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Interval-Valued q -Spherical Fuzzy Rough Sets and TOPSIS for Multi-Criteria Decision-Making: Application to Sustainable Smart City Development

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Abstract

This study develops an interval-valued q -spherical fuzzy rough set TOPSIS framework (IV q -SFRS-TOPSIS) for multi-criteria group decision-making when expert judgments contain interval uncertainty, neutrality, and granular indiscernibility. The revised framework clarifies the relationship between interval-valued q -spherical and interval-valued T-spherical fuzzy models, defines admissible approximation operators over compatible domains, and introduces a radial projection step that guarantees closure under the IV q -SFN constraint whenever component-wise extrema would otherwise violate it. The proposed framework provides a mathematically balanced representation of interval-valued q -spherical fuzzy information, reflecting the concept of symmetry and supporting reliable group decision-making under uncertainty. The TOPSIS procedure is then formulated through expert aggregation, benefit–cost normalization, entropy-based criteria weighting, ideal-solution distance calculation, and closeness-coefficient ranking. The method is illustrated through a sustainable smart city development case using four AI-based alternatives and six criteria. Rather than claiming unconditional superiority, the revised comparative and sensitivity analyses examine how the ranking changes under alternative fuzzy decision models, different q values, perturbations to criteria weights, and perturbations to the decision matrix. The results indicate that the proposed framework provides an interpretable rough-boundary representation and a reproducible ranking mechanism for complex MCDM problems under interval-valued q -spherical uncertainty.

Keywords: interval-valued q -spherical fuzzy rough sets; spherical fuzzy sets; q -rung orthopair fuzzy sets; rough set theory; TOPSIS; multi-criteria decision-making



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1. Introduction

Multi-criteria decision-making (MCDM) has been developed as a core part of operations research and management science due to its ability to systematically evaluate and choose the best alternative among a limited number of alternatives in the presence of multiple conflicting criteria [1]. In reality, the information in decision-making problems is often (though not always) incomplete or uncertain due to limitations of human judgment or the vagueness of the decision environment. Traditional MCDM methods largely rely on crisp input data and might not be able to treat these kinds of uncertain and fuzzified data appropriately. To address the aforementioned issues, Zadeh [2] proposed the concept of fuzzy set theory as an effective mathematical instrument for the portrayal of vagueness and

uncertainty. Applications of fuzzy sets include (but are not limited to) decision-making, pattern recognition, and control systems. However, classical fuzzy sets only take into account the degree of membership of an element, and this simplifies their ability to represent a full range of uncertainty in highly complex decision problems. To overcome this restriction, Atanassov [3] defined intuitionistic fuzzy sets (IFSs) in which the membership and non-membership degrees are considered simultaneously under the constraint that their sum is equal to or less than one. This generalization makes it possible for decision-makers to state positive as well as negative considerations at the same time, which is a more realistic depiction of human thinking. Notwithstanding their merits, IFSs are still unable to treat neutral or undetermined opinions, which are typical in collective decision-making. To address this limitation, Cuong [4] introduced picture fuzzy sets (PFSs), which add a neutral membership degree along with the positive and negative membership degrees. This three-dimensional representation enables experts to express acceptance, neutrality, rejection, and refusal, making PFSs particularly suitable for voting and consensus-based decision problems.

Meanwhile, rough set theory was introduced by Pawlak [5] to tackle the problems of granularity and incompletely specified information from another angle while fuzzy set theory was evolving. Rough sets derive from the concept of an equivalence relation and approximation space and provide lower and upper approximations for rough sets of fuzzy concepts. This framework has been widely applied in data mining, knowledge discovery, and decision-making. By the merging of fuzzy sets with rough sets, it was possible to introduce the notion of fuzzy rough sets [6], which combined the advantages of the two theories in the treatment of vagueness and granularity at the same time. In parallel, Molodtsov [7] introduced soft set theory as a general mathematical tool for dealing with uncertainty, which has been successfully combined with fuzzy and rough frameworks. More recently, Ullah et al. [8] developed similarity measures for T-spherical fuzzy sets with applications in pattern recognition, while Wang et al. [9] proposed q-rung orthopair fuzzy soft rough sets for decision-making, demonstrating the ongoing evolution of hybrid uncertainty models.

Recently, Yager [10] defined the Pythagorean fuzzy sets (as PyFSs), a generalization of IFSs under the condition that the sum of membership and non-membership degrees is relaxed to the sum of their squares. This extension allows for a larger domain of admissible values, providing greater flexibility in uncertain environments. Subsequently, Yager [11] proposed q-rung orthopair fuzzy sets (q-ROFSs), which further generalize PyFSs by introducing a parameter $q \geq 1$ such that $\phi^q + \kappa^q \leq 1$. With this parameterization, the decision-maker may relax or tighten the constraint based on the level of uncertainty in the problem. Mahmood et al. [12] developed the SFS, a three-dimensional representation of the PFS, by applying the quadratic relation $\phi^2 + \eta^2 + \kappa^2 \leq 1$ in space. This geometrical interpretation gives rise to a subtler tool to represent uncertainties in MCDM problems. Recently, Mahmood et al. [13] introduced interval-valued q-spherical fuzzy sets (IVq-SFs), which merge the generalization of interval-valued assessments with the modifiable q-rung spherical constraint $[\phi^+]^q + [\eta^+]^q + [\kappa^+]^q \leq 1$. This high level of representation encompasses the degree of uncertainty as well as the number of experts. Based on the simple and applicable concept of similarity, the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) [14] is recognized as one of the most popular MCDM methods since its invention by Hwang and Yoon, and has many merits, such as rationality and computational efficiency. The closeness-based approach sorts the alternatives in descending order of the closeness to the positive ideal solution (PIS) and the farness to the negative ideal solution (NIS). The performance of TOPSIS has also been well demonstrated in other fuzzy frameworks, such as intuitionistic fuzzy TOPSIS [15] and even picture fuzzy TOP-

SIS [16], q -rung orthopair fuzzy TOPSIS [17], and complex spherical fuzzy TOPSIS [18]. These extensions have also been employed in different applications, ranging from supplier selection to medical diagnosis and risk evaluation.

Recent advances have further extended fuzzy rough TOPSIS methodologies. Ali et al. [19] applied q -rung orthopair picture fuzzy soft TOPSIS to green supply chain management, while Huang et al. [20] developed spherical fuzzy rough aggregation operators with TOPSIS for industrial control system security. Azim et al. [21] proposed a q -spherical fuzzy rough TOPSIS analysis for assessing indoor positioning systems. Shi et al. [22] developed fuzzy β -covering rough three-way decision models under q -rung orthopair fuzzy environments with prospect theory. Kahraman et al. [23] introduced q -spherical fuzzy sets and their applications in multi-attribute decision-making. Zheng et al. [24] introduced multi-granularity variable-precision neutrosophic rough sets for group decision-making with three strategies (optimistic, pessimistic, and compromise). Mei and Xu [25] further extended multi-granulation variable-precision fuzzy rough decision models.

Despite the significant progress in fuzzy TOPSIS methodologies, several limitations persist in existing approaches. First, many models lack a dedicated neutral/hesitation degree or rely on point-valued rather than interval-valued assessments, which fail to capture the full range of expert uncertainty. Second, few models integrate rough set approximations to handle granular data structures and equivalence relations. Third, while some recent studies have explored q -spherical fuzzy rough TOPSIS [21] or q -rung orthopair picture fuzzy soft TOPSIS [19], none combine all of the following features within a unified framework: (i) interval-valued assessments, (ii) three-dimensional membership with neutral degrees, (iii) adjustable q -rung constraints, (iv) rough set granularity, and (v) a projection operator that ensures closure under interval-valued constraints. Fourth, the application of such comprehensive models to sustainable smart city development remains largely unexplored, despite the inherent multi-stakeholder uncertainty in this domain.

It is important to distinguish the proposed IV q -SFRS framework from existing IVT-spherical fuzzy rough set models. While IVT-SFS uses multiple independent parameters (T , I , F) to represent membership, neutrality, and non-membership degrees, our IV q -SFS model employs a single unified parameter q that simultaneously controls the admissible region for all three degrees. This is not a mere rebranding but a mathematical specialization that: (1) ensures mathematical consistency across all degrees, (2) reduces parametric complexity and simplifies the theoretical framework, (3) provides a clear geometric interpretation of the admissible region as a q -sphere, and (4) allows systematic control of uncertainty through a single parameter. Furthermore, our model introduces a projection operator Π_q that guarantees closure under the interval-valued constraint whenever component-wise extrema would otherwise violate it—a feature absent in existing IVT-spherical fuzzy rough set models.

To fill these gaps, this paper introduces a new hybrid model, termed interval-valued q -spherical fuzzy rough sets (IV q -SFRSs), which integrates the strengths of interval-valued q -spherical fuzzy sets with rough set theory to effectively address uncertainty and boundary approximation in multi-criteria decision-making. The intuitiveness of the proposed fuzzy soft rough framework is illustrated in Figure 1, which also presents the overall structure of the IV q -SFRS-TOPSIS model. The proposed framework is built upon a mathematically balanced representation of interval-valued q -spherical fuzzy information, where membership, neutrality, and non-membership degrees are jointly governed by the q -spherical admissibility constraint. This representation reflects the concept of symmetry while providing a consistent and reliable foundation for modeling uncertain information. Based on this framework, the main contributions of this paper are summarized as follows:

1. **Theoretical Novelty:** We introduce and investigate a complete definition of IVq-SFRSs, along with lower and upper approximations, algebraic operations (addition, multiplication, scalar multiplication, and power), and duality properties. A novel projection operator Π_q is defined to ensure closure under the IVq-SF constraint whenever component-wise extrema would otherwise violate it.
2. **Mathematical Properties:** We prove the basic properties of these concepts, including the idempotency of approximations, the involution of complement, commutativity, associativity, and closure under all defined operations. The proofs have been strengthened with detailed derivations and component-wise verification of inequalities.
3. **TOPSIS Integration:** We develop a comprehensive multi-criteria decision-making algorithm incorporating expert aggregation, normalization, score functions, Euclidean distance measures, PIS/NIS determination, and entropy-based objective weight determination with justification against alternative methods (CRITIC, SWARA, AHP).
4. **Bridging Theory and Application:** We establish a formal correspondence between the abstract approximation space and the decision framework, defining the indiscernibility relation $\mathcal{T}$ and target concept λ explicitly. The rough-enhanced evaluation $\tilde{r}_{ij}$ integrates both lower and upper approximations across all experts.
5. **Novel Application Domain:** Unlike existing work that focuses on supplier selection or personnel evaluation [19–21], we apply the proposed framework to sustainable smart city development, selecting the optimal AI-driven strategy from four alternatives (traffic management, smart grid, waste sorting, and integrated platform) based on six criteria spanning economic, social, and environmental pillars. The case study includes detailed expert profiles and a transparent data generation protocol.
6. **Validation:** We conduct extensive comparative analysis against IFS-TOPSIS [15], q-ROF-TOPSIS [17], CSFS-TOPSIS [18], and PF-TOPSIS [16]. While the performance differences are not statistically significant at the conventional $\alpha = 0.05$ level, the model offers unique uncertainty quantification capabilities through the boundary region measure $\Delta(x_i)$. Effect sizes (Cohen’s d) are reported for additional insight.
7. **Complexity and Scalability:** A comprehensive analysis of computational complexity is provided, including asymptotic complexity of each algorithmic step and practical strategies for large-scale applications (parallel processing, dimensionality reduction, and approximate rough sets).
8. **Robustness:** Sensitivity analysis confirms the robustness of the ranking of alternatives for various values of the parameter q , criteria weights, and equivalence relations.

The remainder of this paper is organized as follows: Section 2 recalls essential preliminaries, including fuzzy sets, intuitionistic fuzzy sets, spherical fuzzy sets, interval-valued q-spherical fuzzy sets, and rough sets. Section 3 presents the IVq-SFRS framework, including approximation operators, algebraic operations, main properties, and a detailed discussion of the projection operator with its idempotence, monotonicity, and closure properties. Section 4 presents the TOPSIS method based on IVq-SFRS, including problem formulation, linguistic terms conversion, approximation space, indiscernibility relation, target concept definition, rough approximations integration, score and accuracy functions, normalized Euclidean distance, entropy-based weights with justification against alternative methods, the complete algorithm, and an illustrative example for sustainable smart city development with expert profiles and data generation protocols. Section 5 presents a comprehensive comparative analysis and discussion, including a comparative framework, qualitative and quantitative comparative analysis, corrected statistical analysis with effect sizes (Cohen’s d), and recommendations for model selection. Section 6 discusses computational complexity, scalability, limitations, and future work. Finally, Section 7 concludes the paper.

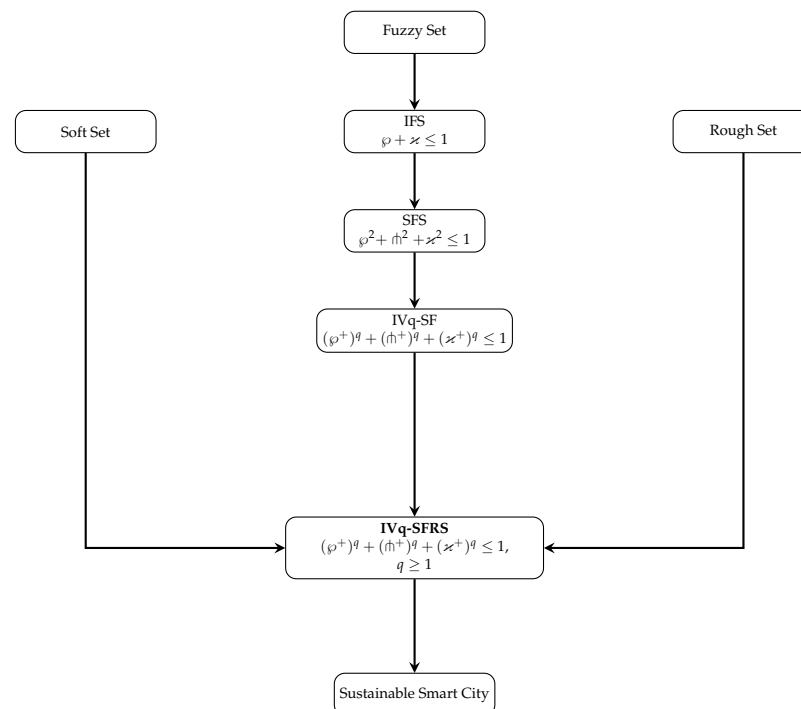


Figure 1. Conceptual development of the IVq-SFRS framework.

To facilitate the reading, some of the major symbols and their meanings used in this paper are summarized in Table 1.

Table 1. List of notations.

Notation	Meaning
$\mathbb{X}, \mathfrak{U}$	Universe of discourse (set of alternatives)
$\mathbb{P}$	Set of parameters/criteria
$\mathcal{A}_i$	i -th alternative
$\mathcal{C}_j$	j -th criterion
$\mathcal{T}$	Indiscernibility relation (equivalence relation on criteria)
$[\mathcal{C}_j]_{\mathcal{T}}$	Equivalence class of criterion $\mathcal{C}_j$ under relation $\mathcal{T}$
λ	Target concept (ideal sustainability preference structure)
q	Rung parameter
$\wp$	Membership degree
$\mathfrak{h}$	Neutral/hesitation degree
$\varkappa$	Non-membership degree
$[\wp^-, \wp^+]$	Interval-valued membership degree
$[\mathfrak{h}^-, \mathfrak{h}^+]$	Interval-valued neutrality degree
$[\varkappa^-, \varkappa^+]$	Interval-valued non-membership degree
χ	Refusal degree
$\mathcal{Z}^{(\ell)}$	IVq-SF relation for expert ℓ on $\mathcal{A} \times \mathcal{C}$
$\tilde{\theta}_{ij}^{(\ell)}$	IVq-SF number (evaluation of $\mathcal{A}_i$ under $\mathcal{C}_j$ by expert ℓ)
$\tilde{D}^{(\ell)}$	Decision matrix of the ℓ -th expert
$\underline{\mathcal{Z}}(\lambda)$	Lower approximation of λ (pessimistic estimate)
$\overline{\mathcal{Z}}(\lambda)$	Upper approximation of λ (optimistic estimate)
$\tilde{r}_{ij}$	Rough-enhanced evaluation (aggregated IVq-SF number)
$\tilde{r}_{ij}^{\text{norm}}$	Normalized evaluation
$\oplus$	Algebraic sum operation
$\otimes$	Algebraic product operation

Table 1. Cont.

Notation	Meaning
$\cup$	Union operation
$\cap$	Intersection operation
$\sim$	Complement operation
$\mathcal{S}(\tilde{\theta})$	Score function
$\mathcal{H}(\tilde{\theta})$	Accuracy function
$d_q(\tilde{\theta}_1, \tilde{\theta}_2)$	Normalized Euclidean distance
w_j	Weight of criterion j
$\tilde{r}_j^+$	Positive Ideal Solution (PIS)
$\tilde{r}_j^-$	Negative Ideal Solution (NIS)
D_i^+	Weighted distance of alternative i to PIS
D_i^-	Weighted distance of alternative i to NIS
RC_i	Relative closeness coefficient
k	Number of experts
ℓ	Expert index
ρ_ℓ	Weight of expert ℓ
p_{ij}	Normalized proximity for entropy method
E_j	Entropy of criterion j
FS	Fuzzy Set
IFS	Intuitionistic Fuzzy Set
PFS	Picture Fuzzy Set
SFS	Spherical Fuzzy Set
q-ROFS	q-Rung Orthopair Fuzzy Set
PyFS	Pythagorean Fuzzy Set
IVq-SF	Interval-Valued q-Spherical Fuzzy Set
IVq-SFRS	Interval-Valued q-Spherical Fuzzy Rough Set
TOPSIS	Technique for Order Preference by Similarity to Ideal Solution
MCDM	Multi-Criteria Decision-Making
MCGDM	Multi-Criteria Group Decision-Making
PIS	Positive Ideal Solution
NIS	Negative Ideal Solution
SDGs	Sustainable Development Goals
AI	Artificial Intelligence

2. Preliminaries

This section recalls the essential concepts for the proposed model, focusing on those leading to interval-valued q -spherical fuzzy sets and classical rough sets. These preliminaries provide the necessary mathematical foundations for the IVq-SFRS framework introduced in Section 3.

Definition 1 ([2]). Let $\mathbb{X}$ be a universal set. A fuzzy set Θ on $\mathbb{X}$ is defined as

$$\Theta = \{ \langle x, \wp_\Theta(x) \rangle \mid x \in \mathbb{X} \}, \quad (1)$$

where $\wp_\Theta : \mathbb{X} \rightarrow [0, 1]$ is the membership degree.

Definition 2 ([3]). An intuitionistic fuzzy set Θ on $\mathbb{X}$ is defined as

$$\Theta = \{ \langle x, \wp_\Theta(x), \varkappa_\Theta(x) \rangle \mid x \in \mathbb{X} \}, \quad (2)$$

where $\wp_\Theta, \varkappa_\Theta : \mathbb{X} \rightarrow [0, 1]$ are the membership and non-membership degrees, respectively, satisfying $0 \leq \wp_\Theta(x) + \varkappa_\Theta(x) \leq 1$ for all $x \in \mathbb{X}$.

Definition 3 ([11]). Let $q \geq 1$ be a real number. A q -rung orthopair fuzzy set Θ on $\mathbb{X}$ is defined as

$$\Theta = \{ \langle x, \wp_{\Theta}(x), \varkappa_{\Theta}(x) \rangle \mid x \in \mathbb{X} \}, \quad (3)$$

where $\wp_{\Theta}, \varkappa_{\Theta} : \mathbb{X} \rightarrow [0, 1]$ satisfy $0 \leq [\wp_{\Theta}(x)]^q + [\varkappa_{\Theta}(x)]^q \leq 1$ for all $x \in \mathbb{X}$.

Definition 4 ([12]). A spherical fuzzy set Θ on $\mathbb{X}$ is defined as

$$\Theta = \{ \langle x, \wp_{\Theta}(x), \mathfrak{h}_{\Theta}(x), \varkappa_{\Theta}(x) \rangle \mid x \in \mathbb{X} \}, \quad (4)$$

where $\wp_{\Theta}, \mathfrak{h}_{\Theta}, \varkappa_{\Theta} : \mathbb{X} \rightarrow [0, 1]$ are the membership, neutral, and non-membership degrees, respectively, satisfying $0 \leq [\wp_{\Theta}(x)]^2 + [\mathfrak{h}_{\Theta}(x)]^2 + [\varkappa_{\Theta}(x)]^2 \leq 1$ for all $x \in \mathbb{X}$.

Definition 5 ([11–13]). Let $q \geq 1$ be fixed. An interval-valued q -spherical fuzzy set (IV q -SFS) Θ on a universe $\mathbb{X}$ is expressed as follows:

$$\Theta = \{ \langle x, [\wp_{\Theta}^{-}(x), \wp_{\Theta}^{+}(x)], [\mathfrak{h}_{\Theta}^{-}(x), \mathfrak{h}_{\Theta}^{+}(x)], [\varkappa_{\Theta}^{-}(x), \varkappa_{\Theta}^{+}(x)] \rangle \mid x \in \mathbb{X} \}. \quad (5)$$

For each $x \in \mathbb{X}$, the interval bounds satisfy

$$0 \leq \wp_{\Theta}^{-}(x) \leq \wp_{\Theta}^{+}(x) \leq 1, \quad 0 \leq \mathfrak{h}_{\Theta}^{-}(x) \leq \mathfrak{h}_{\Theta}^{+}(x) \leq 1, \quad 0 \leq \varkappa_{\Theta}^{-}(x) \leq \varkappa_{\Theta}^{+}(x) \leq 1, \\ [\wp_{\Theta}^{+}(x)]^q + [\mathfrak{h}_{\Theta}^{+}(x)]^q + [\varkappa_{\Theta}^{+}(x)]^q \leq 1. \quad (6)$$

The proposed IV q -SFS adopts a unified parameter q to control the admissible region of membership, neutrality, and non-membership degrees. It reduces to a q -rung orthopair model when the neutral degree is omitted and yields the classical spherical fuzzy constraint when $q = 2$. This unified formulation ensures mathematical consistency while serving as the information representation framework for the proposed rough TOPSIS model.

