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Development of a Heuristic Method for Solving the Problem of Optimizing a Feed Mixture Production Plan with Fuzzy Constraints

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Abstract

The compilation of optimal production plans often requires consideration of the fuzziness in certain initial information. Therefore, a pertinent issue in the theory and practice is the solution of production plan optimization problems under conditions of fuzzy information using mathematical models and tools of fuzzy sets. The research aims to develop various formulations for optimization problems of production plans using the example of a feed production plan. The problem is addressed based on a modified model considering fuzzy constraints by adapting various optimality principles. The research focuses on the processes of feed mixture production at the Makinsk poultry farm, characterized by fuzzy constraints on feed composition. New formulations of optimization problems under fuzzy constraints are obtained in the form of fuzzy mathematical programming problems, and heuristic methods for their solution are proposed. Various scenarios are considered, where the original multi-criteria optimization problem can be reduced to a single-criterion problem. For each formulation of the optimization problem, modifications of Pareto optimality methods and the main criterion are applied, ensuring the effective utilization of expert information and the decision maker's knowledge during the optimization process and the selection of the best solution.

Keywords: optimization; fuzzy constraint; fuzzy mathematical programming problem; decision maker; heuristic method

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1. Introduction

As a rule, optimization problems of production plans in practice are formalized and solved in the form of mathematical programming problems [1–3]. In the formulation and

solution of these problems, difficulties arise due to the multi-criteria nature of the original problem, fuzziness and lack of initial information, which is necessary to build a model of the problem [2,4,5]. The problem of uncertainty in this case may be associated with the randomness or fuzziness of the initial information. In the case of the randomness of the initial information, the uncertainty is reduced based on the methods of probability theory [6]. It should be noted that the use of probabilistic methods is justified when the axioms of the theory of probability are fulfilled—for example, in the case of the statistical stability of an object, the possibility of measuring and collecting statistical data, the possibility of repeated experiments, etc. In practice, some important parameters and indicators affecting the quality of the object's performance are poorly measured or are not measured, and there is no possibility of measuring statistical data, i.e., a problem of uncertainty can be caused due to the fuzziness of the original information. Under these conditions, the use of probabilistic methods for solving uncertainty problems in optimization problems is not permissible. In the presence of experts in the subject area, in these cases it is necessary to take into account and use the initial fuzzy information, i.e., subjective information, or the preferences of an expert or decision maker.

In practice, the complexity and economically inexpediency of collecting and processing statistical data can lead to the ineffective use of statistical methods, even with the possibility of using probabilistic methods. Thus, in the conditions for the inapplicability of probabilistic methods, a large calculation error or the economically inexpedient use of statistical methods require other, for example, non-statistical approaches to the description and solution of real problems. As one of the effective approaches to solving real problems in conditions of uncertainty, one can highlight fuzzy schemes for solving the optimization problem based on the methods of theories of fuzzy sets [7–13].

Thus, in practice, optimization problems are characterized, as a rule, by multi-criteria and fuzziness. Such problems can be formulated in the form of multi-criteria problems of fuzzy mathematical programming (FMP) [10,14–18].

Let us formalize the problem of the multi-criteria optimization of production plans under conditions of the fuzziness of the initial information. Let us introduce the following notations: $f_1(\mathbf{x}), \dots, f_m(\mathbf{x})$ —objective functions that determine the effectiveness of the plan, and constitute a vector $F(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_m(\mathbf{x}))$; $\mathbf{x} = (x_1, \dots, x_n)$ —the vector of the input independent variables affecting the objective functions; and $\mathbf{y} = (y_1, \dots, y_m)$ —the vector of coefficients of relative importance (weight coefficients). Each solution is characterized by the value of the objective functions at the points $\mathbf{x}$: $f_1(\mathbf{x}), \dots, f_m(\mathbf{x})$. Usually, the first step in solving the problem of multi-criteria optimization, and sometimes its final solution, is to determine such values of the vector $\mathbf{x} = (x_1, \dots, x_n)$ identifying a set of effective solutions (Pareto set). In this set, the values of one particular criterion $f_i(\mathbf{x}) \in F(\mathbf{x})$, $i \in I$ cannot be improved without deteriorating the values of the other criteria $f_j(\mathbf{x})$, $j \in I$, $i \neq j$ where $I = \{1, \dots, m\}$ is the set of indices.

By considering the FMP problems, one can focus on the problem of optimizing the objective function in an admissible set, i.e., ensuring that the requirements of all the constraints are met, and such a problem contains elements of fuzziness. The fuzziness in FMP problems can be in the objective function, in the constraints, or in the weight coefficients [10]. In the problems of FMP, the theory of fuzzy sets are used when setting and solving problems in one way or another [15].

Depending on the source of fuzziness, FMP problems can be divided into the following main classes:

1. FMP problems with fuzzy objective functions $\tilde{f}_i(\mathbf{x})$, $i = \overline{1, m}$;
2. FMP problems with fuzzy maximization operators $\widetilde{\max} f_i(\mathbf{x})$;
3. FMP problems with instructions of the following form: "It is desirable that $f_i(\mathbf{x})$, $i = \overline{1, m}$ has higher possible value";

4. FMP problems with fuzzy constraints. Such fuzzy constraints can be formalized as follows: “It is desirable that the values of the constraint function are not less $\varphi_q(x) \lesssim b_q, q = \overline{1, L}$ (not more $\gtrsim$, are approximately equal $\approx$);
5. FMP problems with a fuzzy weight vector, a number of priority criteria and/or constraints;
6. FMP problems with fuzzy constraints on the independent variables $x = (x_1, \dots, x_n)$ of the following type: $x \in \tilde{C}$ where $\tilde{C}$ —fuzzy set;

2. Analysis of Research on Solving Problems of Multi-Criteria Optimization in Conditions of Uncertainty and Problem Statement

Let us consider the results of the analysis of various approaches to solving the problems of multi-criteria optimization in conditions of uncertainty [10,11,19,20]. In Ref. [21], the authors investigated the sensitivity of solving optimal linear programming problems of complex objects and processes under uncertainty. However, this work did not investigate the issues of solving the problems under study in conditions of the fuzziness of the initial information. In works on stochastic programming [22,23], the researchers proposed an approach to solving the problems of uncertainty arising from the presence of stochastic variables. In these works, the problems of analyzing the accuracy of the solution in comparison with the known probabilistic characteristics of the used information are discussed.

In these analyzed works, the optimization process uses the usual, i.e., crisp information, and fuzzy information is not explicitly taken into account in the models. However, in practice, many problems are described by fuzzy information, which is experience, knowledge, and the judgments of subject area experts and decision makers in relation to the problem. This, in turn, reflects their views in the process of making an optimal decision. Moreover, it is often very difficult or impossible to transform such fuzzy information into crisp information. Even if possible, in the process of such a transformation, a significant portion of important information can be lost. Therefore, the need to take into account such fuzzy information stimulates the development of new fuzzy approaches to solving multi-criteria optimization problems in fuzzy environments [7,8,10,13,15,24].

In works [10,15,16,22], the problems of the generalization of questions of mathematical programming to the class of FMP problems are discussed. For example, in the Bellman–Zadeh approach to the solution of the problem, the desired solution as a membership function $\mu_D(x)$ of a fuzzily defined goal is known to be the minimum $\mu_G(x)$ and $\mu_R(x)$ of membership functions for fuzzy goals and fuzzy constraints: $\mu_D(x) = \min\{\mu_G(x), \mu_R(x)\}$.

The main problem in applying this approach is related to the choice of an alternative. Based on the theory of fuzzy sets, the following way of choosing an alternative can be proposed, according to which it should be chosen so that it satisfies the condition of reaching a maximum: $\max_{x \in X} \mu_D(x) = \max_{x \in X} \min\{\mu_G(x), \mu_R(x)\}$.

There is also an approach to solving FMP problems in which, using a set of levels α , the original fuzzy problem at the stage of formulation is replaced by a system of equivalent crisp problems with different α levels [12,25]. This approach allows for the solving a fuzzy optimization problem using well-known crisp methods, but it is characterized by a large dimension, which makes it difficult to solve problems related to optimizing production facilities, requiring large computational costs, and, in this approach, part of the original fuzzy information is lost.

A relatively well-researched and practically important class of FMP problems is fuzzy linear programming (FLP) problems, which are considered in [14,17–19].

In work [16], the authors discussed an example of several FMP problems, the solution for which is the application of the minimax method and the generalization principle.

Variants, approaches and methods are considered that allow for the reduction of the problems of fuzzy mathematics to the known methods for solving problems of mathematical programming. Proofs for a number of theorems on this problem are provided.

In methods for solving FMP problems, as in methods of stochastic programming, there are two fundamentally different approaches to solving the problem. In the first approach, called indirect methods, the membership functions of the fuzzy parameters of the optimization model are considered known. In this approach, it is possible in some cases to replace the original fuzzy problem with an equivalent crisp problem. In direct methods, i.e., in the second approach, it is assumed that an expert or a decision maker can give an estimate of the membership function of a fuzzy variable at any point.

Thus, in all the above methods for solving fuzzy problems, the researchers use the idea of converting a fuzzy problem to several crisp problems solved by the methods of classical mathematics, probability theory and statistics. But, in practice, one often has to face situations where a fuzzy optimization problem cannot be solved, since there is no independent variable x or an alternative that satisfies all of the constraints. In this case, in order to find such a solution to a fuzzy problem, one has to pose a problem, preserving and taking into account the fuzziness of the initial information. Moreover, since there may be no alternative, i.e., the independent variable x , which simultaneously satisfies all the constraints, it is necessary to use compromise schemes to account for various fuzzy constraints [9,10].

Let us analyze the works on the research topic by authors from the CIS and Russia. Methods for the development of mathematical models of production problems are investigated in [9,26–28]. In these works, the researchers propose methods for developing models for solving production problems, which can be used to solve production optimization problems. Optimization problems, including multi-criteria optimization and decision-making, are investigated in works [9,20,27,29,30]. In these works, the authors investigate and solve the multi-criteria optimization problems of the raw materials composition for the manufacture of a workpiece; the problems of multi-criteria selection on a fuzzy set of solutions; and the methods of convolution of criteria in multi-criteria problems and the solution of multi-criteria problems using human-machine procedures and based on expert evaluation methods. Optimization problems under conditions of uncertainty and fuzziness are investigated in [9,10,15,25,31], which propose approaches for solving multi-criteria problems related to the optimization of technological processes, problems of fuzzy optimization, and multi-criteria optimization of the recipe composition of a product using the idea of the Pareto optimality principle.

In the analyzed papers, intended for solving optimization problems under uncertainty conditions, authors consider an approach to solving fuzzy problems based on the use of a set of level α , i.e., the original fuzzy problem is replaced by a set of equivalent original problems. At the next stage, one or another deterministic method is applied to solve the obtained crisp problems. Unfortunately, this approach leads to a decrease in the adequacy of the problem model, and the efficiency of the solution can be quite low. In addition, the analyzed works do not provide examples of solving specific real problems in a fuzzy environment, analysis and comparison of the results of the solutions obtained, or any evaluation of their effectiveness. Thus, there is a problem of ensuring adequacy and effectiveness when solving multi-criteria problems in a fuzzy environment.

