 1.1041]$
λ	0.0193	0.0195	$[0.0180, 0.0233]$	$[0.0000, 0.0439]$
γ	2.4831	2.4836	$[2.1310, 2.6152]$	$[0.0000, 3.4310]$
Acceptance rate (%)	0.00180			

**Figure 14.** Scatter-plot matrix for pairs of the proposed model parameters from the ABC marginal posterior samples for the dielectric insulation data.**Figure 15.** Box plot of the ABC marginal posterior samples for the dielectric insulation data.

5.2. Second Dataset

The data consist of failure times for thirty devices reported in [53]. The box plot in Figure 16 (left) provides an initial summary of the data, while the scaled TTT-transform in Figure 16 (right) exhibits a convex–concave pattern, indicating a bathtub-shaped FR. This behavior suggests that the proposed model is suitable for analyzing the dataset. Table 9 summarizes the parameter estimates obtained using both ABC and MLE methods, along with goodness-of-fit measures, including $L(\Theta)$, AIC, BIC, CAIC, KS, AD, and CVM. The proposed model consistently yields the smallest values across all criteria, demonstrating a superior fit compared with the competing models.

The effectiveness of the ABC approach is demonstrated by the close agreement between its estimates and those obtained via MLE. As reported in Table 9, the proposed model achieves the smallest goodness-of-fit measures, with KS (0.082), AD (0.190), and CVM (0.030) under ABC estimation, and KS (0.088), AD (0.169), and CVM (0.027) under MLE. In addition, the MLE-based information criteria for the proposed model are markedly lower (AIC = 146.23, BIC = 153.24, and CAIC = 148.73) than those of the competing models.

The compatibility analysis displayed in Figure 17 further supports these findings, as the proposed model closely reproduces the observed data with minimal deviation, whereas the alternative models exhibit greater dispersion. These results confirm the superior fitting and predictive capability of the proposed model.

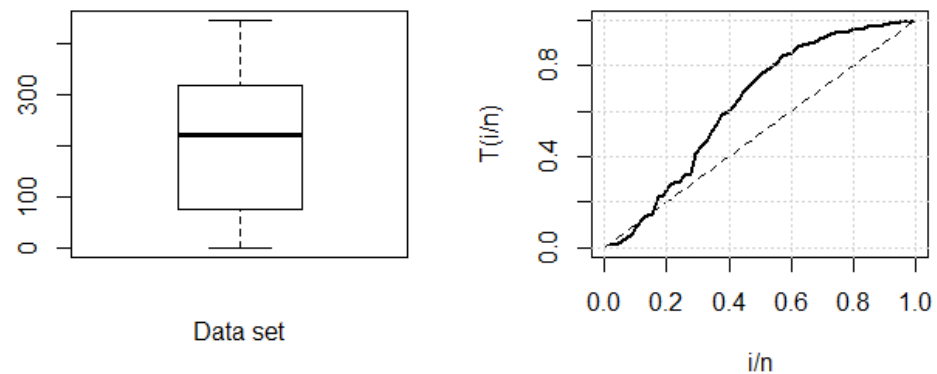


Figure 16. Empirical data box plot (left), and scaled TTT-transform plot (right) for the thirty devices data.

Table 9. Parameter estimates and goodness-of-fit (p -value in parenthesis) measures of the competing models for the thirty devices' data.

