

Article

Dynamic Modeling and Characteristics of a Two-Dimensional Nonlinear Friction-Induced Slider Moving on an Oscillating Belt Based on Stick-Slip Motion

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Abstract

The belt velocity is not strictly constant but exhibits periodic oscillations in certain industrial applications. A lumped mass model of a two-dimensional nonlinear friction-induced slider moving on an oscillating belt is established in this paper. Tangential stick-slip motion and normal separation–re-contact behavior are both considered under a symmetrically and uniformly distributed interface force. The analytical expression of the static friction force is deduced in terms of dynamic equations and stick-slip-separate state transition boundaries of the slider. The sliding friction force is modelled by a dynamic model that accounts for relative velocity. The Runge–Kutta algorithm combining the bisection method to capture the transition point between stick and slip motions is adopted to compute the vibration responses of the slider. The numerical results indicate that the belt's oscillatory angular vibration frequency, the vertical preload, and the nonlinear stiffness have great effects on the dynamic characteristics of the slider, and the slider can experience p -periodic ($p = 1, 2, 3, 4$, ect.) and chaotic vibration due to the non-smooth behavior at the contact interface.

Keywords: nonlinear friction-induced slider; oscillating belt; stick-slip; static friction force; non-smooth

1. Introduction

Dry friction plays a dual role in mechanical systems [1]. On one hand, it serves as the fundamental mechanism enabling power transmission in critical components such as automotive disc brakes, clutches, and actuators. On the other hand, dry friction acts as a primary source of nonlinear vibrations and dynamic instabilities. Although phenomena such as stick-slip motions and friction-induced self-excited vibrations have been extensively studied, the inherent complexity of their underlying mechanisms and the persistent challenge of vibration suppression in industrial environments remain incompletely resolved.

Dry-friction-induced vibrations are widespread in engineering [2,3]; thus, they are of great interest for researchers. Dynamic models play a crucial role in the study of friction-induced vibration; they can help to reveal the physical mechanism. Two discrete and two continuous models of the stick-slip systems with rich bifurcational and chaotic



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behavior were investigated by Popp and Stelter [4]. Effects of damping on mode-coupling instability in friction-induced oscillations were studied using a two-degree-of-freedom minimal model [5]. As there lacks a minimal model to describe the basic behavior of the disk brake squeal, which can easily be associated with an automotive disk brake, a new minimal model of a disk brake, which takes into account the vibration of the disk, was introduced [6]. Elmaian et al. [7] proposed a slip-stick phenomenological model to improve understanding of friction-induced noises in automobiles. They considered three physical mechanisms that can produce these noises: slip-stick, sprag-slip, and mode-coupling instabilities. Li et al. [8] unveiled the nonlinear amplification effect caused by dynamic interfacial separation using a separation-recontact model. Wang et al. [9] described a new approach to study the nonlinear dynamical behaviors of a belt-driven two-coupled friction oscillator. A 4-dof nonlinear self-excited friction oscillator was established for brake squeal analysis [10]. The stochastic planar stick-slip motions were investigated using a slider-on-belt model where the coefficient of friction of the contact interface is modelled as a random field [11]. Wang et al. [12] established a longitudinal-vertical coupled dynamics model of a vehicle-track system with a disc brake system, incorporating disc-pad nonlinear friction using the Stribeck model, with parameters derived from block-on-disc tribological tests.

Stick-slip vibration is one of the significant mechanisms proposed to explain the occurrence of friction-induced self-excited vibration. Stick-slip vibration includes two motion regimes; the stick motion may appear when the relative velocity is zero between the contact surfaces, and the slip motion appears when there is relative motion between them [13,14]. Stick-slip motion is characterized by the alternating occurrence of stick and slip phases and has attracted significant interest from both academic and industrial researchers [15–20]. The slip-stick motion occurs when the static friction coefficient is greater than the kinetic coefficient of friction or the coefficient of friction decreases with relative velocity [4,7,21,22]. The discontinuity of the dry-friction-induced system, which is caused by transformation between stick and slip motions, makes this problem difficult to solve theoretically and practically. The friction-induced slip-stick phenomenon was focused on in [23–27]. Won et al. [28] developed a novel stick-slip criteria for a velocity-dependent friction model. Chen and Zhou [29] performed a time–frequency analysis to analyze time–frequency characteristics of friction-induced vibration. Brunetti et al. [30] proposed a Modal Absorption Index (MAI) for unstable mode selection in squeal prediction by complex eigenvalue analysis. The effect of dynamic normal force on the slip-stick vibration characteristics was studied [31]. The parameters of the Stribeck model were identified to construct a dynamic relationship between working conditions and the coefficient of friction to accurately reveal the stick-slip characteristics of a Disc–Block Friction System [32]. Using commercial brake pads in three different sizes, the size effects of brake pads on stick-slip phenomena were investigated [33]. The phenomenon of brake squeal noise generated by automotive disc brake systems was delved into through numerical analysis [34]. The stability, slip-stick bifurcation characteristics, and key factors affecting the slip-stick vibration of the disc brake system were analyzed through models simulation [35]. Convergence analysis of the numerical algorithm of the dry friction constrained system considering stick-slip motion was performed [36].

While the aforementioned studies provide deep insights into friction-induced vibrations under constant sliding speeds, in many practical engineering applications, the belt velocity is rarely perfectly constant; thus, the influence of variable velocities on dry-friction-induced vibration problems is examined. Acceleration of the disc motion on the frictional instability was investigated [37]. This phenomenon precisely indicates that the spinning speed of the disc has a significant impact on the friction-induced dynamics of the system. The friction-induced vibration of a mass–slider with in-plane and transverse springs and dampers in sliding contact with a spinning elastic disc in three different situations of

spinning speed, i.e., constant deceleration, constant acceleration and constant speed, was explored by Liu and Ouyang [38]. Distinct dynamic behaviors were observed under these three rotational-speed conditions. This phenomenon precisely indicates that the spinning speed of the disc has a significant impact on the friction-induced dynamics of the system.

In the existing literature, the dry-friction-induced vibration problems are usually studied by employing a constant velocity, e.g., constant belt velocity in the slider-belt model, the constant spinning speed of the disc, or linear acceleration or deceleration. However, the belt velocity may not be constant or linear variation, and often the belt translation is always in a periodically oscillating state in industry [39]. In certain industrial applications, the belt velocity is not strictly constant but exhibits periodic oscillations due to factors such as eccentric pulleys, torsional vibrations in the drivetrain, or specific servo-driven indexing tasks. And the research on the dry-friction-induced dynamics under oscillating belt translation is still quite limited. In order to examine the influence of oscillating belt translation on the dynamics of the frictional system in a more realistic model, the nonlinear friction-induced slider moving on the belt at oscillating speeds is explored in this paper.

2. The Mechanical Model

To capture and reveal the properties of the slider that is moving on an oscillating belt, a two-dimensional nonlinear friction-induced slider (2D-NLFIS) model is developed in terms of a minimal model frequently utilized to describe basic behaviors of systems like disc brakes, proposed by Hoffmann et al. [40], as shown in Figure 1. The belt surface traveling with a time varying speed v_b , v_b is defined here based on the work by Luo and Gegg [39], which can be written as

$$v_b = v_v \sin(\omega t + \varphi) + v_0 \quad (1)$$

where ω is the oscillatory angular frequency of the belt, $f_e = \omega / (2\pi)$ denotes the oscillatory frequency of the belt, and v_v and v_0 are oscillatory amplitude and constant of the traveling belt surface, respectively. The slider of mass m is compressed by a vertical compression force N to bring the slider against the rigid belt moving at an oscillating speed; a linear spring k_2 and a nonlinear cubic spring k_2^{NL} are used to model the normal contact stiffness between the slider and the oscillating belt. The cubic nonlinear spring is selected because it is a widely accepted, minimalist mathematical representation used to capture the stiffness hardening effect commonly observed at dry friction contact interfaces. In addition, the slider is constrained by two springs k_1 , k_3 and three dampers c_1 , c_2 , c_3 . In this 2D-NLFIS model, the structural springs k_1 and k_3 constrain the slider's macro-vibrations and represent the surrounding elastic fixtures; thus, the structural springs are represented by linear springs. The inclined spring k_3 and damper c_3 construct off-diagonal cross-coupling terms in the stiffness and damper matrices, which are physically essential to simulate mode-coupling instability—a primary cause of friction-induced vibrations. f_N and f_τ are the normal contact force and friction force acting on the slider, respectively. The normal contact force and friction force are both distributed symmetrically and uniformly over the whole contact surface; thus, friction can be modeled by the macro-slip model. The coordinate system x - y used in this paper is defined with its origin at the position before the vertical preload N is applied and before the springs deform.

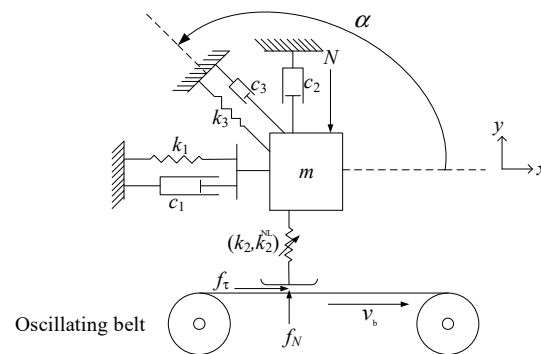


Figure 1. A 2D-NLFIS model moving on an oscillating belt.