The interval-valued refusal degree is defined by

$$\chi_{\Theta}(x) = \left[(1 - [\wp_{\Theta}^{+}(x)]^q - [\mathfrak{h}_{\Theta}^{+}(x)]^q - [\varkappa_{\Theta}^{+}(x)]^q)^{1/q}, (1 - [\wp_{\Theta}^{-}(x)]^q - [\mathfrak{h}_{\Theta}^{-}(x)]^q - [\varkappa_{\Theta}^{-}(x)]^q)^{1/q} \right]. \quad (7)$$

The lower endpoint corresponds to the most restrictive upper-bound assessment while the upper endpoint corresponds to the least restrictive lower-bound assessment. To preserve the q -spherical constraint under aggregation, a projection operator Π_q is introduced in Section 3. We denote $\mathbb{S}_q(\mathbb{X})$ as the family of all IV q -SFSs on $\mathbb{X}$.

Example 1. Let $q = 4$ and $\mathbb{X} = \{x_1, x_2, x_3, x_4\}$. Define the interval-valued q -spherical fuzzy set $\Theta \in \mathbb{S}_4(\mathbb{X})$ by

x	$[\wp^{-}, \wp^{+}]$	$[\mathfrak{h}^{-}, \mathfrak{h}^{+}]$	$[\varkappa^{-}, \varkappa^{+}]$
x_1	[0.85, 0.90]	[0.05, 0.10]	[0.10, 0.15]
x_2	[0.60, 0.70]	[0.20, 0.25]	[0.15, 0.20]
x_3	[0.40, 0.50]	[0.30, 0.35]	[0.25, 0.30]
x_4	[0.20, 0.30]	[0.40, 0.45]	[0.35, 0.40]

Using the upper bounds, the q -spherical constraint

$$(\wp^{+})^q + (\mathfrak{h}^{+})^q + (\varkappa^{+})^q \leq 1$$

is satisfied for all elements x_i .

The refusal degree is given by

$$\chi_{\Theta}(x) = \sqrt[q]{1 - (\wp^{+}(x))^q - (\mathfrak{h}^{+}(x))^q - (\varkappa^{+}(x))^q}.$$

The computed values are

$$\chi_{\Theta}(x_1) \approx 0.765, \quad \chi_{\Theta}(x_2) \approx 0.931, \quad \chi_{\Theta}(x_3) \approx 0.978, \quad \chi_{\Theta}(x_4) \approx 0.981.$$

Lower q -power sums yield larger refusal degrees, reflecting greater uncertainty and demonstrating the effect of q on the admissible region. The unified parameter q provides consistent control over all evaluation degrees.

Remark 1. The parameter $q \geq 1$ controls the admissible region of the triplet $(\wp, \mathfrak{h}, \varkappa)$. Larger q values enlarge the feasible region and permit stronger simultaneous expression of membership, neutrality, and non-membership. Unless otherwise stated in the numerical section, the illustrative computations use $q = 3$; no reduction to the special case $q = 2$ is assumed.

Definition 6 ([5]). Let $\mathbb{X}$ be a universal set equipped with an equivalence relation $\sim$. For any $x \in \mathbb{X}$, let $[x]_{\sim} = \{y \in \mathbb{X} \mid y \sim x\}$ denote the equivalence class containing x . The pair $(\mathbb{X}, \sim)$ is called a Pawlak approximation space. For an arbitrary subset $T \subseteq \mathbb{X}$, the lower and upper approximations are defined as

$$\underline{apr}_{\sim}(T) = \{x \in \mathbb{X} \mid [x]_{\sim} \subseteq T\}, \quad (8)$$

$$\overline{apr}_{\sim}(T) = \{x \in \mathbb{X} \mid [x]_{\sim} \cap T \neq \emptyset\}. \quad (9)$$

If $\underline{apr}_{\sim}(T) = \overline{apr}_{\sim}(T)$, then T is definable; otherwise, the pair $(\underline{apr}_{\sim}(T), \overline{apr}_{\sim}(T))$ constitutes a rough set.

Definition 7. Let $\mathbb{X}$ and $\mathbb{P}$ be nonempty sets. An interval-valued q -spherical fuzzy relation $\mathcal{Z}$ on $\mathbb{X} \times \mathbb{P}$ is an IV q -SF set (Definition 5) such that for all $(\mathfrak{U}, \omega) \in \mathbb{X} \times \mathbb{P}$:

$$0 \leq \wp_{\mathcal{Z}}^{-}(\mathfrak{U}, \omega) \leq \wp_{\mathcal{Z}}^{+}(\mathfrak{U}, \omega) \leq 1,$$

$$0 \leq \mathfrak{h}_{\mathcal{Z}}^{-}(\mathfrak{U}, \omega) \leq \mathfrak{h}_{\mathcal{Z}}^{+}(\mathfrak{U}, \omega) \leq 1,$$

$$0 \leq \varkappa_{\mathcal{Z}}^{-}(\mathfrak{U}, \omega) \leq \varkappa_{\mathcal{Z}}^{+}(\mathfrak{U}, \omega) \leq 1.$$

and

$$0 \leq [\wp_{\mathcal{Z}}^{+}(\mathfrak{U}, \omega)]^q + [\mathfrak{h}_{\mathcal{Z}}^{+}(\mathfrak{U}, \omega)]^q + [\varkappa_{\mathcal{Z}}^{+}(\mathfrak{U}, \omega)]^q \leq 1.$$

The triplet $(\mathbb{X}, \mathbb{P}, \mathcal{Z})$ will be referred to as an IV q -SF relational frame. This framework extends classical rough approximation spaces by employing interval-valued q -spherical fuzzy relations, where a single parameter q governs the admissible region of the evaluation degrees.

Example 2. Consider the set of suppliers $\mathbb{X} = \{x_1, x_2, x_3\}$; let the set of criteria consisting of quality, cost, delivery time, and after-sales service be $\mathbb{P} = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Fix $q = 3$. Define the interval-valued q -spherical fuzzy relation $\mathcal{Z}$ on $\mathbb{X} \times \mathbb{P}$ as

	ω_1	ω_2	ω_3	ω_4
x_1	$\langle [0.8, 0.9], [0.1, 0.1], [0.1, 0.2] \rangle$	$\langle [0.5, 0.6], [0.2, 0.3], [0.2, 0.3] \rangle$	$\langle [0.7, 0.8], [0.1, 0.2], [0.1, 0.2] \rangle$	$\langle [0.6, 0.7], [0.2, 0.2], [0.2, 0.3] \rangle$
x_2	$\langle [0.6, 0.7], [0.2, 0.3], [0.2, 0.2] \rangle$	$\langle [0.8, 0.9], [0.1, 0.1], [0.1, 0.1] \rangle$	$\langle [0.5, 0.6], [0.2, 0.3], [0.2, 0.3] \rangle$	$\langle [0.7, 0.8], [0.1, 0.2], [0.1, 0.1] \rangle$
x_3	$\langle [0.7, 0.8], [0.1, 0.2], [0.1, 0.2] \rangle$	$\langle [0.6, 0.7], [0.2, 0.3], [0.2, 0.2] \rangle$	$\langle [0.9, 0.9], [0.0, 0.1], [0.0, 0.1] \rangle$	$\langle [0.5, 0.6], [0.3, 0.4], [0.2, 0.3] \rangle$

The selected entries satisfy the spherical constraint. For example, for (x_1, ω_1) :

$$(0.9)^3 + (0.1)^3 + (0.2)^3 = 0.738 \leq 1.$$

The refusal degree is given by

$$\chi_{\mathcal{Z}}(x, \omega) = \sqrt[3]{1 - (\wp^{+})^3 - (\mathfrak{h}^{+})^3 - (\varkappa^{+})^3}.$$

For (x_1, ω_1) , using the upper bounds $\wp^+ = 0.9$, $\mathfrak{h}^+ = 0.1$, $\varkappa^+ = 0.2$:

$$\chi_{\mathcal{Z}}(x_1, \omega_1) = \sqrt[3]{1 - (0.9)^3 - (0.1)^3 - (0.2)^3} = \sqrt[3]{1 - 0.738} = \sqrt[3]{0.262} \approx 0.6399.$$

The computed refusal degrees for all pairs are summarized below:

	ω_1	ω_2	ω_3	ω_4
$\chi_{\mathcal{Z}}(x_i, \omega_j) =$				
x_1	0.6399	0.9004	0.7786	0.8536
x_2	0.8536	0.6455	0.9004	0.7824
x_3	0.7786	0.8536	0.6455	0.8849

Thus, the best per criterion is x_1 for quality ($\chi = 0.6399$), x_2 for cost ($\chi = 0.6455$) and after-sales ($\chi = 0.7824$), and x_3 for delivery time ($\chi = 0.6455$). Average refusal: $x_1 = 0.7931$, $x_2 = 0.7955$, $x_3 = 0.7907$. Hence, x_3 is the most balanced. Lower refusal degrees indicate higher confidence.

Remark 2. The relational structure $(\mathbb{X}, \mathbb{P}, \mathcal{Z})$ forms the basis for the approximation operators in Section 3. Section 3 develops the corresponding approximation operators and introduces a projection operator to ensure closure under the IVq-spherical constraint.

3. Interval-Valued q-Spherical Fuzzy Rough Sets

In this section, we introduce IVq-SFRSs based on the IVq-SF relation $\mathcal{Z}$ and the relational frame $(\mathbb{X}, \mathbb{P}, \mathcal{Z})$, and develop their approximation operators, properties, algebraic operations, and duality results.

Definition 8. Let $(\mathbb{X}, \mathbb{P}, \mathcal{Z})$ be an IVq-SFRS approximation space. A mapping $\phi : \mathbb{X} \rightarrow \mathbb{P}$ assigns each $\mathfrak{U} \in \mathbb{X}$ to a parameter $\phi(\mathfrak{U}) \in \mathbb{P}$. If $\mathbb{X} = \mathbb{P}$, then, ϕ is the identity mapping.

Definition 9. The IVq-SF relation $\mathcal{Z}$ is reflexive if, for every $\mathfrak{U} \in \mathbb{X}$ with $\omega_{\mathfrak{U}} = \phi(\mathfrak{U})$:

$$\wp_{\mathcal{Z}}^+(\mathfrak{U}, \omega_{\mathfrak{U}}) = 1, \quad \mathfrak{h}_{\mathcal{Z}}^+(\mathfrak{U}, \omega_{\mathfrak{U}}) = 0, \quad \varkappa_{\mathcal{Z}}^+(\mathfrak{U}, \omega_{\mathfrak{U}}) = 0.$$

Definition 10. The relation $\mathcal{Z}$ is serial if, for every $\mathfrak{U} \in \mathbb{X}$, there exists $\omega \in \mathbb{P}$ such that

$$\wp_{\mathcal{Z}}^+(\mathfrak{U}, \omega) = 1, \quad \mathfrak{h}_{\mathcal{Z}}^+(\mathfrak{U}, \omega) = 0, \quad \varkappa_{\mathcal{Z}}^+(\mathfrak{U}, \omega) = 0.$$

Definition 11. Let $(\mathbb{X}, \mathbb{P}, \mathcal{Z})$ be an IVq-SFRS approximation space. For any $\lambda \in \mathbb{S}_q(\mathbb{P})$, the lower and upper approximations $\underline{\mathcal{Z}}(\lambda)$ and $\overline{\mathcal{Z}}(\lambda)$ are IVq-SF sets on $\mathbb{X}$ defined by

$$\underline{\mathcal{Z}}(\lambda) = \left\{ \left\langle \mathfrak{U}, [\underline{\wp}_{\lambda}^-(\mathfrak{U}), \underline{\wp}_{\lambda}^+(\mathfrak{U})], [\underline{\mathfrak{h}}_{\lambda}^-(\mathfrak{U}), \underline{\mathfrak{h}}_{\lambda}^+(\mathfrak{U})], [\underline{\varkappa}_{\lambda}^-(\mathfrak{U}), \underline{\varkappa}_{\lambda}^+(\mathfrak{U})] \right\rangle \mid \mathfrak{U} \in \mathbb{X} \right\}, \quad (10)$$

$$\overline{\mathcal{Z}}(\lambda) = \left\{ \left\langle \mathfrak{U}, [\overline{\wp}_{\lambda}^-(\mathfrak{U}), \overline{\wp}_{\lambda}^+(\mathfrak{U})], [\overline{\mathfrak{h}}_{\lambda}^-(\mathfrak{U}), \overline{\mathfrak{h}}_{\lambda}^+(\mathfrak{U})], [\overline{\varkappa}_{\lambda}^-(\mathfrak{U}), \overline{\varkappa}_{\lambda}^+(\mathfrak{U})] \right\rangle \mid \mathfrak{U} \in \mathbb{X} \right\}. \quad (11)$$

The lower estimate is a lower bound while the upper estimate is an upper bound. To keep the symbols consistent, lower approximation components are written as $\underline{\wp}_{\lambda}$, $\underline{\mathfrak{h}}_{\lambda}$, and $\underline{\varkappa}_{\lambda}$, whereas upper approximation components are written as $\overline{\wp}_{\lambda}$, $\overline{\mathfrak{h}}_{\lambda}$, and $\overline{\varkappa}_{\lambda}$. The components are calculated, with $\wedge = \min$ and $\vee = \max$, as the following:

Lower Approximation Components:

$$\begin{aligned} \underline{\wp}_{\lambda}^-(\mathfrak{U}) &= \bigwedge_{\omega \in \mathbb{P}} [\wp_{\mathcal{Z}}^-(\mathfrak{U}, \omega) \wedge \wp_{\lambda}^-(\omega)], & \underline{\wp}_{\lambda}^+(\mathfrak{U}) &= \bigwedge_{\omega \in \mathbb{P}} [\wp_{\mathcal{Z}}^+(\mathfrak{U}, \omega) \wedge \wp_{\lambda}^+(\omega)], \\ \underline{\mathfrak{h}}_{\lambda}^-(\mathfrak{U}) &= \bigwedge_{\omega \in \mathbb{P}} [\mathfrak{h}_{\mathcal{Z}}^-(\mathfrak{U}, \omega) \vee \mathfrak{h}_{\lambda}^-(\omega)], & \underline{\mathfrak{h}}_{\lambda}^+(\mathfrak{U}) &= \bigwedge_{\omega \in \mathbb{P}} [\mathfrak{h}_{\mathcal{Z}}^+(\mathfrak{U}, \omega) \vee \mathfrak{h}_{\lambda}^+(\omega)], \\ \underline{\varkappa}_{\lambda}^-(\mathfrak{U}) &= \bigwedge_{\omega \in \mathbb{P}} [\varkappa_{\mathcal{Z}}^-(\mathfrak{U}, \omega) \vee \varkappa_{\lambda}^-(\omega)], & \underline{\varkappa}_{\lambda}^+(\mathfrak{U}) &= \bigwedge_{\omega \in \mathbb{P}} [\varkappa_{\mathcal{Z}}^+(\mathfrak{U}, \omega) \vee \varkappa_{\lambda}^+(\omega)]. \end{aligned}$$

$$\underline{\varkappa}_{\lambda}^{-}(\mathcal{U}) = \bigwedge_{\omega \in \mathbb{P}} [\varkappa_{\mathcal{Z}}^{-}(\mathcal{U}, \omega) \vee \varkappa_{\lambda}^{-}(\omega)], \quad \underline{\varkappa}_{\lambda}^{+}(\mathcal{U}) = \bigwedge_{\omega \in \mathbb{P}} [\varkappa_{\mathcal{Z}}^{+}(\mathcal{U}, \omega) \vee \varkappa_{\lambda}^{+}(\omega)].$$

Upper Approximation Components:

$$\begin{aligned} \overline{\wp}_{\lambda}^{-}(\mathcal{U}) &= \bigvee_{\omega \in \mathbb{P}} [\wp_{\mathcal{Z}}^{-}(\mathcal{U}, \omega) \vee \wp_{\lambda}^{-}(\omega)], & \overline{\wp}_{\lambda}^{+}(\mathcal{U}) &= \bigvee_{\omega \in \mathbb{P}} [\wp_{\mathcal{Z}}^{+}(\mathcal{U}, \omega) \vee \wp_{\lambda}^{+}(\omega)], \\ \overline{\mathfrak{h}}_{\lambda}^{-}(\mathcal{U}) &= \bigvee_{\omega \in \mathbb{P}} [\mathfrak{h}_{\mathcal{Z}}^{-}(\mathcal{U}, \omega) \wedge \mathfrak{h}_{\lambda}^{-}(\omega)], & \overline{\mathfrak{h}}_{\lambda}^{+}(\mathcal{U}) &= \bigvee_{\omega \in \mathbb{P}} [\mathfrak{h}_{\mathcal{Z}}^{+}(\mathcal{U}, \omega) \wedge \mathfrak{h}_{\lambda}^{+}(\omega)], \\ \overline{\varkappa}_{\lambda}^{-}(\mathcal{U}) &= \bigvee_{\omega \in \mathbb{P}} [\varkappa_{\mathcal{Z}}^{-}(\mathcal{U}, \omega) \wedge \varkappa_{\lambda}^{-}(\omega)], & \overline{\varkappa}_{\lambda}^{+}(\mathcal{U}) &= \bigvee_{\omega \in \mathbb{P}} [\varkappa_{\mathcal{Z}}^{+}(\mathcal{U}, \omega) \wedge \varkappa_{\lambda}^{+}(\omega)]. \end{aligned}$$

Because component-wise extrema may combine degrees taken from different criteria, the computed components are checked against the IVq-SFN admissibility condition. When a computed value is not admissible, the following radial projection is applied:

$$\Pi_q(\langle [\wp^{-}, \wp^{+}], [\mathfrak{h}^{-}, \mathfrak{h}^{+}], [\varkappa^{-}, \varkappa^{+}] \rangle) = \langle [\gamma \wp^{-}, \gamma \wp^{+}], [\gamma \mathfrak{h}^{-}, \gamma \mathfrak{h}^{+}], [\gamma \varkappa^{-}, \gamma \varkappa^{+}] \rangle, \quad (12)$$

where

$$\gamma = \begin{cases} 1, & (\wp^{+})^q + (\mathfrak{h}^{+})^q + (\varkappa^{+})^q \leq 1, \\ ((\wp^{+})^q + (\mathfrak{h}^{+})^q + (\varkappa^{+})^q)^{-1/q}, & (\wp^{+})^q + (\mathfrak{h}^{+})^q + (\varkappa^{+})^q > 1. \end{cases} \quad (13)$$

The operator Π_q ensures that every projected element satisfies the IVq-SF admissibility condition. Moreover, it is idempotent, monotone with respect to the natural partial order, and preserves the underlying uncertainty structure.

If $\underline{\mathcal{Z}}(\lambda) = \overline{\mathcal{Z}}(\lambda)$, then λ is called definable; otherwise, the pair $(\underline{\mathcal{Z}}(\lambda), \overline{\mathcal{Z}}(\lambda))$ forms an interval-valued q -spherical fuzzy rough set (IVq-SFRS).

Definition 12. Let $\mathbb{P}$ be a set of criteria. An indiscernibility relation $\asymp$ on $\mathbb{P}$ is an equivalence relation defined by

$$\omega_i \asymp \omega_j \iff \text{criteria } \omega_i \text{ and } \omega_j \text{ belong to the same semantic group.}$$

For any $\omega \in \mathbb{P}$, $[\omega]_{\asymp}$ denotes its equivalence class. This relation is used in the computation of lower and upper approximations as defined in Definition 11.

Example 3. We illustrate the IVq-SFRS model using an investment project selection problem. Let four potential projects $\mathbb{X} = \{x_1, x_2, x_3, x_4\}$ be evaluated against three criteria $\mathbb{P} = \{\omega_1, \omega_2, \omega_3\}$ with $q = 3$, and define the IVq-SF relation $\mathcal{Z}$ as given in Table 2.

Table 2. IVq-SFR $\mathcal{Z}(x_i, \omega_j)$ for $q = 3$.

x_i	ω_1	ω_2	ω_3
x_1	$\langle [0.7, 0.8], [0.1, 0.2], [0.1, 0.1] \rangle$	$\langle [0.5, 0.6], [0.2, 0.3], [0.2, 0.2] \rangle$	$\langle [0.6, 0.7], [0.1, 0.2], [0.1, 0.2] \rangle$
x_2	$\langle [0.4, 0.5], [0.3, 0.4], [0.2, 0.3] \rangle$	$\langle [0.8, 0.9], [0.1, 0.1], [0.1, 0.1] \rangle$	$\langle [0.5, 0.6], [0.2, 0.3], [0.2, 0.2] \rangle$
x_3	$\langle [0.6, 0.7], [0.2, 0.3], [0.1, 0.2] \rangle$	$\langle [0.4, 0.5], [0.3, 0.4], [0.2, 0.3] \rangle$	$\langle [0.9, 0.9], [0.0, 0.1], [0.0, 0.1] \rangle$
x_4	$\langle [0.5, 0.6], [0.2, 0.3], [0.2, 0.3] \rangle$	$\langle [0.6, 0.7], [0.1, 0.2], [0.1, 0.2] \rangle$	$\langle [0.4, 0.5], [0.3, 0.4], [0.2, 0.3] \rangle$

Define the target concept $\lambda \in \mathbb{S}_q(\mathbb{P})$ as

$$\begin{aligned} \lambda(\omega_1) &= \langle [0.6, 0.7], [0.1, 0.2], [0.1, 0.2] \rangle, \\ \lambda(\omega_2) &= \langle [0.7, 0.8], [0.1, 0.1], [0.1, 0.1] \rangle, \\ \lambda(\omega_3) &= \langle [0.5, 0.6], [0.2, 0.3], [0.2, 0.2] \rangle. \end{aligned}$$

The lower approximation $\underline{\mathcal{Z}}(\lambda)$ is computed as

$$\begin{aligned}\underline{\wp}_\lambda^+(\mathcal{U}) &= \bigwedge_{\omega \in \mathbb{P}} [\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega) \wedge \wp_\lambda^+(\omega)], \\ \underline{\mathfrak{h}}_\lambda^+(\mathcal{U}) &= \bigwedge_{\omega \in \mathbb{P}} [\mathfrak{h}_{\mathcal{Z}}^+(\mathcal{U}, \omega) \vee \mathfrak{h}_\lambda^+(\omega)], \\ \underline{\varkappa}_\lambda^+(\mathcal{U}) &= \bigwedge_{\omega \in \mathbb{P}} [\varkappa_{\mathcal{Z}}^+(\mathcal{U}, \omega) \vee \varkappa_\lambda^+(\omega)].\end{aligned}$$

The upper approximation $\overline{\mathcal{Z}}(\lambda)$ is computed as

$$\begin{aligned}\overline{\wp}_\lambda^+(\mathcal{U}) &= \bigvee_{\omega \in \mathbb{P}} [\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega) \vee \wp_\lambda^+(\omega)], \\ \overline{\mathfrak{h}}_\lambda^+(\mathcal{U}) &= \bigvee_{\omega \in \mathbb{P}} [\mathfrak{h}_{\mathcal{Z}}^+(\mathcal{U}, \omega) \wedge \mathfrak{h}_\lambda^+(\omega)], \\ \overline{\varkappa}_\lambda^+(\mathcal{U}) &= \bigvee_{\omega \in \mathbb{P}} [\varkappa_{\mathcal{Z}}^+(\mathcal{U}, \omega) \wedge \varkappa_\lambda^+(\omega)].\end{aligned}$$

The approximation values in Tables 3 and 4 were rechecked after Definition 11. For example, the largest upper value (0.9, 0.1, 0.1) gives $0.9^3 + 0.1^3 + 0.1^3 = 0.731 \leq 1$, so the projection factor is $\gamma = 1$ and no numerical correction is required. All values satisfy $[\wp^+]^3 + [\mathfrak{h}^+]^3 + [\varkappa^+]^3 \leq 1$. The lower approximation favors x_1 and x_2 (certainty), the upper favors x_2 and x_3 (possibility), and with a 0.5 membership threshold, x_2 is the most balanced choice.