Model	Estimated Parameters	$\ell(\Theta)$	AIC	BIC	CAIC	KS	AD	CVM
New model _{ABC}	$\hat{\alpha} = 0.0014, \hat{\beta} = 0.7818$ $\hat{\theta} = 0.0759, \hat{\lambda} = 0.0833, \hat{\gamma} = 0.0217$	-	-	-	-	0.082 (0.9772)	0.190 (0.8903)	0.030 (0.839)
New model _{MLE}	$\hat{\alpha} = 0.0010, \hat{\beta} = 0.7986$ $\hat{\theta} = 0.0799, \hat{\lambda} = 0.0963, \hat{\gamma} = 0.0166$	-68.12	146.23	153.24	148.73	0.088 (0.9583)	0.169 (0.926)	0.027 (0.977)
MWE	$\hat{\alpha} = 1.5938, \hat{\beta} = 0.7914$ $\hat{\lambda} = 0.0073$	-77.092	160.18	164.39	161.11	0.141 (0.543)	1.089 (0.006)	0.164 (0.014)
ACW	$\hat{\alpha} = 0.0889, \hat{\beta} = 5.6347$ $\hat{\lambda} = 0.0486, \hat{\gamma} = 0.1965$	-70.561	149.12	154.73	150.72	0.095 (0.925)	0.228 (0.794)	0.035 (0.755)
EMWE	$\hat{\alpha} = 0.0426, \hat{\beta} = 0.5253$ $\hat{\lambda} = 4.26e-8, \hat{\gamma} = 0.2295$	-72.907	153.81	159.42	155.41	0.141 (0.542)	0.476 (0.223)	0.066 (0.3063)
AMW	$\hat{\alpha} = 0.0181, \hat{\beta} = 15.117, \hat{\theta} = 0.3316$ $\hat{\lambda} = 0.2060, \hat{\gamma} = 0.1078$	-73.168	156.34	163.34	158.84	0.128 (0.659)	0.552 (0.142)	0.082 (0.190)
EAddW	$\hat{\alpha} = 4.1e^{-5}, \hat{\beta} = 4.3560, \hat{\theta} = 19.240$ $\hat{\lambda} = 0.0489, \hat{\gamma} = 2.1330$	-70.088	150.18	157.18	152.68	0.093 (0.934)	0.219 (0.823)	0.034 (0.778)

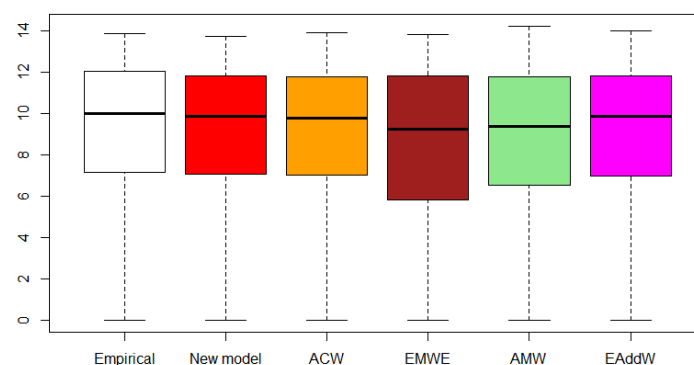


Figure 17. Empirical and fitted models' box plots for compatibility test regarding the thirty devices' data.

Figure 18 compares the empirical and fitted FR functions, showing that the proposed model accurately captures the critical region from the useful-life to the wear-out phase. The Kaplan–Meier reliability curve in Figure 19 further demonstrates the flexibility and

superior fit of the proposed model relative to competing alternatives. Likewise, Figure 20 shows that the proposed MRL function closely follows the empirical MRL and accurately identifies its turning point, supporting the suitability of the model for practical reliability analysis. ABC estimates are obtained using the algorithm in Algorithm 2 with gamma prior distributions specified by $\kappa_1 = 6$, $\eta_1 = 5$, $\kappa_2 = 9$, $\eta_2 = 5$, $\kappa_3 = 6$, $\eta_3 = 5$, $\kappa_4 = 9$, $\eta_4 = 6$, and $\kappa_5 = 6$, $\eta_5 = 9$, and a tolerance level $\epsilon = 0.1$. Figure 21 presents the scatter-plot matrix of the ABC posterior samples, indicating generally weak parameter correlations that do not affect estimation accuracy. The posterior distributions are further illustrated by the box plots in Figure 22. Finally, Table 10 reports the posterior summaries (mean, median, and 95% HPD credible intervals) for the ABC estimates, together with the 95% asymptotic confidence intervals of the MLEs, confirming the strong performance of both estimation methods for the proposed model.