The proposed work develops a model of the optimization problem for the production plan of feed mixture for poultry at the Makinsk poultry farm LLP (Akmola region), with the aim of minimizing its cost and taking into account the requirements of various nutrients and other restrictions, which are characterized by fuzziness.

3. The Purpose and Objectives of the Study

The purposes of the work are the production optimization of feed mixture, taking into account the fuzziness of the constraints, the formulation of a mathematical model of the problem being solved, the development of a heuristic method for solving it and the solution of the formulated problem using different approaches.

Let us give a qualitative formulation of the problem in the Makinsk poultry farm, where there are 10,850 birds. Each bird requires at least 1.0 kg of additional feed mixture during the week. In the production of the feed mixture, ingredients are used that contain the following nutrients: calcium, protein, and fiber. The feed ingredients are limestone, soybeans and wheat. Each of these ingredients contains specific levels of the listed nutrients. Table 1 shows data that characterizes the nutrient content of wheat, soybeans and limestone, as well as the unit cost of each ingredient.

The following requirements are imposed on the composition of the feed mixture, which are expressed by fuzzy constraints:

- (1) The calcium content must be not less than 0.8%;
- (2) The protein content must be not less than 22%;
- (3) The fiber content should be not less than 5%.

Table 1. The content of calcium, protein and fiber in the ingredients of the feed mixture and the cost of 1 kg of each ingredient.

Feed Mix Ingredients	Nutrient Content (1 kg)			Cost, 1 kg, in \$
	Calcium	Protein	Fiber	
Limestone	0.38	-	-	0.27
Soybean	0.001	0.09	0.02	0.56
Wheat	0.002	0.50	0.08	0.44

It is necessary to draw up an optimal plan for the production of feed mixture with a minimum cost and fulfillment of all requirements for the content of nutrients. Thus, it is required to determine the amount of each of the ingredients in the weekly ration, provided that the cost of the feed with the necessary nutrient contents is minimized.

To achieve the research purpose, the following tasks are set and solved:

- Build a mathematical model of the optimization problem for the production of feed mixture;
- Develop heuristic methods for solving the optimization problem with fuzzy constraints;
- Solve the optimization problem of the feed production plan in a fuzzy environment based on different (deterministic and fuzzy) approaches.

The methodological framework of the study is based on fuzzy set theory, fuzzy logic methods, the principles of multi-criteria optimization, and heuristic approaches to decision-making under uncertainty. The study relies on the following methodological principles:

1. The principle of accounting for the fuzziness of initial information.
Real-world production problems are often characterized by incompleteness, uncertainty, and the linguistic nature of the initial data. Therefore, the parameters of the problem and its constraints are considered not as exact values, but as fuzzy sets described by membership functions.
2. The principle of representing constraints in fuzzy form.
The constraints of the problem are modeled by means of fuzzy relations and linguistic variables, which make it possible to account for an acceptable degree of deviation

of parameters from standard values. Constraint satisfaction is interpreted through a degree of fulfillment rather than through the binary criterion of “satisfied / not satisfied.”

3. The principle of heuristic search for compromise solutions.
Unlike classical mathematical programming methods aimed at finding a strictly optimal solution, the heuristic approach involves generating rational solutions based on expert knowledge, preference rules, and iterative search within the space of feasible alternatives.
4. The principle of adaptability of heuristic procedures.
Heuristic methods allow for the modification of search algorithms depending on the structure of the problem, the degree of uncertainty, and changes in external conditions, which ensures their applicability to weakly structured systems.
5. The principle of interpretability of results.
The obtained solutions should be practically explainable and make it possible to interpret the influence of fuzzy constraints and preferences on the final choice of an alternative.

The study proceeds from the assumption that decision-making problems under fuzzy constraints can be solved more effectively through heuristic methods that combine the formal apparatus of fuzzy logic, expert knowledge, and procedures for searching for compromise solutions, thereby ensuring the greater adaptability, robustness, and practical applicability of models under conditions of uncertainty.

4. Research Results Related to the Creation of a Mathematical Model of the Optimization Problem, the Development of Heuristic Methods for Solving and Solutions Using Various Approaches

4.1. Construction of a Mathematical Model of the Problem of Optimizing Feed Mixture

Let us construct a mathematical model of the problem of optimizing the plan for the production of feed mixture in the Makinsk poultry farm. Let us consider a situation where several optimization criteria (profit, costs, etc.) can be integrated into one criterion, for example, the cost of the produced feed mixture. At this stage, the problem of optimizing the production plan is reduced to the problem of minimizing the cost of feed while meeting all the requirements for the composition of the feed in terms of the nutrient content.

For this, we introduce the following notation:

x_1 —The amount of limestone in the feed mixture, in kg;

x_2 —The amount of soybeans in the feed, in kg;

x_3 —The amount of wheat in the feed mixture, in kg.

1. The minimum total weight of the feed mixture, which is consumed per week when feeding the poultry, based on the given data, is determined as follows:

$$10,850 \times 1.0 \text{ kg} = 10,850 \text{ kg.}$$

Based on the result obtained, one can formulate the requirements for the total weight of the feed mixture, i.e., the entire feed mixture must be at least 10,850 kg:

$$x_1 + x_2 + x_3 \geq 10,850$$

2. The calcium content must be at least 0.8%. In accordance with the initial data (Table 1), the calcium content is $0.38x_1 + 0.001x_2 + 0.002x_3$. Then the restrictions associated with the calcium content in the feed mixture “The calcium content must be at least 0.8%” can be written in the following form:

$$0.38x_1 + 0.001x_2 + 0.002x_3 \geq 0.008(x_1 + x_2 + x_3);$$

Rewrite the obtained fuzzy constraints in a simpler form:

$$0.372x_1 - 0.007x_2 - 0.006x_3 \gtrless 0$$

Similarly, the mathematical formulation of a fuzzy limitation on the content of protein and fiber in the feed mixture is determined.

The protein content in the feed must be at least $0.22(x_1 + x_2 + x_3)$. Taking into account the initial data, the protein content is $0.00x_1 + 0.09x_2 + 0.50x_3$, and the mathematical formulation of the fuzzy constraint “The protein content must be not less than 22%” can be written as follows:

$$0.00x_1 + 0.09x_2 + 0.50x_3 \gtrless 0.22(x_1 + x_2 + x_3)$$

Write the obtained fuzzy constraint in a simplified form:

$$-0.22x_1 - 0.130x_2 + 0.280x_3 \gtrless 0$$

The mathematical formulation of the fuzzy constraint “Fiber content should be not less than 5%” can be defined as follows:

$$0.00x_1 + 0.02x_2 + 0.08x_3 \gtrless 0.05(x_1 + x_2 + x_3) = 0.05x_1 - 0.03x_2 + 0.03x_3 \gtrless 0$$

Based on the initial data on the costs of the ingredients shown in Table 1, the objective function, which expresses the cost of the feed mixture that must be minimized, can be written as follows:

$$f(\mathbf{x}) = 0.27x_1 + 0.56x_2 + 0.44x_3 \rightarrow \min$$

In addition to the cost of ingredients, it is necessary to add the costs associated with the production of feed itself to the objective function. Based on normative and average statistical data, this part of the consumption is approximately 0.5% of the total amount of nutrients, i.e., the objective function, taking into account production costs, can be written as $f(\mathbf{x}) = 0.27x_1 + 0.56x_2 + 0.44x_3 + 0.005(x_1 + x_2 + x_3)$. Then, in a simplified form, the objective function can be written as follows:

$$f(\mathbf{x}) = 0.275x_1 + 0.565x_2 + 0.445x_3 \rightarrow \min$$

Based on the results obtained, one can obtain a mathematical model of the problem of optimizing the plan for the production of feed mixture (1)–(6):

$$f(\mathbf{x}) = 0.275x_1 + 0.565x_2 + 0.445x_3 \rightarrow \min, \quad (1)$$

$$x_1 + x_2 + x_3 \geq 10,850 - \text{minimum weekly ration}, \quad (2)$$

$$0.372x_1 - 0.007x_2 - 0.006x_3 \gtrless 0 - \text{calcium content}, \quad (3)$$

$$0.22x_1 - 0.130x_2 + 0.280x_3 \gtrless 0 - \text{protein content}, \quad (4)$$

$$0.050x_1 - 0.030x_2 + 0.030x_3 \gtrless 0 - \text{fiber content}, \quad (5)$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \quad (6)$$

The problem under study, i.e., the problem of optimizing the production plan in a fuzzy information environment and in conditions of multi-criteria, can be formulated as an FMP task in the following manner:

Find a vector of independent variables that satisfy the decision maker (DM): $\mathbf{x}^* = (x_1^*, \dots, x_m^*)$, which provides optimal, for example, maximum, values of the objective functions under the conditions of fulfilling the requirements of the imposed constraints:

$$\max_{\mathbf{x} \in X} \tilde{f}_i(\mathbf{x}), \quad i = \overline{1, m} \quad (7)$$

$$X = \{\mathbf{x} \in \Omega, \varphi_q(\mathbf{x}) \lesssim b_q, q = \overline{1, L}\} \quad (8)$$

In the given formulation $\tilde{f}_i(\mathbf{x})$ are fuzzy objective functions that assess qualities, i.e., the optimal solution; $\varphi_q(\mathbf{x}) \lesssim b_q, q = \overline{1, L}$, are fuzzy constraints defining the admissible domain Ω of the solution to the FMP problem (1) and (2); and $b_q, q = \overline{1, L}$, are numbers that define the boundary of the limitation; they may be fuzzy.

In the above formulation of the optimization problem of the production plan (1)–(6), the situation is considered when there is one integrated crisp criterion and several fuzzy constraints. Accordingly, the general formulation of the multi-objective problem of fuzzy optimization (7) and (8) can be written in the following manner, in a more simplified form:

$$\min_{\mathbf{x} \in X} f(\mathbf{x}), \quad (9)$$

$$X = \{\mathbf{x} \in \Omega, \varphi_q(\mathbf{x}) \lesssim (\lesseqgtr) 0, q = \overline{1, 3}\} \quad (10)$$

In the obtained optimization problem with fuzzy constraints (9) and (10) $\mathbf{x} = (x_1, x_2, x_3)$, it also must be considered that $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ and $x_1 + x_2 + x_3 \geq 10,850$.

4.2. Development of Heuristic Methods for Solving Optimization Problems in Fuzzy Constraints

A heuristic method is proposed for solving the initial fuzzy optimization problem in a fuzzy environment with the maximum use of the initial fuzzy information. These methods are based on the use of the experience, knowledge and intuition of decision makers, and expert specialists. Moreover, this approach requires some effort from them and, if necessary, their preliminary preparation for solving the problem.