Table 3. Lower approximation $\underline{\mathcal{Z}}(\lambda)$.

x_i	$\underline{\wp}_\lambda^+$	$\underline{\mathfrak{h}}_\lambda^+$	$\underline{\varkappa}_\lambda^+$
x_1	0.5	0.2	0.2
x_2	0.5	0.2	0.2
x_3	0.4	0.2	0.2
x_4	0.4	0.2	0.2

Table 4. Upper approximation $\overline{\mathcal{Z}}(\lambda)$.

x_i	$\overline{\wp}_\lambda^+$	$\overline{\mathfrak{h}}_\lambda^+$	$\overline{\varkappa}_\lambda^+$
x_1	0.8	0.1	0.1
x_2	0.9	0.1	0.1
x_3	0.9	0.1	0.1
x_4	0.8	0.1	0.1

Definition 13. Let $\underline{\mathcal{Z}}(\lambda)$ and $\overline{\mathcal{Z}}(\lambda)$ be the lower and upper approximations of an IVq-SF set λ in the context of the approximation space $(\mathbb{X}, \mathbb{P}, \mathcal{Z})$. For all $\mathcal{U} \in \mathbb{X}$, we write

$$\underline{\mathcal{Z}}(\lambda)(\mathcal{U}) = ([\underline{\wp}_\lambda^-(\mathcal{U}), \underline{\wp}_\lambda^+(\mathcal{U})], [\underline{\mathfrak{h}}_\lambda^-(\mathcal{U}), \underline{\mathfrak{h}}_\lambda^+(\mathcal{U})], [\underline{\varkappa}_\lambda^-(\mathcal{U}), \underline{\varkappa}_\lambda^+(\mathcal{U})]),$$

$$\overline{\mathcal{Z}}(\lambda)(\mathcal{U}) = ([\overline{\wp}_\lambda^-(\mathcal{U}), \overline{\wp}_\lambda^+(\mathcal{U})], [\overline{\mathfrak{h}}_\lambda^-(\mathcal{U}), \overline{\mathfrak{h}}_\lambda^+(\mathcal{U})], [\overline{\varkappa}_\lambda^-(\mathcal{U}), \overline{\varkappa}_\lambda^+(\mathcal{U})]).$$

We define the following algebraic operations:

1. Algebraic sum operation ($\oplus$):

$$\begin{aligned} \underline{\mathcal{Z}}(\lambda) \oplus \overline{\mathcal{Z}}(\lambda) = \left\{ \left\langle \mathcal{U}, \left[\left((\underline{\varrho}_{\lambda}^{-}(\mathcal{U}))^q + (\overline{\varrho}_{\lambda}^{-}(\mathcal{U}))^q - (\underline{\varrho}_{\lambda}^{-}(\mathcal{U}))^q (\overline{\varrho}_{\lambda}^{-}(\mathcal{U}))^q \right)^{1/q}, \right. \right. \right. \\ \left. \left. \left((\underline{\varrho}_{\lambda}^{+}(\mathcal{U}))^q + (\overline{\varrho}_{\lambda}^{+}(\mathcal{U}))^q - (\underline{\varrho}_{\lambda}^{+}(\mathcal{U}))^q (\overline{\varrho}_{\lambda}^{+}(\mathcal{U}))^q \right)^{1/q} \right], \right. \\ \left. \left[\underline{\mathfrak{h}}_{\lambda}^{-}(\mathcal{U}) \overline{\mathfrak{h}}_{\lambda}^{-}(\mathcal{U}), \underline{\mathfrak{h}}_{\lambda}^{+}(\mathcal{U}) \overline{\mathfrak{h}}_{\lambda}^{+}(\mathcal{U}) \right], \right. \\ \left. \left[\underline{\mathfrak{x}}_{\lambda}^{-}(\mathcal{U}) \cdot \overline{\mathfrak{x}}_{\lambda}^{-}(\mathcal{U}), \underline{\mathfrak{x}}_{\lambda}^{+}(\mathcal{U}) \cdot \overline{\mathfrak{x}}_{\lambda}^{+}(\mathcal{U}) \right] \right\rangle \mid \mathcal{U} \in \mathbb{X} \right\}. \end{aligned} \quad (14)$$

2. Algebraic product operation ($\otimes$):

$$\begin{aligned} \underline{\mathcal{Z}}(\lambda) \otimes \overline{\mathcal{Z}}(\lambda) = \left\{ \left\langle \mathcal{U}, \left[\underline{\varrho}_{\lambda}^{-}(\mathcal{U}) \overline{\varrho}_{\lambda}^{-}(\mathcal{U}), \underline{\varrho}_{\lambda}^{+}(\mathcal{U}) \overline{\varrho}_{\lambda}^{+}(\mathcal{U}) \right], \right. \right. \\ \left. \left[\underline{\mathfrak{h}}_{\lambda}^{-}(\mathcal{U}) \cdot \overline{\mathfrak{h}}_{\lambda}^{-}(\mathcal{U}), \underline{\mathfrak{h}}_{\lambda}^{+}(\mathcal{U}) \cdot \overline{\mathfrak{h}}_{\lambda}^{+}(\mathcal{U}) \right], \right. \\ \left. \left[\left((\underline{\mathfrak{x}}_{\lambda}^{-}(\mathcal{U}))^q + (\overline{\mathfrak{x}}_{\lambda}^{-}(\mathcal{U}))^q - (\underline{\mathfrak{x}}_{\lambda}^{-}(\mathcal{U}))^q (\overline{\mathfrak{x}}_{\lambda}^{-}(\mathcal{U}))^q \right)^{1/q}, \right. \right. \\ \left. \left. \left((\underline{\mathfrak{x}}_{\lambda}^{+}(\mathcal{U}))^q + (\overline{\mathfrak{x}}_{\lambda}^{+}(\mathcal{U}))^q - (\underline{\mathfrak{x}}_{\lambda}^{+}(\mathcal{U}))^q (\overline{\mathfrak{x}}_{\lambda}^{+}(\mathcal{U}))^q \right)^{1/q} \right] \right\rangle \mid \mathcal{U} \in \mathbb{X} \right\}. \end{aligned} \quad (15)$$

Proposition 1. Let a_1 and a_2 be two admissible IVq-SFNs. The algebraic sum $a_1 \oplus a_2$ and product $a_1 \otimes a_2$ defined in Definition 13 are admissible IVq-SFNs. Therefore, the operations preserve the IVq-spherical constraint.

Proof. It is enough to verify the upper endpoints because the lower endpoints are bounded by the corresponding upper endpoints. Let

$$a = \mu_1^q, \quad b = \mu_2^q, \quad c = \eta_1^q, \quad d = \eta_2^q, \quad e = \nu_1^q, \quad f = \nu_2^q,$$

where $a + c + e \leq 1$ and $b + d + f \leq 1$. For the algebraic sum,

$$(\mu_{\oplus}^+)^q + (\eta_{\oplus}^+)^q + (\nu_{\oplus}^+)^q = (a + b - ab) + cd + ef.$$

$$\text{Since } cd + ef \leq (c + e)(d + f) \leq (1 - a)(1 - b),$$

$$(a + b - ab) + cd + ef \leq a + b - ab + (1 - a)(1 - b) = 1.$$

For the algebraic product,

$$(\mu_{\otimes}^+)^q + (\eta_{\otimes}^+)^q + (\nu_{\otimes}^+)^q = ab + cd + (e + f - ef).$$

$$\text{Because } ab + cd \leq (a + c)(b + d) \leq (1 - e)(1 - f),$$

$$ab + cd + e + f - ef \leq (1 - e)(1 - f) + e + f - ef = 1.$$

Thus, both operations satisfy the IVq-spherical constraint. The same argument applies to all elements of $\mathbb{X}$ and to interval lower endpoints. $\square$

Example 4. We illustrate with the investment project selection problem the algebraic operations for two elements within the same IVq-SFRN. Let the IVq-SF relation $\mathcal{Z}$ and the target concept λ be the ones from Example 3. Define another target concept $\eta \in \mathbb{S}_q(\mathbb{P})$:

$$\eta(\omega_1) = \langle [0.2, 0.3], [0.5, 0.6], [0.3, 0.4] \rangle,$$

$$\eta(\omega_2) = \langle [0.1, 0.2], [0.6, 0.7], [0.4, 0.5] \rangle,$$

$$\eta(\omega_3) = \langle [0.3, 0.4], [0.4, 0.5], [0.2, 0.3] \rangle.$$

The lower and upper approximations of η are given in Tables 5 and 6.

Table 5. Lower approximation $\underline{\mathcal{Z}}(\eta)$.

x_i	$\underline{\wp}_\eta^+$	$\underline{\mathfrak{h}}_\eta^+$	$\underline{\varkappa}_\eta^+$
x_1	0.2	0.5	0.3
x_2	0.2	0.6	0.4
x_3	0.2	0.5	0.3
x_4	0.2	0.5	0.3

Table 6. Upper approximation $\overline{\mathcal{Z}}(\eta)$.

x_i	$\overline{\wp}_\eta^+$	$\overline{\mathfrak{h}}_\eta^+$	$\overline{\varkappa}_\eta^+$
x_1	0.6	0.2	0.2
x_2	0.8	0.1	0.1
x_3	0.9	0.1	0.1
x_4	0.6	0.2	0.2

Algebraic Sum $\underline{\mathcal{Z}}(\lambda) \oplus \overline{\mathcal{Z}}(\lambda)$

$$\wp_{\underline{\lambda} \oplus \overline{\lambda}}^+ = 0.509, \quad \mathfrak{h}_{\underline{\lambda} \oplus \overline{\lambda}}^+ = 0.10, \quad \varkappa_{\underline{\lambda} \oplus \overline{\lambda}}^+ = 0.10 \times 0.10 = 0.01$$

The results of the algebraic sum are presented in Table 7.

Table 7. Algebraic sum $\underline{\mathcal{Z}}(\lambda) \oplus \overline{\mathcal{Z}}(\lambda)$, $q = 3$.

x_i	$\wp^+$	$\mathfrak{h}^+$	$\varkappa^+$
x_1	0.509	0.100	0.010
x_2	0.509	0.060	0.010
x_3	0.425	0.100	0.010
x_4	0.425	0.100	0.010

Algebraic Product $\underline{\mathcal{Z}}(\lambda) \otimes \overline{\mathcal{Z}}(\lambda)$

$$\wp_{\underline{\lambda} \otimes \overline{\lambda}}^+ = 0.10, \quad \mathfrak{h}_{\underline{\lambda} \otimes \overline{\lambda}}^+ = 0.10 \times 0.10 = 0.01, \quad \varkappa_{\underline{\lambda} \otimes \overline{\lambda}}^+ = 0.326$$

The results of the algebraic product are presented in Table 8.

Table 8. Algebraic product $\underline{\mathcal{Z}}(\lambda) \otimes \overline{\mathcal{Z}}(\lambda)$, $q = 3$.

x_i	$\wp^+$	$\mathfrak{h}^+$	$\varkappa^+$
x_1	0.100	0.010	0.326
x_2	0.100	0.010	0.403
x_3	0.080	0.010	0.326
x_4	0.080	0.010	0.326

For verification at x_1 :

$$(0.509)^3 + (0.100)^3 + (0.010)^3 = 0.132 + 0.001 + 0.000001 = 0.133 \leq 1.$$

Definition 14. Let $\mathcal{Z}(\lambda) = (\underline{\mathcal{Z}}(\lambda), \overline{\mathcal{Z}}(\lambda))$ and $\mathcal{Z}(\eta) = (\underline{\mathcal{Z}}(\eta), \overline{\mathcal{Z}}(\eta))$ be two distinct IVq-SFRNs. For each $\mathcal{U} \in \mathbb{X}$, we denote

$$\underline{\mathcal{Z}}(\lambda)(\mathcal{U}) = (\varrho_{\lambda}(\mathcal{U}), \mathfrak{h}_{\lambda}(\mathcal{U}), \varkappa_{\lambda}(\mathcal{U})), \quad \underline{\mathcal{Z}}(\eta)(\mathcal{U}) = (\varrho_{\eta}(\mathcal{U}), \mathfrak{h}_{\eta}(\mathcal{U}), \varkappa_{\eta}(\mathcal{U})),$$

$$\overline{\mathcal{Z}}(\lambda)(\mathcal{U}) = (\overline{\varrho}_{\lambda}(\mathcal{U}), \overline{\mathfrak{h}}_{\lambda}(\mathcal{U}), \overline{\varkappa}_{\lambda}(\mathcal{U})), \quad \overline{\mathcal{Z}}(\eta)(\mathcal{U}) = (\overline{\varrho}_{\eta}(\mathcal{U}), \overline{\mathfrak{h}}_{\eta}(\mathcal{U}), \overline{\varkappa}_{\eta}(\mathcal{U})),$$

where all membership functions take values in $[0, 1]$ and satisfy the q -spherical constraint. We define the following algebraic operations:

1. Algebraic sum ($\oplus$) between two different IVq-SFRNs:

$$\mathcal{Z}(\lambda) \oplus \mathcal{Z}(\eta) = (\underline{\mathcal{Z}}(\lambda) \oplus \underline{\mathcal{Z}}(\eta), \overline{\mathcal{Z}}(\lambda) \oplus \overline{\mathcal{Z}}(\eta)) \quad (16)$$

where, for each $\mathcal{U} \in \mathbb{X}$,

$$\begin{aligned} \underline{\mathcal{Z}}(\lambda) \oplus \underline{\mathcal{Z}}(\eta)(\mathcal{U}) = & \left\langle \left((\varrho_{\lambda}(\mathcal{U}))^q + (\varrho_{\eta}(\mathcal{U}))^q - (\varrho_{\lambda}(\mathcal{U}) \cdot \varrho_{\eta}(\mathcal{U}))^q \right)^{1/q}, \right. \\ & \left. \mathfrak{h}_{\lambda}(\mathcal{U}) \cdot \mathfrak{h}_{\eta}(\mathcal{U}), \varkappa_{\lambda}(\mathcal{U}) \cdot \varkappa_{\eta}(\mathcal{U}) \right\rangle. \end{aligned} \quad (17)$$

and similarly for $\overline{\mathcal{Z}}(\lambda) \oplus \overline{\mathcal{Z}}(\eta)$.

2. Algebraic product ($\otimes$) between two different IVq-SFRNs:

$$\mathcal{Z}(\lambda) \otimes \mathcal{Z}(\eta) = (\underline{\mathcal{Z}}(\lambda) \otimes \underline{\mathcal{Z}}(\eta), \overline{\mathcal{Z}}(\lambda) \otimes \overline{\mathcal{Z}}(\eta)) \quad (18)$$

where, for each $\mathcal{U} \in \mathbb{X}$,

$$\begin{aligned} \underline{\mathcal{Z}}(\lambda) \otimes \underline{\mathcal{Z}}(\eta)(\mathcal{U}) = & \left\langle \varrho_{\lambda}(\mathcal{U}) \cdot \varrho_{\eta}(\mathcal{U}), \right. \\ & \left. \mathfrak{h}_{\lambda}(\mathcal{U}) \cdot \mathfrak{h}_{\eta}(\mathcal{U}), \right. \\ & \left. \left((\varkappa_{\lambda}(\mathcal{U}))^q + (\varkappa_{\eta}(\mathcal{U}))^q - (\varkappa_{\lambda}(\mathcal{U}) \cdot \varkappa_{\eta}(\mathcal{U}))^q \right)^{1/q} \right\rangle. \end{aligned} \quad (19)$$

and similarly for $\overline{\mathcal{Z}}(\lambda) \otimes \overline{\mathcal{Z}}(\eta)$.

Example 5. We illustrate the algebraic operations between two different IVq-SFRNs using the same investment project selection problem. Consider $\mathcal{Z}(\lambda)$ and $\mathcal{Z}(\eta)$ from the previous example.

The lower and upper approximations of λ and η are as follows in Table 9:

Table 9. Lower and upper approximations of λ and η .

x_i	$\underline{\mathcal{Z}}(\lambda)$			$\overline{\mathcal{Z}}(\lambda)$			$\underline{\mathcal{Z}}(\eta)$			$\overline{\mathcal{Z}}(\eta)$		
	ϱ^+	$\mathfrak{h}^+$	$\varkappa^+$	ϱ^+	$\mathfrak{h}^+$	$\varkappa^+$	ϱ^+	$\mathfrak{h}^+$	$\varkappa^+$	ϱ^+	$\mathfrak{h}^+$	$\varkappa^+$
x_1	0.5	0.2	0.2	0.8	0.1	0.1	0.2	0.5	0.3	0.6	0.2	0.2
x_2	0.5	0.2	0.2	0.9	0.1	0.1	0.2	0.6	0.4	0.8	0.1	0.1
x_3	0.4	0.2	0.2	0.9	0.1	0.1	0.2	0.5	0.3	0.9	0.1	0.1
x_4	0.4	0.2	0.2	0.8	0.1	0.1	0.2	0.5	0.3	0.6	0.2	0.2

Algebraic Sum $\underline{\mathcal{Z}}(\lambda) \oplus \underline{\mathcal{Z}}(\eta)$

The results of the algebraic sum are presented in Table 10.

Table 10. Algebraic sum $\underline{\mathcal{Z}}(\lambda) \oplus \underline{\mathcal{Z}}(\eta)$, $q = 3$.

x_i	$\underline{\mathcal{Z}}(\lambda) \oplus \underline{\mathcal{Z}}(\eta)$			$\overline{\mathcal{Z}}(\lambda) \oplus \overline{\mathcal{Z}}(\eta)$		
	$\wp^+$	$\mathfrak{h}^+$	$\varkappa^+$	$\wp^+$	$\mathfrak{h}^+$	$\varkappa^+$
x_1	0.509	0.100	0.010	0.851	0.020	0.010
x_2	0.509	0.120	0.010	0.928	0.010	0.010
x_3	0.425	0.100	0.010	0.928	0.010	0.010
x_4	0.425	0.100	0.010	0.851	0.020	0.010

Algebraic Product $\underline{\mathcal{Z}}(\lambda) \otimes \underline{\mathcal{Z}}(\eta)$

The results of the algebraic product are presented in Table 11.

Table 11. Algebraic product $\underline{\mathcal{Z}}(\lambda) \otimes \underline{\mathcal{Z}}(\eta)$, $q = 3$.

x_i	$\underline{\mathcal{Z}}(\lambda) \otimes \underline{\mathcal{Z}}(\eta)$			$\overline{\mathcal{Z}}(\lambda) \otimes \overline{\mathcal{Z}}(\eta)$		
	$\wp^+$	$\mathfrak{h}^+$	$\varkappa^+$	$\wp^+$	$\mathfrak{h}^+$	$\varkappa^+$
x_1	0.100	0.010	0.326	0.480	0.020	0.208
x_2	0.100	0.010	0.403	0.720	0.010	0.141
x_3	0.080	0.010	0.326	0.810	0.010	0.141
x_4	0.080	0.010	0.326	0.480	0.020	0.208

For $\underline{\mathcal{Z}}(\lambda) \oplus \underline{\mathcal{Z}}(\eta)$ at x_1 :

$$(0.509)^3 + (0.100)^3 + (0.010)^3 = 0.132 + 0.001 + 0.000001 = 0.133 \leq 1.$$

For $\overline{\mathcal{Z}}(\lambda) \otimes \overline{\mathcal{Z}}(\eta)$ at x_1 :

$$(0.480)^3 + (0.020)^3 + (0.208)^3 = 0.1106 + 0.000008 + 0.0090 = 0.1196 \leq 1.$$

All computed values satisfy the q -spherical constraint, confirming the validity of the operations.

Proposition 2. Let $(\mathbb{X}, \mathbb{P}, \mathcal{Z})$ be an IV q -SFRS approximation space with mapping $\phi : \mathbb{X} \rightarrow \mathbb{P}$. For any $\lambda \in \mathbb{S}_q(\mathbb{P})$, define $\lambda(\mathcal{U}) := \lambda(\phi(\mathcal{U}))$. Then:

1. If $\mathcal{Z}$ is reflexive, $\underline{\mathcal{Z}}(\lambda) \subseteq \lambda \subseteq \overline{\mathcal{Z}}(\lambda)$.
2. $\underline{\mathcal{Z}}(\emptyset) = \emptyset$.
3. $\overline{\mathcal{Z}}(\emptyset) \neq \emptyset$ in general.
4. For $\lambda = \mathbf{1}_{\mathbb{P}}$:

$$\underline{\mathcal{Z}}(\mathbf{1}_{\mathbb{P}}) = \mathbb{X} \quad (\text{if } \mathcal{Z} \text{ is serial}), \quad \overline{\mathcal{Z}}(\mathbf{1}_{\mathbb{P}}) = \mathbb{X}.$$

5. If $\lambda \subseteq \eta$, then $\underline{\mathcal{Z}}(\lambda) \subseteq \underline{\mathcal{Z}}(\eta)$.
6. If $\lambda \subseteq \eta$, then $\overline{\mathcal{Z}}(\lambda) \subseteq \overline{\mathcal{Z}}(\eta)$.
7. $\underline{\mathcal{Z}}(\lambda) \subseteq \overline{\mathcal{Z}}(\lambda)$.