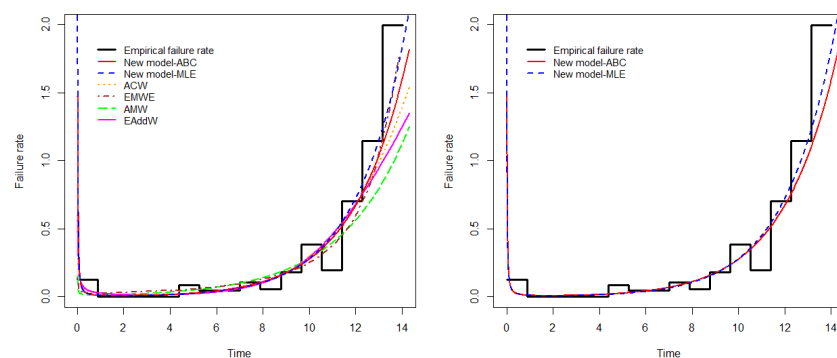


Figure 18. Plots of the empirical and fitted FR functions of the competing models (**left**), and empirical and proposed model FR functions only (**right**) for the thirty devices data.

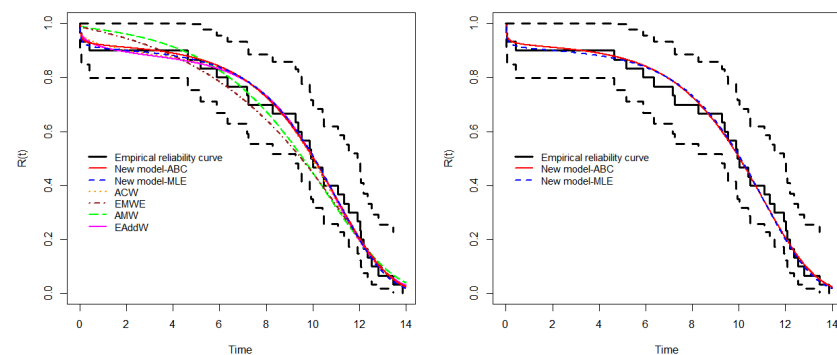


Figure 19. Plots of the empirical and fitted R functions of the competing models (**left**), and empirical and proposed model R functions only (**right**) for the thirty devices' data.