For a unified description of particular criteria, the range of changes in their values will be reduced to the interval $[0, 1]$, i.e., objective functions are presented in a normalized form. This normalized form of objective functions allows one to compare their values with the values of the membership functions.

- It is necessary to apply mathematical programming methods adapted to work in a fuzzy environment based on the methods of fuzzy set theories to obtain an adequate formulation and effective solution of multi-criteria optimization problems under conditions of fuzzy initial information. At the same time, in order to select a compromise solution, the convolution of local criteria, the narrowing of the solution area, and various compromise schemes are used [10,24,26], as well as methods of multi-criteria assessment of alternatives, methods of fuzzy logic, etc. [12,13,31].
- Depending on the type of the formulated mathematical programming problem, the appropriate methods can be used: methods of linear, discrete, and fuzzy programming [3,14–16]. In the conditions of a shortage of initial information for the development of a model and solutions to the problem of multi-criteria optimization, it is necessary to apply hybrid methods that allow solutions to be obtained for the problem based on information of a different nature [9,27,32,33].
- Since the problem of optimizing the plan for the production of feed mixture, taking into account the fuzzy constraints, solved in this work, is formulated in the form of a linear fuzzy programming problem, the fuzzy linear programming method is used to solve it [14,17–19,22].

Let us present a description of the main stages of the heuristic methods proposed in this work for solving the posed optimization problem with fuzzy constraints. The developed heuristic method, based on the idea of the Pareto optimality principle, for solving the optimization problem (9)–(10) consists of the following main points:

4.2.1. FPS Method (Fuzzy Pareto Set)

1. The number of optimization steps is set p_q , $q = \overline{1, L}$, for the q -th coordinate of the fuzzy constraints of the optimization problem;
2. The value of optimization steps is calculated $h_q = \frac{1}{p_q}$, $q = \overline{1, L}$ —for each coordinate of constraints;
3. For fuzzy constraints, a set of weighting factors is determined, $\beta^1, \beta^2, \dots, \beta^N$, $N = (p_1 + 1) \times (p_2 + 1) \times \dots \times (p_L + 1)$, by changing the coordinate in the range $[0, 1]$ with a step h_q ;
4. Based on the methods of expert assessments, the term set $T(X, Y)$ is determined and the membership functions are constructed to describe fuzzy constraints $\mu_q(x)$, $q = \overline{1, L}$;
5. The formulated optimization problem with weight coefficients is solved, $\beta^1, \beta^2, \dots, \beta^N$, and solutions are determined: $x(\beta^t)$, $t = \overline{1, N}$, are the values of the input, the operating parameters at which the optimal value of the objective function is achieved $\mu_0(x(\beta^t))$, $t = \overline{1, N}$, and the maximum values of MF of fuzzy constraints are $\mu_1(x(\beta^t)), \dots, \mu_L(x(\beta^t))$, $q = \overline{1, N}$.
6. The decision maker analyzes the results obtained in the previous paragraph and, based on their preference, chooses the best solution. If the current solutions satisfy the decision maker, then they choose the final, best solution and proceeds to the next item. If the results obtained at the previous point do not satisfy the decision maker, then, in order to improve the solution, they change the values of β and the search for the best solution is repeated starting from step 4.
7. The final results, selected by the decision maker as the best solutions, are displayed: $x^*(\beta^t)$ are the values of the independent variables (in this case, the optimal amounts of nutrients); the optimal (minimum) value of the objective function, i.e., the feed cost $\mu_0(x^*(\beta^t))$ and the maximum membership function values that estimate the degree of fulfillment of fuzzy constraints $\mu_1(x^*(\beta^t)), \dots, \mu_L(x^*(\beta^t))$, are achieved at $x^*(\beta^t)$.

In the developed heuristic FPS Pareto method, the optimal solution is determined by approximating N points to which the solution is sought. In this case, one can apply special interactive methods for finding an effective solution, i.e., the Pareto solution in the dialog mode between the decision maker and computer [24,27,31].

Let us consider an approach to the formulation and solution of the optimization problem in the presence of fuzzy constraints using the modified method of the main criterion. For this, introduce the following notation: $\mu_0(x)$ is the optimization criterion after normalization; $I_R = \{1, \dots, L\}$ is a priority series for constraints that are known or specified by experts; and μ_q^R , $q = \overline{1, L}$, are boundary values of the limitation assigned by the decision maker. Then the statement of the optimization problem can be rewritten as the following FMP problem:

$$\max_{x \in X} \mu_0(x), \quad (11)$$

$$X = \{x: x \in \Omega \wedge \arg(\mu_q(x) \gtrsim \mu_q^R), q = \overline{1, L}\}. \quad (12)$$

Independent variables, i.e., the required optimization parameters $x(\mu_1^R, \dots, \mu_L^R)$ of this problem, depend on $\mu_1^R, \dots, \mu_L^R$, which are appointed by the decision maker. For the purpose of a more reasonable appointment of a decision maker μ_q^R , $q = \overline{1, L}$, it is necessary to organize and apply a special dialog procedure that allows decision makers to quickly analyze and correct the boundary values. At the same time, due to the increase in the amount of work of the decision maker, it is possible to study the sensitivity to μ_q^R , $q = \overline{1, L}$ different solutions, to influence the quality of solutions and to improve them [28].

The proposed heuristic method for solving the FMP problem when setting (11) and (12) has the following structure:

4.2.2. FPMMC Method (Fuzzy Principle Modified Main Criterion Method)

1. With the involvement of decision makers and experts, a number of priority of fuzzy restrictions is determined: $I_R = \{1, \dots, L\}$.
2. On the basis of expert assessment, a term set $T(X, Y)$ is specified, which describes fuzzy constraints, and membership functions are constructed: $\mu_q(\mathbf{x})$, $q = \overline{1, L}$, evaluating the degree of fulfillment of fuzzy constraints.
3. The initial values of the boundary values of the limitation are assigned: $\mu_q^{R(l)}$, $q = \overline{1, L}$, $l = 1$.
4. On the set of feasible solutions X (expression (12)), the optimum of the objective function is determined: $\mu_0(x)$ (11); i.e., the current values of the solutions are determined: $\mathbf{x}(\mu_q^{R(l)}), \mu_0(\mathbf{x}(\mu_q^{R(l)})), \mu_1(\mathbf{x}(\mu_1^{R(l)})), \dots, \mu_L(\mathbf{x}(\mu_L^{R(l)}))$.
5. The obtained values of the current solutions are presented to the decision maker for the analysis and selection of the final, best solution. If the decision maker is satisfied with the solution, then they choose the final solution. To deduce the best solution go to step 7; otherwise, in order to continue with the best solution, go to the next step.
6. The decision maker assigns new values of the boundary values for all constraints: $\mu_q^{R(l)}$, $l = l + 1$, and the search for the best solution is repeated starting from point 4.
7. The search for a solution stops and the best solutions selected by the decision maker are displayed, as follows: $\mathbf{x}^*(\mu_q^{R(l)}), q = \overline{1, L}$, are the best values for the independent variables, i.e., the required optimized parameters; $\mu_0(\mathbf{x}^*(\mu_q^{R(l)}))$ is the optimal value of the objective function obtained with these arguments; and $\mu_1(\mathbf{x}^*(\mu_q^{R(l)})), \dots, \mu_L(\mathbf{x}^*(\mu_q^{R(l)}))$ are the maximum values of membership functions, assessing the degree of fulfillment of fuzzy constraints.

Authors have considered the FMP problem for the case when the optimization problem can be reduced to a problem with one objective function and several fuzzy constraints. But, in practice, it is not always possible to represent the optimization problem in this form. For example, objective functions can have different units of measurement and a different physical nature; this complicates or makes it impossible to convolve them to one integrated objective function. It is also sometimes impossible to identify the main criterion from the set of criteria. In such situations, it is necessary to formulate and solve the problem of multi-criteria optimization with fuzzy elements. Such tasks can be formulated in the form of multi-criteria FMP tasks.

Let us formulate the FMP problem with several objective functions and several fuzzy constraints and consider an approach to its solution.

For this, let us introduce the following notation: $\mu_0(\mathbf{x}) = (\mu_0^1(\mathbf{x}), \dots, \mu_0^m(\mathbf{x}))$ is the vector of normalized objective functions, which evaluates the effectiveness of the production plan; $\varphi_q(\mathbf{x}) \lesseqgtr b_q$, $q = \overline{1, L}$ are fuzzy constraints; $\mu_q(\mathbf{x}), q = \overline{1, L}$, are the membership functions of fuzzy constraints; $\gamma = (\gamma_1, \dots, \gamma_m)$ are the weight vector and coefficients of importance of local criteria; and $\beta = (\beta_1, \dots, \beta_L)$ are the weight vector and coefficients of importance of constraints.

Taking into account the above formalization, the FMP problem with several objective functions and fuzzy constraints can be written in the following manner:

$$\max_{\mathbf{x} \in X} \mu_0^i(\mathbf{x}), \quad i = \overline{1, m},$$

$$X = \left\{ \mathbf{x}: \arg \max_{\mathbf{x} \in \Omega} \mu_q(\mathbf{x}), \quad q = \overline{1, L} \right\}$$

Applying the Pareto principle of optimality, the given problem can be written in the next formulation:

$$\max_{\mathbf{x} \in X} \left(\sum_{i=1}^m \gamma_i \mu_0^i(\mathbf{x}) \right), \quad i = \overline{1, m} \quad (13)$$

$$X = \left\{ \mathbf{x} : \arg \max_{\mathbf{x} \in \Omega} \sum_{q=1}^L \beta_q \mu_q(\mathbf{x}) \wedge \sum_{q=1}^L \beta_q = 1 \wedge \beta_q \geq 0, \quad q = \overline{1, L} \right\} \quad (14)$$

where $\wedge$ is the logical operator “and” sign; the rest of the designations are described above. The operator $\wedge$ means that all statements associated with it are true.

In the obtained FMP problem (13) and (14), the search for an effective set of solutions is performed at the points of the Pareto set formed by the objective functions and constraints. To solve this multi-criteria FMP problem, a heuristic method can be developed, similarly to the above heuristic methods.

4.3. Solution to the Optimization Problem of the Feed Production Plan in a Fuzzy Environment Based on Various (Deterministic and Fuzzy) Approaches

The formulated problem of the production plan of a feed mixture is solved using different methods: reducing the problem to a crisp problem of mathematical programming and solving it using well-known deterministic methods and using the proposed heuristic method. It has been established that when solving the given problem using a fuzzy approach, it is possible to draw up a more optimal plan for the production of feed, and, accordingly, to improve the solutions.

Let us consider various approaches (crisp and fuzzy) to solving the formulated problem of optimizing (minimizing) the cost of the feed mixture at the Makinsk poultry farm with the implementation of all fuzzy constraints on the composition of nutrients.