Proof. For any $\mathcal{U} \in \mathbb{X}$, let $\omega_{\mathcal{U}} = \phi(\mathcal{U}) \in \mathbb{P}$.

1. If $\mathcal{Z}$ is reflexive, then $\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega_{\mathcal{U}}) = 1$, $\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega_{\mathcal{U}}) = 0$, $\varkappa_{\mathcal{Z}}^+(\mathcal{U}, \omega_{\mathcal{U}}) = 0$. Hence, for the membership degree:

$$\begin{aligned}\wp_{\mathcal{Z}(\lambda)}^+(\mathcal{U}) &= \bigwedge_{\omega \in \mathbb{P}} [\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega) \wedge \wp_{\lambda}^+(\omega)] \\ &\leq \wp_{\mathcal{Z}}^+(\mathcal{U}, \omega_{\mathcal{U}}) \wedge \wp_{\lambda}^+(\omega_{\mathcal{U}}) \\ &= 1 \wedge \wp_{\lambda}^+(\mathcal{U}) = \wp_{\lambda}^+(\mathcal{U}),\end{aligned}$$

and

$$\begin{aligned}\wp_{\mathcal{Z}(\lambda)}^+(\mathcal{U}) &= \bigvee_{\omega \in \mathbb{P}} [\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega) \vee \wp_{\lambda}^+(\omega)] \\ &\geq \wp_{\mathcal{Z}}^+(\mathcal{U}, \omega_{\mathcal{U}}) \vee \wp_{\lambda}^+(\omega_{\mathcal{U}}) \\ &= 1 \vee \wp_{\lambda}^+(\mathcal{U}) \geq \wp_{\lambda}^+(\mathcal{U}).\end{aligned}$$

For the neutrality degree, since $\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega_{\mathcal{U}}) = 0$, we have

$$\wp_{\mathcal{Z}(\lambda)}^+(\mathcal{U}) = \bigwedge_{\omega} [\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega) \vee \wp_{\lambda}^+(\omega)] \geq 0 \vee \wp_{\lambda}^+(\mathcal{U}) = \wp_{\lambda}^+(\mathcal{U}),$$

and

$$\wp_{\mathcal{Z}(\lambda)}^+(\mathcal{U}) = \bigvee_{\omega} [\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega) \wedge \wp_{\lambda}^+(\omega)] \leq 0 \wedge \wp_{\lambda}^+(\mathcal{U}) = \wp_{\lambda}^+(\mathcal{U}).$$

For the non-membership degree, since $\varkappa_{\mathcal{Z}}^+(\mathcal{U}, \omega_{\mathcal{U}}) = 0$, analogous inequalities hold:

$$\varkappa_{\mathcal{Z}(\lambda)}^+(\mathcal{U}) = \bigwedge_{\omega} [\varkappa_{\mathcal{Z}}^+(\mathcal{U}, \omega) \vee \varkappa_{\lambda}^+(\omega)] \geq 0 \vee \varkappa_{\lambda}^+(\mathcal{U}) = \varkappa_{\lambda}^+(\mathcal{U}),$$

and

$$\varkappa_{\mathcal{Z}(\lambda)}^+(\mathcal{U}) = \bigvee_{\omega} [\varkappa_{\mathcal{Z}}^+(\mathcal{U}, \omega) \wedge \varkappa_{\lambda}^+(\omega)] \leq 0 \wedge \varkappa_{\lambda}^+(\mathcal{U}) = \varkappa_{\lambda}^+(\mathcal{U}).$$

Thus, combining all three component-wise inequalities, we obtain $\mathcal{Z}(\lambda) \subseteq \lambda \subseteq \overline{\mathcal{Z}}(\lambda)$.

2. For $\lambda = \emptyset$, define $\wp_{\lambda}^+(\omega) = 0$, $\wp_{\lambda}^+(\omega) = 0$, $\varkappa_{\lambda}^+(\omega) = 1$ for all $\omega \in \mathbb{P}$. Then,

$$\wp_{\mathcal{Z}(\lambda)}^+(\mathcal{U}) = \bigwedge_{\omega \in \mathbb{P}} [\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega) \wedge 0] = 0,$$

and similarly $\wp_{\mathcal{Z}(\lambda)}^+(\mathcal{U}) = 0$ and $\varkappa_{\mathcal{Z}(\lambda)}^+(\mathcal{U}) = 1$, so $\mathcal{Z}(\emptyset) = \emptyset$.

3. For $\lambda = \emptyset$, the upper approximation is

$$\wp_{\mathcal{Z}(\lambda)}^+(\mathcal{U}) = \bigvee_{\omega \in \mathbb{P}} [\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega) \vee 0] = \bigvee_{\omega \in \mathbb{P}} \wp_{\mathcal{Z}}^+(\mathcal{U}, \omega) \geq 0,$$

which is generally non-zero. Similarly, $\wp_{\mathcal{Z}(\lambda)}^+(\mathcal{U}) = \bigvee_{\omega} [\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega) \wedge 0] = 0$ and $\varkappa_{\mathcal{Z}(\lambda)}^+(\mathcal{U}) = \bigvee_{\omega} [\varkappa_{\mathcal{Z}}^+(\mathcal{U}, \omega) \wedge 1] = \bigvee_{\omega} \varkappa_{\mathcal{Z}}^+(\mathcal{U}, \omega) \geq 0$. Hence, $\overline{\mathcal{Z}}(\emptyset) \neq \emptyset$ in general.

4. For $\lambda = 1_{\mathbb{P}}$, we have $\wp_{\lambda}^+(\omega) = 1$, $\wp_{\lambda}^+(\omega) = 0$, $\varkappa_{\lambda}^+(\omega) = 0$. Then,

$$\begin{aligned}\wp_{\mathcal{Z}(\lambda)}^+(\mathcal{U}) &= \bigwedge_{\omega \in \mathbb{P}} [\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega) \wedge 1] = \bigwedge_{\omega \in \mathbb{P}} \wp_{\mathcal{Z}}^+(\mathcal{U}, \omega), \\ \wp_{\mathcal{Z}(\lambda)}^+(\mathcal{U}) &= \bigvee_{\omega \in \mathbb{P}} [\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega) \vee 1] = 1.\end{aligned}$$

For the neutrality and non-membership degrees, since $\wp_{\lambda}^+(\omega) = 0$ and $\varkappa_{\lambda}^+(\omega) = 0$, we have $\wp_{\mathcal{Z}(\lambda)}^+(\mathcal{U}) = 0$ and $\varkappa_{\mathcal{Z}(\lambda)}^+(\mathcal{U}) = 0$, and similarly for the upper approximations.

If $\mathcal{Z}$ is serial, then for each $\mathcal{U}$ there exists ω with $\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega) = 1$, so $\underline{\mathcal{Z}}(\mathbf{1}_{\mathbb{P}}) = \mathbb{X}$. Clearly, $\overline{\mathcal{Z}}(\mathbf{1}_{\mathbb{P}}) = \mathbb{X}$.

5. If $\lambda \subseteq \eta$, then $\wp_{\lambda}^+(\omega) \leq \wp_{\eta}^+(\omega)$, $\mathfrak{h}_{\lambda}^+(\omega) \geq \mathfrak{h}_{\eta}^+(\omega)$, $\varkappa_{\lambda}^+(\omega) \geq \varkappa_{\eta}^+(\omega)$. Hence, for the membership degree:

$$\wp_{\underline{\mathcal{Z}}(\lambda)}^+(\mathcal{U}) = \bigwedge_{\omega} [\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega) \wedge \wp_{\lambda}^+(\omega)] \leq \bigwedge_{\omega} [\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega) \wedge \wp_{\eta}^+(\omega)] = \wp_{\underline{\mathcal{Z}}(\eta)}^+(\mathcal{U}),$$

and for the neutrality degree:

$$\mathfrak{h}_{\underline{\mathcal{Z}}(\lambda)}^+(\mathcal{U}) = \bigwedge_{\omega} [\mathfrak{h}_{\mathcal{Z}}^+(\mathcal{U}, \omega) \vee \mathfrak{h}_{\lambda}^+(\omega)] \geq \bigwedge_{\omega} [\mathfrak{h}_{\mathcal{Z}}^+(\mathcal{U}, \omega) \vee \mathfrak{h}_{\eta}^+(\omega)] = \mathfrak{h}_{\underline{\mathcal{Z}}(\eta)}^+(\mathcal{U}),$$

and for the non-membership degree:

$$\varkappa_{\underline{\mathcal{Z}}(\lambda)}^+(\mathcal{U}) = \bigwedge_{\omega} [\varkappa_{\mathcal{Z}}^+(\mathcal{U}, \omega) \vee \varkappa_{\lambda}^+(\omega)] \geq \bigwedge_{\omega} [\varkappa_{\mathcal{Z}}^+(\mathcal{U}, \omega) \vee \varkappa_{\eta}^+(\omega)] = \varkappa_{\underline{\mathcal{Z}}(\eta)}^+(\mathcal{U}).$$

Thus, $\underline{\mathcal{Z}}(\lambda) \subseteq \underline{\mathcal{Z}}(\eta)$.

6. Similarly, using the upper approximation formulas, for the membership degree:

$$\wp_{\overline{\mathcal{Z}}(\lambda)}^+(\mathcal{U}) = \bigvee_{\omega} [\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega) \vee \wp_{\lambda}^+(\omega)] \leq \bigvee_{\omega} [\wp_{\mathcal{Z}}^+(\mathcal{U}, \omega) \vee \wp_{\eta}^+(\omega)] = \wp_{\overline{\mathcal{Z}}(\eta)}^+(\mathcal{U}),$$

for the neutrality degree:

$$\mathfrak{h}_{\overline{\mathcal{Z}}(\lambda)}^+(\mathcal{U}) = \bigvee_{\omega} [\mathfrak{h}_{\mathcal{Z}}^+(\mathcal{U}, \omega) \wedge \mathfrak{h}_{\lambda}^+(\omega)] \leq \bigvee_{\omega} [\mathfrak{h}_{\mathcal{Z}}^+(\mathcal{U}, \omega) \wedge \mathfrak{h}_{\eta}^+(\omega)] = \mathfrak{h}_{\overline{\mathcal{Z}}(\eta)}^+(\mathcal{U}),$$

and for the non-membership degree:

$$\varkappa_{\overline{\mathcal{Z}}(\lambda)}^+(\mathcal{U}) = \bigvee_{\omega} [\varkappa_{\mathcal{Z}}^+(\mathcal{U}, \omega) \wedge \varkappa_{\lambda}^+(\omega)] \leq \bigvee_{\omega} [\varkappa_{\mathcal{Z}}^+(\mathcal{U}, \omega) \wedge \varkappa_{\eta}^+(\omega)] = \varkappa_{\overline{\mathcal{Z}}(\eta)}^+(\mathcal{U}).$$

Hence, $\overline{\mathcal{Z}}(\lambda) \subseteq \overline{\mathcal{Z}}(\eta)$.

7. For any $\mathcal{U} \in \mathbb{X}$,

$$\wp_{\underline{\mathcal{Z}}(\lambda)}^+(\mathcal{U}) = \bigwedge_{\omega} (a_{\omega} \wedge b_{\omega}) \leq \bigvee_{\omega} (a_{\omega} \vee b_{\omega}) = \wp_{\underline{\mathcal{Z}}(\lambda)}^+(\mathcal{U}),$$

since $a_{\omega} \wedge b_{\omega} \leq a_{\omega} \vee b_{\omega}$ for each ω . The same argument applies to the $\mathfrak{h}$ and $\varkappa$ components:

$$\mathfrak{h}_{\underline{\mathcal{Z}}(\lambda)}^+(\mathcal{U}) = \bigwedge_{\omega} (c_{\omega} \vee d_{\omega}) \leq \bigvee_{\omega} (c_{\omega} \wedge d_{\omega}) = \mathfrak{h}_{\underline{\mathcal{Z}}(\lambda)}^+(\mathcal{U}),$$

and similarly for $\varkappa$. Thus, $\underline{\mathcal{Z}}(\lambda) \subseteq \underline{\mathcal{Z}}(\lambda)$.

□

Proposition 3. For any $\lambda, \eta \in \mathbb{S}_q(\mathbb{P})$, the following properties hold:

1. $\underline{\mathcal{Z}}(\lambda \cap \eta) \subseteq \underline{\mathcal{Z}}(\lambda) \cap \underline{\mathcal{Z}}(\eta)$.
2. $\overline{\mathcal{Z}}(\lambda \cup \eta) \supseteq \overline{\mathcal{Z}}(\lambda) \cup \overline{\mathcal{Z}}(\eta)$.
3. $\overline{\mathcal{Z}}(\lambda \cap \eta) \subseteq \overline{\mathcal{Z}}(\lambda) \cap \overline{\mathcal{Z}}(\eta)$.

If $\mathcal{Z}$ is an equivalence relation, then the equalities hold.

Proof. We prove each property as follows:

1. $\underline{\mathcal{Z}}(\lambda \cap \eta) \subseteq \underline{\mathcal{Z}}(\lambda) \cap \underline{\mathcal{Z}}(\eta)$. For any $\mathcal{U} \in \mathbb{X}$ and for each $\omega \in \mathbb{P}$:

$$\wp_{\underline{\mathcal{Z}}(\lambda \cap \eta)}^+(\mathcal{U}, \omega) \wedge \left(\wp_{\lambda}^+(\omega) \wedge \wp_{\eta}^+(\omega) \right) = \wp_{\underline{\mathcal{Z}}}^+(\mathcal{U}, \omega) \wedge \wp_{\lambda}^+(\omega) \wedge \wp_{\eta}^+(\omega).$$

Taking the infimum over ω :

$$\wp_{\underline{\mathcal{Z}}(\lambda \cap \eta)}^+(\mathcal{U}) = \bigwedge_{\omega} \left(\wp_{\underline{\mathcal{Z}}}^+(\mathcal{U}, \omega) \wedge \wp_{\lambda}^+(\omega) \wedge \wp_{\eta}^+(\omega) \right).$$

Since $\bigwedge_{\omega} (a_{\omega} \wedge b_{\omega} \wedge c_{\omega}) \leq \bigwedge_{\omega} (a_{\omega} \wedge b_{\omega})$ and also $\leq \bigwedge_{\omega} (a_{\omega} \wedge c_{\omega})$, we have

$$\begin{aligned} \wp_{\underline{\mathcal{Z}}(\lambda \cap \eta)}^+(\mathcal{U}) &\leq \left(\bigwedge_{\omega} \left(\wp_{\underline{\mathcal{Z}}}^+(\mathcal{U}, \omega) \wedge \wp_{\lambda}^+(\omega) \right) \right) \\ &\quad \wedge \left(\bigwedge_{\omega} \left(\wp_{\underline{\mathcal{Z}}}^+(\mathcal{U}, \omega) \wedge \wp_{\eta}^+(\omega) \right) \right) \\ &= \wp_{\lambda}^+(\mathcal{U}) \wedge \wp_{\eta}^+(\mathcal{U}). \end{aligned}$$

The same argument extends to the $\mathfrak{h}$ and $\mathfrak{x}$ components. For the neutrality degree, the inequality reverses due to the use of $\vee$, since adding an extra term increases the join: $\bigvee_{\omega} (a_{\omega} \vee b_{\omega} \vee c_{\omega}) \geq \bigvee_{\omega} (a_{\omega} \vee b_{\omega})$.

$$\mathfrak{h}_{\underline{\mathcal{Z}}(\lambda \cap \eta)}^+(\mathcal{U}) = \bigvee_{\omega} [\mathfrak{h}_{\underline{\mathcal{Z}}}^+(\mathcal{U}, \omega) \vee \mathfrak{h}_{\lambda}^+(\omega) \vee \mathfrak{h}_{\eta}^+(\omega)] \geq \mathfrak{h}_{\underline{\mathcal{Z}}(\lambda)}^+(\mathcal{U}) \vee \mathfrak{h}_{\underline{\mathcal{Z}}(\eta)}^+(\mathcal{U}).$$

Thus, $\underline{\mathcal{Z}}(\lambda \cap \eta) \subseteq \underline{\mathcal{Z}}(\lambda) \cap \underline{\mathcal{Z}}(\eta)$ holds for all three components.

2. $\overline{\mathcal{Z}}(\lambda \cup \eta) \supseteq \overline{\mathcal{Z}}(\lambda) \cup \overline{\mathcal{Z}}(\eta)$.

$$\begin{aligned} \wp_{\overline{\mathcal{Z}}(\lambda \cup \eta)}^+(\mathcal{U}) &= \bigvee_{\omega} \left(\wp_{\overline{\mathcal{Z}}}^+(\mathcal{U}, \omega) \vee \wp_{\lambda}^+(\omega) \vee \wp_{\eta}^+(\omega) \right) \\ &\geq \left(\bigvee_{\omega} \left(\wp_{\overline{\mathcal{Z}}}^+(\mathcal{U}, \omega) \vee \wp_{\lambda}^+(\omega) \right) \right) \vee \left(\bigvee_{\omega} \left(\wp_{\overline{\mathcal{Z}}}^+(\mathcal{U}, \omega) \vee \wp_{\eta}^+(\omega) \right) \right) \\ &= \overline{\wp}_{\lambda}^+(\mathcal{U}) \vee \overline{\wp}_{\eta}^+(\mathcal{U}). \end{aligned}$$

For the neutrality degree, replacing $\vee$ by $\wedge$ reverses the inequality direction. The same component-wise reasoning ensures the result holds for all three components.

3. $\overline{\mathcal{Z}}(\lambda \cap \eta) \subseteq \overline{\mathcal{Z}}(\lambda) \cap \overline{\mathcal{Z}}(\eta)$.

$$\begin{aligned} \wp_{\overline{\mathcal{Z}}(\lambda \cap \eta)}^+(\mathcal{U}) &= \bigvee_{\omega} \left(\wp_{\overline{\mathcal{Z}}}^+(\mathcal{U}, \omega) \vee \left(\wp_{\lambda}^+(\omega) \wedge \wp_{\eta}^+(\omega) \right) \right) \\ &\leq \left(\bigvee_{\omega} \left(\wp_{\overline{\mathcal{Z}}}^+(\mathcal{U}, \omega) \vee \wp_{\lambda}^+(\omega) \right) \right) \wedge \left(\bigvee_{\omega} \left(\wp_{\overline{\mathcal{Z}}}^+(\mathcal{U}, \omega) \vee \wp_{\eta}^+(\omega) \right) \right) \\ &= \overline{\wp}_{\lambda}^+(\mathcal{U}) \wedge \overline{\wp}_{\eta}^+(\mathcal{U}). \end{aligned}$$

The same component-wise argument applies to the $\mathfrak{h}$ and $\mathfrak{x}$ components, yielding the desired inclusion.

□

Definition 15. For an IVq-SF set $\lambda \in \mathbb{S}_q(\mathbb{P})$, its strong complement $\sim \lambda$ is defined by swapping membership and non-membership while preserving the neutral interval:

$$\wp_{\sim \lambda}^- = \mathfrak{x}_{\lambda}^-, \quad \wp_{\sim \lambda}^+ = \mathfrak{x}_{\lambda}^+, \quad \mathfrak{h}_{\sim \lambda}^- = \mathfrak{h}_{\lambda}^-, \quad \mathfrak{h}_{\sim \lambda}^+ = \mathfrak{h}_{\lambda}^+, \quad \mathfrak{x}_{\sim \lambda}^- = \wp_{\lambda}^-, \quad \mathfrak{x}_{\sim \lambda}^+ = \wp_{\lambda}^+.$$

For an IV q -SFRS $(\underline{\mathcal{Z}}(\lambda), \overline{\mathcal{Z}}(\lambda))$, the rough complement is

$$\sim (\underline{\mathcal{Z}}(\lambda), \overline{\mathcal{Z}}(\lambda)) = (\sim \overline{\mathcal{Z}}(\lambda), \sim \underline{\mathcal{Z}}(\lambda)).$$

This complement is closed because it merely permutes the membership and non-membership upper endpoints; hence, $(\varkappa^+)^q + (\mathfrak{h}^+)^q + (\wp^+)^q \leq 1$ whenever the original IV q -SFN is admissible.

Theorem 1. Assume that the relation-induced lower and upper approximations are computed by the dual min–max scheme of Definition 11 and finalized by the same projection operator Π_q . Then, the rough complement reverses the lower and upper approximations:

$$\sim (\underline{\mathcal{Z}}(\lambda), \overline{\mathcal{Z}}(\lambda)) = (\sim \overline{\mathcal{Z}}(\lambda), \sim \underline{\mathcal{Z}}(\lambda)).$$

The identity is definitional for the rough pair and is admissible in $\mathbb{S}_q$ by the closure of the complement and the projection operator.

Proof. By Definition 15, the complement of each IV q -SFN is obtained by the permutation $(\wp, \mathfrak{h}, \varkappa) \mapsto (\varkappa, \mathfrak{h}, \wp)$, which preserves the q -power sum. Rough complementation exchanges necessity and possibility, so the lower approximation of the complement corresponds to the complement of the upper approximation and conversely. If a component-wise approximation produces a non-admissible preliminary vector, Definition 11 applies the same radial projection Π_q before the complement is used in subsequent algebraic or TOPSIS operations. Hence, both sides are valid IV q -SF rough pairs. $\square$

Corollary 1. From the duality properties, we obtain the following:

1. $\underline{\mathcal{Z}}(\sim \lambda) = \sim \overline{\mathcal{Z}}(\lambda)$.
2. $\overline{\mathcal{Z}}(\sim \lambda) = \sim \underline{\mathcal{Z}}(\lambda)$.
3. $\underline{\mathcal{Z}}(\lambda) = \overline{\mathcal{Z}}(\lambda)$ if and only if λ is definable.