3. Criterion for Stick-Slip-Separate Transition and the Dynamic Equations

From a practical point of view, vertical separate motion and horizontal stick-slip motion may happen between the contact surfaces. Accordingly, the equations of motion of the slider could be written as

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_{\tau} \\ f_N - N \end{bmatrix} \quad (2)$$

where the coefficients of the stiffness matrix and the damping matrix can be described as

$$\begin{aligned} k_{11} &= k_1 + k_3 \cos^2 \alpha \\ k_{12} &= k_3 \cos \alpha \sin \alpha \\ k_{21} &= k_3 \sin \alpha \cos \alpha \\ k_{22} &= k_3 \sin^2 \alpha \end{aligned} \quad (3)$$

$$\begin{aligned} c_{11} &= c_1 + c_3 \cos^2 \alpha \\ c_{12} &= c_3 \cos \alpha \sin \alpha \\ c_{21} &= c_3 \sin \alpha \cos \alpha \\ c_{22} &= c_2 + c_3 \sin^2 \alpha \end{aligned} \quad (4)$$

As shown in Figure 1, the normal motion of the slider will lead to the variation of the normal contact force and two stages of separate and contact states between the belt and the slider. The normal contact force f_N is expressed as

$$f_N = \begin{cases} -k_2 y - k_2^{\text{NL}} y^3 & y < 0 \\ 0 & y \geq 0 \end{cases} \quad \begin{matrix} \text{contact state} \\ \text{separate state} \end{matrix} \quad (5)$$

When $y > 0$, the belt and the slider are in the separate state, $f_N = 0$, $f_{\tau} = 0$. When $y \leq 0$, the belt and the slider are in the contact state, $f_N = -k_2 y - k_2^{\text{NL}} y^3$. The current continuous contact model is applicable to the study of stick-slip vibration caused by friction because it belongs to the category of microscopic contact. However, this continuous contact model will fail under macroscopic large separation. The contact interface may undergo stick or slip motion due to the relative tangential motion between the belt and the slider, which is determined by the relative tangential displacement, velocity and acceleration between the belt and the slider. The relative tangential displacement between the belt and the slider can be described as

$$x_r = x_b - x \quad (6)$$

Based on Equation (1), the tangential displacement of the belt can be written as

$$x_b = -\frac{v_v}{\omega} \cos(\omega t + \varphi) + v_0 t + c_0 \quad (7)$$

where c_0 is the integration constant that can be determined by the initial condition. And corresponding relative tangential velocity and acceleration between the slider and the oscillating belt could be separately written as

$$v_r = \dot{x}_r = \dot{x}_b - \dot{x} = v_b - \dot{x} = v_v \sin(\omega t + \varphi) + v_0 - \dot{x} \quad (8)$$

$$a_r = \ddot{x}_r = \ddot{x}_b - \ddot{x} = \dot{v}_b - \ddot{x} = v_v \omega \cos(\omega t + \varphi) - \ddot{x} \quad (9)$$

An exponential dynamic friction model [28], which can consider the effect of the relative velocity, is adopted in this study, and its coefficient function is described as

$$\mu(v_r) = \mu_k + (\mu_s - \mu_k)e^{-\lambda|v_r|} \quad (10)$$

where μ_k and μ_s are the minimum dynamic friction coefficient and the maximum static friction coefficient, respectively, λ is the design parameter to control the decline rate of the curve, and v_r is the relative tangential velocity between the contact surfaces.

According to the Coulomb friction model, the friction force f_τ can be expressed as

$$f_\tau = \begin{cases} \mu(v_r)f_N & v_r > 0 & \text{slip 1 state} \\ -\mu(v_r)f_N & v_r < 0 & \text{slip 2 state} \\ -\mu_s f_N \leq f_\tau \leq \mu_s f_N & v_r = 0 & \text{stick state} \end{cases} \quad (11)$$

As shown in Equation (11), the analytical expression of the sliding friction force is determined, but the static friction force in the stick state is between the negative maximum static friction and the maximum static friction, and its exact value needs to be defined. Next, the stick-slip state transition boundaries will be constructed and the analytical expression of the friction force in the stick state will be deduced for calculating its exact value.

(1) If $v_r \neq 0$, the belt and the slider are in the slip state. In terms of Equation (11), we can obtain

$$f_\tau = \mu(v_r)f_N \text{sgn}(v_r) \quad (12)$$

(2) If $v_r = 0$, the relative tangential displacement between the belt and the slider will remain unchanging, and, obviously, a_r is also equal to zero. By denoting c_r as the tangential displacement difference between the belt and the slider at the end of the previous slip phase, we can get

$$x_r = -\frac{v_v}{\omega} \cos(\omega t + \varphi) + v_0 t + c_0 - x = c_r \quad (13)$$

Due to $v_r = 0$, $x_r = c_r$, $a_r = 0$, based on Equation (2), the static friction force f_τ can be formulated as

$$f_\tau = mv_v \omega \cos(\omega t + \varphi) + c_{11}[v_v \sin(\omega t + \varphi) + v_0] + c_{12}\dot{y} + k_{11}\left[-\frac{v_v}{\omega} \cos(\omega t + \varphi) + v_0 t + c_0 - c_r\right] + k_{12}y \quad (14)$$

If $|f_\tau| \leq \mu_s f_N$ (f_τ is obtained by Equation (14)), the slider will stick relative to the belt. If $|f_\tau| > \mu_s f_N$ (f_τ is obtained by Equation (14)), the friction force will reverse its direction and the slider will still slip relative to the belt.

4. Numerical Investigation

As the state of the slider-belt system switches between separate and contact phases, and between stick and slip phases, this brings out difficulties in the numerical calculations. The Runge–Kutta method is used to solve the non-smooth dynamic equations in this paper. The states between the belt and the slider, including stick-slip-separate phases, are monitored at each time step. In order to improve the calculation accuracy, the dichotomous method is used to accurately capture the stick-slip-separate state transition point of the slider-belt system. The values for the coefficient of friction are determined with reference to the work in Reference [28], which is commonly used for contact between solid surfaces in dry conditions. And the main system parameter values used in the numerical examples are provided with reference to the work in Reference [41]. The values for the coefficient of friction and the main system are all listed in Table 1, and the values of other parameters are given in each sub-section.

Table 1. Values of the main system parameters [28,41].

Argument	Value	Argument	Value
m	0.1 kg	c_1	0.5 N · s/m
c_2	0.5 N · s/m	c_3	0.5 N · s/m
k_1	1×10^4 N/m	k_2	0.5×10^4 N/m
k_3	4×10^4 N/m	v_0	1 m/s
v_v	4 m/s	φ	0
α	$2\pi/3$	μ_s	0.5
μ_k	0.2	λ	3 s/m

4.1. The Sensitivity of the Solutions to the Choice of Time Step

Denote n_p as the number of iteration steps of one oscillation period of the belt; the time step dt of the numerical algorithm can be written as $dt = \frac{2\pi}{\omega \cdot n_p}$. $\omega = 30$ rad/s, $k_2^{NL} = 5000$ N/m³, $N = 160$ N are selected in this study. The transient responses of the slider under different numbers of iteration steps are displayed in Figure 2.

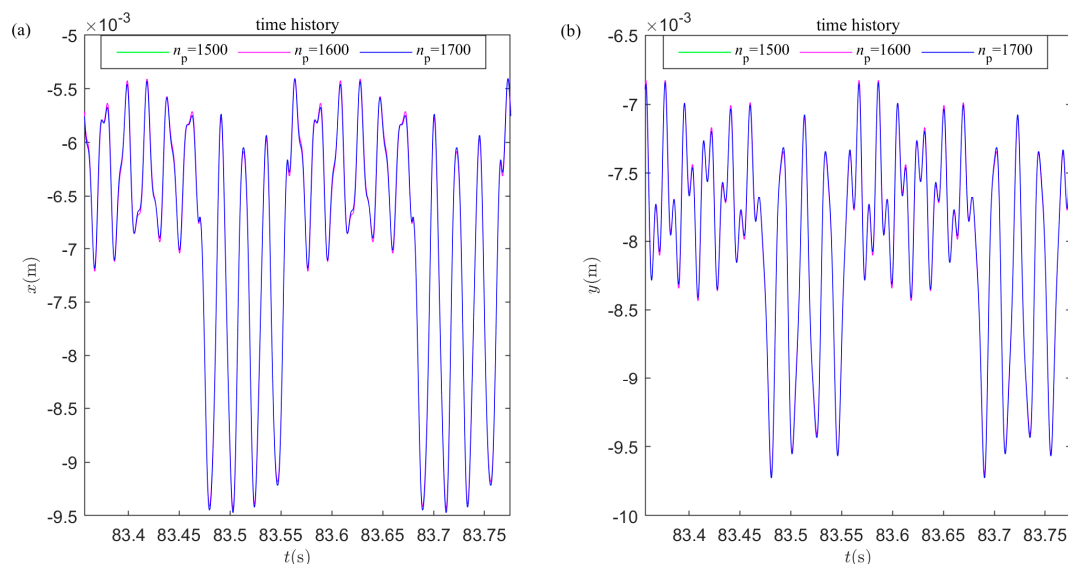


Figure 2. The transient responses of the slider under different numbers of iteration steps: (a) x direction; (b) y direction.

As illustrated in Figure 2, the displacement curves for three different iterative steps almost perfectly overlap. No visible deviations or phase shifts are observed. This highly

consistent behavior visually demonstrates that the numerical solution has already reached an extremely stable state when the number of iteration steps is 1500. Considering both the stability and computational efficiency of the numerical algorithm, the time step for the numerical simulation is defined as $dt = 2\pi/\omega/1500$; in other words, one thousand five hundred data points are sampled for each period of the belt oscillation.