1. Deterministic approach. In this approach, by transforming the original fuzzy constraints into strict (crisp) constraints, one can represent the optimization problem as the following problem (15)–(20):

$$f(\mathbf{x}) = 0.275x_1 + 0.565x_2 + 0.445x_3 \rightarrow \min, \quad (15)$$

$$x_1 + x_2 + x_3 \geq 10,850 \quad (16)$$

$$0.372x_1 - 0.007x_2 - 0.006x_3 \geq 0 \quad (17)$$

$$0.220x_1 - 0.130x_2 + 0.280x_3 \geq 0 \quad (18)$$

$$0.050x_1 - 0.030x_2 + 0.030x_3 \geq 0 \quad (19)$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \quad (20)$$

The resulting linear programming (LP) problem can be solved using one of the methods for solving linear programming problems.

To solve the formulated LP problem, we used the MANAGER software package, which implements various methods for solving mathematical programming problems. In solving the formulated LP problem, the simplex method was used.

The results of inputting the initial data of the LP problem being solved in the interactive mode are shown in Figure 1.

```

D:\FOA5~1\493B~1\MANAGER\MANAGER.EXE
Linear Prog. II - PAGE 1 -
***** INPUT DATA *****
Min Z = 0.275 x 1 + 0.565 x 2 + 0.445 x 3
Constraints
C 1 1 x 1 + 1 x 2 + 1 x 3 >= 10,850
C 2 -0.372 x 1 - 0.007 x 2 - 0.006 x 3 >= 0
C 3 -0.22 x 1 + 0.13 x 2 - 0.28 x 3 <= 0
C 4 0.05 x 1 - 0.03 x 2 - 0.03 x 3 >= 0
F1 GO F2 CHANGE F3 RERUN F4 EXIT COMMAND ->

```

Figure 1. Results input of initial data for solving LP problem (15)–(20) in the MANAGER software package in interactive mode.

As a result of solving the problem after the fourth iteration, the optimal solution is obtained, shown in Figure 2.

```

D:\FOA5~1\4938~1\MANAGER\MANAGER.EXE
Linear Prog. II - PAGE 1 -
***** RESULT *****
Optimal solution obtained in 4 iterations
Optimal Z = 3847.80000
Optimal solution in row order
      Solution
1.    x 1    6160.00000
2.    x 2    0000.00000
3.    x 3    4840.00000
F1 LOOK F2 PRINT F3 STORE F4 RERUN F5 EXIT COMMAND ->

```

Figure 2. Results of solving the LP problem (16)–(20) with simplex method using the MANAGER program.

Thus, the optimal solution, i.e., the minimum value of the objective function of the LP problem being solved, is achieved for the following values of independent variables: $x_1 = 6160$, $x_2 = 0$ and $x_3 = 4840$. For these values of the independent variables, the minimum value of the objective function is achieved, i.e., the cost of producing the required volume for the poultry farm, i.e., 10850 kg of the produced feed mixture, is equal to \$3847.8.

Discussion of the results. The results obtained, in the face of strict constraints on the content of ingredients, show the following: if the poultry farm produces a feed mixture from ingredients $x_1 = 6160$ kg and $x_3 = 4840$ kg, where the ingredient $x_2 = 0$ is not used, then the cost of the required amount of feed will be minimal (\$3847.8) and all the conditions for the weekly diet and nutrient content are met. With any other variant of the feed production plan, the cost of the produced feed mixture will be higher.

2. Fuzzy approach. Now, taking into account the fuzziness of the constraint, one can consider a fuzzy approach to solving the problem of minimizing the cost of a feed mixture using FMP methods. Since the constraints are fuzzy, i.e., approximate, their violation is allowed to the permissible value. Based on the formulation of the optimization problem taking into account the fuzzy constraint (9)–(10), it can be written in the form of (21)–(26):

$$\min_{\mathbf{x} \in X} f(\mathbf{x}) \quad (21)$$

$$\varphi_0(\mathbf{x}) \geq 10,850 \quad (22)$$

$$\varphi_1(\mathbf{x}) \lesssim 0, \quad (23)$$

$$\varphi_2(\mathbf{x}) \lesssim 0, \quad (24)$$

$$\varphi_3(\mathbf{x}) \lesssim 0, \quad (25)$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \quad (26)$$

In the resulting model of the FMP problem (21)–(26), $\mathbf{x} = (x_1, x_2, x_3)$ is a vector of nutrients in the feed mixture.

Let us assume that the membership functions are determined based on expert assessment $\mu_q(x)$, $q = 1, 2, 3$ estimating the degree of fulfillment of fuzzy constraints $\varphi_q(\mathbf{x}) \lesssim 0$, $q = \overline{1, 3}$. On the basis of expert assessment, a number of priorities for fuzzy constraints is determined: $I = \{1, 3, 2\}$, i.e., the first constraint has the highest priority and the next priority is the third constraint, whereas the second constraint has the lowest priority. Consequently, applying the idea of the main criterion method, problem (21)–(26) can be written as the following FMP problem:

$$\min_{x_i \in X} f(x_i), \quad i = \overline{1, 3}$$

$$X = \{x_i: x_i \in \Omega \wedge \arg(\mu_q(x_i) \geq \mu_q^R), \quad i = \overline{1, 3}, q = \overline{1, 3}\}$$

We normalize the objective function in a mathematically correct manner as follows:

$$\mu_0(\mathbf{x}) = \frac{\sup_{\mathbf{x} \in X} f(\mathbf{x}) - f(\mathbf{x})}{\sup_{\mathbf{x} \in X} f(\mathbf{x}) - \inf_{\mathbf{x} \in X} f(\mathbf{x})} = \frac{3850 - f(\mathbf{x})}{3850 - 0} = 1 - \frac{f(\mathbf{x})}{3850}$$

where 3850 is the upper limit (sup) for the value of the production cost of the feed mixture.

This objective function normalization procedure ensures the following principle: the lower the cost of feed mixture production, the higher the degree of satisfaction. This is explained by the fact that the objective function represents the cost of the produced feed mixture, which must be minimized; accordingly, the lower the feed cost, the higher the degree of satisfaction of the solution.

Thus, the optimization problem of production plan for feed mixture with an objective of cost minimization of feed production per week under fuzzy restrictions can be written in the following manner:

$$\min_{\mathbf{x} \in X} f(\mathbf{x}) = \max_{\mathbf{x} \in X} \left(1 - \frac{f(\mathbf{x})}{3850} \right),$$

$$X = \{\mathbf{x}: \mathbf{x} \in \Omega = [x_i \geq 0, \quad i = \overline{1, 3}] \wedge \arg(\mu_1(\mathbf{x}) = 1) \wedge \arg(\mu_2(\mathbf{x}) \geq 0.9) \wedge \arg(\mu_3(\mathbf{x}) \geq 1)\}.$$

This formulation of the problem is fully consistent with the logic of fuzzy mathematical programming: the degree of satisfaction of the objective is maximized, rather than the

cost itself. Moreover, this makes the objective function consistent with the constraints defined through the membership functions $\mu_1(\mathbf{x})$, $\mu_2(\mathbf{x})$, $\mu_3(\mathbf{x})$.

To solve the obtained FMP problem, the above proposed heuristic FPMMC method was chosen, i.e., based on the modification of the idea of the main criterion method for fuzzy constraints, the following main points are implemented:

4.3.1. With the Involvement of Decision Makers and Subject Area Experts

A number of priorities for fuzzy restrictions is identified: $I_R = \{1, 3, 2\}$. Thus, the fuzzy limit has values of priorities 1, 2 and 3 on calcium content (3), on fiber content (5), and on protein content (4), respectively.

At this stage, expert judgments play an important role, since they make it possible to take into account the requirements of the fuzzy constraints and to solve the problem effectively in a fuzzy environment. Based on the expert selection procedure, a group of eight experts was formed. The core of the expert group consisted of six practitioner experts—process engineers with extensive experience and high qualifications in poultry feed production technology. In addition, the panel included two academic researchers studying feed mixture production problems. Consensus of opinion was achieved using the Delphi method; after the sixth round, the calculated coefficient of concordance, which measures the degree of agreement among expert opinions, reached 0.90. The subjective preferences of the experts and the decision maker were transformed into membership functions and boundary values by selecting the parameters of the membership functions while taking into account the technological requirements of feed mixture production.

Expert Validation of Fuzzy Priorities Using the Delphi Method

The procedure of expert validation of the priorities of fuzzy constraints. To determine the priorities of the fuzzy constraints related to the calcium, protein, and fiber content in the feed mixture, the Delphi method was used, which makes it possible to achieve consistency of expert assessments under conditions of uncertainty.

1. Expert Selection Criteria

The expert panel was formed on the basis of the following criteria:

- At least 5 years of professional experience in feed production, poultry nutrition, veterinary science, or agricultural biotechnology;
- Practical experience in formulating feeding rations or in quality control of feed mixtures;
- Participation in research studies, publications, or industrial projects related to the nutritional value of feed;
- Independence of expert judgments and the absence of direct influence on one another.
- A high value of the competence coefficient, determined according to the established methodology.

The study involved eight experts. To ensure representativeness, the expert panel included specialists from both academic and industrial fields.

2. Structure of the Delphi Survey

The experts were asked to rank the importance of the fuzzy constraints: C_1—calcium content; C_2—protein content; and C_3—fiber content.

The assessment was performed using a rank-order scale, where one represented the highest priority; two was medium priority; and three was lowest priority.

Each expert independently determined the relative importance of the constraints from the standpoint of their influence on the safety and nutritional value of the feed mixture.

The survey was conducted over six rounds. After each round, the experts were provided with aggregated information about the group opinion (median ranks and the range of deviations), after which they could review, revise, adjust, or justify their previous assessments.

3. Assessment of the Agreement Among Expert Opinions

To quantitatively assess the degree of agreement, Kendall's coefficient of concordance was used:

$$W = \frac{12S}{m^2(n^2 - n)},$$

where $m = 8$ is the number of experts; $n = 3$ is the number of evaluated constraints; and S is the sum of squared deviations of the ranks from the mean value:

$$S = \sum_{i=1}^n (R_i - \bar{R})^2,$$

where R_i is the sum of ranks for the i -th constraint; $\bar{R} = \frac{m(n+1)}{2}$ is the mean value of the total ranks.

The coefficient $W \in [0, 1]$ was interpreted as follows: $W < 0.4$ represents weak agreement; $0.4 \leq W < 0.6$ represents moderate agreement; $0.6 \leq W < 0.9$ represents sufficient agreement; and $W \geq 0.9$ represents high agreement.

The Delphi rounds were terminated when a stable level of agreement was reached ($W = 0.9$) and no substantial changes in the ranks were observed between successive rounds.

4. Determination of Constraint Priorities

After completion of the Delphi procedure, the final ranks were determined on the basis of the median assessments and the level of agreement among the experts. The constraint with the lowest mean rank was assigned the highest priority.

According to the results of the expert procedure, the following priorities were established:

- The calcium-content constraint—priority 1;
- The fiber-content constraint—priority 2;
- The protein-content constraint—priority 3.

The obtained priorities were used in forming the sequence for satisfying the fuzzy constraints in the FPMMC heuristic algorithm.