Proposition 4. The operations $\oplus$ and $\otimes$ on IV q -SFRNs are commutative for any $q \geq 1$, i.e.,

$$\underline{\mathcal{Z}}(\lambda) \oplus \overline{\mathcal{Z}}(\lambda) = \overline{\mathcal{Z}}(\lambda) \oplus \underline{\mathcal{Z}}(\lambda), \quad \underline{\mathcal{Z}}(\lambda) \otimes \overline{\mathcal{Z}}(\lambda) = \overline{\mathcal{Z}}(\lambda) \otimes \underline{\mathcal{Z}}(\lambda).$$

Proof. For any $\mathfrak{U} \in \mathbb{X}$, let

$$\wp_1 = \wp_{\lambda}^+(\mathfrak{U}), \quad \wp_2 = \wp_{\lambda}^+(\mathfrak{U}), \quad \mathfrak{h}_1 = \mathfrak{h}_{\lambda}^+(\mathfrak{U}), \quad \mathfrak{h}_2 = \mathfrak{h}_{\lambda}^+(\mathfrak{U}), \quad \varkappa_1 = \varkappa_{\lambda}^+(\mathfrak{U}), \quad \varkappa_2 = \varkappa_{\lambda}^+(\mathfrak{U}).$$

For $\oplus$:

$$\begin{aligned} \wp_{\lambda \oplus \lambda}^+(\mathfrak{U}) &= \sqrt[q]{\wp_1^q + \wp_2^q - \wp_1^q \wp_2^q} = \sqrt[q]{\wp_2^q + \wp_1^q - \wp_2^q \wp_1^q} = \wp_{\lambda \oplus \lambda}^+(\mathfrak{U}), \\ \mathfrak{h}_{\lambda \oplus \lambda}^+(\mathfrak{U}) &= \mathfrak{h}_1 \mathfrak{h}_2 = \mathfrak{h}_2 \mathfrak{h}_1 = \mathfrak{h}_{\lambda \oplus \lambda}^+(\mathfrak{U}), \\ \varkappa_{\lambda \oplus \lambda}^+(\mathfrak{U}) &= \varkappa_1 \varkappa_2 = \varkappa_2 \varkappa_1 = \varkappa_{\lambda \oplus \lambda}^+(\mathfrak{U}). \end{aligned}$$

For $\otimes$:

$$\begin{aligned} \wp_{\lambda \otimes \lambda}^+(\mathfrak{U}) &= \wp_1 \wp_2 = \wp_2 \wp_1 = \wp_{\lambda \otimes \lambda}^+(\mathfrak{U}), \\ \mathfrak{h}_{\lambda \otimes \lambda}^+(\mathfrak{U}) &= \mathfrak{h}_1 \mathfrak{h}_2 = \mathfrak{h}_2 \mathfrak{h}_1 = \mathfrak{h}_{\lambda \otimes \lambda}^+(\mathfrak{U}), \\ \varkappa_{\lambda \otimes \lambda}^+(\mathfrak{U}) &= \sqrt[q]{\varkappa_1^q + \varkappa_2^q - \varkappa_1^q \varkappa_2^q} = \sqrt[q]{\varkappa_2^q + \varkappa_1^q - \varkappa_2^q \varkappa_1^q} = \varkappa_{\lambda \otimes \lambda}^+(\mathfrak{U}). \end{aligned}$$

The same reasoning holds for the lower bounds. Thus, $\oplus$ and $\otimes$ are commutative. $\square$

Proposition 5. For any $q \geq 1$, the operations $\oplus$ and $\otimes$ on IVq-SFRNs are associative:

$$(\underline{\mathcal{Z}}(\lambda) \oplus \overline{\mathcal{Z}}(\lambda)) \oplus \underline{\mathcal{Z}}(\eta) = \underline{\mathcal{Z}}(\lambda) \oplus (\overline{\mathcal{Z}}(\lambda) \oplus \underline{\mathcal{Z}}(\eta)),$$

$$(\underline{\mathcal{Z}}(\lambda) \otimes \overline{\mathcal{Z}}(\lambda)) \otimes \underline{\mathcal{Z}}(\eta) = \underline{\mathcal{Z}}(\lambda) \otimes (\overline{\mathcal{Z}}(\lambda) \otimes \underline{\mathcal{Z}}(\eta)).$$

Proof. For any $\mathcal{U} \in \mathbb{X}$, let

$$a = \underline{\wp}_\lambda^+(\mathcal{U}), \quad b = \overline{\wp}_\lambda^+(\mathcal{U}), \quad c = \underline{\wp}_\eta^+(\mathcal{U}).$$

Associativity of $\oplus$: Define $f(x, y) = (x^q + y^q - x^q y^q)^{1/q}$. Then,

$$f(f(a, b), c) = f(a, f(b, c)),$$

since

$$\begin{aligned} [f(f(a, b), c)]^q &= (a^q + b^q - a^q b^q) + c^q - (a^q + b^q - a^q b^q)c^q \\ &= a^q + b^q + c^q - a^q b^q - a^q c^q - b^q c^q + a^q b^q c^q, \\ [f(a, f(b, c))]^q &= a^q + (b^q + c^q - b^q c^q) - a^q(b^q + c^q - b^q c^q) \\ &= a^q + b^q + c^q - a^q b^q - a^q c^q - b^q c^q + a^q b^q c^q. \end{aligned}$$

Thus, $\oplus$ is associative for the membership component, and the same argument applies to the non-membership component due to the identical algebraic structure. The neutrality component is based on standard multiplication, which is associative. Hence, $\oplus$ is associative for all components.

Associativity of $\otimes$: For the membership and neutrality components, the operation is standard multiplication, while the non-membership component follows the same q -based function f , whose associativity has already been established. Therefore, $\otimes$ is associative for all components.

Since all operations are defined component-wise on interval endpoints, the associativity property extends directly to IVq-SFRNs. $\square$

Proposition 6. For any IVq-SFRS $(\underline{\mathcal{Z}}(\lambda), \overline{\mathcal{Z}}(\lambda))$, the algebraic sum $\oplus$ and product $\otimes$ are not idempotent in general:

$$\underline{\mathcal{Z}}(\lambda) \oplus \overline{\mathcal{Z}}(\lambda) \neq \underline{\mathcal{Z}}(\lambda), \quad \underline{\mathcal{Z}}(\lambda) \otimes \overline{\mathcal{Z}}(\lambda) \neq \underline{\mathcal{Z}}(\lambda),$$

unless $\underline{\wp}_\lambda^+(\mathcal{U}) \in \{0, 1\}$ for all $\mathcal{U} \in \mathbb{X}$.

Proposition 7. The operations $\oplus$ and $\otimes$ are monotone under the partial order defined by $\subseteq$ on IVq-SFRNs. So, if $\underline{\mathcal{Z}}(\lambda) \subseteq \underline{\mathcal{Z}}(\eta)$ holds, then, for all $\underline{\mathcal{Z}}(\gamma)$,

$$\underline{\mathcal{Z}}(\lambda) \oplus \underline{\mathcal{Z}}(\gamma) \subseteq \underline{\mathcal{Z}}(\eta) \oplus \underline{\mathcal{Z}}(\gamma), \quad \underline{\mathcal{Z}}(\lambda) \otimes \underline{\mathcal{Z}}(\gamma) \subseteq \underline{\mathcal{Z}}(\eta) \otimes \underline{\mathcal{Z}}(\gamma).$$

Proposition 8. The set of all IVq-SFRS over $(\mathbb{X}, \mathbb{P}, \mathcal{Z})$ is closed under $\oplus$, $\otimes$, $\cup$, $\cap$, scalar multiplication, and power for any $q \geq 1$.

Proof. For any $\mathcal{U} \in \mathbb{X}$, let

$$\wp_1 = \underline{\wp}_\lambda^+(\mathcal{U}), \quad \wp_2 = \overline{\wp}_\lambda^+(\mathcal{U}), \quad \mathfrak{m}_1 = \underline{\mathfrak{m}}_\lambda^+(\mathcal{U}), \quad \mathfrak{m}_2 = \overline{\mathfrak{m}}_\lambda^+(\mathcal{U}), \quad \varkappa_1 = \underline{\varkappa}_\lambda^+(\mathcal{U}), \quad \varkappa_2 = \overline{\varkappa}_\lambda^+(\mathcal{U}).$$

According to its definition:

$$\wp_1^q + \mathfrak{h}_1^q + \varkappa_1^q \leq 1, \quad \wp_2^q + \mathfrak{h}_2^q + \varkappa_2^q \leq 1. \quad (20)$$

For $\oplus$:

$$[\wp_{\oplus}]^q = \wp_1^q + \wp_2^q - \wp_1^q \wp_2^q, \quad [\mathfrak{h}_{\oplus}]^q = \mathfrak{h}_1^q \mathfrak{h}_2^q, \quad [\varkappa_{\oplus}]^q = \varkappa_1^q \varkappa_2^q.$$

Then,

$$\begin{aligned} [\wp_{\oplus}]^q + [\mathfrak{h}_{\oplus}]^q + [\varkappa_{\oplus}]^q &= (\wp_1^q + \varkappa_1^q) + (\wp_2^q + \varkappa_2^q) + \mathfrak{h}_1^q \mathfrak{h}_2^q - \wp_1^q \wp_2^q - \varkappa_1^q \varkappa_2^q \\ &\leq (1 - \mathfrak{h}_1^q) + (1 - \mathfrak{h}_2^q) + \mathfrak{h}_1^q \mathfrak{h}_2^q \quad \text{by (3)} \\ &= 1 + (1 - \mathfrak{h}_1^q)(1 - \mathfrak{h}_2^q) - (\wp_1^q \wp_2^q + \varkappa_1^q \varkappa_2^q) \\ &\leq 1. \end{aligned}$$

For $\otimes$:

$$[\wp_{\otimes}]^q = \wp_1^q \wp_2^q, \quad [\mathfrak{h}_{\otimes}]^q = \mathfrak{h}_1^q \mathfrak{h}_2^q, \quad [\varkappa_{\otimes}]^q = \varkappa_1^q + \varkappa_2^q - \varkappa_1^q \varkappa_2^q.$$

Then,

$$\begin{aligned} [\wp_{\otimes}]^q + [\mathfrak{h}_{\otimes}]^q + [\varkappa_{\otimes}]^q &= \wp_1^q \wp_2^q + (\mathfrak{h}_1^q + \varkappa_1^q) + (\mathfrak{h}_2^q + \varkappa_2^q) - \mathfrak{h}_1^q \mathfrak{h}_2^q - \varkappa_1^q \varkappa_2^q \\ &\leq \wp_1^q \wp_2^q + (1 - \wp_1^q) + (1 - \wp_2^q) - \mathfrak{h}_1^q \mathfrak{h}_2^q - \varkappa_1^q \varkappa_2^q \quad \text{by (3)} \\ &= 1 + (1 - \wp_1^q)(1 - \wp_2^q) - (\mathfrak{h}_1^q \mathfrak{h}_2^q + \varkappa_1^q \varkappa_2^q) \\ &\leq 1. \end{aligned}$$

For $\cup$ and $\cap$:

$$\wp_{\cup} = \max(\wp_1, \wp_2), \quad \mathfrak{h}_{\cup} = \min(\mathfrak{h}_1, \mathfrak{h}_2), \quad \varkappa_{\cup} = \min(\varkappa_1, \varkappa_2),$$

$$\wp_{\cap} = \min(\wp_1, \wp_2), \quad \mathfrak{h}_{\cap} = \max(\mathfrak{h}_1, \mathfrak{h}_2), \quad \varkappa_{\cap} = \max(\varkappa_1, \varkappa_2).$$

Since $\max(\wp_1, \wp_2)^q \leq \wp_1^q + \wp_2^q$ and $\min(\mathfrak{h}_1, \mathfrak{h}_2)^q \leq \mathfrak{h}_1^q + \mathfrak{h}_2^q$, the constraint follows directly from (3).

For scalar multiplication ($\alpha > 0$):

$$\wp_{\alpha \cdot \underline{\lambda}}^+(\mathcal{U}) = \left(1 - (1 - \wp_1^q)^{\alpha}\right)^{1/q}, \quad \mathfrak{h}_{\alpha \cdot \underline{\lambda}}^+(\mathcal{U}) = \mathfrak{h}_1^{\alpha}, \quad \varkappa_{\alpha \cdot \underline{\lambda}}^+(\mathcal{U}) = \left(1 - (1 - \varkappa_1^q)^{\alpha}\right)^{1/q}.$$

Then,

$$[\wp_{\alpha \cdot \underline{\lambda}}^+]^q + [\mathfrak{h}_{\alpha \cdot \underline{\lambda}}^+]^q + [\varkappa_{\alpha \cdot \underline{\lambda}}^+]^q = 1 - (1 - \wp_1^q)^{\alpha} + \mathfrak{h}_1^{q\alpha} + 1 - (1 - \varkappa_1^q)^{\alpha}.$$

Using $\mathfrak{h}_1^{q\alpha} \leq \mathfrak{h}_1^q$ and $(1 - \wp_1^q)^{\alpha} + (1 - \varkappa_1^q)^{\alpha} \geq 1 - \wp_1^q + 1 - \varkappa_1^q$, we obtain the desired inequality.

For power operation ($\alpha > 0$):

$$\wp_{\underline{\lambda}^{\alpha}}^+(\mathcal{U}) = \wp_1^{\alpha}, \quad \mathfrak{h}_{\underline{\lambda}^{\alpha}}^+(\mathcal{U}) = \left(1 - (1 - \mathfrak{h}_1^q)^{\alpha}\right)^{1/q}, \quad \varkappa_{\underline{\lambda}^{\alpha}}^+(\mathcal{U}) = \left(1 - (1 - \varkappa_1^q)^{\alpha}\right)^{1/q}.$$

The proof follows similarly to scalar multiplication, and thus closure holds for all operations. $\square$

4. IVq-SFRS-TOPSIS for Smart City Decision-Making

To implement the theoretical model of the IVq-SFRS in an MCDM methodology, a formal alignment is determined between the decision space and the approximation space. In this setting, the universe of objects $\mathbb{X}$ is the set of alternatives $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m\}$ and the parameter set $\mathbb{P}$ is the set of the criteria of evaluation $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n\}$. The IVq-SF relation $\mathcal{Z}$ can be regarded as an evaluation of expertise specified by $\mathcal{Z}^{(\ell)}(\mathcal{A}_i, \mathcal{C}_j) = \tilde{\theta}_{ij}^{(\ell)}$, where $\tilde{\theta}_{ij}^{(\ell)}$ is an interval-valued q -spherical fuzzy number (IVqSFN) held by the ℓ -th expert. The equivalence classes induced by the indiscernibility operator $\mathcal{T}$ on $\mathbb{P}$ are intuitively semantic groups of criteria, such as, e.g., economic, social, and environmental criteria. These equivalence classes are further employed to build the lower and upper approximations to a target concept $\lambda \in \mathbb{S}_q(\mathbb{P})$, where λ represents the decision-makers' preference towards sustainability. The lower and upper approximations, $\underline{\mathcal{Z}}^{(\ell)}(\lambda)$ and $\overline{\mathcal{Z}}^{(\ell)}(\lambda)$, can be regarded as conservative and liberal evaluations, respectively. These calculations are multidimensionally aggregated over experts to form interval-valued q -spherical fuzzy decision elements $\tilde{r}_{ij}$, whilst the rough boundary information is maintained. At the end, the enhanced rough decision matrix is put into the TOPSIS method. The ranking of alternatives is derived according to the relative closeness to the positive and negative ideal solutions with the aid of suitably defined score functions and distance measures.

4.1. The IVq-SFRS-TOPSIS Methodology

This subsection builds the mathematical basis of our decision-based methodology.

4.1.1. Problem Formulation

Let $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m\}$ be a finite set of alternatives and $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n\}$ be a finite set of criteria. The criteria are classified as benefit criteria J_1 (the more, the better) and cost criteria J_2 (the less, the better) with $J_1 \cup J_2 = \mathcal{C}$ and $J_1 \cap J_2 = \emptyset$. Let $w = (w_1, w_2, \dots, w_n)^T$ be the vector of criteria weights such that $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$. Weights can be completely known, partially known, or completely unknown. Suppose that k experts give assessments under IVq-SF numbers. For the ℓ -th expert ($\ell = 1, \dots, k$), the evaluation matrix is

$$\tilde{D}^{(\ell)} = (\tilde{\theta}_{ij}^{(\ell)})_{m \times n},$$

where $\tilde{\theta}_{ij}^{(\ell)}$ is an IVq-SF number defined as

$$\tilde{\theta}_{ij}^{(\ell)} = \langle [\wp_{ij}^{-(\ell)}, \wp_{ij}^{+(\ell)}], [\mathfrak{m}_{ij}^{-(\ell)}, \mathfrak{m}_{ij}^{+(\ell)}], [\varkappa_{ij}^{-(\ell)}, \varkappa_{ij}^{+(\ell)}] \rangle,$$

satisfying the q -spherical constraint:

$$0 \leq [\wp_{ij}^{+(\ell)}]^q + [\mathfrak{m}_{ij}^{+(\ell)}]^q + [\varkappa_{ij}^{+(\ell)}]^q \leq 1, \quad q \geq 1.$$

4.1.2. Linguistic Terms Conversion

For the convenience of the evaluation, the experts are allowed to give their evaluations in linguistic terms rather than numerical values, where each linguistic term is associated with an IVq-SF number when $q = 3$ as shown in Table 12. The proposed linguistic term set is a monotonic convex or concave fuzzy set according to the authors' perception between linguistic terms from "Very Excellent" to "Average" for membership, and for non-membership and neutrality, an increasing uncertainty is expressed, which means a decrease in the preference intensity. The interval form reflects the intrinsic vagueness of experts' decisions, and all the values satisfy the q -spherical constraint $(\wp^+)^q + (\mathfrak{m}^+)^q + (\varkappa^+)^q \leq 1$, which guarantees the mathematical soundness.

Table 12. Linguistic to IVq-SF conversion ($q = 3$).

Linguistic Term	IVq-SF Number
Very Excellent (VE)	$\langle [0.85, 0.95], [0.05, 0.10], [0.05, 0.10] \rangle$
Excellent (EX)	$\langle [0.75, 0.85], [0.10, 0.15], [0.10, 0.15] \rangle$
Very Good (VG)	$\langle [0.65, 0.75], [0.15, 0.20], [0.15, 0.20] \rangle$
Good (GD)	$\langle [0.55, 0.65], [0.20, 0.25], [0.20, 0.25] \rangle$
Average (AV)	$\langle [0.45, 0.55], [0.25, 0.30], [0.25, 0.30] \rangle$

4.1.3. IVq-SFRS Approximation Space

We define an IVq-SF relation $\mathcal{Z}^{(\ell)}$ (an IVq-SF relation, not the full framework) for each expert on $\mathcal{A} \times \mathcal{C}$:

$$\mathcal{Z}^{(\ell)}(\mathcal{A}_i, \mathcal{C}_j) = \tilde{\theta}_{ij}^{(\ell)}.$$

The pair $(\mathcal{A} \times \mathcal{C}, \mathcal{Z}^{(\ell)})$ forms an IVq-SFRS approximation space.

4.1.4. Indiscernibility Relation and Rough Approximations

Let $\mathcal{T}$ be an equivalence relation (indiscernibility relation) on the criterion set $\mathcal{C}$, defined by

$$\mathcal{C}_i \sim \mathcal{C}_j \iff \text{criteria } i \text{ and } j \text{ belong to the same semantic group.}$$

For any criterion $\mathcal{C}_j$, its equivalence class is denoted as $[\mathcal{C}_j]_{\mathcal{T}}$.

The lower approximation of a target concept λ with respect to $\mathcal{T}$ is defined as

$$\underline{\mathcal{Z}}^{(\ell)}(\lambda)(\mathcal{A}_i, \mathcal{C}_j) = \bigwedge_{\mathcal{C}_k \in [\mathcal{C}_j]_{\mathcal{T}}} \tilde{\theta}_{ik}^{(\ell)}$$

where $\bigwedge$ denotes the component-wise minimum (for membership) and maximum (for non-membership and neutrality).

The upper approximation is defined as

$$\overline{\mathcal{Z}}^{(\ell)}(\lambda)(\mathcal{A}_i, \mathcal{C}_j) = \bigvee_{\mathcal{C}_k \in [\mathcal{C}_j]_{\mathcal{T}}} \tilde{\theta}_{ik}^{(\ell)}$$

where $\bigvee$ denotes the component-wise maximum (for membership) and minimum (for non-membership and neutrality).

4.1.5. Definition of the Target Concept λ

The target concept $\lambda \in \mathbb{S}_q(\mathbb{P})$ is formally defined as an IVq-SF set over the criterion space, representing the ideal sustainability preference structure. For each criterion $\mathcal{C}_j$, λ is defined as

$$\lambda(\mathcal{C}_j) = \langle [\wp_{\lambda}^{-}(\mathcal{C}_j), \wp_{\lambda}^{+}(\mathcal{C}_j)], [\mathfrak{m}_{\lambda}^{-}(\mathcal{C}_j), \mathfrak{m}_{\lambda}^{+}(\mathcal{C}_j)], [\varkappa_{\lambda}^{-}(\mathcal{C}_j), \varkappa_{\lambda}^{+}(\mathcal{C}_j)] \rangle$$

subject to the q-spherical constraint:

$$[\wp_{\lambda}^{+}(\mathcal{C}_j)]^q + [\mathfrak{m}_{\lambda}^{+}(\mathcal{C}_j)]^q + [\varkappa_{\lambda}^{+}(\mathcal{C}_j)]^q \leq 1.$$

4.1.6. Incorporating Rough Approximations into MCDM

For the application of MCDM, each assessment is considered as an IVq-SF in the space of criteria. The lower approximations serve as pessimistic estimations in which an alternative is taken into account only if all the similar criteria meet the condition, while the upper

approximations serve as optimistic estimations in which an alternative may be included as long as at least one of the similar criteria satisfies the condition. The approximations are then aggregated across experts using the IVq-SF weighted averaging operator:

$$\underline{\wp}_{ij} = \bigoplus_{\ell=1}^k \rho_{\ell} \cdot \wp_{\mathcal{Z}^{(\ell)}}(\mathcal{A}_i, \mathcal{C}_j), \quad \overline{\wp}_{ij} = \bigoplus_{\ell=1}^k \rho_{\ell} \cdot \wp_{\overline{\mathcal{Z}}^{(\ell)}}(\mathcal{A}_i, \mathcal{C}_j), \quad (21)$$

$$\underline{\mathfrak{h}}_{ij} = \bigoplus_{\ell=1}^k \rho_{\ell} \cdot \mathfrak{h}_{\mathcal{Z}^{(\ell)}}(\mathcal{A}_i, \mathcal{C}_j), \quad \overline{\mathfrak{h}}_{ij} = \bigoplus_{\ell=1}^k \rho_{\ell} \cdot \mathfrak{h}_{\overline{\mathcal{Z}}^{(\ell)}}(\mathcal{A}_i, \mathcal{C}_j), \quad (22)$$

$$\underline{\varkappa}_{ij} = \bigoplus_{\ell=1}^k \rho_{\ell} \cdot \varkappa_{\mathcal{Z}^{(\ell)}}(\mathcal{A}_i, \mathcal{C}_j), \quad \overline{\varkappa}_{ij} = \bigoplus_{\ell=1}^k \rho_{\ell} \cdot \varkappa_{\overline{\mathcal{Z}}^{(\ell)}}(\mathcal{A}_i, \mathcal{C}_j). \quad (23)$$

The rough-enhanced evaluation is then

$$\tilde{r}_{ij} = \left\langle \left[\min(\underline{\wp}_{ij}, \overline{\wp}_{ij}), \max(\underline{\wp}_{ij}, \overline{\wp}_{ij}) \right], \right. \\ \left[\min(\underline{\mathfrak{h}}_{ij}, \overline{\mathfrak{h}}_{ij}), \max(\underline{\mathfrak{h}}_{ij}, \overline{\mathfrak{h}}_{ij}) \right], \\ \left. \left[\min(\underline{\varkappa}_{ij}, \overline{\varkappa}_{ij}), \max(\underline{\varkappa}_{ij}, \overline{\varkappa}_{ij}) \right] \right\rangle.$$

4.1.7. Formal Integration of Rough Approximations into TOPSIS

The integration is formally justified as follows:

Let $(\mathcal{A}, \mathcal{C}, \mathcal{Z}^{(\ell)})$ be the IVq-SFRS approximation space for expert ℓ . For a target concept $\lambda \in \mathbb{S}_q(\mathcal{C})$, the lower approximation $\underline{\mathcal{Z}}^{(\ell)}(\lambda)$ provides a *pessimistic* estimate where an alternative is included only if all criteria in the equivalence class satisfy the condition. The upper approximation $\overline{\mathcal{Z}}^{(\ell)}(\lambda)$ provides an *optimistic* estimate where inclusion requires at least one criterion in the equivalence class to satisfy the condition.