4.2. The Effect of the Oscillatory Angular Frequency ω of the Belt on Vibration Responses

The influence of the oscillatory angular frequency ω of the belt on the vibration response of the slider is studied in this sub-section, and $N = 500$ N, $k_2^{\text{NL}} = 5000$ N/m³ are taken in this study. The bifurcation diagrams and their locally enlarged view of the vibration displacements of the slider in the x and y directions versus the oscillatory angular frequency ω of the belt are displayed in Figure 3. As can be seen from Figure 3, the chaotic motion occurs in certain oscillatory angular frequency bands, which are separated by bands with p -periodic motions ($p = 1, 2, 3, 6$, ect.). From the locally enlarged view of the bifurcation diagrams, it can be seen that within a very narrow range of oscillatory angular frequency variation, the dynamic characteristics of the slider undergo significant changes. This strongly indicates that the dynamic characteristics of the slider are highly sensitive to the variation of oscillatory angular frequency. With the increase of ω , the period-doubling bifurcation occurs when $\omega > 27.28$ rad/s.

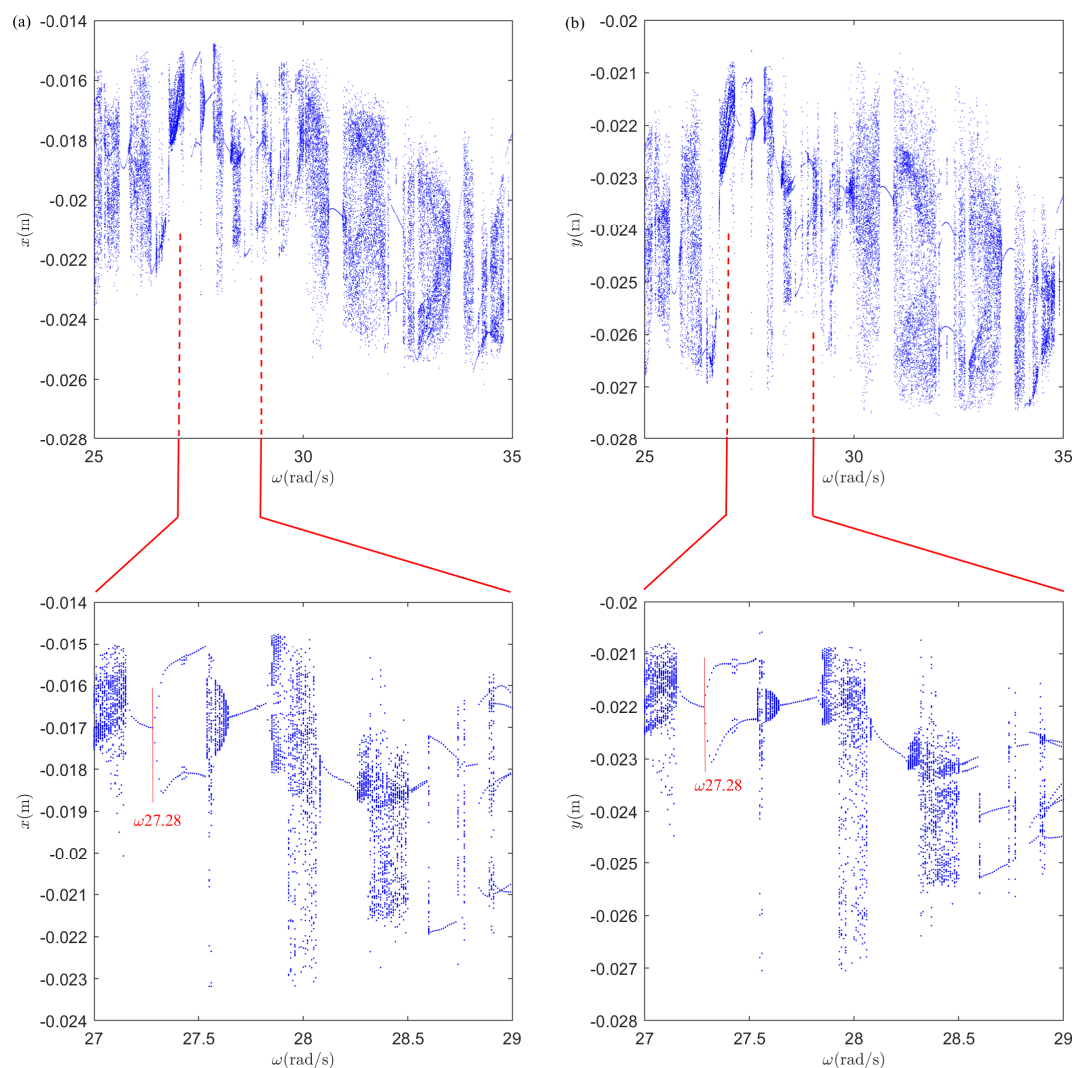


Figure 3. Bifurcation diagrams and their locally enlarged view of the vibration displacements of the slider versus the oscillatory angular frequency ω of the belt: (a) x direction; (b) y direction.

Vibration responses of the slider at $\omega = 27.2 \text{ rad/s}$, 27.5 rad/s , 28 rad/s , 28.85 rad/s , 29 rad/s are provided in Figures 4–8, respectively. When $\omega = 27.2 \text{ rad/s}$, the dominated motion of the slider is the period-one (p1) motion, the time response is regular, there is only one isolated point on the Poincaré map, the fundamental frequency and super-harmonic frequencies ($2f_e$, $3f_e$, ect.) of the oscillatory frequency of the oscillating belt can be observed from the frequency spectrums in Figure 4, and the vibration responses of the slider are still periodic (whose period is equal to the period of the oscillating belt). Higher-order harmonics are primarily caused by the strong non-smooth nonlinearities inherent in the system. Specifically, the discontinuous nature of the alternating stick-slip transitions in the tangential direction, combined with the separation-recontact impacts in the normal direction. The period-two (p2) motion appears at $\omega = 27.5 \text{ rad/s}$; in addition to the fundamental frequency and some super-harmonic frequencies, some fractional frequencies ($f_e/2$, $3f_e/2$, ect.) can be found. The reason for p2 motion is that the periods of the friction force and the normal contact force are both two times of the period of the oscillating belt. Compared with the p1 motion in which only one isolated point appears on the Poincaré map (see Figure 4), two isolated points appear on the Poincaré map (see Figure 5). Figure 6 displays the chaotic motion of the slider; the correlation dimension of the displacement response data is 2.80 in terms of the Grassberger–Procaccia algorithm [42]. From the phase diagram, it can be seen that the trajectory is not closed; instead, it spreads densely within a certain limited area, forming a complex cluster-like region. From the frequency spectrum, continuous spectrums appear. The time response is irregular, and there are scattered Poincaré points. The period-three (p3) motion and period-six (p6) motion that appear at $\omega = 28.85 \text{ rad/s}$, 29 rad/s are illustrated in Figures 7 and 8, respectively; three isolated points and six isolated points appear on the Poincaré map, respectively. In addition to the fundamental frequency and some multiple frequencies, some fractional frequencies can also be observed. The reasons for the appearance of the p3 motion and p6 motion are similar to the p2 motion.

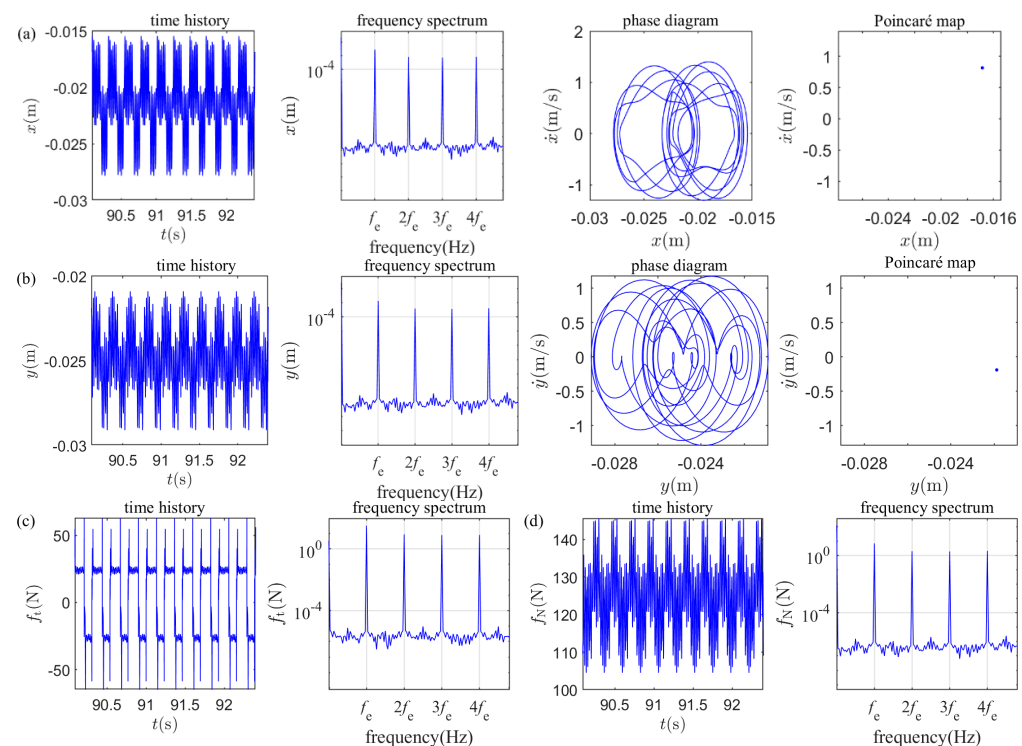


Figure 4. Vibration responses of the slider ($\omega = 27.2 \text{ rad/s}$): (a) displacement in the x direction, (b) displacement in the y direction, (c) friction force, and (d) normal contact force.