The main results of the Delphi procedure are presented in Table 2.

Table 2. Main results of the Delphi procedure.

Round	Kendall W	p -Value	Interpretation
1	0.45	<0.07	moderate agreement
2	0.50	<0.05	moderate agreement
3	0.62	<0.03	sufficient agreement
4	0.70	<0.02	sufficient agreement
5	0.81	<0.01	sufficient agreement
6	0.90	<0.01	high and stable agreement

In this table, the p -value is the level of statistical significance of Kendall's coefficient of concordance W , indicating how likely it is that the observed agreement among the experts arose by chance.

The following statistical hypothesis is tested:

H1. There is no agreement among the experts, and the coincidence of ranks is random;

H2. There is a statistically significant agreement among the experts.

If $p < 0.05$, the agreement is considered statistically significant, and the null hypothesis is rejected.

If $p > 0.05$, one cannot claim that the agreement is non-random.

In our case,

Round 1: $p < 0.07$ —the statistical significance is still weak (usually, $0.07 > 0.05$, which means that the agreement is not yet sufficiently confirmed);

Round 2: $p < 0.05$ —statistically significant agreement begins to appear;

Round 5–6: $p < 0.01$ —there is a very high statistical significance of expert agreement.

4.3.2. By Means of Expert Assessment

A term set describing fuzzy constraints is determined, and membership functions are constructed to assess the degree of fulfillment of fuzzy constraints: $\mu_q(x_i)$, $q = \overline{2,4}$, and $i = \overline{1,3}$.

The term set possesses the following categories “low degree of fulfillment”; “sufficient degree of fulfillment”; and “high or full degree of fulfillment”.

As the most appropriate type of membership function for the proposed problem, an analytical expression of exponential type can be written as follows [9]:

$$\mu_q(x_i) = \exp(Q_q(x_i - x_i^{md})),$$

where Q_q is the parameter characterizing the degree of fuzziness, and its value is identified when constructing the membership function; x_i^{md} is a fuzzy variable that more closely matches the selected term and it is determined by the expression $\mu_q(x_i^{md}) = \max \mu_q(x_i)$.

Thus, the membership functions for assessing the degree of fulfillment of fuzzy constraints are constructed in the following form: $\mu_q(x_i) = \exp(Q_q(x_i - x_i^{md}))$. Consequently, the membership function for assessing the degree of fulfillment of the fuzzy constraint “Calcium content must be not less than 0.8%”, with priority 1, must correspond to the term “high or full degree of fulfillment” and it has the following form:

$$\mu_1(x_1) = \begin{cases} \exp(1.0(x_1 - 0.8)), & \text{if } x_1 = [0.1 - 0.8] \\ 1, & \text{if } x_1 > 0.8 \end{cases}.$$

The membership function for assessing the degree of fulfillment of the fuzzy constraint “Protein content must be not less than 22%”, which has the lowest priority (3), must correspond to the term “high or full degree of fulfillment” and it has the following form:

$$\mu_2(x_2) = \begin{cases} 0, & \text{if } x_2 = (0 - 10) \\ \exp(0.1(x_2 - 22)), & \text{if } x_2 = [10 - 22] \\ 1, & \text{if } x_2 > 22 \end{cases}$$

Analogously, the membership function for assessing the degree of fulfillment of the fuzzy restriction “Fiber content should be no more than 5%”, with priority 2, must correspond to the term “high or full degree of fulfillment” and it has the following form:

$$\mu_3(x_3) = \begin{cases} 0 & \text{if } x_3 = [0, 5) \\ \exp(-0.5(x_3 - 5.0)), & \text{if } x_3 = [5 - 10] \\ 1 & \text{if } x_3 > 10 \end{cases}$$

1. With the involvement of a decision maker, the initial boundary values of the constraints are set: $\mu_q^{R(l)}$, $q = \overline{1,3}$, $l = 1$: $\mu_1^{R(1)} = 1.0$; $\mu_2^{R(1)} = 0.90$; and $\mu_3^{R(1)} = 1.0$.
2. The problem of determining the minimum of the objective function is solved, i.e., $\min_{x \in X} \mu_0(x)$, estimating the cost of producing a feed mixture per week on an admissible set X , and a set of following solutions is determined: $x(\mu_q^{R(l)}), \mu_0(x(\mu_q^{R(l)})), \mu_2(x(\mu_q^{R(l)})), \mu_3(x(\mu_q^{R(l)})), \mu_4(x(\mu_q^{R(l)}))$, $q = 1,2,3$, $l = 1,2,\dots$.
3. The current solutions are presented to decision makers for analysis and selection of final solutions.
4. The decision maker corrected the values of the boundary values of fuzzy constraints four times and returned to step 4 in order to find the best solutions.
5. In the fifth iteration with $\mu_1^{R(1)} = 1.0$; $\mu_2^{R(1)} = 0.85$; and $\mu_3^{R(1)} = 0.95$, taking into account the fuzziness and priorities of the constraints, the decision maker selected the optimal values $x^* = (x_1^*, x_2^*, x_3^*)$, i.e., the amount of limestone (x_1^*), soy beans (x_2^*) and wheat (x_3^*), ensuring the minimum value of the objective function, provided that the specified fuzzy constraints are met, considering their priorities. In this case, the decision maker selected the following values: $x_1^* = 6100$; $x_2^* = 0$; and $x_3^* = 4750$.

The objective function takes the following values at the selected solutions:

$$f(x) = 0.275x_1^* + 0.565x_2^* + 0.445x_3^* = 0.275 \cdot 6100 + 0.565 \cdot 0 + 0.445 \cdot 4750 = 3791.25\$.$$

Thus, a fuzzy approach to solving the posed problem of optimizing the production plan of a feed mixture, taking into account the fuzzy constraints, provides an improvement in the value of the objective function. As can be seen from the results of the solution, the cost of feed production with a fuzzy approach decreases by \$ 56.55 per week ($3847.8 - 3791.25 = 56.55$), i.e., the objective function value is improved by 1.5%.

At the same time, the fuzzy limitation “Calcium content $\gtrsim 0.08\%$: $0.372x_1 - 0.007x_2 - 0.006x_3 \gtrsim 0$ ”, with priority 1, fully corresponds to the term “high or full degree of fulfillment” for the following reason:

$$0.372x_1^* - 0.007x_2^* - 0.006x_3^* = 0.372 \cdot 6100 - 0.007 \cdot 0 - 0.006 \cdot 4750 = 2240.7 > 0.$$

This implies that the membership function

$$\mu_1(x_1) = \begin{cases} \exp(1.0(x_1 - 0.8)), & \text{if } x_1 = [0.1 - 0.8] \\ 1, & \text{if } x_1 > 0.8 \end{cases}$$

has a value equal to $\mu_2(x_1) = 1.0$ and the condition of this fuzzy constraint “high or full degree of fulfillment” is satisfied.

On the basis of expert judgment, it has been established that a fuzzy with (the lowest) priority 3 must correspond to the term “sufficient degree of fulfillment”. Fuzzy constraint “Protein content $\gtrsim 22\%$ ” with the lowest priority should be satisfied with “sufficient degree of fulfillment”. The necessary condition for this is “Protein content $\gtrsim 21\%$ ”.

Then, the mathematical expression of the fuzzy constraint “Protein content $\gtrsim 21\%$ ” has the following form:

$$0.00x_1 + 0.09x_2 + 0.50x_3 \gtrsim 0.21(x_1 + x_2 + x_3) = -0.210x_1 - 0.120x_2 + 0.29x_3 \gtrsim 0 \text{ or} \\ 0.210x_1^* + 0.120x_2^* - 0.290x_3^* \gtrsim 0 = 0.210 \cdot 6100 + 0.120 \cdot 0 - 0.290 \cdot 4750 = -96.5 < 0.$$

This means that there is a “sufficient degree of fulfillment”, i.e., the conditions of this fuzzy restriction are satisfied. Accordingly, the fuzzy constraint “Protein content $\gtrsim 21\%$ ” has the following form:

$$\mu_2(x_2) = \begin{cases} 0, & \text{if } x_2 = (0 - 10) \\ \exp(0.1(x_2 - 21)), & \text{if } x_2 = [10 - 21] \\ 1, & \text{if } x_2 > 21 \end{cases}$$

This membership function at $x_2 = 21$ has the maximum value equal to $\mu_3(x_2) = \exp(0.1(21 - 21)) = \exp(0) = 1$, i.e., the fuzzy constraint is met with a “sufficient degree of fulfillment”.

In contrast, the membership function for assessing the degree of fulfillment of the fuzzy constraint “Protein content ≥ 22 with the lowest priority (3) will be equal to

$$\mu_2(x_2) = \begin{cases} \exp(0.1(21 - 22)), & \text{if } x_2 = [19 - 22] \\ 1, & \text{if } x_2 > 22 \end{cases} = \exp(-0.1) = 0.91$$

Although the protein content constraint was not fully satisfied and was achieved only at a “sufficient” level, the resulting fuzzy solution remains practically feasible and meets the requirements of feed production. The use of fuzzy constraints made it possible to account for the natural variability in raw material quality and to ensure a compromise between nutritional value and economic efficiency. The results showed that a slight relaxation of the protein constraint does not lead to a critical deterioration in feed mixture quality, while reducing production costs by 1.5%.

From the perspective of fuzzy optimization theory, this is regarded not as a violation of constraints, but as a controlled reduction in the degree of satisfaction, permissible within the framework of the specified membership function and the expert-defined quality thresholds.

Now, let us check the degree of fulfillment of the fuzzy constraint “Fiber content $\geq 0.05\%$: $-0.05x_1 - 0.03x_2 + 0.03x_3 \geq 0$ with priority 2:

$$-0.05x_1 - 0.03x_2 + 0.03x_3 = -0.05 \cdot 6100 - 0.03 \cdot 0 + 0.03 \cdot 4750 = -162.5 < 0.$$

This means that there is a “high or full degree of fulfillment”, i.e., conditions of this fuzzy constraint are satisfied.

Thus, this fuzzy constraint is fully satisfied and it corresponds to the term “high or full degree of fulfillment”, and its membership function will be equal to one:

$$\mu_3(x_3) = \begin{cases} 0 & \text{if } x_3 = [0, 5) \\ \exp(-0.5(x_3 - 5)), & \text{if } x_3 = [5 - 10]. \text{ At } x_3 = 5, \mu_3(x_3) = \exp(0) = 1. \\ 1, & \text{if } x_3 > 10 \end{cases}$$

The nonlinear form of the Gaussian-type membership function makes it possible to represent expert judgments more adequately, while the parameters of the function were determined by practitioner experts who took into account nutritional standards and the results of sensitivity analysis.

Thus, with a fuzzy approach to solving the problem of optimizing the plan for the production of a feed mixture, by taking into account the fuzzy constraints and their priorities, it is possible to improve the plan. At the same time, the weekly cost of production of the feed mixture is reduced by \$ 56.55, and the monthly savings will accordingly be more than \$226.