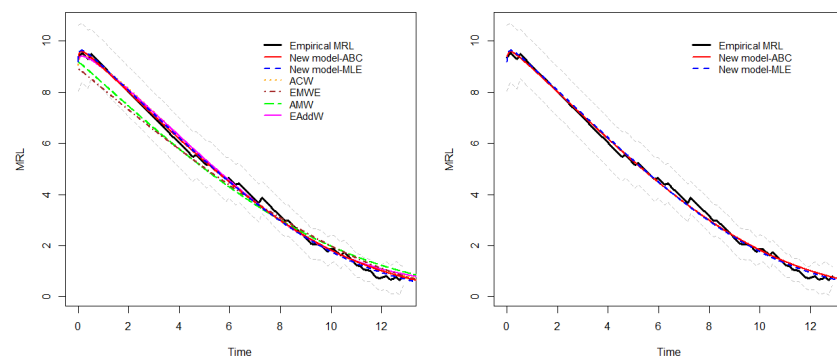
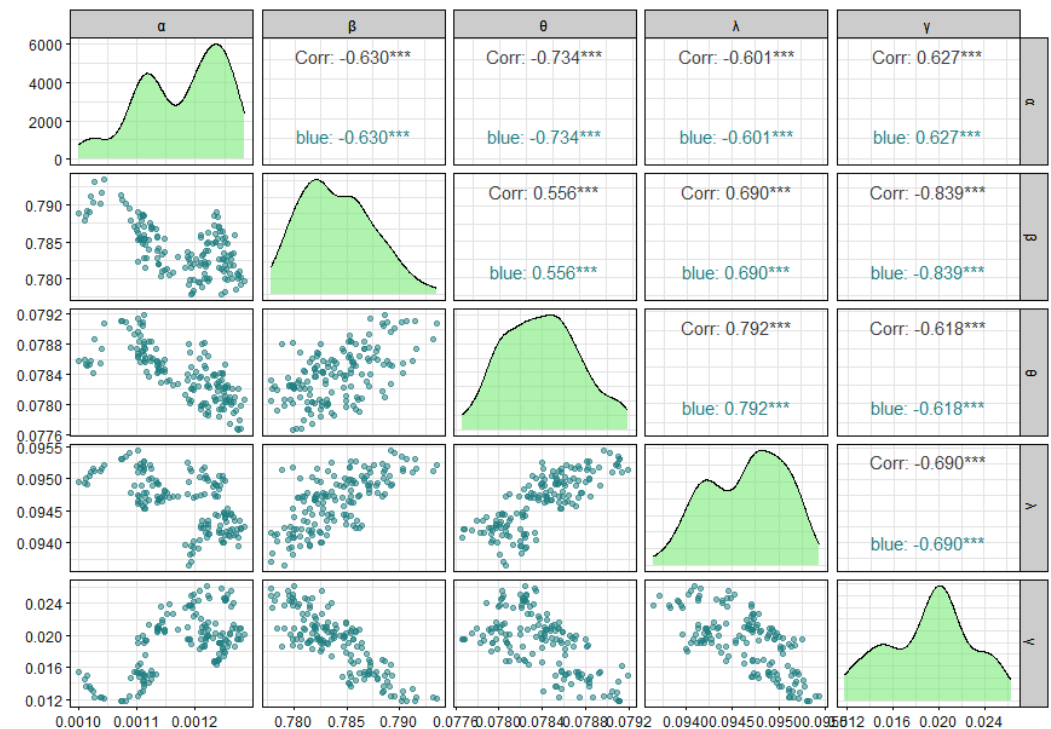
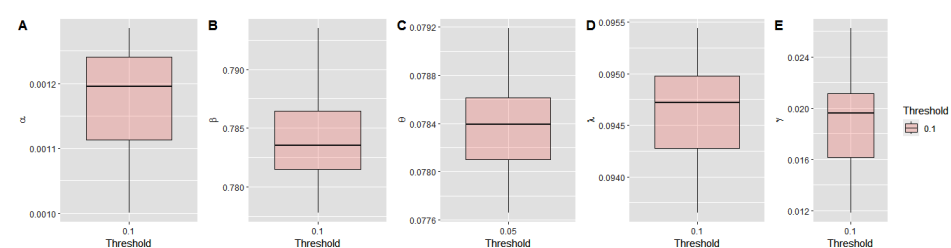


Figure 20. Plots of the empirical and fitted MRL of the competing models (**left**), and empirical and proposed model MRL functions only (**right**) for the thirty devices' data.

Table 10. Confidence intervals of the estimates of the proposed model for the thirty devices' data.

Parameters	Mean	Median	95% HPD	95% ACI
α	0.0014	0.0011	[0.0010, 0.0017]	[0.0000, 0.0029]
β	0.7818	0.7678	[0.7778, 0.7993]	[0.7049, 0.8931]
θ	0.0759	0.0793	[0.0704, 0.0791]	[0.0000, 0.3508]
λ	0.0833	0.0940	[0.0803, 0.0953]	[0.0000, 0.2093]
γ	0.0217	0.0432	[0.0121, 0.0252]	[0.0000, 0.0450]
Acceptance rate (%)	0.1531			

**Figure 21.** Scatter-plot matrix for pairs of the proposed model parameters from the ABC marginal posterior samples for the thirty devices' data.**Figure 22.** Box plot of the ABC marginal posterior samples for the thirty devices' data.