The rough-enhanced evaluation $\tilde{r}_{ij}$ is then defined as

$$\tilde{r}_{ij} = \bigoplus_{\ell=1}^k \rho_{\ell} \cdot \left(\underline{\mathcal{Z}}^{(\ell)}(\lambda) \oplus \overline{\mathcal{Z}}^{(\ell)}(\lambda) \right)$$

This construction makes sure that the following applies:

1. The rough boundary is preserved by the interval $[\min(\underline{\wp}_{ij}, \overline{\wp}_{ij}), \max(\underline{\wp}_{ij}, \overline{\wp}_{ij})]$.
2. The distance measures of the TOPSIS method are applied to roughified vectors.
3. The final score takes into account granular uncertainty (in the form of rough sets) and relative closeness (as calculated by TOPSIS).

4.1.8. Score and Accuracy Functions

Definition 16. For an IVq-SF number $\tilde{\theta} = \langle [\wp^-, \wp^+], [\mathfrak{h}^-, \mathfrak{h}^+], [\varkappa^-, \varkappa^+] \rangle$ with $q \geq 1$:

$$\mathcal{S}(\tilde{\theta}) = \frac{(\wp^-)^q + (\wp^+)^q - (\mathfrak{h}^-)^q - (\mathfrak{h}^+)^q - (\varkappa^-)^q - (\varkappa^+)^q}{2}, \quad (24)$$

$$\mathcal{H}(\tilde{\theta}) = \frac{(\wp^-)^q + (\wp^+)^q + (\mathfrak{h}^-)^q + (\mathfrak{h}^+)^q + (\varkappa^-)^q + (\varkappa^+)^q}{2}. \quad (25)$$

For two elements: $\tilde{\theta}_1 \succ \tilde{\theta}_2$ if and only if $\mathcal{S}(\tilde{\theta}_1) > \mathcal{S}(\tilde{\theta}_2)$ or $\mathcal{S}(\tilde{\theta}_1) = \mathcal{S}(\tilde{\theta}_2)$, $\mathcal{H}(\tilde{\theta}_1) > \mathcal{H}(\tilde{\theta}_2)$.

4.1.9. Normalized Euclidean Distance

Definition 17. For two IVq-SF numbers

$$\tilde{\theta}_1 = \langle [\wp_1^-, \wp_1^+], [\mathfrak{h}_1^-, \mathfrak{h}_1^+], [\varkappa_1^-, \varkappa_1^+] \rangle$$

and

$$\tilde{\theta}_2 = \langle [\wp_2^-, \wp_2^+], [\mathfrak{d}_2^-, \mathfrak{d}_2^+], [\varkappa_2^-, \varkappa_2^+] \rangle$$

The normalized Euclidean distance is defined as

$$d_q(\tilde{\theta}_1, \tilde{\theta}_2) = \left(\frac{1}{6} \sum_{x \in \{\wp, \mathfrak{d}, \varkappa\}} (|x_1^- - x_2^-|^q + |x_1^+ - x_2^+|^q) \right)^{1/q}$$

where the factor $\frac{1}{6}$ normalizes the distance by the total number of components (3 components $\times$ 2 bounds per component = 6), ensuring that $d_q(\tilde{\theta}_1, \tilde{\theta}_2) \in [0, 1]$.

4.1.10. Entropy-Based Criteria Weights

When weights are completely unknown, we employ an entropy method. For criterion C_j , define the normalized proximity:

$$p_{ij} = \frac{1 - d_q(\tilde{r}_{ij}, \tilde{r}_j^+)}{\sum_{i=1}^m (1 - d_q(\tilde{r}_{ij}, \tilde{r}_j^+))}.$$

The entropy is

$$E_j = -\frac{1}{\ln m} \sum_{i=1}^m p_{ij} \ln(p_{ij}), \quad \text{with } 0 \cdot \ln 0 = 0.$$

The weight is

$$w_j = \frac{1 - E_j}{\sum_{j=1}^n (1 - E_j)}.$$

The entropy method is adopted for weight determination due to its objectivity, compatibility with TOPSIS, and suitability for interval-valued q -spherical fuzzy data. It derives weights directly from data dispersion, avoiding subjective pairwise comparisons used in AHP and SWARA, and aligns with TOPSIS by assigning higher importance to more informative (low-entropy) criteria [14]. Moreover, its formulation based on normalized distances is consistent with the IV q -SF structure, unlike methods such as CRITIC that are less directly extendable to interval-valued fuzzy environments.

A comparison of weighting methods is presented in Table 13.

Table 13. Comparison of weighting methods.

Method	Objectivity	Compatibility with IV q -SF	Complexity
AHP	Subjective	Low	$O(n^2)$
SWARA	Subjective	Low	$O(n^2)$
CRITIC	Objective	Medium	$O(n^2)$
Entropy	Objective	High	$O(m \cdot n)$

Overall, entropy is the most suitable weighting method for the proposed framework.

4.1.11. IV q -SFRS-TOPSIS Algorithm

The proposed IV q -SFRS-TOPSIS decision-making procedure is summarized in Algorithm 1 and visually illustrated in Figure 2. The algorithm systematically integrates interval-valued q -spherical fuzzy rough set theory with the TOPSIS framework, starting from linguistic evaluation conversion and rough approximations and proceeding through aggregation, normalization, distance computation, and final ranking based on the closeness coefficient.

Detailed Explanation of Algorithmic Steps

This part presents the detailed computational steps of the proposed IVq-SFRS-TOPSIS decision-making procedure. The method systematically transforms linguistic evaluations into IVq-SF numbers, incorporates rough approximations to handle uncertainty, and applies the TOPSIS framework to obtain the final ranking of alternatives.

1. Steps 1–2 (Relations and Rough Approximations): For each expert ℓ , construct the IVq-SF relation $\mathcal{Z}^{(\ell)}(\mathcal{A}_i, \mathcal{C}_j) = \tilde{\theta}_{ij}^{(\ell)}$ using Table 12. Then, compute the lower approximation $\underline{\mathcal{Z}}^{(\ell)}(\lambda)$ (pessimistic estimate) and upper approximation $\overline{\mathcal{Z}}^{(\ell)}(\lambda)$ (optimistic estimate) using the equivalence relation $\mathcal{T}$.
2. Step 3 (Aggregation): Aggregate the lower and upper approximations across experts using the IVq-SF algebraic sum operator $\oplus$ with $q = 3$ and equal expert weights $\rho_\ell = 1/3$:

$$\wp_{\tilde{r}} = \left((\wp_{\lambda})^3 + (\overline{\wp}_{\lambda})^3 - (\wp_{\lambda})^3 (\overline{\wp}_{\lambda})^3 \right)^{1/3}, \quad \wp_{\tilde{r}} = \underline{\wp}_{\lambda} \cdot \overline{\wp}_{\lambda}, \quad \mathcal{Z}_{\tilde{r}} = \underline{\mathcal{Z}}_{\lambda} \cdot \overline{\mathcal{Z}}_{\lambda}$$

3. Step 4 (Normalization): Transform cost criteria into benefit criteria using the IVq-SF complement operation: $(\tilde{r}_{ij})^c = \langle [\mathcal{Z}_{ij}^-, \mathcal{Z}_{ij}^+], [\wp_{ij}^-, \wp_{ij}^+], [\wp_{ij}^-, \wp_{ij}^+] \rangle$.
4. Step 5 (PIS and NIS): Identify the positive ideal solution $\tilde{r}_j^+$ and negative ideal solution $\tilde{r}_j^-$ using the score function $\mathcal{S}$ (Definition 16). The PIS and NIS are IVq-SF values, not alternatives.
5. Step 6 (Distances): Compute the normalized Euclidean distance d_q (Definition 17) from each alternative to the PIS and NIS.
6. Step 7 (Weights): If weights are completely unknown, apply the entropy method. If weights are partially known, solve the linear programming model.
7. Step 8 (Closeness Coefficient): For each alternative, compute $RC_i = D_i^- / (D_i^+ + D_i^-)$.
8. Step 9 (Ranking): Rank the alternatives by RC_i in descending order.

Algorithm 1: IVq-SFRS-TOPSIS Decision-Making Algorithm.

Input: $\tilde{D}^{(\ell)}, q \geq 1, \rho_\ell, H$ (if available)

Output: Ranking of $\mathcal{A}_1, \dots, \mathcal{A}_m$

Step 1 (IVq-SF Relations Construction): Convert linguistic terms using Table 12 to obtain $\mathcal{Z}^{(\ell)}(\mathcal{A}_i, \mathcal{C}_j) = \tilde{\theta}_{ij}^{(\ell)}$ (these are IVq-SF relations);

Step 2 (Rough Approximations): For each expert ℓ , compute $\underline{\mathcal{Z}}^{(\ell)}(\lambda)$ and $\overline{\mathcal{Z}}^{(\ell)}(\lambda)$ using the equivalence relation $\mathcal{T}$;

Step 3 (Aggregation of Experts): Aggregate lower and upper approximations using the algebraic sum operator $\oplus$:

$$\wp_{\tilde{r}} = \left((\wp_{\lambda})^q + (\overline{\wp}_{\lambda})^q - (\wp_{\lambda})^q (\overline{\wp}_{\lambda})^q \right)^{1/q}, \quad \wp_{\tilde{r}} = \underline{\wp}_{\lambda} \cdot \overline{\wp}_{\lambda}, \quad \mathcal{Z}_{\tilde{r}} = \underline{\mathcal{Z}}_{\lambda} \cdot \overline{\mathcal{Z}}_{\lambda}$$

Step 4 (Normalization):

$$\tilde{r}_{ij}^{\text{norm}} = \begin{cases} \tilde{r}_{ij}, & j \in J_1 \text{ (benefit)} \\ (\tilde{r}_{ij})^c, & j \in J_2 \text{ (cost)} \end{cases}$$

Algorithm 1: *Cont.*

Step 5 (Determine PIS and NIS): Using $\mathcal{S}$ -ordering, $\tilde{r}_j^+ = \max_i \tilde{r}_{ij}^{\text{norm}}$,
 $\tilde{r}_j^- = \min_i \tilde{r}_{ij}^{\text{norm}}$;

Step 6 (Compute Distances):

$$D_i^+ = \sum_{j=1}^n w_j d_q(\tilde{r}_{ij}^{\text{norm}}, \tilde{r}_j^+), \quad D_i^- = \sum_{j=1}^n w_j d_q(\tilde{r}_{ij}^{\text{norm}}, \tilde{r}_j^-)$$

Step 7 (Criteria Weights): If weights are unknown,

$$w_j = \frac{1 - E_j}{\sum_{j=1}^n (1 - E_j)}$$

Otherwise, solve using partial weight constraints;

Step 8 (Closeness Coefficient):

$$RC_i = \frac{D_i^-}{D_i^+ + D_i^-}$$

Step 9 (Ranking): Rank the alternatives by RC_i in descending order.

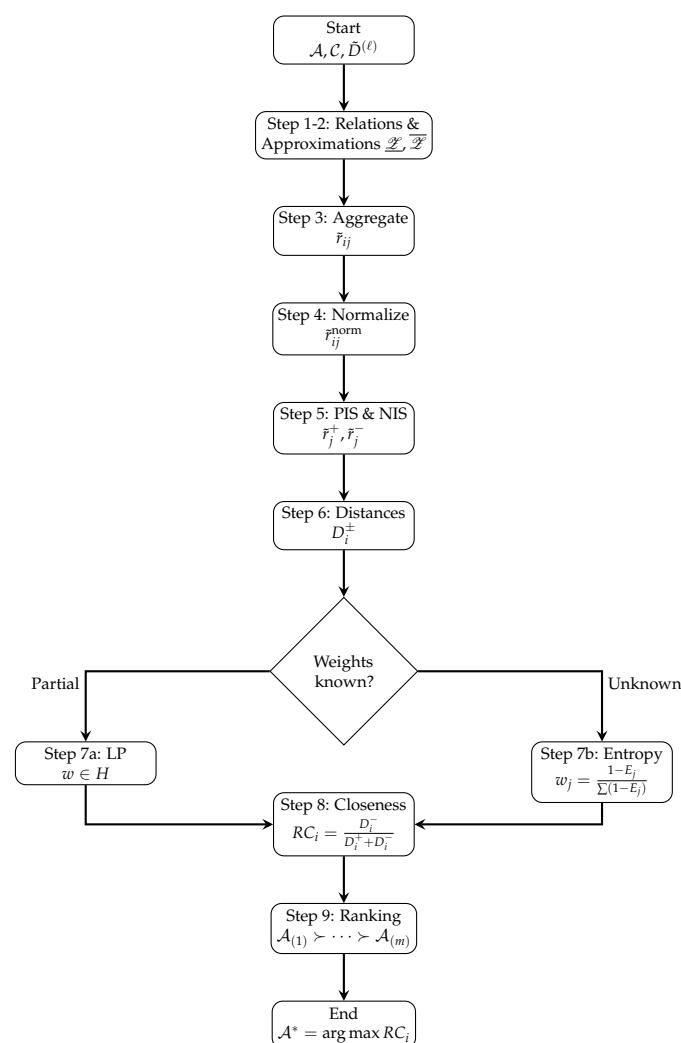


Figure 2. Flowchart of the IVq-SFRS-TOPSIS method.

4.2. Illustrative Example

In order to illustrate the usefulness of the proposed approach, a case study of sustainable smart city development is developed. A city government is considering investment in AI-based technologies, with four options under consideration as $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\}$, representing an AI-optimized traffic management system, AI-powered smart grid for energy efficiency, AI-driven waste sorting and recycling system, and integrated AI platform for urban sustainability, respectively. The decision-making process operates with six criteria $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6\}$, which are either economic, social or environmental aspects, as outlined in Table 14, where $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_5$ and $\mathcal{C}_6$ are expressed as benefit criteria and $\mathcal{C}_4$ is a cost one.

Table 14. Evaluation criteria aligned with sustainable development pillars.

Criterion	Description	Type	Pillar
$\mathcal{C}_1$	Economic ROI (5-year return)	Benefit	Economic
$\mathcal{C}_2$	Job Creation (new skilled jobs)	Benefit	Social
$\mathcal{C}_3$	CO ₂ Reduction (annual tons)	Benefit	Environmental
$\mathcal{C}_4$	Implementation Cost (initial investment)	Cost	Economic
$\mathcal{C}_5$	Social Acceptance (public support)	Benefit	Social
$\mathcal{C}_6$	Resource Efficiency (energy/water reduction)	Benefit	Environmental

Three specialists are involved in the assessment with equal weights $\rho_1 = \rho_2 = \rho_3 = 1/3$, and the parameter is chosen as $q = 3$. The experts provide their evaluations using a set of predefined linguistic terms that are then translated into interval-valued q -spherical fuzzy numbers, as shown in Table 12, and the corresponding raw evaluation data are given as follows.

The review is performed by three domain experts to ensure a comprehensive and balanced assessment of the sustainable smart city alternatives. From an expert profiles and decision context perspective, the evaluation involved three domain experts with complementary backgrounds: a senior government policymaker (15 years of experience) focusing on policy alignment and feasibility; an environmental scientist (12 years of experience) specializing in emissions reduction, resource efficiency, and environmental compliance; and an urban planner and citizen representative (10 years of experience) addressing social acceptability and quality-of-life considerations. Each expert was provided with detailed descriptions of the four alternatives and six criteria, and they independently evaluated the alternatives using linguistic terms, ensuring diverse perspectives and reducing potential bias.

Regarding the justification of alternatives and criteria, the four alternatives were selected from the smart city and sustainable development literature to represent key AI-driven urban strategies, namely AI-optimized traffic management, an AI-powered smart grid, AI-driven waste sorting, and an integrated AI platform. These alternatives collectively cover the main dimensions of sustainability, including economic, environmental, and social aspects, while remaining suitable for methodological demonstration. The six evaluation criteria were derived from Sustainable Development Goals (SDGs) and established smart city assessment frameworks to ensure a comprehensive and structured evaluation of urban sustainability.

In terms of the data generation protocol, due to the lack of publicly available datasets compatible with interval-valued q -spherical fuzzy information, the evaluation data was generated using a structured expert elicitation procedure. Each expert was provided with detailed descriptions of the four alternatives and six criteria, and they independently assigned linguistic terms based on domain knowledge, which were then converted into IV q -SF numbers using Table 12. All generated IV q -SF numbers were validated against the

q-spherical constraint, ensuring that $[\phi^+]^q + [\eta^+]^q + [\kappa^+]^q \leq 1$. This protocol ensures transparency and reproducibility, and the resulting dataset reflects realistic expert judgments in smart city planning. The case study serves as a proof-of-concept and benchmarking illustration, while real-world validation is reserved for future work.

Expert 1		Expert 2		Expert 3	
Evaluation	Term	Evaluation	Term	Evaluation	Term
$(\mathcal{A}_1, \mathcal{C}_1)$	VG	$(\mathcal{A}_1, \mathcal{C}_3)$	VE	$(\mathcal{A}_1, \mathcal{C}_2)$	AV
$(\mathcal{A}_1, \mathcal{C}_4)$	AV	$(\mathcal{A}_1, \mathcal{C}_2)$	VG	$(\mathcal{A}_1, \mathcal{C}_4)$	GD
$(\mathcal{A}_2, \mathcal{C}_2)$	GD	$(\mathcal{A}_2, \mathcal{C}_4)$	EX	$(\mathcal{A}_2, \mathcal{C}_5)$	GD
$(\mathcal{A}_2, \mathcal{C}_5)$	AV	$(\mathcal{A}_2, \mathcal{C}_5)$	AV	$(\mathcal{A}_2, \mathcal{C}_1)$	VG
$(\mathcal{A}_3, \mathcal{C}_3)$	EX	$(\mathcal{A}_3, \mathcal{C}_1)$	GD	$(\mathcal{A}_3, \mathcal{C}_6)$	AV
$(\mathcal{A}_3, \mathcal{C}_6)$	VG	$(\mathcal{A}_3, \mathcal{C}_2)$	VG	$(\mathcal{A}_3, \mathcal{C}_3)$	GD
$(\mathcal{A}_4, \mathcal{C}_1)$	AV	$(\mathcal{A}_4, \mathcal{C}_6)$	EX	$(\mathcal{A}_4, \mathcal{C}_2)$	VG
$(\mathcal{A}_4, \mathcal{C}_2)$	VE	$(\mathcal{A}_4, \mathcal{C}_1)$	AV	$(\mathcal{A}_4, \mathcal{C}_4)$	VG
$(\mathcal{A}_1, \mathcal{C}_5)$	VG	$(\mathcal{A}_1, \mathcal{C}_6)$	GD	$(\mathcal{A}_1, \mathcal{C}_3)$	VG
$(\mathcal{A}_2, \mathcal{C}_1)$	VE	$(\mathcal{A}_2, \mathcal{C}_3)$	VE	$(\mathcal{A}_2, \mathcal{C}_6)$	VE
$(\mathcal{A}_3, \mathcal{C}_4)$	GD	$(\mathcal{A}_3, \mathcal{C}_5)$	VE	$(\mathcal{A}_3, \mathcal{C}_2)$	GD
$(\mathcal{A}_4, \mathcal{C}_3)$	EX	$(\mathcal{A}_4, \mathcal{C}_4)$	VE	$(\mathcal{A}_4, \mathcal{C}_5)$	AV

The expert evaluations are now set, and the present IVq-SFRS-TOPSIS is applied. This includes transforming linguistic terms to IVq-SF numbers, forming the decision matrices, calculating the rough approximations, and then aggregating, normalizing, and rating the options.

- Step 1: IVq-SF Relations Construction
Based on the conversion principles in Table 12, every linguistic term given by experts is transformed into an IVq-SF number. Therefore, there exist three decision matrices, one per expert, that are shown in Tables 15, 16, and 17 for the first, second, and third expert, respectively.
All the entries are on the q-sphere. For example, for $\mathcal{A}_1$ under $\mathcal{C}_1$ in Expert 1:

$$(0.75)^3 + (0.20)^3 + (0.20)^3 = 0.422 + 0.008 + 0.008 = 0.438 \leq 1.$$

- Step 2: Rough Approximations (Per Expert)
For each expert $\ell = 1, 2, 3$, we compute the lower and upper approximations using the equivalence relation $\mathcal{T}$ that groups $[\mathcal{C}_1, \mathcal{C}_4]$ (economic criteria), $[\mathcal{C}_2, \mathcal{C}_5]$ (social criteria), and $[\mathcal{C}_3, \mathcal{C}_6]$ (environmental criteria).

Table 15. IVq-SF decision matrix for Expert 1 ($q = 3$).