As a result of optimizing the plan for the production of feed mixture, the following optimal solution is obtained:

$$x_1^*(\mu_q^{R(l)}) = 61,000, \quad x_2^*(\mu_q^{R(l)}) = 0 \text{ and } x_3^*(\mu_q^{R(l)}) = 4750$$

Respectively, the optimal amount of limestone, soybeans and wheat in the feed mixture, in kg,

$$\mu_0(x^*(\mu_q^{R(l)})) = \frac{3791.25}{3850} \cdot 0.98 \Rightarrow f(x^*) = 3791.25$$

and the optimal (minimum) value of the objective function,

$$\varphi_0(x^*) \geq 10,850 = x_1 + x_2 + x_3 = 6100 + 0 + 4750 = 10,850,$$

i.e., the constraint conditions (22), are satisfied.

$$\mu_1(x_1^*(\mu_q^{R(l)})) = 1.0, \mu_2(x_2^*(\mu_q^{R(l)})) = 0.91 \text{ and } \mu_3(x^*(\mu_q^{R(l)})) = 1.0$$

The maximum values of the function, i.e., membership, describing the degree of fulfillment of fuzzy constraints, i.e., the requirements of these constraints, are also fully met.

Since the fuzziness actually relates to the nutritional constraints (calcium, protein, and fiber), which are calculated through the total contribution of the ingredients to the final mixture, let us interpret the results by moving from the ingredient level to the level of the aggregated nutritional indicators of the mixture.

The results of solving the feed mixture production optimization problem with fuzzy constraints show that the application of the proposed heuristic approach made it possible to obtain an economically efficient formulation with a satisfactory level of compliance with the nutritional requirements of the final mixture. At the same time, the membership functions were formed not directly with respect to the decision variables x_1, x_2, x_3 , which characterize the quantities of the individual ingredients, but with respect to the final nutritional characteristics of the mixture—namely, the calcium, protein, and fiber contents determined through the combined contribution of all ingredients.

Thus, the degree of satisfaction of the fuzzy constraints is evaluated on the basis of the values of the aggregated functions:

$$Ca(x) = \sum_{i=1}^3 a_i x_i, Pr(x) = \sum_{i=1}^3 b_i x_i, Fib(x) = \sum_{i=1}^3 c_i x_i,$$

where a_i, b_i, c_i are the concentrations of calcium, protein, and fiber in the i -th ingredient, respectively, and x_i is the proportion of the ingredient in the formulation.

Therefore, the obtained solution is interpreted not as the direct satisfaction of the requirements by the individual ingredients, but as ensuring the required degree of membership with respect to the aggregated nutritional indicators of the final feed. This means that changes in the values of the variables x_1, x_2, x_3 affect the feasibility of the solution indirectly—through changes in the final concentrations of nutrients in the mixture.

Under the fuzzy formulation of the problem, the feasibility of the solution is determined by the degree to which the nutrient constraints are satisfied, which makes it possible to take into account technological uncertainty, variability in raw material composition, and expert-defined intervals of acceptable values. Relaxation of the rigid bounds on calcium, protein, and fiber content provides a more flexible search for feasible solutions and makes it possible to reduce the cost of the feed mixture while maintaining an acceptable quality level.

Thus, the membership functions determine the degree of satisfaction of the calcium, protein, and fiber constraints, which are calculated through linear combinations of the decision variables x_1, x_2, x_3 , characterizing the contribution of the ingredients to the final mixture. The mixture formed by these ingredients satisfies the corresponding nutrient constraint with a certain degree of membership.

Figure 3 presents the results of solving the poultry feed production optimization problem with fuzzy constraints (21)–(26) based on the fuzzy approach using the heuristic FPMC method:

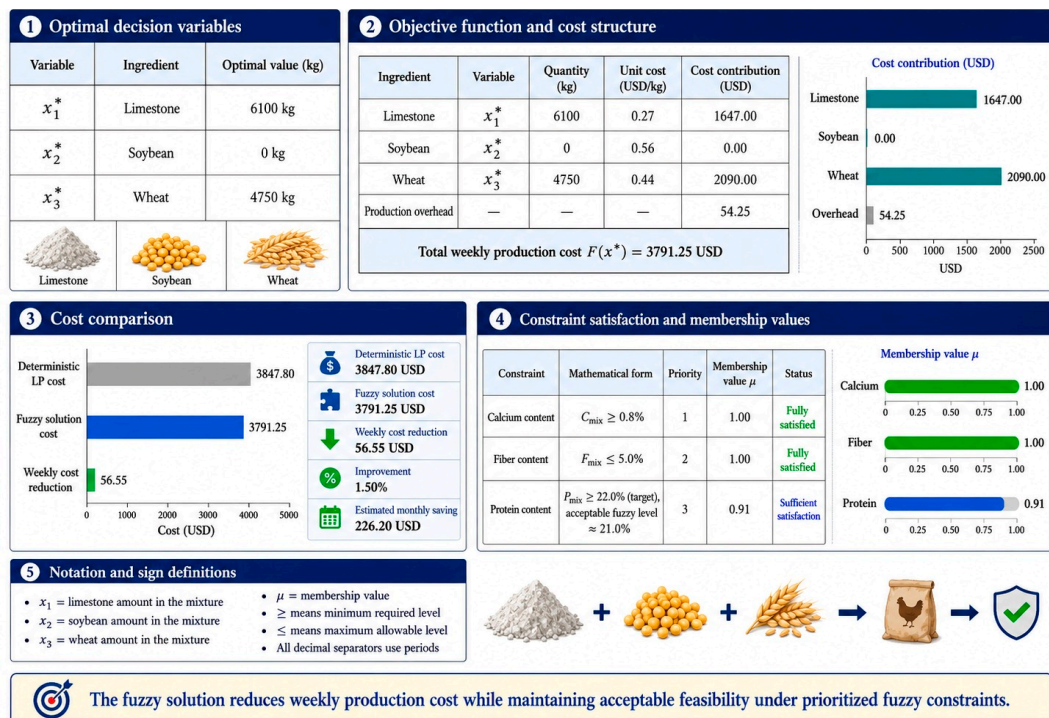


Figure 3. Numerical and visual representation of feed mixture optimization results under fuzzy constraints.

Let us present the main results of the quantitative comparison between the proposed and applied FPMMC method for solving the optimization problem with fuzzy constraints and the conventional fuzzy linear programming model (the Bellman–Zadeh/Zimmermann max–min approach) for the same feed mixture optimization problem and using the same input data set. For ease of analysis, the comparison results are presented in Table 3.

For comparison, the classical fuzzy linear programming model was used, in which the minimum degree of constraint satisfaction is maximized:

$$\max \lambda \quad \text{subject to: } \mu_q(x) \geq \lambda, q = 1, 2, 3, x \in \Omega, x_i \geq 0,$$

where λ is the integral degree of solution feasibility.

Unlike FLP, the proposed FPMMC heuristic method takes into account the priority of the fuzzy constraints, which makes it possible to relax lower-rank constraints without violating the most important requirements.

Table 3. Comparison of optimization results based on FPMMC and the classical fuzzy linear programming model.

Indicator	FPMMC (Proposed Method)	Classical FLP (Zimmermann Max–Min)
Model type	Priority-based fuzzy mathematical programming	Standard fuzzy linear programming
Consideration of constraint priorities	Yes ($I_R = \{1, 3, 2\}$)	No
Optimal solution	$x_1^* = 6100, x_2^* = 0, x_3^* = 4750$	$x_1^* = 5980, x_2^* = 320, x_3^* = 4550$
Objective function value	3791.25\$	3847.80\$
Cost improvement	1.5%	–
Integral measure of solution feasibility	$\mu^* = 0.95$	$\lambda^* = 0.90$
Satisfaction of the calcium constraint	$\mu_1 = 1.00$	$\mu_1 = 0.92$
Satisfaction of the protein constraint	$\mu_2 = 0.91$	$\mu_2 = 0.90$
Satisfaction of the fiber constraint	$\mu_3 = 1.00$	$\mu_3 = 0.90$

Number of iterations until the final solution	5	1
Computer processing time	0.17 c	0.09 c
Sensitivity to changes in expert assessments	Low	Medium
Change in cost when expert parameters are varied by $\pm 5\%$	$\leq 1.8\%$	up to 4.7%
Change in the solution structure when expert weights are modified	Slight	Moderate

The numerical values of the compared methods, FPMMC and FLP, were obtained using the same set of input data, and the comparison was performed under identical computational conditions. The result values for FLP were reproduced through a computational experiment in the MATLAB 5 environment.

The pseudocode of the fuzzy heuristic FPMMC algorithm for feed mixture optimization under fuzzy constraints follows:

Algorithm 1. Pseudocode of the fuzzy heuristic FPMMC algorithm for feed mixture optimization under fuzzy constraints.

BEGIN multi-criteria optimization with fuzzy constraints

* Input of initial data

** Objective function for minimizing the cost of the feed mixture:

$[F(x_1, x_2, x_3) \rightarrow \min]$

*** Problem constraints:

*** deterministic constraints:

$[x_i \geq 0, i = \overline{1,3}]$

*** fuzzy constraints:

$[\varphi_q(x_1, x_2, x_3) \gtrless (\lesseqgtr), q = \overline{1,3}]$

*** Priorities of fuzzy constraints: Based on expert evaluation and with the involvement of the decision maker, determine the order of importance of the fuzzy constraints:

$[P = \{p_1, p_2, p_3\}]$

*** Term set: based on expert evaluation, determine the terms that fuzzily describe the degrees of satisfaction of the fuzzy constraints

$[T = \{\{\text{low degree of satisfaction}\}; \{\text{sufficient degree of satisfaction}\}; \{\text{high (full) degree of satisfaction}\}\}]$

*** Membership functions: Formation of Gaussian-type membership functions that formally describe the terms of the term set

$[\mu_q(x_i), q = \overline{1,3}, i = \overline{1,3}]$

** Initial boundary values: Initialization, set the initial boundary values of the fuzzy constraints:

$[\mu_q^{R(l)}, q = \overline{1,3}, l = 1]$

*** Solving the objective function minimization problem

$[F(x_1, x_2, x_3) \rightarrow \min \text{ на допустимом множестве}]$

$X = \{\mathbf{x}: \mathbf{x} \in \Omega \wedge \arg(\mu_q(\mathbf{x}) \gtrless \mu_q^R), q = 2,4 \wedge \mu_3(\mathbf{x}) \gtrless \mu_3^R\}$

taking into account the current boundary values of the fuzzy constraints.