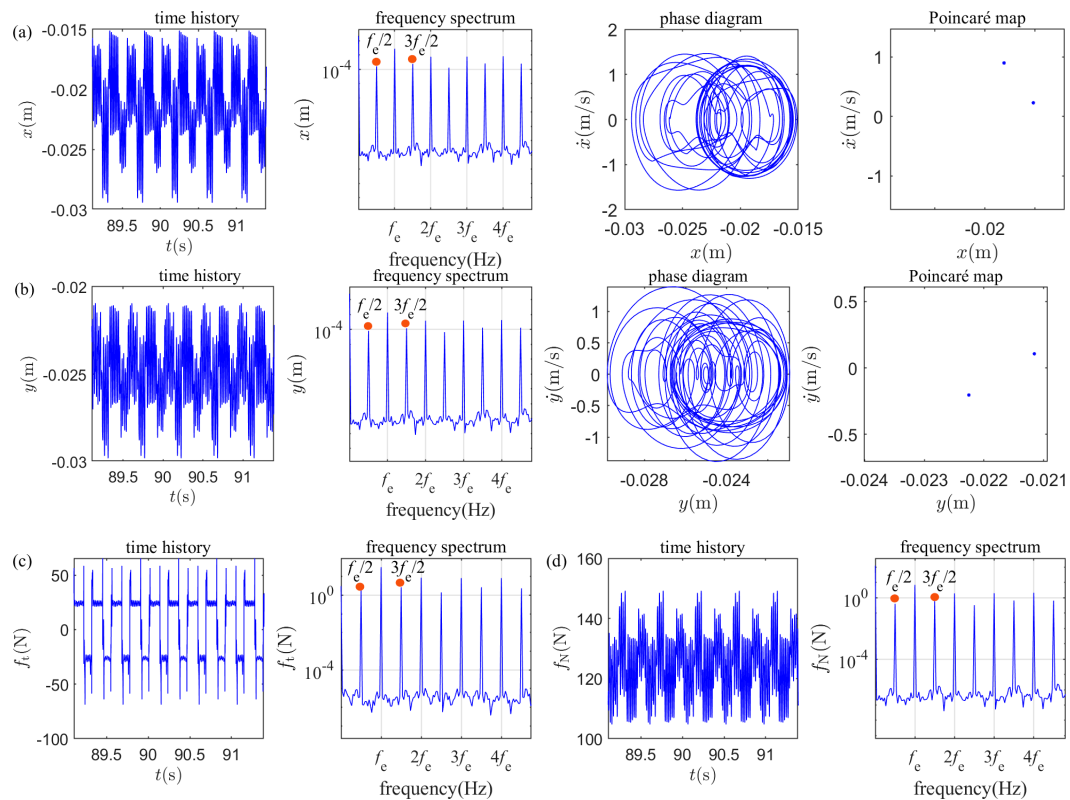


Figure 5. Vibration responses of the slider ($\omega = 27.5$ rad/s): (a) displacement in the x direction, (b) displacement in the y direction, (c) friction force, and (d) normal contact force.

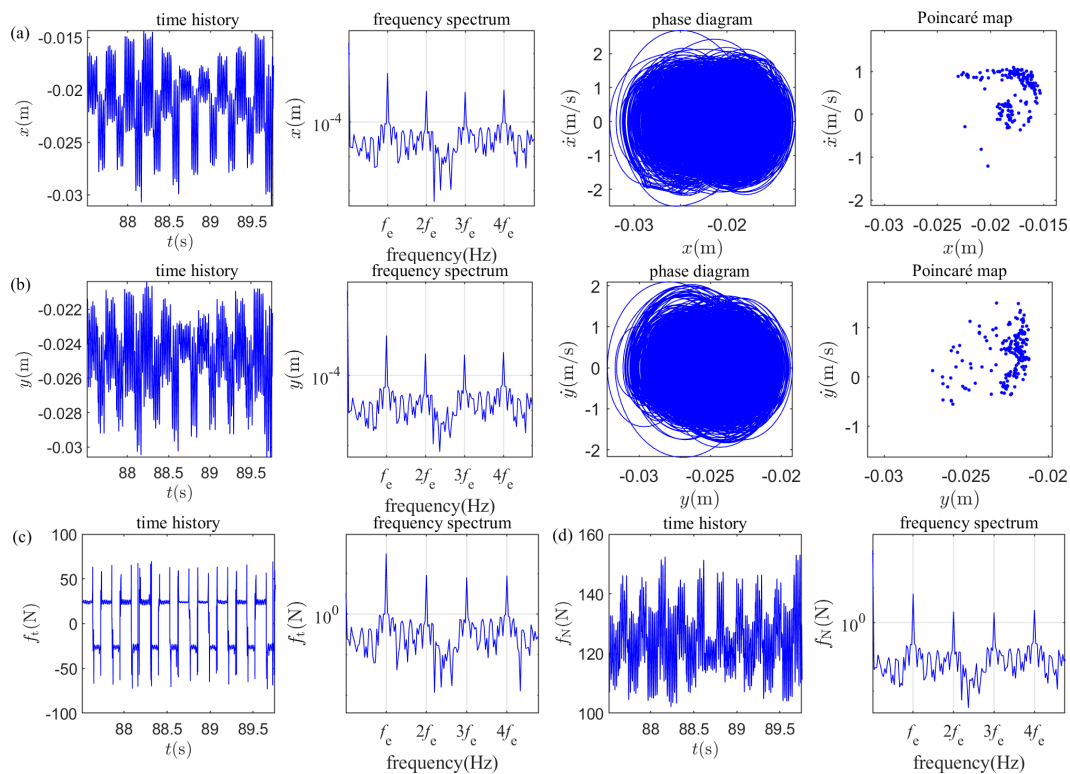


Figure 6. Vibration responses of the slider ($\omega = 28$ rad/s): (a) displacement in the x direction, (b) displacement in the y direction, (c) friction force, and (d) normal contact force.

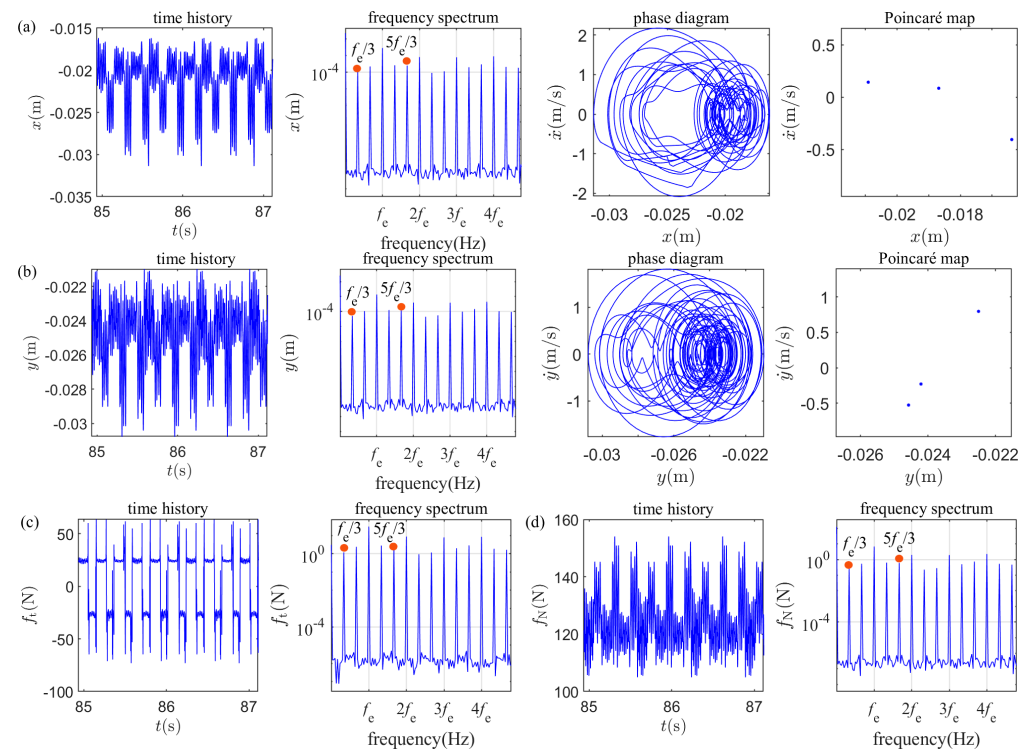


Figure 7. Vibration responses of the slider ($\omega = 28.85$ rad/s): (a) displacement in the x direction, (b) displacement in the y direction, (c) friction force, and (d) normal contact force.