6. Conclusions

In this paper, an ABC-Gibbs estimation framework is developed using the Kolmogorov–Smirnov statistic as the distance measure. Through extensive simulation studies, the proposed ABC-Gibbs approach is shown to be robust, accurate, and computationally efficient, with posterior approximations that converge toward the true posterior as the tolerance level decreases. The use of the KS distance, a well-established and distribution-free statistic, enhances the reliability of the inference procedure and provides a transparent and interpretable mechanism for threshold selection within the unit interval. In addition, a novel two-component additive lifetime model with a flexible bathtub-shaped FR is introduced. The model, constructed by combining extended Weibull and Chen distributions, involves

five parameters and exhibits substantial adaptability in modeling skewed and complex lifetime behaviors. A detailed analytical investigation of key reliability characteristics, including the reliability function, FR function, and MRL, demonstrates the model's ability to capture a wide range of FR patterns, particularly bathtub and non-monotone shapes. Applications to two real datasets further confirm the practical effectiveness and superior fitting performance of the proposed model when compared with competing alternatives.

Although the proposed ABC-Gibbs framework demonstrates strong performance, the current analysis nevertheless focuses mainly on complete lifetime data. Future work may extend the proposed ABC-Gibbs estimation method to censored, progressively censored, and interval-censored reliability data, which are commonly encountered in engineering and biomedical applications. In addition, the present study considers a univariate competing-risk lifetime model. Future research may develop multivariate or dependent-risk extensions of the proposed model by incorporating copula structures or other dependence mechanisms. This would allow the model to describe systems with correlated components or multiple dependent failure causes, thereby broadening its applicability to complex engineering systems.

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Appendix A

The basic ABC algorithm in Algorithm A1.

Algorithm A1 Basic ABC-Algorithm

1. Choose: prior distribution $\pi(\Theta)$, $s(\cdot)$ summary statistics, distance function $\rho(\cdot, \cdot)$, tolerance threshold ϵ , model $M(\cdot)$, and iteration number T
 2. **for** $t = 1$ to T **do**
 3. **repeat**
 4. Generate Θ^* from the $\pi(\Theta)$
 5. Generate data set, say, $s(D^*)$ from the forward model $M(\Theta^*)$
 6. Accept Θ^* if $\rho(s(D^*), s(D_{obs})) \leq \epsilon$
 7. **until** Acceptance
 8. Set $\Theta^{(t)} = \Theta^*$
 9. **end for**
-

Appendix B

The ABC-Gibbs algorithm in Algorithm A2.

Algorithm A2 ABC-Gibbs Algorithm

1. Input: prior distribution $\pi(\Theta)$, distance function $\rho(\cdot, \cdot)$, tolerance threshold ϵ , model $M(\cdot)$, $s(D_{obs})$ summary statistics, and iteration number K
 2. Start: input $\Theta^{(0)}$ preferably from prior $\pi(\Theta)$
 3. **for** $k = 1$ to T **do**
 4. For each parameter in Θ_i in the Θ **do**
 5. Sample: $\Theta_i^* \sim \pi_i(\Theta_i | s(D_{obs}), \Theta_{-i}^{k-1})$, marginal posterior
 6. Generate data set, say, $s(D^*)$ from the forward model $M(\Theta_i^*)$
 7. Compute $\rho(s(D^*), s(D_{obs}))$
 8. **if** $\rho \leq \epsilon$, **then**
 9. Acceptance Θ_i^* and set $\Theta_i^{(k)} = \Theta_i^*$
 10. **else**
 11. reject Θ_i^* and set $\Theta_i^{(k)} = \Theta_i^{(k-1)}$
 12. **end if**
 13. **end for**
 14. update $\Theta^{(k)} = (\Theta_1^{(k)}, \Theta_2^{(k)}, \dots, \Theta_n^{(k)})$
 15. **end for**
-

Appendix C. The Derivation of MRL and Its Asymptote

Lemma A1 ([54]). Let $z \in \mathbb{R}$ and $\vartheta > 0$; let $\gamma(-z, \vartheta)$ be an incomplete gamma function defined by

$$\gamma(-z, \vartheta) = \int_0^\vartheta v^{-z-1} e^{-v} dv, \quad v > 0.$$

Then,

$$\gamma(-z, \vartheta) = \begin{cases} \Gamma(-z) - \Gamma(-z, \vartheta) & \text{for } z \notin \mathbb{N}, \\ \sum_{\substack{l=0 \\ l \neq z}}^{\infty} \frac{(-1)^l z^{l-z}}{l!(l-z)} + \frac{(-1)^z}{z!} \ln(\vartheta) & \text{for } z \in \mathbb{N}, \\ \sum_{l=1}^{\infty} \frac{(-\vartheta)^l}{l(l!)} + \ln(\vartheta) & \text{for } z = 0, \end{cases}$$

where $\Gamma(-z, \vartheta)$ is an incomplete gamma function.