*** Evaluation of the degree of satisfaction of fuzzy constraints:

For each fuzzy constraint, calculate:

$[\mu_q(x_i), q = \overline{1,3}, i = \overline{1,3}]$

*** determination of the best solution

$[\mathbf{x}^* = (x_1^*, x_2^*, x_3^*)]$

And the value:

$[F^*(\mathbf{x})]$

** Maximum number of iterations:

$[N_{max}]$

** Allowable accuracy of change in the objective function:

$[\varepsilon]$

END optimization taking into account the fuzziness of constraints based on the heuristic FPMMC method

The stopping criteria of the FPMMC heuristic algorithm may be as follows:

1. Achievement of an acceptable degree of satisfaction of priority constraints. For all fuzzy constraints, the following holds:

$$[\mu_q(\mathbf{x}) \gtrsim \mu_q^R],$$

where μ_q^R is the minimum acceptable satisfaction level established by the decision maker with due regard to the priority of the constraint.

2. Stabilization of the objective function. The change in the value of the objective function between neighboring iterations becomes insignificant:

$$[F^{t+1}(\mathbf{x}) - F^t(\mathbf{x})] < \varepsilon,$$

where ε is a pre-specified small value.

3. No improvement in the solution. Over several consecutive iterations, there is no improvement in cost:

$$[F^{t+1}(\mathbf{x}) \approx F^t(\mathbf{x})]$$

4. Decision maker satisfaction. The decision maker accepts the obtained solution as satisfactory, taking into account economic and technological requirements.
5. Reaching the maximum number of iterations, N_{max} , which prevents an infinite iterative process.

When solving the stated feed mixture production optimization problem, the algorithm terminated at the fifth iteration, since an acceptable solution satisfying the priority fuzzy constraints had been achieved, and further adjustment of the parameters did not lead to a significant improvement in the value of the objective function.

Let us demonstrate how the obtained solution, $x_1 = 6100$, $x_2 = 0$, $x_3 = 4750$, for the feed production optimization problem with fuzzy constraints ensures the corresponding membership values for each membership function, namely $\mu_1(x_3) = 1.0$; $\mu_2(x_3) = 0.91$ and $\mu_3(x_3) = 1.0$. For this purpose, we perform an independent verification of the solution in the form of a step-by-step calculation, where, for the obtained values $x_1 = 6100$, $x_2 = 0$, and $x_3 = 4750$, the actual nutrient contents are first calculated, and then the values of the membership functions are determined.

1. The substitution of the obtained solution into the original nutrient balance equations and the determination of the actual values of calcium, protein, and fiber are performed:

$$Ca^* = Ca(6100, 0, 4750) = 0.92;$$

$$Pr^* = Pr(6100, 0, 4750) = 18.1;$$

$$Fib^* = Fib(6100, 0, 4750) = 5.2.$$

2. Using the analytical forms of the linear-type membership functions, for example, for calcium we obtain

$$\mu_{Ca}(x) = \begin{cases} 0, & Ca(\mathbf{x}) < L_{Ca} \\ \frac{Ca(\mathbf{x}) - L_{Ca}}{U_{Ca} - L_{Ca}}, & L_{Ca} \leq Ca(\mathbf{x}) \leq U_{Ca}, \\ 1, & Ca(\mathbf{x}) \geq U_{Ca} \end{cases}$$

where L_{Ca} , U_{Ca} are the lower and upper bounds for calcium.

Analogous membership functions were used for protein and fiber.

3. By substituting the calculated values into the membership functions, an independent verification was carried out; for example, for calcium $Ca^* = 0.92$,

$$\mu_{Ca}(x) = \frac{0.92 - 0.85}{0.92 - 0.85} = 1.$$

For protein, the membership function has the form

$$\mu_{Pr}(x) = \frac{Pr(x) - 17}{18.2 - 17}$$

Then

$$\mu_{Pr}(x) = \frac{18.1 - 17}{18.2 - 17} = \frac{1.1}{1.2} = 0.917 \approx 0.91$$

For fiber $Fib^* = 5.2$,

$$\mu_{Fib}(x) = \frac{5.2 - 5.0}{5.2 - 5.0} = 1.$$

Substituting the obtained solution $x_1 = 6100$, $x_2 = 0$, $x_3 = 4750$ into the original nutrient content equations showed that the calcium and fiber constraints are fully satisfied ($\mu = 1$), whereas the protein constraint is satisfied at the level $\mu = 0.91$, which reflects an acceptable compromise resulting from the minimization of the feed mixture cost.

5. Discussion of Results

As can be seen, when solving the optimization problem based on a fuzzy approach, taking into account the fuzzy constraint, the best solutions have been obtained. The objective function value improves, i.e., the cost of producing feed mixture per week is reduced by \$56.55. Production savings is at least \$ 226 per month.

Since $\mu_1(\mathbf{x}) = 1$, the fuzzy constraint on calcium content $0.372x_1 - 0.007x_2 - 0.006x_3 \geq 0$ with priority 1 is fully satisfied and it has a “high or full degree of fulfillment”. Membership function assessing the degree of fulfillment of a fuzzy restriction on fiber content $-0.05x_1 - 0.03x_2 + 0.03x_3 \geq 0$ with priority 2 is also fully satisfied, i.e., $\mu_3(\mathbf{x}) = 1$. The membership function assessing the degree of fulfillment of a fuzzy constraint on the protein content $0.220x_1 + 0.130x_2 - 0.280x_3 \geq 0$ with the lowest priority is satisfied with “sufficient degree of fulfillment”, since it has the value $\mu_2(\mathbf{x}) = 0.91$. Thus, with a fuzzy approach to solving the formulated problem of optimizing feed production, the requirements of all fuzzy constraints are met, and the value of the objective function improves. This shows that taking into account the fuzziness of the constraint provides an improvement in the values of the objective function.

The difference between deterministic and fuzzy approaches to poultry feed production planning becomes especially evident under conditions of uncertainty in raw material composition, price fluctuations, quality requirements, and imposed constraints on feed composition. Below is a systematic comparison with an emphasis on a 1.5% cost reduction achieved through the relaxation of fuzzy constraints.

In the deterministic approach, all parameters are assumed to be precisely known and fixed: the nutrient content of feed mixture components is specified by exact values; constraints on calcium, protein, fiber, and cost are rigid; and feed production parameters such as raw material availability and equipment capacity are assumed to remain unchanged.

The drawback of the deterministic approach is that the solution proves to be overly rigid. Even slight deviations in raw material quality may make the optimal plan practically infeasible or economically inefficient.

In the fuzzy approach, constraints are flexible due to fuzziness described by membership functions, and partial deviation from standards is allowed within an acceptable degree of satisfaction. The idea of this approach is that feed composition requirements need not be absolutely rigid, but may allow for a certain range of deviations without significant loss of feed mixture quality.

The key difference in the context of a 1.5% cost reduction lies in the flexibility of constraints. In the deterministic approach, the system strictly complies with all standards; more expensive components are often used to guarantee satisfaction of the constraints; and the solution is conservative. In the fuzzy approach, controlled relaxation of certain constraints is allowed, for example, a slight reduction in protein or calcium content within permissible limits; this creates the possibility of including less expensive ingredients; and the cost of the formulation decreases.

If, as a result of relaxing fuzzy constraints, costs decreased by 1.5%, this means that the system found a compromise between cost and the degree of fulfillment of the requirements. Thus, when solving the feed production optimization problem, a 1.5% improvement in the objective function is achieved by relaxing the protein constraint from 22% to a “sufficient degree of satisfaction” at a level of about 21%. This relaxation of the protein constraint is practically acceptable for poultry feed production and fully satisfies nutritional requirements.

Unlike the deterministic approach, which assumes the strict fixation of parameters and constraints, the fuzzy approach makes it possible to account for uncertainty in raw material parameters and flexibility in feed mixture composition requirements. The relaxation of fuzzy constraints expands the feasible solution space and makes it possible to find a more economically efficient formulation. The modeling results showed that the application of fuzzy optimization reduced total production costs by 1.5% compared with the deterministic model while maintaining an acceptable level of requirement satisfaction.

For the feed mixture optimization problem with fuzzy constraints, sensitivity and robustness analysis is highly important, since it shows how well the obtained solution is preserved when expert assumptions change, namely the parameters of membership functions, the priorities of criteria, and the boundary values of the constraints. The purpose of the sensitivity analysis is to show how stable the optimal feed mixture composition is, whether the economic effect changes, for example, the 1.5% cost reduction, and whether the practical applicability of the solution is preserved;

To validate the robustness of the proposed heuristic method with quantitative evidence, we present the ranges of parameter variation, specific scenarios, the obtained solutions, verification of the feasibility of each constraint, and a numerical measure of sensitivity/robustness.

For a formal definition of the sensitivity analysis, we explicitly state that the robustness of the FPMCM method was investigated with respect to three groups of parameters: $\Theta = \{Q_q, \mu_q^R, I_R\}$, where Q_q are the parameters of the membership functions; μ_q^R are the boundary satisfaction levels; and I_R is the priority order of the fuzzy constraints.

5.1. Ranges of Parameter Variation

To vary the ranges of the above-described parameters, we specify them explicitly.

5.1.1. Membership-Function Parameters

For each constraint, a variation of $\pm 20\%$ is considered, as shown in Table 4: $Q_q^{var} \in [0.8Q_q, 1.2Q_q]$:

Table 4. Variation interval of membership-function parameters.

Constraint	Base	Variation Interval
Calcium	1.0	[0.8; 1.2]
Protein	0.10	[0.08; 0.12]
Fiber	0.50	[0.40; 0.60]

5.1.2. Boundary Satisfaction Levels

The range is presented in Table 5. $\mu_q^R \in [\mu_q^R - 0.10, \mu_q^R + 0.10]$.

Table 5. Tested range of boundary satisfaction levels.

Constraint	Base	Tested Range
Calcium	1.00	[0.90; 1.00]
Protein	0.85	[0.75; 0.95]
Fiber	0.95	[0.85; 1.00]

5.1.3. Priorities

Together with the basic priority order, five main scenarios were tested:

$$\{1,3,2\}, \{1,2,3\}, \{2,1,3\}, \{2,3,1\}, \{3,1,2\}$$

5.2. Numerical Scenarios

The calculation results are presented in Table 6.

Table 6. Sensitivity analysis results.

Scenario	Modified Parameter	x_1	x_2	x_3	Cost (\$)	μ_1	μ_2	μ_3	Feasible
Base	—	6100	0	4750	3791.25	1.00	0.91	1.00	yes
S1	$Q_1 = -20\%$	6020	0	4830	3794.4	0.98	0.92	1.00	yes
S2	$Q_1 = +20\%$	6170	0	4680	3788.6	1.00	0.90	0.99	yes
S3	$\mu_2^R = 0.95$	5750	450	4650	3815.1	1.00	0.95	1.00	yes
S4	priority = {2, 1, 3}	5900	600	4350	3832.8	1.00	1.00	0.97	yes

5.3. Verification of the Feasibility of Each Constraint

The actual satisfaction of the calcium, protein, and fiber constraints is presented in Table 7.

Table 7. Verification of the feasibility of the nutritional constraints.