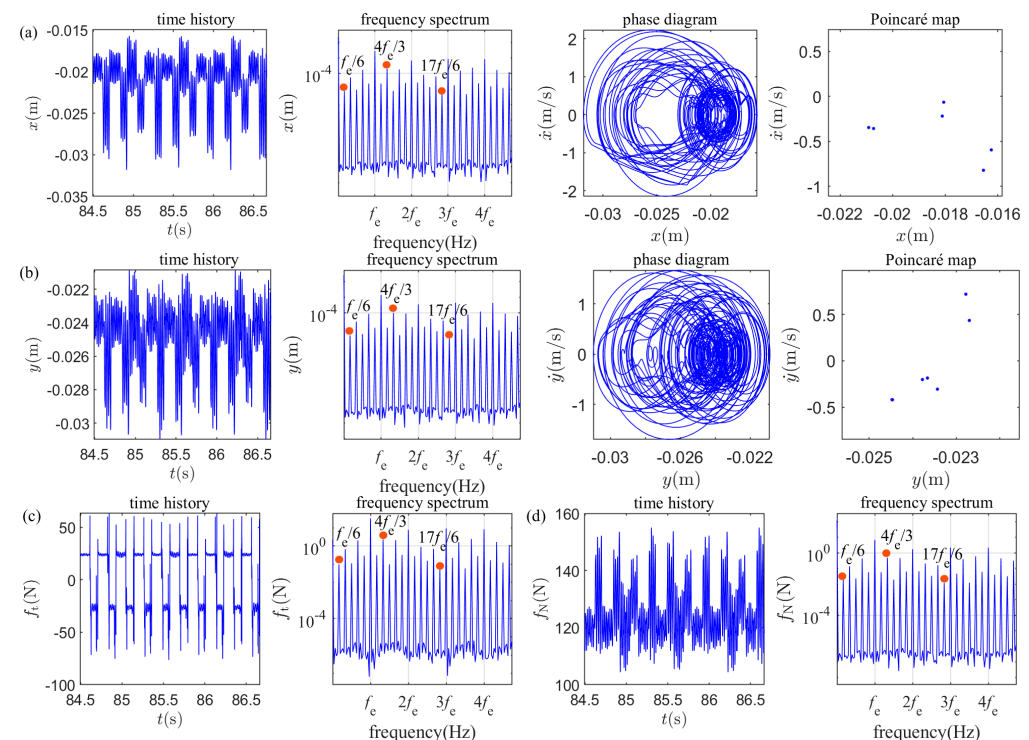


Figure 8. Vibration responses of the slider ($\omega = 29$ rad/s): (a) displacement in the x direction, (b) displacement in the y direction, (c) friction force, and (d) normal contact force.

4.3. The Effect of the Vertical Preload N on Vibration Responses

In this sub-section, the influence of the vertical preload N on the vibration response of the slider is explored, and $\omega = 30$ rad/s, $k_2^{\text{NL}} = 5000$ N/m³ are selected in this study. The bifurcation diagrams of the vibration displacements of the slider in the x and y directions versus the vertical preload N are shown in Figure 9. With the increase of the vertical

preload, the slider alternately gives rise to periodic motion and chaotic motion. It can be seen that within a very brand range (for instance, 150–202, 467–483.5, ect.) of the vertical preload variation, the motion of the slider is always period-one motion. This demonstrates that the influence of the vertical preload on the dynamic characteristics of the slider is not as sensitive as the oscillatory angular frequency. Around $N = 202$ N, a single point suddenly transforms into a point cloud. A significant displacement jump occurred at $N = 431.5$ N; the motion of the slider suddenly changed from period-three to period-one. When N increases to 492.9 N, the point cloud suddenly converges into five discrete lines; the slider then enters the p5 motion window.

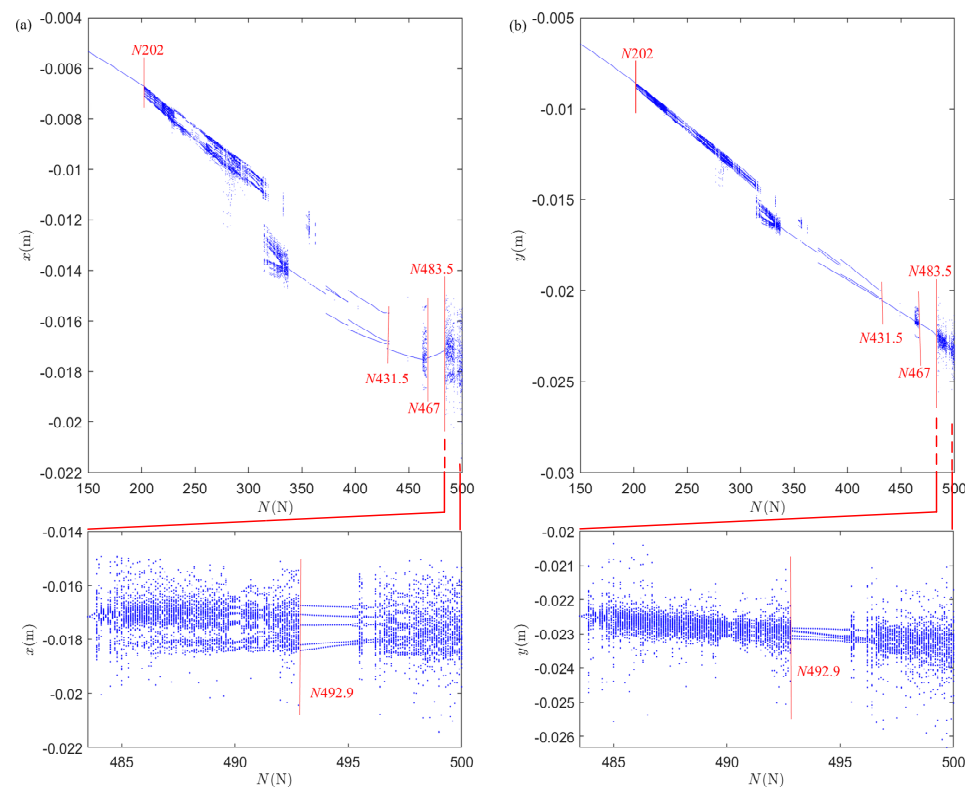


Figure 9. Bifurcation diagrams and its locally enlarged view of the vibration displacements of the slider versus the vertical preload N : (a) x direction; (b) y direction.

Vibration responses of the slider at four typical vertical preloads ($N = 160$ N, 280 N, 410 N, 494 N) are given in Figures 10–13, respectively. Figure 10 shows the period-one (p1) motion that appears at $N = 160$ N, whose time-domain waveform exhibits a highly regular, continuously repeating oscillatory pattern. The phase diagram is a closed curve with only one isolated point on the Poincaré map. Due to the presence of some higher-order harmonics ($2f_e$, $3f_e$, ect.), the phase trajectory here is not a simple ellipse but a complex closed shape with multiple intersections and folds. Figure 11 illustrates the chaotic motion of the slider that occurs at $N = 280$ N; the correlation dimension of the displacement response data is 2.65 in terms of the Grassberger–Procaccia algorithm [42]. The points on the Poincaré map form a two-dimensional pattern, and the motions between the slider and the belt become more complex with the appearance of the continuous spectrums. The trajectory looks like an extremely dense tangle; although it is generally confined within a roughly circular area, the internal trajectory lines are endlessly intertwined with each other. Figures 7 and 8 display the period-three (p3) motion and period-five (p5) motion, which appear at $N = 410$ N and $N = 497$ N, respectively. Three isolated points and six isolated points appear on the Poincaré map, respectively; in addition to the fundamental frequency and some multiple frequencies, some fractional frequencies can also be observed.

According to the well-known Li-Yorke theorem, p3 implies chaos. The reason for p3 motion is that the periods of the friction force and the normal contact force are both three times the period of the oscillating belt.

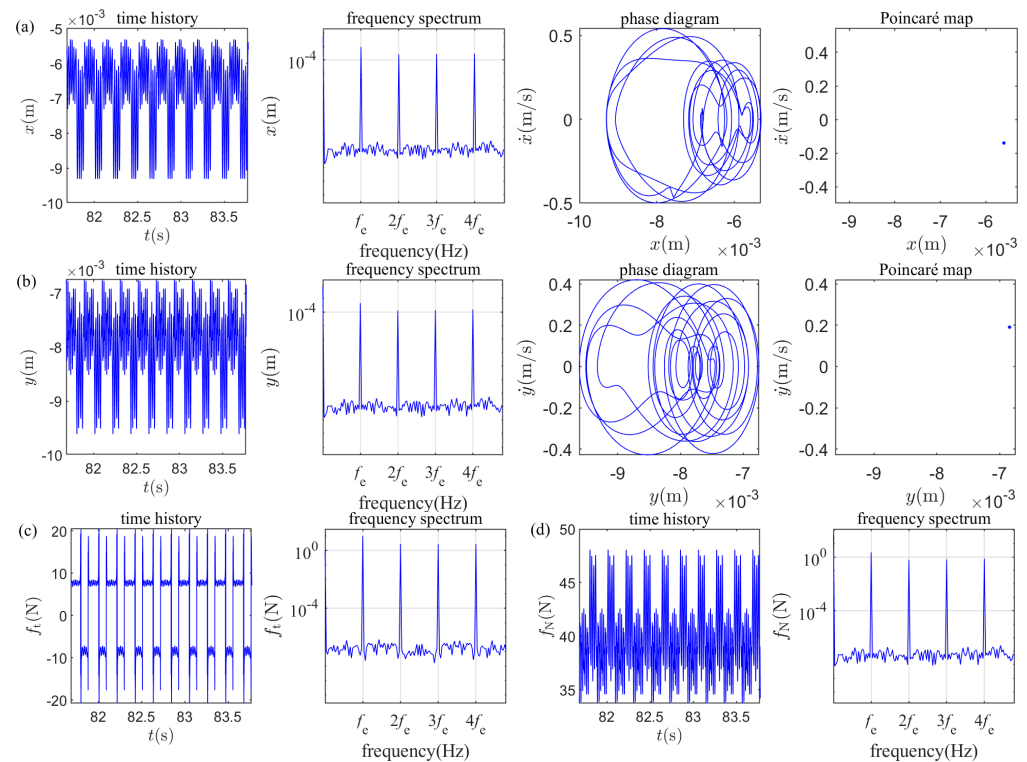


Figure 10. Vibration responses of the slider ($N = 160$ N): (a) displacement in the x direction, (b) displacement in the y direction, (c) friction force, and (d) normal contact force.