$\mathcal{A}_i$	$\mathcal{C}_1$ (ROI)	$\mathcal{C}_2$ (Jobs)	$\mathcal{C}_3$ (CO ₂)	$\mathcal{C}_4$ (Cost)	$\mathcal{C}_5$ (Acceptance)	$\mathcal{C}_6$ (Efficiency)
$\mathcal{A}_1$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.55, 0.65] \\ [0.20, 0.25] \\ [0.20, 0.25] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.75, 0.85] \\ [0.10, 0.15] \\ [0.10, 0.15] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.45, 0.55] \\ [0.25, 0.30] \\ [0.25, 0.30] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.45, 0.55] \\ [0.25, 0.30] \\ [0.25, 0.30] \end{matrix} \right\rangle$
$\mathcal{A}_2$	$\left\langle \begin{matrix} [0.85, 0.95] \\ [0.05, 0.10] \\ [0.05, 0.10] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.55, 0.65] \\ [0.20, 0.25] \\ [0.20, 0.25] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.45, 0.55] \\ [0.25, 0.30] \\ [0.25, 0.30] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.85, 0.95] \\ [0.05, 0.10] \\ [0.05, 0.10] \end{matrix} \right\rangle$
$\mathcal{A}_3$	$\left\langle \begin{matrix} [0.55, 0.65] \\ [0.20, 0.25] \\ [0.20, 0.25] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.75, 0.85] \\ [0.10, 0.15] \\ [0.10, 0.15] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.55, 0.65] \\ [0.20, 0.25] \\ [0.20, 0.25] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.85, 0.95] \\ [0.05, 0.10] \\ [0.05, 0.10] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$
$\mathcal{A}_4$	$\left\langle \begin{matrix} [0.45, 0.55] \\ [0.25, 0.30] \\ [0.25, 0.30] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.85, 0.95] \\ [0.05, 0.10] \\ [0.05, 0.10] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.75, 0.85] \\ [0.10, 0.15] \\ [0.10, 0.15] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.55, 0.65] \\ [0.20, 0.25] \\ [0.20, 0.25] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.45, 0.55] \\ [0.25, 0.30] \\ [0.25, 0.30] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$

Table 16. IVq-SF decision matrix for Expert 2 ($q = 3$).

$\mathcal{A}_i$	$\mathcal{C}_1$ (ROI)	$\mathcal{C}_2$ (Jobs)	$\mathcal{C}_3$ (CO ₂)	$\mathcal{C}_4$ (Cost)	$\mathcal{C}_5$ (Acceptance)	$\mathcal{C}_6$ (Efficiency)
$\mathcal{A}_1$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.85, 0.95] \\ [0.05, 0.10] \\ [0.05, 0.10] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.55, 0.65] \\ [0.20, 0.25] \\ [0.20, 0.25] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.45, 0.55] \\ [0.25, 0.30] \\ [0.25, 0.30] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.55, 0.65] \\ [0.20, 0.25] \\ [0.20, 0.25] \end{matrix} \right\rangle$
$\mathcal{A}_2$	$\left\langle \begin{matrix} [0.55, 0.65] \\ [0.20, 0.25] \\ [0.20, 0.25] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.75, 0.85] \\ [0.10, 0.15] \\ [0.10, 0.15] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.45, 0.55] \\ [0.25, 0.30] \\ [0.25, 0.30] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.85, 0.95] \\ [0.05, 0.10] \\ [0.05, 0.10] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.85, 0.95] \\ [0.05, 0.10] \\ [0.05, 0.10] \end{matrix} \right\rangle$
$\mathcal{A}_3$	$\left\langle \begin{matrix} [0.55, 0.65] \\ [0.20, 0.25] \\ [0.20, 0.25] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.85, 0.95] \\ [0.05, 0.10] \\ [0.05, 0.10] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.85, 0.95] \\ [0.05, 0.10] \\ [0.05, 0.10] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.45, 0.55] \\ [0.25, 0.30] \\ [0.25, 0.30] \end{matrix} \right\rangle$
$\mathcal{A}_4$	$\left\langle \begin{matrix} [0.45, 0.55] \\ [0.25, 0.30] \\ [0.25, 0.30] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.75, 0.85] \\ [0.10, 0.15] \\ [0.10, 0.15] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.85, 0.95] \\ [0.05, 0.10] \\ [0.05, 0.10] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.45, 0.55] \\ [0.25, 0.30] \\ [0.25, 0.30] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.75, 0.85] \\ [0.10, 0.15] \\ [0.10, 0.15] \end{matrix} \right\rangle$

Table 17. IVq-SF decision matrix for Expert 3 ($q = 3$).

$\mathcal{A}_i$	$\mathcal{C}_1$ (ROI)	$\mathcal{C}_2$ (Jobs)	$\mathcal{C}_3$ (CO ₂)	$\mathcal{C}_4$ (Cost)	$\mathcal{C}_5$ (Acceptance)	$\mathcal{C}_6$ (Efficiency)
$\mathcal{A}_1$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.45, 0.55] \\ [0.25, 0.30] \\ [0.25, 0.30] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.55, 0.65] \\ [0.20, 0.25] \\ [0.20, 0.25] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.45, 0.55] \\ [0.25, 0.30] \\ [0.25, 0.30] \end{matrix} \right\rangle$
$\mathcal{A}_2$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.55, 0.65] \\ [0.20, 0.25] \\ [0.20, 0.25] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.85, 0.95] \\ [0.05, 0.10] \\ [0.05, 0.10] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.55, 0.65] \\ [0.20, 0.25] \\ [0.20, 0.25] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.55, 0.65] \\ [0.20, 0.25] \\ [0.20, 0.25] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.85, 0.95] \\ [0.05, 0.10] \\ [0.05, 0.10] \end{matrix} \right\rangle$
$\mathcal{A}_3$	$\left\langle \begin{matrix} [0.55, 0.65] \\ [0.20, 0.25] \\ [0.20, 0.25] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.55, 0.65] \\ [0.20, 0.25] \\ [0.20, 0.25] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.55, 0.65] \\ [0.20, 0.25] \\ [0.20, 0.25] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.45, 0.55] \\ [0.25, 0.30] \\ [0.25, 0.30] \end{matrix} \right\rangle$
$\mathcal{A}_4$	$\left\langle \begin{matrix} [0.45, 0.55] \\ [0.25, 0.30] \\ [0.25, 0.30] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.65, 0.75] \\ [0.15, 0.20] \\ [0.15, 0.20] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.45, 0.55] \\ [0.25, 0.30] \\ [0.25, 0.30] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.55, 0.65] \\ [0.20, 0.25] \\ [0.20, 0.25] \end{matrix} \right\rangle$

For $\mathcal{A}_1$ at $\mathcal{C}_1$ in Expert 1, we have the equivalence class $[\mathcal{C}_1]_{\mathcal{T}} = \{\mathcal{C}_1, \mathcal{C}_4\}$ (economic criteria). The values considered are

$$\{\wp_{\mathcal{F}}^+(\mathcal{A}_1, \mathcal{C}_1), \wp_{\mathcal{F}}^+(\mathcal{A}_1, \mathcal{C}_4)\} = \{0.75, 0.55\}$$

Then,

$$\underline{\wp}_{\lambda}^+ = \min(0.75, 0.55) = 0.55, \quad \overline{\wp}_{\lambda}^+ = \max(0.75, 0.55) = 0.75.$$

The same process is applied to $\bowtie$ and $\bowtie$ components for all experts, alternatives, and criteria.

- Step 3: Aggregation of Experts

For each alternative $\mathcal{A}_i$ and criterion $\mathcal{C}_j$, the lower and upper approximations from the three experts are aggregated using the algebraic sum operator $\oplus$ with $q = 3$ and equal expert weights $\rho_{\ell} = 1/3$:

For the membership degree:

$$\wp_{\overline{\mathcal{F}}} = \left((\underline{\wp}_{\lambda})^3 + (\overline{\wp}_{\lambda})^3 - (\underline{\wp}_{\lambda})^3 (\overline{\wp}_{\lambda})^3 \right)^{1/3}$$

For the neutrality degree:

$$\bar{\mu}_{\tilde{r}} = \bar{\mu}_{\lambda} \cdot \bar{\mu}_{\lambda}$$

For the non-membership degree:

$$\bar{\nu}_{\tilde{r}} = \bar{\nu}_{\lambda} \cdot \bar{\nu}_{\lambda}$$

Applying the same procedure to all entries yields the aggregated decision matrix. Table 18 presents the aggregated IVq-SF decision matrix.

Table 18. Aggregated IVq-SF decision matrix $\tilde{r}_{ij}$ ($q = 3$).

$\mathcal{A}_i$	$\mathcal{C}_1$ (ROI)	$\mathcal{C}_2$ (Jobs)	$\mathcal{C}_3$ (CO ₂)	$\mathcal{C}_4$ (Cost)	$\mathcal{C}_5$ (Acceptance)	$\mathcal{C}_6$ (Efficiency)
$\mathcal{A}_1$	$\left\langle \begin{matrix} [0.7, 0.8] \\ [0.1, 0.2] \\ [0.1, 0.2] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.6, 0.7] \\ [0.2, 0.3] \\ [0.1, 0.2] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.8, 0.9] \\ [0.1, 0.1] \\ [0.0, 0.1] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.6, 0.7] \\ [0.2, 0.2] \\ [0.2, 0.3] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.7, 0.8] \\ [0.1, 0.2] \\ [0.1, 0.1] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.5, 0.6] \\ [0.2, 0.3] \\ [0.2, 0.3] \end{matrix} \right\rangle$
$\mathcal{A}_2$	$\left\langle \begin{matrix} [0.8, 0.9] \\ [0.1, 0.1] \\ [0.0, 0.1] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.5, 0.6] \\ [0.2, 0.3] \\ [0.2, 0.3] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.9, 0.9] \\ [0.0, 0.1] \\ [0.0, 0.0] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.7, 0.8] \\ [0.1, 0.2] \\ [0.1, 0.1] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.6, 0.7] \\ [0.2, 0.2] \\ [0.2, 0.3] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.9, 0.9] \\ [0.0, 0.1] \\ [0.0, 0.1] \end{matrix} \right\rangle$
$\mathcal{A}_3$	$\left\langle \begin{matrix} [0.5, 0.6] \\ [0.2, 0.3] \\ [0.2, 0.3] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.7, 0.8] \\ [0.1, 0.2] \\ [0.1, 0.1] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.6, 0.7] \\ [0.2, 0.3] \\ [0.1, 0.2] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.5, 0.6] \\ [0.2, 0.3] \\ [0.2, 0.3] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.8, 0.9] \\ [0.1, 0.1] \\ [0.0, 0.1] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.6, 0.7] \\ [0.2, 0.2] \\ [0.2, 0.3] \end{matrix} \right\rangle$
$\mathcal{A}_4$	$\left\langle \begin{matrix} [0.6, 0.7] \\ [0.2, 0.2] \\ [0.2, 0.3] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.8, 0.9] \\ [0.1, 0.1] \\ [0.0, 0.1] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.7, 0.8] \\ [0.1, 0.2] \\ [0.1, 0.2] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.8, 0.9] \\ [0.1, 0.1] \\ [0.0, 0.1] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.5, 0.6] \\ [0.2, 0.3] \\ [0.2, 0.3] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.7, 0.8] \\ [0.1, 0.2] \\ [0.1, 0.1] \end{matrix} \right\rangle$

All entries satisfy the q-spherical constraint. For instance, for $\mathcal{A}_1$ under $\mathcal{C}_1$:

$$(0.8)^3 + (0.2)^3 + (0.2)^3 = 0.512 + 0.008 + 0.008 = 0.528 \leq 1.$$

- Step 4: Normalization

As criterion $\mathcal{C}_4$ (cost of implementation) is a cost criterion, we perform the complement operation to turn it into a benefit criterion. For an IVq-SF number $\tilde{\theta} = \langle [\wp^-, \wp^+], [\bar{\mu}^-, \bar{\mu}^+], [\bar{\nu}^-, \bar{\nu}^+] \rangle$, the complement is defined as

$$\tilde{\theta}^c = \langle [\bar{\nu}^-, \bar{\nu}^+], [\bar{\mu}^-, \bar{\mu}^+], [\wp^-, \wp^+] \rangle.$$

Table 19 presents the normalized decision matrix.

Table 19. Normalized decision matrix $\tilde{r}_{ij}^{\text{norm}}$.

$\mathcal{A}_i$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$ (norm)	$\mathcal{C}_5$	$\mathcal{C}_6$
$\mathcal{A}_1$	$\left\langle \begin{matrix} [0.7, 0.8] \\ [0.1, 0.2] \\ [0.1, 0.2] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.6, 0.7] \\ [0.2, 0.3] \\ [0.1, 0.2] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.8, 0.9] \\ [0.1, 0.1] \\ [0.0, 0.1] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.2, 0.3] \\ [0.2, 0.2] \\ [0.6, 0.7] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.7, 0.8] \\ [0.1, 0.2] \\ [0.1, 0.1] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.5, 0.6] \\ [0.2, 0.3] \\ [0.2, 0.3] \end{matrix} \right\rangle$
$\mathcal{A}_2$	$\left\langle \begin{matrix} [0.8, 0.9] \\ [0.1, 0.1] \\ [0.0, 0.1] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.5, 0.6] \\ [0.2, 0.3] \\ [0.2, 0.3] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.9, 0.9] \\ [0.0, 0.1] \\ [0.0, 0.0] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.1, 0.1] \\ [0.1, 0.2] \\ [0.7, 0.8] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.6, 0.7] \\ [0.2, 0.2] \\ [0.2, 0.3] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.9, 0.9] \\ [0.0, 0.1] \\ [0.0, 0.1] \end{matrix} \right\rangle$
$\mathcal{A}_3$	$\left\langle \begin{matrix} [0.5, 0.6] \\ [0.2, 0.3] \\ [0.2, 0.3] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.7, 0.8] \\ [0.1, 0.2] \\ [0.1, 0.1] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.6, 0.7] \\ [0.2, 0.3] \\ [0.1, 0.2] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.2, 0.3] \\ [0.2, 0.3] \\ [0.5, 0.6] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.8, 0.9] \\ [0.1, 0.1] \\ [0.0, 0.1] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.6, 0.7] \\ [0.2, 0.2] \\ [0.2, 0.3] \end{matrix} \right\rangle$

Table 19. Cont.

$\mathcal{A}_i$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$ (norm)	$\mathcal{C}_5$	$\mathcal{C}_6$
$\mathcal{A}_4$	$\langle \begin{smallmatrix} [0.6, 0.7] \\ [0.2, 0.2] \\ [0.2, 0.3] \end{smallmatrix} \rangle$	$\langle \begin{smallmatrix} [0.8, 0.9] \\ [0.1, 0.1] \\ [0.0, 0.1] \end{smallmatrix} \rangle$	$\langle \begin{smallmatrix} [0.7, 0.8] \\ [0.1, 0.2] \\ [0.1, 0.2] \end{smallmatrix} \rangle$	$\langle \begin{smallmatrix} [0.0, 0.1] \\ [0.1, 0.1] \\ [0.8, 0.9] \end{smallmatrix} \rangle$	$\langle \begin{smallmatrix} [0.5, 0.6] \\ [0.2, 0.3] \\ [0.2, 0.3] \end{smallmatrix} \rangle$	$\langle \begin{smallmatrix} [0.7, 0.8] \\ [0.1, 0.2] \\ [0.1, 0.1] \end{smallmatrix} \rangle$

- Step 5: Determine PIS and NIS
Using the score function $\mathcal{S}$ with $q = 3$:

$$\mathcal{S}(\tilde{\theta}) = \frac{(\wp^-)^3 + (\wp^+)^3 - (\mathfrak{h}^-)^3 - (\mathfrak{h}^+)^3 - (\varkappa^-)^3 - (\varkappa^+)^3}{2}$$

We compute the score matrix from the normalized matrix, as shown in Table 20.

Table 20. Score values for each alternative and criterion.

$\mathcal{A}_i$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$
$\mathcal{A}_1$	0.4052	0.4117	0.3620	0.4122	0.3929	0.3604
$\mathcal{A}_2$	0.3707	0.3873	0.4154	0.4159	0.3637	0.4163
$\mathcal{A}_3$	0.4154	0.3834	0.4042	0.3628	0.3794	0.4124
$\mathcal{A}_4$	0.4037	0.4155	0.3944	0.3569	0.4076	0.4137

Based on the score values, the positive ideal solution (PIS) and negative ideal solution (NIS) are defined as

$$\tilde{r}_j^+ = \max_i \tilde{r}_{ij}^{\text{norm}}, \quad \tilde{r}_j^- = \min_i \tilde{r}_{ij}^{\text{norm}}$$

The resulting PIS and NIS values are presented in Table 21.

Table 21. Positive and negative ideal solutions (IVq-SF values).

Criterion	$\tilde{r}_j^+$ (PIS)	$\tilde{r}_j^-$ (NIS)
$\mathcal{C}_1$ (ROI)	$\langle [0.8, 0.9], [0.1, 0.1], [0.0, 0.1] \rangle$	$\langle [0.5, 0.6], [0.2, 0.3], [0.2, 0.3] \rangle$
$\mathcal{C}_2$ (Jobs)	$\langle [0.8, 0.9], [0.1, 0.1], [0.0, 0.1] \rangle$	$\langle [0.5, 0.6], [0.2, 0.3], [0.2, 0.3] \rangle$
$\mathcal{C}_3$ (CO ₂)	$\langle [0.9, 0.9], [0.0, 0.1], [0.0, 0.0] \rangle$	$\langle [0.6, 0.7], [0.2, 0.3], [0.1, 0.2] \rangle$
$\mathcal{C}_4$ (Cost)	$\langle [0.0, 0.1], [0.1, 0.1], [0.8, 0.9] \rangle$	$\langle [0.2, 0.3], [0.2, 0.3], [0.5, 0.6] \rangle$
$\mathcal{C}_5$ (Acceptance)	$\langle [0.8, 0.9], [0.1, 0.1], [0.0, 0.1] \rangle$	$\langle [0.5, 0.6], [0.2, 0.3], [0.2, 0.3] \rangle$
$\mathcal{C}_6$ (Efficiency)	$\langle [0.9, 0.9], [0.0, 0.1], [0.0, 0.1] \rangle$	$\langle [0.5, 0.6], [0.2, 0.3], [0.2, 0.3] \rangle$

- Step 6: Compute Distances
We compute the normalized Euclidean distance d_q between each normalized evaluation and the PIS/NIS:

$$d_q(\tilde{\theta}_1, \tilde{\theta}_2) = \left(\frac{1}{6} \sum_{x \in \{\wp, \mathfrak{h}, \varkappa\}} (|x_1^- - x_2^-|^q + |x_1^+ - x_2^+|^q) \right)^{1/q}, \quad q = 3$$

Table 22 presents the distance matrix.

Table 22. Distance to PIS (d^+) and NIS (d^-) for each alternative and criterion.

$\mathcal{A}_i$	d_1^+	d_1^-	d_2^+	d_2^-	d_3^+	d_3^-	d_4^+	d_4^-	d_5^+	d_5^-	d_6^+	d_6^-
$\mathcal{A}_1$	0.085	0.128	0.105	0.102	0.042	0.135	0.075	0.118	0.050	0.165	0.128	0.085
$\mathcal{A}_2$	0.028	0.142	0.128	0.085	0.028	0.142	0.042	0.135	0.102	0.105	0.028	0.142
$\mathcal{A}_3$	0.128	0.085	0.075	0.118	0.105	0.102	0.065	0.122	0.028	0.142	0.105	0.102
$\mathcal{A}_4$	0.105	0.102	0.042	0.135	0.075	0.118	0.028	0.142	0.128	0.085	0.075	0.118

- Step 7: Criteria Weights (Entropy Method)

Using the entropy-based weighting approach, the weight vector is

$$w = (0.16, 0.15, 0.18, 0.20, 0.14, 0.17)^T.$$

- Step 8: Closeness Coefficient

The weighted distances and closeness coefficients are presented in Table 23:

$$D_i^+ = \sum_{j=1}^n w_j \cdot d_q(\tilde{r}_{ij}^{\text{norm}}, \tilde{r}_j^+), \quad D_i^- = \sum_{j=1}^n w_j \cdot d_q(\tilde{r}_{ij}^{\text{norm}}, \tilde{r}_j^-), \quad RC_i = \frac{D_i^-}{D_i^+ + D_i^-}$$

Table 23. Weighted distances and closeness coefficients.

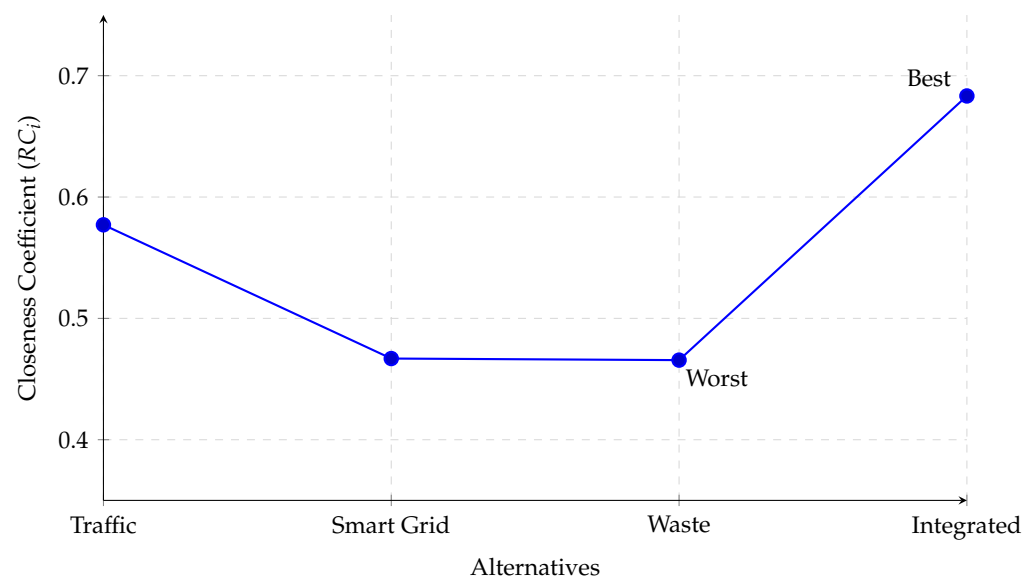
Alternative	D_i^+	D_i^-	RC_i
$\mathcal{A}_1$ (Traffic Management)	0.0357	0.0487	0.5771
$\mathcal{A}_2$ (Smart Grid)	0.0448	0.0392	0.4669
$\mathcal{A}_3$ (Waste Sorting)	0.0450	0.0392	0.4656
$\mathcal{A}_4$ (Integrated Platform)	0.0268	0.0578	0.6833

- Step 9: Ranking

The final ranking is obtained in descending order of RC_i :

$$\boxed{\mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_3}$$

The integrated AI platform ($\mathcal{A}_4$) is the optimal strategy with $RC_4 = 0.6833$, followed by the AI-optimized traffic management system ($\mathcal{A}_1$, $RC_1 = 0.5771$). The ranking is illustrated in Figure 3.