Now,

$$\mu_T(x) = \int_0^\infty \frac{R(t+x)}{R(x)} dt,$$

and we can obtain the MRL as

$$\mu_T(x) = \frac{1}{R(x)} \sum_{i,j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{i+j} \alpha^i \lambda^j i^k e^\alpha}{i! j! k!} \int_0^\infty (x+t)^{\theta j + \beta k} e^{-\frac{j\gamma}{x+t}} dt.$$

Let $u = \frac{j\gamma}{(x+t)}$; then, $\mu_T(x)$ becomes

$$\mu_T(x) = \frac{1}{R(x)} \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j} \alpha^i \lambda^j i^k (j\gamma)^{\theta j + \beta k + 1} e^\alpha}{i! j! k!} \int_0^{\frac{j\gamma}{x}} u^{-(\theta j + \beta k + 1) - 1} e^{-u} du,$$

thus, by using the Lemma A1, we get

$$\mu_T(x) = \frac{1}{R(x)} \sum_{i,j,k=0}^{\infty} \phi_{i,j,k}(\alpha, \lambda, \theta, \gamma) \gamma(-(\theta j + \beta k + 1), \frac{j\gamma}{x}),$$

$$\text{where } \phi_{i,j,k}(\alpha, \lambda, \theta, \gamma) = \frac{(-1)^{i+j} \alpha^i \lambda^j i^k (j\gamma)^{\theta j + \beta k + 1} e^\alpha}{i! j! k!}.$$

In order to prove Theorem 1, we need the following lemma.

Lemma A2. Let $\lambda > 0, \theta > 0, \gamma > 0, \beta > 0$, and $x > 0$ be fixed constants. Define

$$R_{EW}(x; t) := \exp \left[-\lambda \left((t+x)^\theta e^{-\gamma/(t+x)} - t^\theta e^{-\gamma/t} \right) \right],$$

$$R_{CH}(x; t) := \exp \left[\alpha \left(e^{t^\beta} - e^{(t+x)^\beta} \right) \right].$$

Then, as $t \rightarrow \infty$, we have the following:

$$\begin{aligned} 1. \quad \lim_{t \rightarrow \infty} R_{EW}(x; t) &= \begin{cases} 1, & 0 < \theta < 1, \\ \exp(-\lambda x), & \theta = 1, \\ 0, & \theta > 1. \end{cases} \\ 2. \quad \lim_{t \rightarrow \infty} R_{CH}(x; t) &= \exp \left[-\beta x t^{\beta-1} \exp(t^\beta) \right]. \end{aligned}$$

Proof. We provide the proof of (1); the proof of (2) follows analogously. Although the result could be obtained using a Taylor series expansion, we instead present a rigorous proof based on the Intermediate Value Theorem (I.V.T). Define the function

$$f(y) = y^\theta e^{-\gamma/y}, \quad y > 0.$$

By the I.V.T, there exists a point $\tau_t \in (t, t+x)$ such that $f(t+x) - f(t) = f'(\tau_t) x$. A direct computation of the derivative gives $f'(y) = e^{-\gamma/y} y^{\theta-1} (\theta + \gamma/y)$, so that

$$(t+x)^\theta e^{-\gamma/(t+x)} - t^\theta e^{-\gamma/t} = f'(\tau_t) x = x \tau_t^{\theta-1} \left(\theta + \frac{\gamma}{\tau_t} \right) e^{-\gamma/\tau_t}.$$