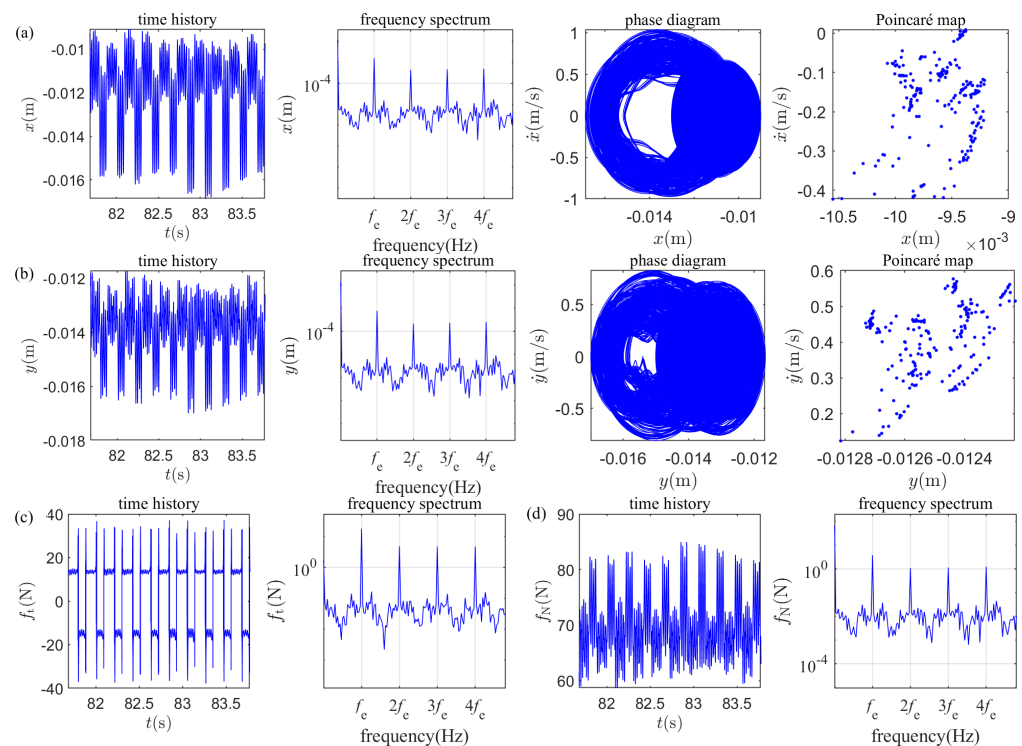


Figure 11. Vibration responses of the slider ($N = 280$ N): (a) displacement in the x direction, (b) displacement in the y direction, (c) friction force, and (d) normal contact force.

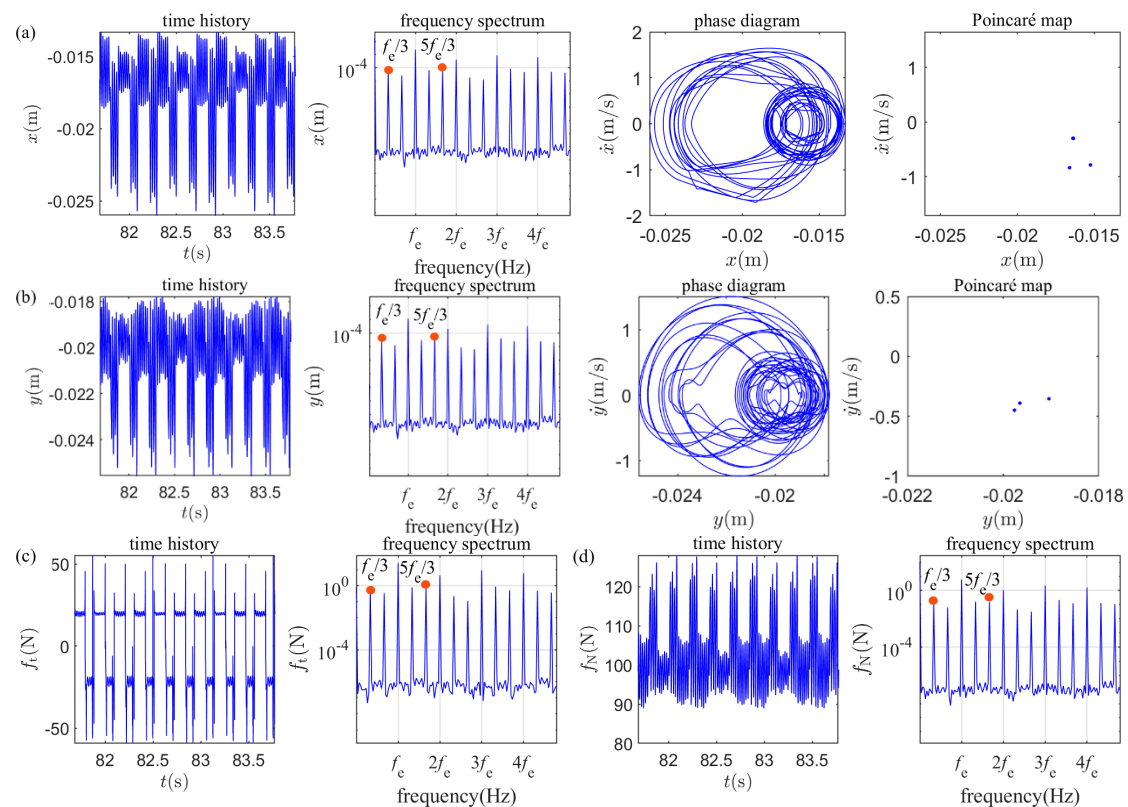


Figure 12. Vibration responses of the slider ($N = 410$ N): (a) displacement in the x direction, (b) displacement in the y direction, (c) friction force, and (d) normal contact force.

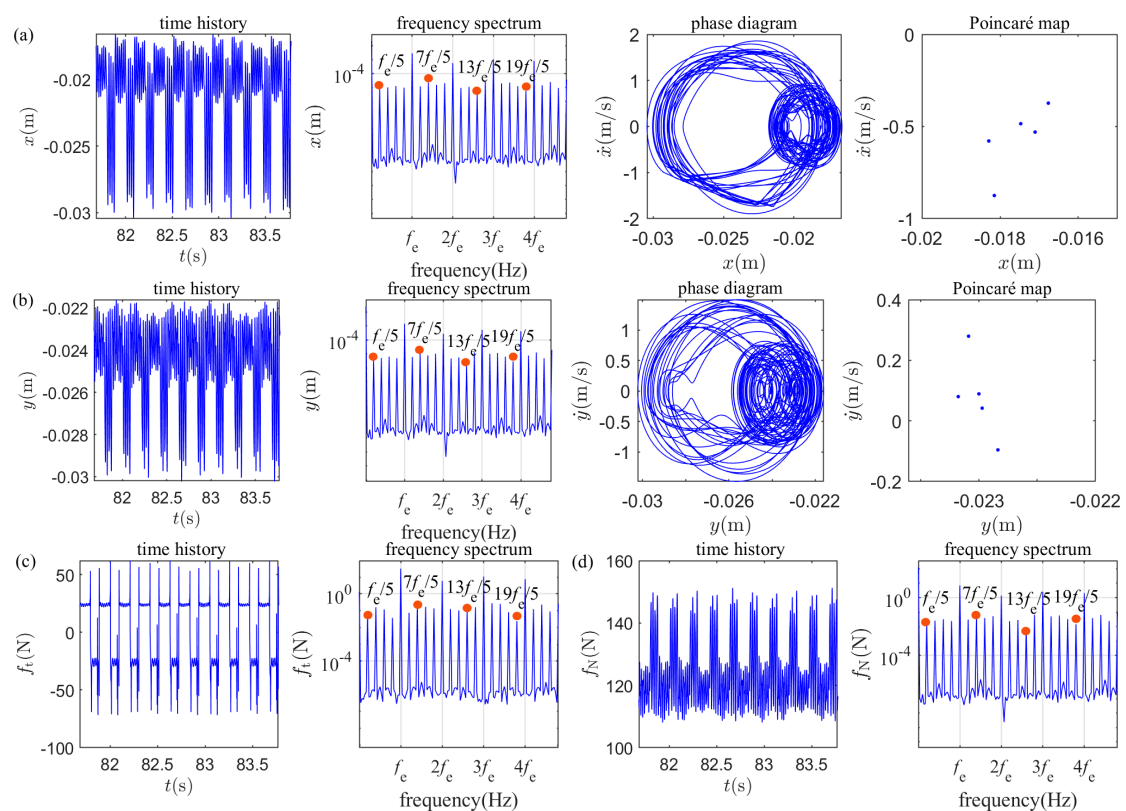


Figure 13. Vibration responses of the slider ($N = 494$ N): (a) displacement in the x direction, (b) displacement in the y direction, (c) friction force, and (d) normal contact force.