Scenario	Calcium	Protein	Fiber	Constraint Status
Base	2240.7	−96.5	−162.5	satisfied
S1	2218.4	−88.2	−151.1	satisfied
S3	2120.6	−10.1	−132.8	satisfied

5.4. Quantitative Indicator of Method Robustness

Let us present a quantitative indicator of robustness. As a single numerical index, the robustness index is used:

$$RI = 1 - \frac{1}{N} \sum_k \left| \frac{f_k - f_0}{f_0} \right|,$$

where f_0 is the baseline solution; f_k is the cost in scenario k ; and N is the number of scenarios.

Interpretation: $RI > 0.95$ – highly robust; $0.90 < RI \leq 0.95$ – robust; $0.80 < RI \leq 0.9$ – *moderately* robust; and $RI < 0.80$ – *sensitive*.

In our case $RI = 1 - \frac{0.0329}{5} = 0.99342 \Rightarrow$ highly robust method.

The solution variability indicator is defined as . For , the corresponding variation values are presented in Table 8. $VI_j = \frac{\max(x_j) - \min(x_j)}{x_j^{base}} \times 100\%$. x_1, x_2, x_3

Table 8. Variation values of the solution variables.

Variable	Variation
limestone x_1	6.8%
soybean x_2	9.2%
wheat x_3	5.4%

To verify the robustness of the proposed FPMMC method, a sensitivity analysis was carried out with respect to the following factors: the parameters of the membership functions, the boundary satisfaction levels, and the priority order of the fuzzy constraints.

The parameters of the membership functions were varied within $\pm 20\%$ of their baseline values, while the boundary satisfaction levels were varied within ± 0.10 of their initial values. In addition, the main permutations of the priority order of the fuzzy constraints were considered.

For each scenario, the optimization problem was solved, and the obtained solution was evaluated according to the following parameters: (1) feed composition, (2) production cost, (3) the degree of membership satisfaction, and (4) the feasibility of all nutritional constraints.

The robustness of the proposed method was quantitatively assessed using the robustness index RI. The obtained results showed that variations in the fuzzy parameters led only to moderate changes in the optimal feed composition and in the value of the objective function. The calculated robustness index $RI = 0.99342$ indicates a high degree of stability of the proposed FPMMC method under conditions of parameter uncertainty.

As can be seen from the comparison results presented in Table 3, the proposed FPMMC heuristic method makes it possible to take into account the priorities of the fuzzy constraints, provides a better value of the objective function, and improves the cost by 1.5%. In addition, the proposed heuristic method demonstrates greater robustness to uncertainty in expert assessments, since changes in the parameters lead to smaller fluctuations in the final solution.

Thus, in order to quantitatively assess the effectiveness of the proposed FPMMC method, a comparative analysis was carried out with a well-known fuzzy linear programming model based on the classical Bellman–Zadeh/Zimmermann approach, using the same input data set. In the course of the comparison, the value of the objective function, the feasibility indicator, the computation time, the number of iterations required to obtain the final solution, and the sensitivity of the results to changes in expert assessments were taken into account.

The obtained results show that the proposed FPMMC heuristic method provides a lower production cost of the feed mixture, equal to 3791.25 USD/week, compared with 3847.80 USD/week obtained using the classical fuzzy linear programming model. Thus, the proposed method improves the value of the objective function by approximately 1.5%.

In addition, the FPMMC method demonstrated a higher degree of feasibility of the final solution due to taking into account the priorities of the fuzzy constraints. Unlike the classical fuzzy linear programming model, in which all constraints are treated equally, the proposed approach makes it possible to relax lower-priority constraints in a controlled manner while ensuring strict satisfaction of high-priority nutritional requirements.

In addition, a sensitivity analysis was carried out by varying the expert-defined parameters and the admissibility threshold values within $\pm 5\%$. The obtained results showed that the variation in the objective function when using the FPMMC method remained within 0.8–1.8%, whereas for the classical fuzzy linear programming model the deviation reached 4.7%. This confirms the greater robustness of the proposed method to uncertainty in expert assessments. Thus, the comparative analysis confirms that the proposed FPMMC method is not only computationally feasible, but also provides better economic performance and greater robustness under fuzzy constraints than the traditional fuzzy linear programming approach.

The fuzzy method leads to changes in the structure of the optimal ingredient combination in the following directions: the use of cheaper components increases if a slight deviation from target nutrient levels is considered acceptable; dependence on costly protein and energy ingredients decreases; and the sensitivity of the solution to fluctuations in raw material quality is reduced. Thus, the application of the fuzzy method changes the formulation strategy: instead of searching for an absolutely strict solution, the model seeks a rational balance between cost, nutritional adequacy, and technological feasibility.

The relaxation of constraints contributes to finding stable and practically feasible solutions; however, it requires careful adjustment of admissibility parameters. As a result, a rational compromise is achieved between cost minimization and compliance with nutritional requirements, which increases the practical applicability of the model under industrial feed production conditions.

6. Conclusions

Mathematical models are developed for the plan for the production of feed mixture on the example of the Makinsk poultry farm. A fuzzy mathematical programming problem is formulated to find the optimal plan for the production of feed mixtures in the presence of fuzzy constraints, and a heuristic approach to its solution is proposed. Heuristic methods *FPS* and *FPMC* are developed by modifying the Pareto principle of optimality and the method of the main criterion taking into account fuzziness. The results of solving the problem of optimizing the plan for the production of feed mixture on the basis of a deterministic and fuzzy approach are presented and compared. It is shown that in the presence of fuzzy initial information, the use of a fuzzy approach provides a more efficient solution.

Thus, in the work in accordance with the goal and objectives, the following main results are obtained:

1. A mathematical model has been built for the optimization problem on the production of feed mixture, taking into account the fuzziness of some constraints;
2. Heuristic methods are developed for solving the formulated optimization problems with fuzzy constraints;
3. The optimization problem has been solved for the poultry production plan in a fuzzy environment on the basis of deterministic and fuzzy approaches.

The originality and novelty of the proposed formalization, the problem formulation with fuzzy constraints, and the heuristic methods for solving them, in contrast to known methods, lie in the maximum use of the available fuzzy information, which consists of the experience, knowledge, and expertise of the decision maker and experts. Since such information contains a non-formalizable, fuzzy relationship between the input and output parameters of the problem being solved, it makes it possible to obtain a more adequate and better solution to the original production problem in a fuzzy environment.

Unlike known methods of fuzzy linear programming, in the proposed heuristic approach the fuzzy elements of the problem are formalized in the form of adequate

membership functions, which, due to their continuity, make it possible to maximally use the experience, knowledge, and intuition of experts. This, in turn, allows a more effective and more adequate solution to be obtained in a fuzzy environment.

The proposed heuristic method for solving problems with fuzzy constraints differs from known methods of fuzzy linear programming (FLP) and Pareto-based compromise methods primarily in the way uncertainty is taken into account, in the flexibility of solution selection, and in the procedure for searching for the optimum.

FLP methods usually assume the following: finding a solution by transforming the fuzzy problem into an equivalent deterministic form, and applying the principle of maximizing the degree of constraint satisfaction. In contrast, the proposed heuristic approach makes it possible to solve the problem directly in a fuzzy environment by formalizing the fuzzy elements of the problem through their membership functions. In FLP methods, the use of expert knowledge is limited, whereas in heuristic methods it is extensive. FLP methods operate under formalized constraints, and the search for a solution is carried out analytically, while heuristic methods can operate under poorly formalized, fuzzy constraints, and the search process is iterative and heuristic. Compared with FLP methods, the proposed heuristic methods are more robust under incomplete data.

The heuristic approach makes it possible to integrate fuzzy rules, expert knowledge, empirical relationships, and ANFIS-type models, and it also allows work with non-formalizable constraints. In the absence of precise statistical data, heuristic methods make it possible to use expert assessments instead of rigid model coefficients, that is, they ensure robustness under incomplete and fuzzy information.

Unlike the Pareto approach, the proposed heuristic method directly generates a preferred solution by taking into account the weights of the criteria, the degree of constraint satisfaction, and expert priorities.

The heuristic approach makes it possible to integrate fuzziness directly into the optimization procedure. In the classical Pareto approach, constraints are usually specified in a rigid form: $\varphi_i(x) \leq b_i$, whereas in the heuristic approach fuzzy constraints are used, formalized in the form of membership functions $\mu_i(x) \in [0, 1]$. This makes it possible to account for permissible deviations of parameters. In addition, the heuristic approach allows adaptation to the preferences of the decision maker, enables easy modification of the search strategy, and has higher applicability for solving real-world problems.

Thus, the key advantages of heuristic methods include the following:

1. Accounting for fuzziness in constraints and expert uncertainty.
2. The integration of expert knowledge into the optimization procedure.
3. Obtaining one practically acceptable solution rather than merely a set of alternatives.
4. Flexibility in adjusting criterion weights in multi-criteria optimization.
5. Robustness under incomplete or noisy data.
6. High adaptability for intelligent process control systems.

The proposed heuristic methods FPS and FPMMC differ in relation to the following:

- The first method is intended for solving single-criterion problems with fuzzy constraints;
- The second method is intended for solving multi-criteria problems with fuzzy constraints by obtaining an integrated criterion based on the Pareto principle.

The proposed heuristic approach is not limited to a direct adaptation of the known ideas of the fuzzy Pareto set and the principal criterion method, but includes their modification with respect to the problem of production planning under fuzzy constraints of different significance and with the use of membership functions constructed with expert participation.

The key novelty of the heuristic approach lies in the fact that the constraints are considered hierarchically rather than as equally significant, for example: calcium is critically important; fiber is of medium importance; and protein is less critical. This makes it possible to search for a solution not merely according to the principle of maximizing the overall degree of membership, but with due regard to the actual technological importance of the constraints. Such an approach is closer to real decision-making practice in compound feed production.

Unlike known methods, where violation of a constraint may render a solution infeasible, the heuristic approach allows for controlled relaxation of constraints; that is, small deviations are permitted, the degree of admissibility of the violation is assessed, and the technological feasibility of the formulation is preserved. This is especially important under conditions of ingredient shortages or price fluctuations.

In existing methods of fuzzy mathematical programming, the role of the expert is limited to specifying parameters. In the proposed heuristic approach, the expert performs a much broader function: they establish the priorities of constraints, determine admissible intervals, participate in tuning the membership functions, and adjust the compromise between cost and quality. Thus, the proposed heuristic approach may be positioned as a hybrid heuristic mechanism of multi-criteria fuzzy choice with priority-based processing of constraints and adaptive relaxation of admissible solution regions.

The scientific novelty of the obtained research result lies in the development of a heuristic method for solving the poultry feed optimization problem under fuzzily specified constraints. This method differs from existing methods of fuzzy mathematical programming by using a priority structure of nutritional constraints, a mechanism for adaptive relaxation of admissible boundaries, and an iterative search for a compromise solution between cost minimization and the degree of satisfaction of feed nutritional value requirements.

The theoretical contribution of the work is determined in the development of heuristic methods for solving fuzzy mathematical programming problems. The practical significance of the work lies in the fact that the proposed approach allows one to obtain more effective solutions to real production problems in conditions of fuzziness of the initial information.

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