Hence, the expression in the limit can be rewritten as

$$\exp \left[-\lambda \left((t+x)^\theta e^{-\gamma/(t+x)} - t^\theta e^{-\gamma/t} \right) \right] = \exp \left[-\lambda x \tau_t^{\theta-1} \left(\theta + \frac{\gamma}{\tau_t} \right) e^{-\gamma/\tau_t} \right].$$

Since $\tau_t \in (t, t+x)$, we have $\tau_t \sim t$ as $t \rightarrow \infty$, and therefore $e^{-\gamma/\tau_t} \rightarrow 1, \gamma/\tau_t \rightarrow 0$ and $\tau_t^{\theta-1} \sim t^{\theta-1}$. Consequently, for large t ,

$$-\lambda x \tau_t^{\theta-1} \left(\theta + \frac{\gamma}{\tau_t} \right) e^{-\gamma/\tau_t} \sim -\lambda x \theta t^{\theta-1},$$

and we now consider three cases. If $0 < \theta < 1$, then $t^{\theta-1} \rightarrow 0$ as $t \rightarrow \infty$, and consequently $\lim_{t \rightarrow \infty} \exp(-\lambda x \theta t^{\theta-1}) = 1$. If $\theta = 1$, then $t^{\theta-1} = 1$, which yields $\lim_{t \rightarrow \infty} \exp(-\lambda x \theta) = \exp(-\lambda x)$. Finally, if $\theta > 1$, then $t^{\theta-1} \rightarrow \infty$ as $t \rightarrow \infty$, and hence $\lim_{t \rightarrow \infty} \exp(-\lambda x \theta t^{\theta-1}) = 0$. Combining these three cases yields the desired result. $\square$

Proof of Theorem 1. We have that

$$\begin{aligned}\mu_T(t) &= \int_t^\infty \frac{R(x)}{R(t)} dx = \int_0^\infty \frac{R(x+t)}{R(t)} dx \\ &= \int_0^\infty \exp\left[\alpha\left(e^{t^\beta} - e^{(t+x)^\beta}\right)\right] \exp\left[-\lambda\left((t+x)^\theta e^{-\gamma/(x+t)} - t^\theta e^{-\gamma/t}\right)\right] dx \\ &= \int_0^\infty R_{CH}(x; t) R_{EW}(x; t) dx.\end{aligned}$$

For $\lambda, \alpha, \gamma, \beta > 0$ and $0 < \theta < 1$, it then follows from Lemma A2 that, for large t , we have

$$\begin{aligned}\mu_T(t) &\sim \int_0^\infty \exp\left[-\beta x t^{\beta-1} \exp(t^\beta)\right] dx \\ &= \frac{1}{\alpha\beta} t^{1-\beta} \exp\left[-t^\beta\right]\end{aligned}$$

which yields the desired result. The remaining cases can be treated in a similar manner. $\square$

Appendix D. The Derivation of CRE

The computation of this quantity is based on Lemmas A1 and A3.

Lemma A3 ([55], p. 346). Let $x, \vartheta, z > 0$. Define

$$\varphi(z, \vartheta) = \int_\vartheta^\infty x^{-z} e^{-x} dx, \quad \text{then,} \quad \varphi(z, \vartheta) = \vartheta^{-\frac{z}{2}} e^{-\frac{\vartheta}{2}} W_{-\frac{z}{2}, \frac{1-z}{2}}(\vartheta),$$

where $W_{\cdot, \cdot}(\cdot)$ is a Whittaker function [56].