4.4. The Effect of the Nonlinear Stiffness k_2^{NL} on Vibration Responses

In this sub-section, the influence of the nonlinear stiffness k_2^{NL} on the vibration response of the slider is examined, and $N = 500$ N, $\omega = 26$ rad/s are chosen in this study. The bifurcation diagrams of the vibration displacements of the slider in the x and y directions versus the nonlinear stiffness k_2^{NL} are exhibited in Figure 14. With variation of the nonlinear stiffness, period-two (p2) motion, period-four (p4) motion, period-eight (p8) motion and chaotic motion of the slider can be found. Throughout the entire range of k_2^{NL} from 0 N/m³ to 350,000 N/m³, the slider does not have a single stable state that can remain unchanged for a long period. The state transitions are extremely frequent. This shows that the nonlinear stiffness has a great influence on the dynamic characteristics of the slider.

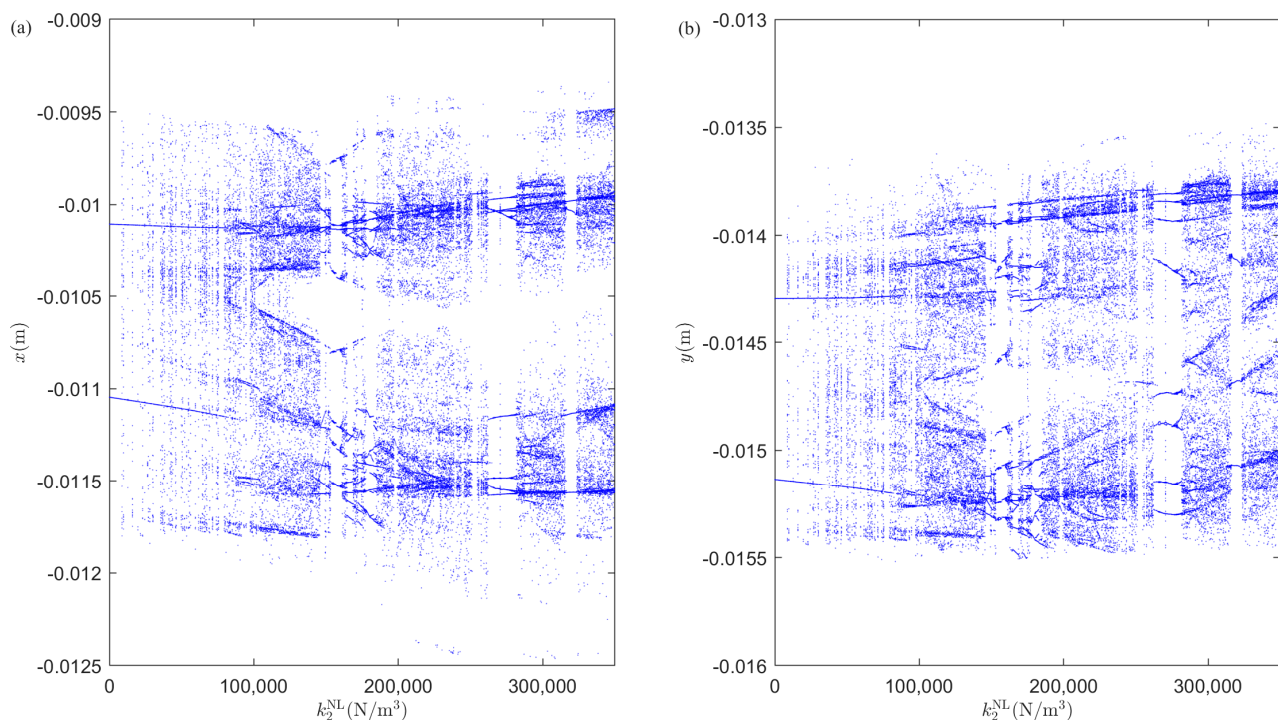


Figure 14. Bifurcation diagrams of the vibration displacements of the slider versus the nonlinear stiffness η : (a) x direction; (b) y direction.

Vibration responses of the slider at different stiffness ratios are provided in Figures 15–18. The period-two (p2) motion that appears at $k_2^{NL} = 0$ N/m³ is exposed at Figure 15 with the appearance of some fractional frequencies ($f_e/2$, $3f_e/2$, etc.) on the frequency spectrums, whose phase diagram is a closed curve with only two isolated points on the Poincarémap. The period-eight (p8) motion and period-four (p4) motion that appear at $k_2^{NL} = 158,000$ N/m³ and $k_2^{NL} = 320,000$ N/m³ are illustrated in Figures 16 and 17, respectively. Eight isolated points and four isolated points appear on the Poincaré map, respectively, in Figures 16 and 17; the fundamental frequency, as well as some multiple and fractional frequencies of the oscillatory frequency of the oscillating belt, all exist. Figure 18 shows the chaotic motion of the slider that occurs at $k_2^{NL} = 340,000$ N/m³ with the appearance of the continuous spectrums; the correlation dimension of the displacement response data is 2.26 in terms of the Grassberger–Procaccia algorithm [42].

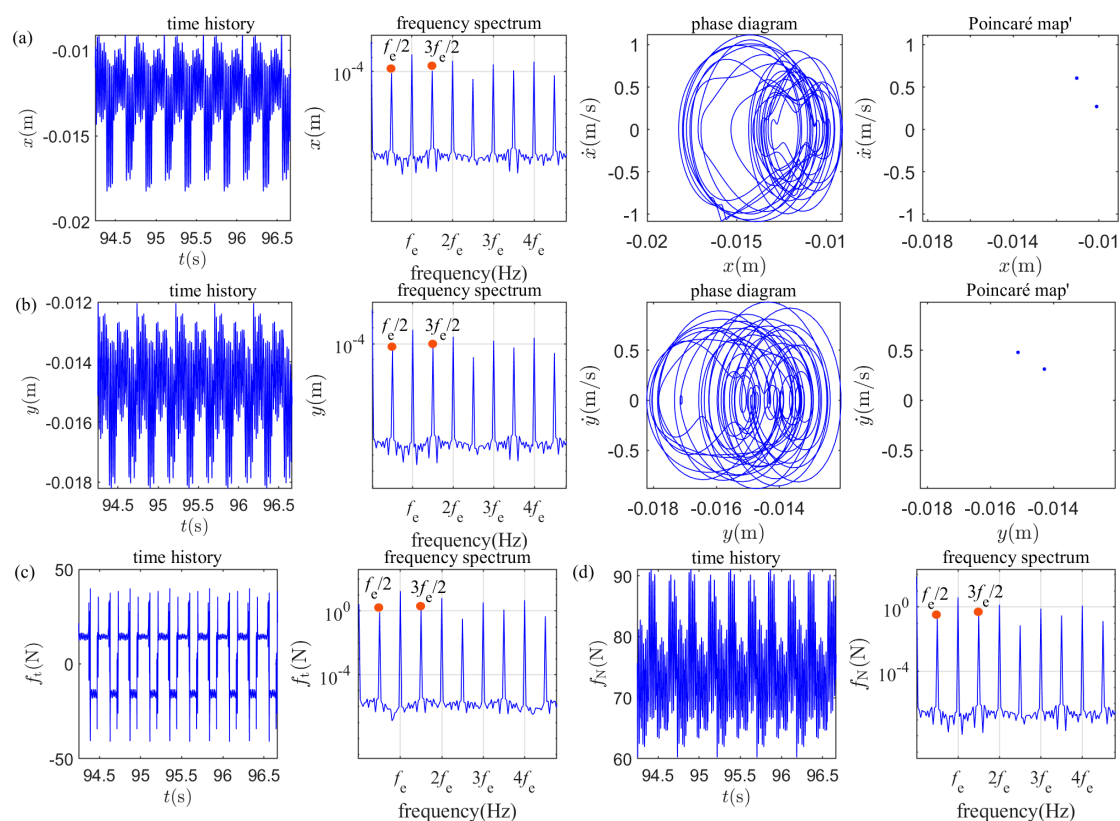


Figure 15. Vibration responses of the slider ($k_2^{\text{NL}} = 0 \text{ N/m}^3$): (a) displacement in the x direction, (b) displacement in the y direction, (c) friction force, and (d) normal contact force.