The CRE is derived as

$$CRE = - \int_0^\infty e^{\alpha(1-e^{t^\beta}) - \lambda t^\theta e^{-\frac{\gamma}{t}}} \left(\alpha(1-e^{t^\beta}) - \lambda t^\theta e^{-\frac{\gamma}{t}} \right) dt.$$

By applying exponential series representation on the integrand, we get

$$\begin{aligned}& e^\alpha e^{-\alpha e^{t^\beta}} e^{-\lambda t^\theta e^{-\frac{\gamma}{t}}} \left(\alpha - \alpha e^{t^\beta} - \lambda t^\theta e^{-\frac{\gamma}{t}} \right) \\ &= e^\alpha \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{(-\alpha)^n (-\lambda)^m}{n! m!} e^{nt^\beta} t^{m\theta} e^{-m\frac{\gamma}{t}} \\ &\quad \times \left(\alpha - \alpha e^{t^\beta} - \lambda t^\theta e^{-\frac{\gamma}{t}} \right).\end{aligned}$$

Therefore,

$$\begin{aligned}CRE &= -\alpha e^\alpha \sum_{n,m,k=0}^\infty \frac{(-\alpha)^n (-\lambda)^m n^k}{n! m! k!} \int_0^\infty t^{m\theta+k\beta} e^{-m\frac{\gamma}{t}} dt \\ &\quad + \alpha e^\alpha \sum_{n,m,k=0}^\infty \frac{(-\alpha)^n (-\lambda)^m (n+1)^k}{n! m! k!} \int_0^\infty t^{m\theta+k\beta} e^{-m\frac{\gamma}{t}} dt \\ &\quad + \lambda e^\alpha \sum_{n,m,k=0}^\infty \frac{(-\alpha)^n (-\lambda)^m n^k}{n! m! k!} \int_0^\infty t^{(m+1)\theta+k\beta} e^{-(m+1)\frac{\gamma}{t}} dt.\end{aligned}$$

Applying the transformation $v = \frac{m\gamma}{t}$ in the first and second term, and $v = \frac{(m+1)\gamma}{t}$ in the third term, we have

$$\begin{aligned} CRE &= -\alpha e^{\alpha} \sum_{n,m,k=0}^{\infty} \frac{(-\alpha)^n (-\lambda)^m n^k}{n! m! k!} (m\gamma)^{m\theta+k\beta+1} \int_0^{\infty} v^{-(m\theta+k\beta)-2} e^{-v} dv \\ &\quad + \alpha e^{\alpha} \sum_{n,m,k=0}^{\infty} \frac{(-\alpha)^n (-\lambda)^m (n+1)^k}{n! m! k!} (m\gamma)^{m\theta+k\beta+1} \int_0^{\infty} v^{-(m\theta+k\beta)-2} e^{-v} dv \\ &\quad + \lambda e^{\alpha} \sum_{n,m,k=0}^{\infty} \frac{(-\alpha)^n (-\lambda)^m n^k}{n! m! k!} ((m+1)\gamma)^{m\theta+k\beta+1} \int_0^{\infty} v^{-((m+1)\theta+k\beta)-2} e^{-v} dv. \end{aligned}$$

Hence, by splitting the integral in to two parts for any $\vartheta > 0$, we can use Lemma A1 and Lemma A3; thus,

$$\begin{aligned} CRE &= -\alpha e^{\alpha} \sum_{n,m,k=0}^{\infty} \frac{(-\alpha)^n (-\lambda)^m n^k}{n! m! k!} (m\gamma)^{m\theta+k\beta+1} \\ &\quad \times \left[\gamma(-(m\theta+k\beta+1), \vartheta) + \varphi(m\theta+k\beta+2, \vartheta) \right] \\ &\quad + \alpha e^{\alpha} \sum_{n,m,k=0}^{\infty} \frac{(-\alpha)^n (-\lambda)^m (n+1)^k}{n! m! k!} (m\gamma)^{m\theta+k\beta+1} \\ &\quad \times \left[\gamma(-(m\theta+k\beta+1), \vartheta) + \varphi(m\theta+k\beta+2, \vartheta) \right] \\ &\quad + \lambda e^{\alpha} \sum_{n,m,k=0}^{\infty} \frac{(-\alpha)^n (-\lambda)^m n^k}{n! m! k!} ((m+1)\gamma)^{m\theta+k\beta+1} \\ &\quad \times \left[\gamma(-((m+1)\theta+k\beta+1), \vartheta) + \varphi((m+1)\theta+k\beta+2, \vartheta) \right]. \end{aligned}$$

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