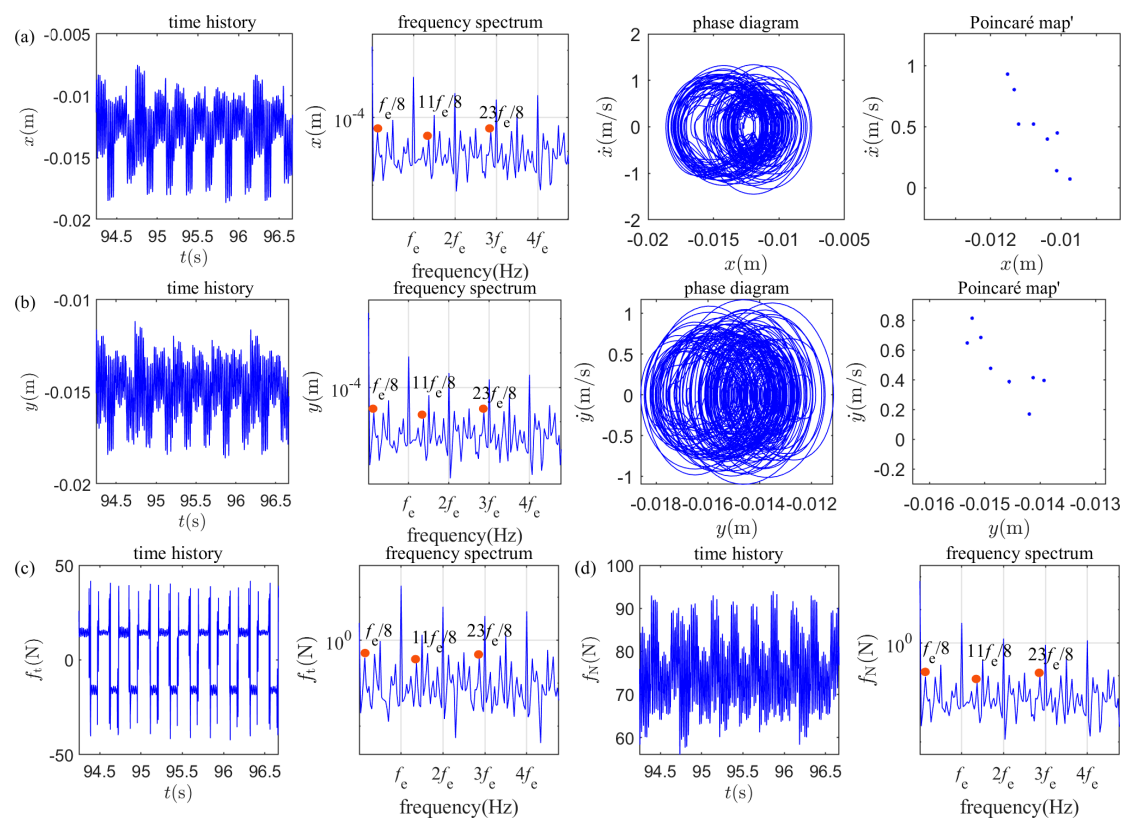


Figure 16. Vibration responses of the slider ($k_2^{\text{NL}} = 158,000 \text{ N/m}^3$): (a) displacement in the x direction, (b) displacement in the y direction, (c) friction force, and (d) normal contact force.

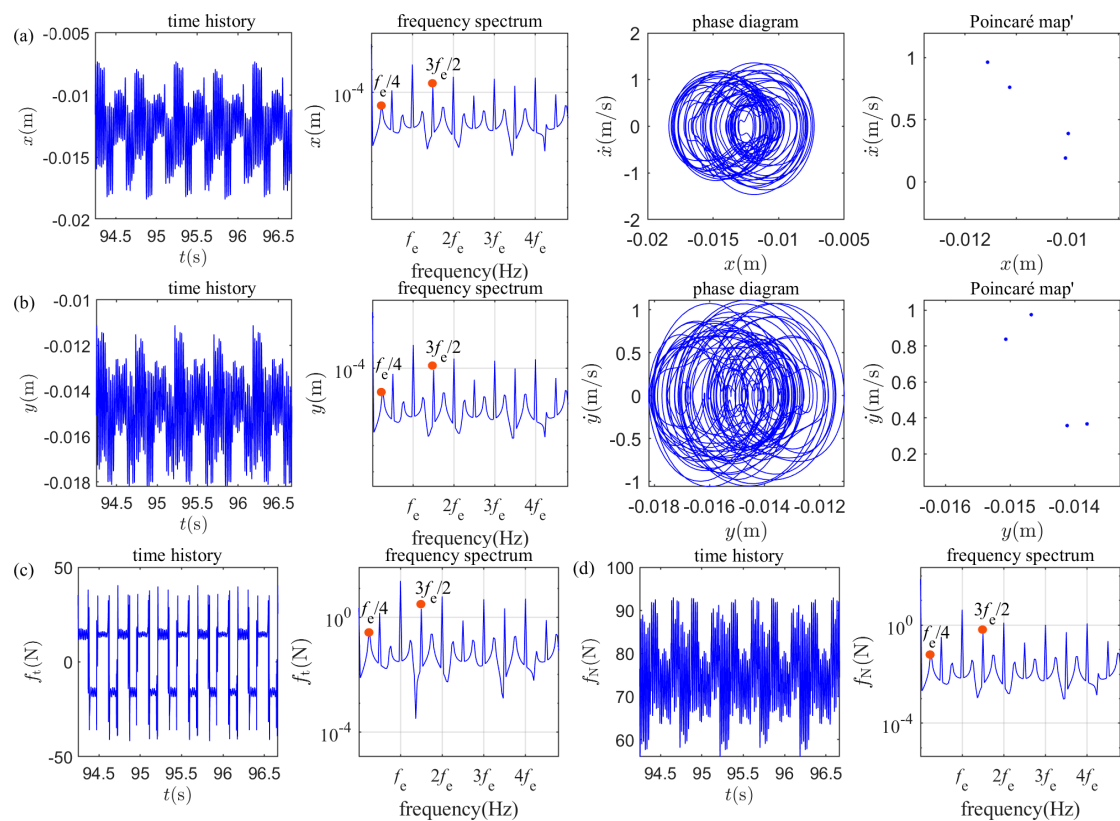


Figure 17. Vibration responses of the slider ($k_2^{\text{NL}} = 320,000 \text{ N/m}^3$): (a) displacement in the x direction, (b) displacement in the y direction, (c) friction force, and (d) normal contact force.

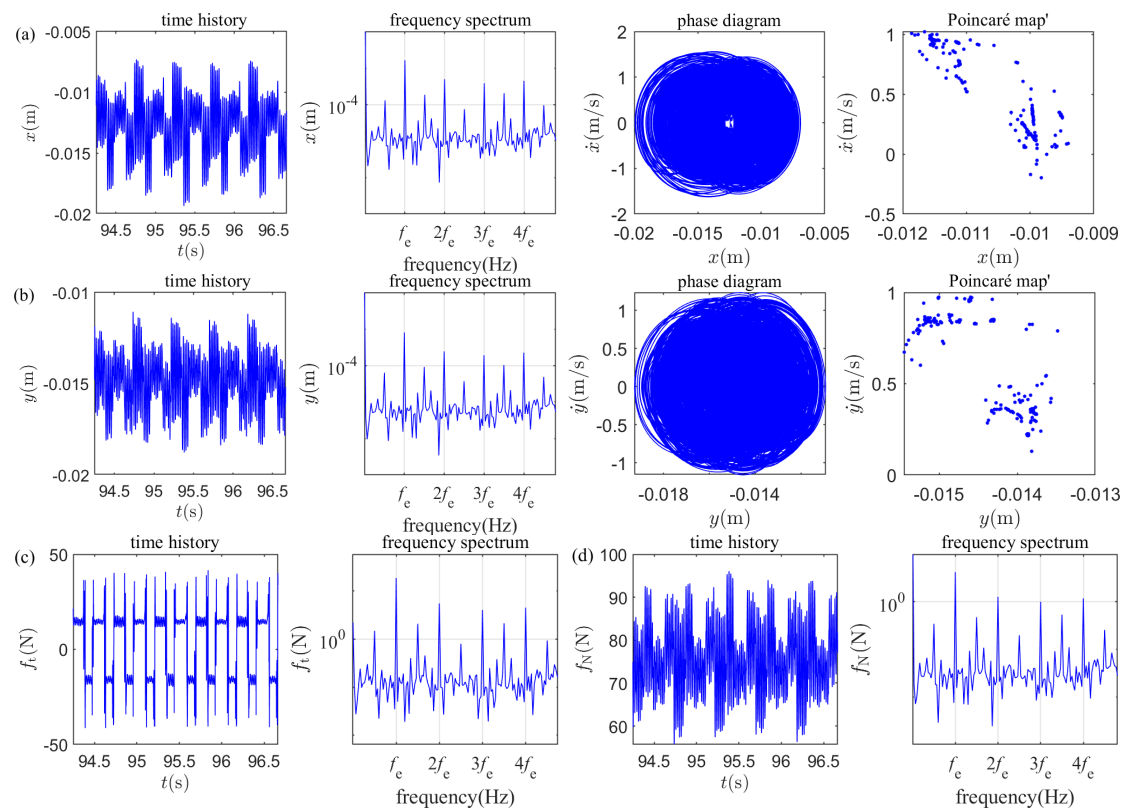


Figure 18. Vibration responses of the slider ($k_2^{\text{NL}} = 340,000 \text{ N/m}^3$): (a) displacement in the x direction, (b) displacement in the y direction, (c) friction force, and (d) normal contact force.

5. Conclusions

In this paper, nonlinear vibration characteristics of the slider moving on an oscillating belt are investigated. The main conclusions are as follows:

(1) Within a very narrow range of oscillatory angular frequency variation, the dynamic characteristics of the slider undergo significant changes, indicating that the dynamic characteristics of the slider are highly sensitive to the variation of oscillatory angular frequency. When designing a slider-on-belt system, the oscillation frequency of the belt should be strictly controlled to prevent the system frequency from falling into an extremely narrow sensitive band that could trigger period-doubling bifurcation.

(2) Given that the slider can maintain a stable p1 motion within a relatively wide range of the preload, engineers can make the slider safely operate within these stable “windows” by actively adjusting or calibrating the vertical preload.

(3) The nonlinear stiffness k_2^{NL} has a great influence on the dynamic characteristics of the slider. It is necessary to carefully select the contact materials and buffering structures to minimize nonlinear stiffness hardening and avoid structural parameters from entering a dangerous range that compromises long-term stability.

(4) While the current study provides theoretical insights into the non-smooth dynamics of the slider, the lack of experimental validation for the current work remains a limitation. Future work will focus on an experimental study.

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Abbreviations

The following abbreviations are used in this manuscript:

2D-NLFIS Two-dimensional nonlinear friction-induced slider

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