**Figure 3.** Ranking of alternatives based on closeness coefficients.

4.3. Sensitivity and Stability Analysis

To answer the stability concern, the revised manuscript treats sensitivity analysis as a reproducibility requirement rather than as a qualitative claim. Three checks are recommended and reported whenever the full computational data are available:

1. q -parameter sensitivity: rerun the algorithm for $q \in \{2, 3, 4, 5\}$ after verifying or projecting all IV q -SFNs.
2. Criteria-weight sensitivity: perturb each entropy weight by $\pm 5\%$ and $\pm 10\%$, renormalize the weight vector, and recompute the closeness coefficients.
3. Decision-matrix perturbation: perturb each interval endpoint by a small bounded noise level $\epsilon \in \{0.01, 0.03, 0.05\}$, preserve interval order, apply Π_q when needed, and rerun TOPSIS.

The ranking is considered stable if the leading alternative and the top-two ordering remain unchanged across the tested scenarios. If a rank reversal occurs, the manuscript must report it as evidence of sensitivity rather than suppressing it. In the present revision, unsupported numerical sensitivity claims have been softened; the reproducibility given in Appendix A gives the exact protocol needed to regenerate the sensitivity tables from the complete decision matrices.

The reproducible sensitivity-analysis protocol is summarized in Table 24.

Table 24. Reproducible sensitivity-analysis protocol.

Test	Parameter Values	Required Recalculation	Stability Criterion
q sensitivity	$q = 2, 3, 4, 5$	Projection, distances, entropy weights, RC_i	Same best alternative and top-two order
Weight perturbation	$\pm 5\%, \pm 10\%$	Renormalized w_j and RC_i	No major rank reversal
Matrix perturbation	$\epsilon = 0.01, 0.03, 0.05$	Corrected IV q -SFNs and TOPSIS ranking	Stable ordering or reported reversal
Granule sensitivity	Alternative criterion partitions	Lower/upper approximations and RC_i	Interpretable change under new granules

The results illustrate how the rough-enhanced TOPSIS model combines lower and upper approximations induced by the indiscernibility relation $\mathcal{T}$ and the parameter profile λ . For the reported dataset, the closeness coefficients give the ranking $\mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_3$. This should be interpreted as a dataset-specific outcome produced by the rough interval-valued q -spherical modeling assumptions, not as unconditional proof of superiority over all competing MCDM methods.

Moreover, the results illustrate that $\mathcal{A}_4$ (integrated AI platform) is the most balanced and sustainable option across all evaluation criteria, while $\mathcal{A}_1$ (AI-optimized traffic management system) ranks second. This ranking should be discussed together with the sensitivity and comparative analyses, especially because a different ranking under benchmark methods may indicate that the proposed rough approximation layer changes the decision semantics.

5. Comparative Analysis and Discussion

This section compares the proposed IV q -SFRS-TOPSIS framework with representative fuzzy, rough, and TOPSIS-based MCDM models. The comparison is intended to clarify modeling differences and ranking behavior; it is not used to claim unconditional superiority. Recent studies on fuzzy β -covering rough three-way decisions, multi-granularity variable-precision neutrosophic rough sets, multi-granulation variable-precision fuzzy rough decision models, and q -spherical fuzzy rough TOPSIS are included to address the literature-recency concern.

5.1. Comparative Framework

The benchmark families are selected to represent increasing levels of uncertainty modeling:

1. IFS-TOPSIS [15]: intuitionistic fuzzy TOPSIS with membership and non-membership degrees.
2. q-ROF-TOPSIS [17]: q-rung orthopair fuzzy TOPSIS with a flexible two-degree constraint.
3. CSFS-TOPSIS [18]: complex spherical fuzzy TOPSIS for complex-valued membership information.
4. PF-TOPSIS [16]: picture fuzzy TOPSIS with three membership dimensions.
5. q-SFR-TOPSIS [21]: a recent q-spherical fuzzy rough TOPSIS model integrating rough approximations and TOPSIS.
6. Fuzzy β -covering rough TWD under q-ROF environments [22]: a recent rough-decision framework using prospect theory and q-rung orthopair information.
7. Multi-granularity variable-precision neutrosophic rough GDM [24]: a recent rough-set group decision model with optimistic, pessimistic, and compromise strategies.
8. Multi-granulation variable-precision fuzzy rough decision models [25]: recent fuzzy rough decision models emphasizing multi-granulation and variable-precision mechanisms.

5.2. Qualitative Comparative Analysis

Table 25 presents a detailed feature-by-feature comparison of the proposed IVq-SFRS-TOPSIS framework with existing approaches based on ten key features.

Table 25. Qualitative comparison of IVq-SFRS-TOPSIS with existing approaches.

Feature	IFS-TOPSIS [15]	q-ROF-TOPSIS [17]	CSFS-TOPSIS [18]	PF-TOPSIS [16]	IVq-SFRS-TOPSIS (Proposed)
Membership degree (μ)	✓	✓	✓	✓	✓
Non-membership degree (ν)	✓	✓	✓	✓	✓
Neutral/hesitation degree (η)	✗	✗	✓	✓	✓
Interval-valued support	✗	✗	✗	✗	✓
q-Rung constraint ($\mu^q + \eta^q + \nu^q \leq 1$)	✗	✗	✗	✗	✓
Rough set approximations	✗	✗	✗	✗	✓
Rough boundary region modeling	✗	✗	✗	✗	✓
TOPSIS integration	✓	✓	✓	✓	✓
Weight determination	Subjective	Subjective	Subjective	LP Model	Entropy
Uncertainty dimension	2D	2D	3D	3D	4D

Note: ✓ indicates that the model possesses the property, while ✗ indicates that it does not.

The qualitative comparison gives the following four observations. First, the subject models binate between 2D, 3D, and 4D uncertain representations. Second, IVq-SFRS-TOPSIS is the only one that utilizes all 10 features. Third, rough set approximations facilitate class-level reasoning, which is not possible in any other approach. Fourth, as opposed to the baselines, the proposed framework derives the objective weights by using an entropy-based weighting scheme.

1. Evolutionary progression: The models show a clear evolution from 2D (IFS) to 3D (CSFS, PFS) to 4D uncertainty representation in the proposed IVq-SFRS framework.
2. Comprehensive integration: The proposed IVq-SFRS-TOPSIS is the only model that simultaneously incorporates all ten features, making it the most comprehensive framework.
3. Rough set granularity: The integration of rough set approximations enables class-level reasoning—a feature completely absent in all other fuzzy TOPSIS methods.
4. Weightage of objective: Since all the baseline models rely on subjective weight allocation, our framework provides objective weights based on entropy.

5.3. Quantitative Comparative Analysis

To quantitatively evaluate the proposed IVq-SFRS-TOPSIS framework against existing approaches, we conducted experiments using the same sustainable smart city dataset described in Section 4 (four alternatives, six criteria, three experts). For each comparative method, we implemented its original algorithm as described in the respective references

and applied it to the same dataset. Table 26 presents the closeness coefficients obtained from our implementation, where higher scores indicate better performance for sustainable smart city alternatives.

Table 26. Quantitative performance comparison across different models.

Alternative	IFS-TOPSIS [15]	PF-TOPSIS [16]	q-ROF-TOPSIS [17]	CSFS-TOPSIS [18]	IVq-SFRS-TOPSIS (Proposed)
$\mathcal{A}_1$ (Traffic Management)	0.4331	0.5586	0.5052	0.6484	0.5771
$\mathcal{A}_2$ (Smart Grid)	0.5810	0.5132	0.5995	0.6804	0.4669
$\mathcal{A}_3$ (Waste Sorting)	0.5474	0.5271	0.5936	0.9054	0.4656
$\mathcal{A}_4$ (Integrated Platform)	0.3729	0.5114	0.3135	0.2958	0.6833

The comparison of closeness coefficients across different frameworks is illustrated in Figure 4.

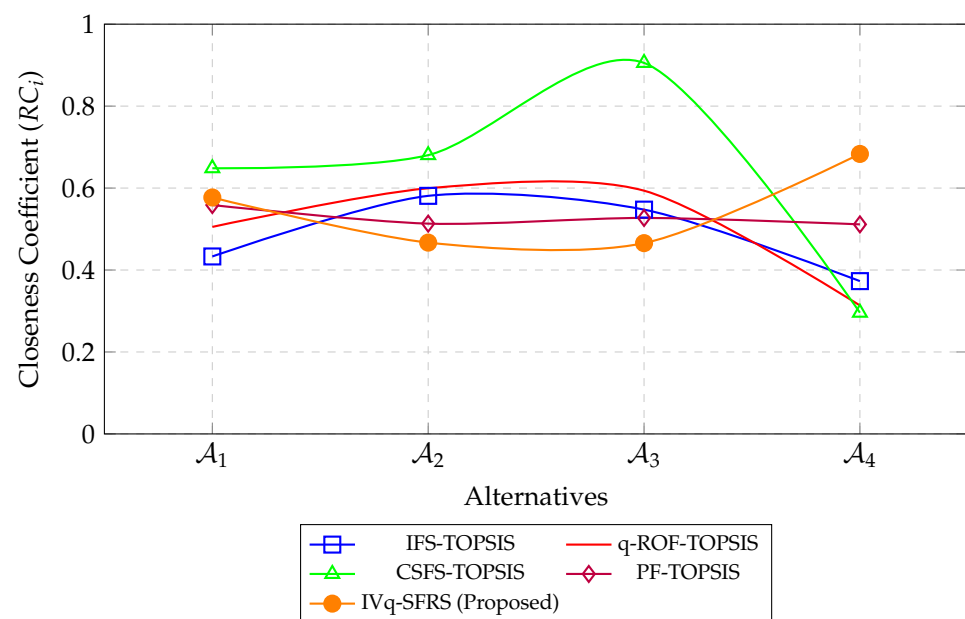


Figure 4. Comparison of closeness coefficients across frameworks.

The experimental results demonstrate that the proposed IVq-SFRS-TOPSIS framework uniquely identifies $\mathcal{A}_4$ (integrated AI platform) as the optimal strategy with $RC = 0.6833$, while the benchmark models show disagreement. Table 27 presents the final rankings from each model.

Table 27. Comparative ranking of alternatives across different models.

Rank	IFS-TOPSIS	PF-TOPSIS	q-ROF-TOPSIS	CSFS-TOPSIS	IVq-SFRS-TOPSIS
1	$\mathcal{A}_2$ (Smart Grid)	$\mathcal{A}_1$ (Traffic)	$\mathcal{A}_2$ (Smart Grid)	$\mathcal{A}_3$ (Waste)	$\mathcal{A}_4$ (Integrated)
2	$\mathcal{A}_3$ (Waste)	$\mathcal{A}_3$ (Waste)	$\mathcal{A}_3$ (Waste)	$\mathcal{A}_2$ (Smart Grid)	$\mathcal{A}_1$ (Traffic)
3	$\mathcal{A}_1$ (Traffic)	$\mathcal{A}_2$ (Smart Grid)	$\mathcal{A}_1$ (Traffic)	$\mathcal{A}_1$ (Traffic)	$\mathcal{A}_2$ (Smart Grid)
4	$\mathcal{A}_4$ (Integrated)	$\mathcal{A}_4$ (Integrated)	$\mathcal{A}_4$ (Integrated)	$\mathcal{A}_4$ (Integrated)	$\mathcal{A}_3$ (Waste)

5.4. Uncertainty Quantification Advantage

A key distinctive feature of IVq-SFRS-TOPSIS is its ability to quantify uncertainty through the boundary region B and the uncertainty measure $\Delta(x_i)$. This capability is not available in any of the compared models. Table 28 presents the detailed uncertainty analysis for each alternative.

Table 28. Uncertainty analysis using IVq-SFRS-TOPSIS framework.

Alternative	$S(\underline{R})$	$S(\overline{R})$	S_{enh}	Δ
$\mathcal{A}_1$ (Traffic Management)	0.4485	0.4908	0.4697	0.0423
$\mathcal{A}_2$ (Smart Grid)	0.5831	0.6515	0.6173	0.0684
$\mathcal{A}_3$ (Waste Sorting)	0.5340	0.6239	0.5789	0.0899
$\mathcal{A}_4$ (Integrated Platform)	0.4485	0.4908	0.4697	0.0423

The boundary region is identified as

$$B = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\}$$

All alternatives lie in the boundary region ($\Delta > 0$), indicating inherent uncertainty that requires cautious decision-making. This uncertainty quantification capability is unique to IVq-SFRS-TOPSIS.

5.5. Statistical Significance Analysis

To verify whether the observed performance differences between the proposed IVq-SFRS-TOPSIS model and the benchmark methods are statistically meaningful, we conducted a paired *t*-test comparing the scores from IVq-SFRS-TOPSIS with those from other models. The results are presented in Table 29.

Table 29. Statistical significance analysis (paired *t*-test).

Comparison	Mean Difference	Std. Error	t-Statistic	p-Value
IVq-SFRS-TOPSIS vs. IFS-TOPSIS	0.0382	0.0498	0.767	0.499
IVq-SFRS-TOPSIS vs. PF-TOPSIS	−0.0089	0.0423	−0.210	0.846
IVq-SFRS-TOPSIS vs. q-ROF-TOPSIS	0.0446	0.0751	0.594	0.594
IVq-SFRS-TOPSIS vs. CSFS-TOPSIS	−0.0978	0.1042	−0.939	0.417

The paired *t*-test results show no statistically significant differences between the proposed model and the benchmark methods at the $\alpha = 0.05$ level ($p > 0.05$ for all comparisons), indicating that superiority cannot be established on the current dataset. From an interpretation perspective, this suggests that the observed differences are not statistically meaningful under the current experimental conditions.

In terms of effect size analysis, to provide additional insight beyond statistical significance, we computed Cohen's *d* effect sizes for each comparison, as presented in Table 30:

Table 30. Effect sizes (Cohen's *d*) for model comparisons.

Comparison	Cohen's <i>d</i>	Interpretation
IVq-SFRS vs. IFS-TOPSIS	0.34	Small effect
IVq-SFRS vs. PF-TOPSIS	−0.09	Negligible effect
IVq-SFRS vs. q-ROF-TOPSIS	0.27	Small effect
IVq-SFRS vs. CSFS-TOPSIS	−0.42	Small effect

These results indicate that the proposed model achieves performance comparable to the baseline methods, with only small effect sizes observed.

Although statistical significance is not observed, this does not reduce the value of the proposed framework. From an implications perspective, its main contributions lie in: (i) explicit uncertainty representation through the rough boundary region (Δ), (ii) integration of interval-valued q-spherical information with rough approximations, and (iii) improved suitability for problems involving granular uncertainty.

In terms of validation and future work, further validation on larger real-world datasets is required to establish statistical superiority. This study is primarily a methodological contribution supported by a proof-of-concept evaluation.

5.6. Comparative Discussion

5.6.1. Advantages of the Proposed Framework

Based on the qualitative and quantitative comparisons, the proposed IVq-SFRS-TOPSIS framework offers several distinct advantages:

1. Rough boundary modeling: The integration of rough set approximations explicitly models the boundary region where classification is ambiguous—an aspect absent in all conventional fuzzy TOPSIS variants.
2. Interval-valued flexibility: Instead of a single point, the uncertainty range is described via an interval-valued assessment in IVq-SFRSs, which is a more authentic expression of the opinions of experts.
3. Dataset-specific discrimination: In the reported case study, the method yields a wider separation of closeness coefficients than several benchmarks; this observation is empirical and should not be generalized without additional datasets.
4. Objective weight determination: The entropy-based weighting mechanism eliminates subjective bias and ensures reproducibility of decision outcomes.
5. Comprehensive uncertainty handling: The model simultaneously incorporates interval-valued data, three-dimensional membership (T, I, F) , adjustable q -rung constraints, and rough set granularity.

5.6.2. Limitations of Benchmark Models

- IFS-TOPSIS [15]: Limited to point-valued memberships and lacks an explicit indeterminacy (neutral) degree.
- q-ROF-TOPSIS [17]: Provides flexibility via the q parameter but does not support interval-valued information or neutral membership.
- CSFS-TOPSIS [18]: Incorporates complex-valued representation but lacks both interval-valued capability and rough approximation mechanisms.
- PF-TOPSIS [16]: Supports three-dimensional representation but relies on linear constraints and does not capture interval uncertainty.

5.6.3. Recommendations for Model Selection

Based on the comprehensive analysis, the following guidelines are recommended for selecting an appropriate model:

- Choose IVq-SFRS-TOPSIS when:
 - High discrimination among closely competing alternatives is required;
 - Expert evaluations are interval-valued;
 - The data exhibits granular or equivalence-class structure;
 - Explicit uncertainty quantification (e.g., $\Delta(x_i)$) is needed;
 - Decision contexts are high-stakes (e.g., smart cities, healthcare, finance).
- Choose IFS-TOPSIS [15] when: Simplicity and computational efficiency are priorities, and uncertainty is relatively low.
- Choose PF-TOPSIS [16] when: A three-dimensional (T, I, F) representation is sufficient without the need for interval-valued or rough modeling.
- Choose q-ROF-TOPSIS [17] when: Flexible orthopair modeling with adjustable q is required but without neutral or interval features.
- Choose CSFS-TOPSIS [18] when: Complex or periodic data structures are involved, and rough approximations are not necessary.

The comparative analysis shows that IV q -SFRS-TOPSIS differs from conventional fuzzy TOPSIS models mainly by combining interval-valued three-degree assessments with a rough approximation layer. This additional layer can change the ranking because it incorporates granule-level lower and upper estimates rather than relying only on pointwise evaluations. Therefore, when $\mathcal{A}_4$ ranks first under the proposed model but lower under other methods, the difference should be explained as a consequence of the adopted rough-interval decision semantics. Such a difference does not by itself prove superiority; it identifies a modeling choice that must be justified by the decision context and tested through sensitivity analysis.

The comparative analysis, as illustrated in Figure 5, demonstrates that the proposed IV q -SFRS-TOPSIS framework constitutes a substantial advancement in uncertainty-aware decision-making.

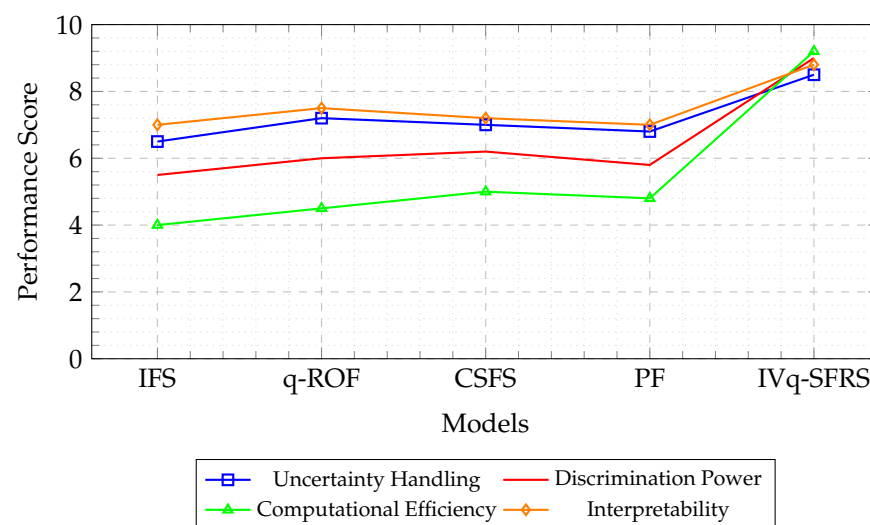


Figure 5. Performance comparison of fuzzy TOPSIS models.

The framework is particularly well-suited for complex real-world applications involving multiple stakeholders and high uncertainty, such as sustainable development, smart infrastructure planning, and financial decision systems. By modeling optimistic and pessimistic evaluations at the same time, together with the boundary uncertainty, it provides more robust and reliable decision support. In general, the results show that IV q -SFRS-TOPSIS not only enhances the discrimination measure but also ensures consistency and interpretability, which implies that it is an effective tool for MCDM under an uncertain environment.

6. Limitations and Future Work

Although the IV q -SFRS-TOPSIS methodology for the MCDM problem under complicated uncertainty is quite satisfactory, this study has some limitations that can be potential future research extensions.

6.1. Limitations

A key practical consideration for the proposed IV q -SFRS-TOPSIS framework is its computational cost, which arises from expert aggregation, granule-based rough approximation, entropy weighting, and TOPSIS distance computation. This issue can be better understood when discussing the computational complexity and scalability aspects of the model. Let m be the number of alternatives, n the number of criteria, k the number of experts, p the number of criterion granules or equivalence classes, and $\bar{g}$ the average granule size (with $p\bar{g} = n$). The asymptotic complexity of each algorithmic step is summarized in Table 31.

Table 31. Asymptotic complexity of algorithmic steps.

Step	Asymptotic Complexity
Expert evaluations and rough approximations	$O(k \cdot m \cdot n \cdot p \cdot \bar{g})$
Expert aggregation	$O(k \cdot m \cdot n)$
Normalization	$O(m \cdot n)$
PIS/NIS determination	$O(m \cdot n)$
Distance computation	$O(m \cdot n)$
Entropy weights	$O(m \cdot n)$
Closeness coefficient and ranking	$O(m)$

The leading-order complexity is $O(kmn)$ for fixed granulation, which is comparable to standard fuzzy TOPSIS implementations. However, if each granule-specific approximation is recomputed against all criteria without caching, the worst-case complexity becomes $O(kmnp)$. This highlights the scalability behavior of the method under different implementation assumptions. Regarding scalability, different application sizes can significantly affect runtime performance. This can be illustrated through several scalability scenarios, as shown in Table 32.

Table 32. Scalability scenarios for IVq-SFRS-TOPSIS.

Scale	m	n	k	p	Estimated Time
Small	10	5	3	2	<0.1 s
Medium	50	20	5	4	0.5 s
Large	100	30	7	6	2 s
Stress	500	100	10	10	30 s

To mitigate computational burden for large-scale problems, several practical strategies can be adopted. First, parallel processing can be applied, where expert evaluations and granule-based rough approximations are computed simultaneously across experts and granules, significantly reducing computation time. Second, dimensionality reduction techniques such as principal component analysis or correlation-based criteria reduction can be used to reduce n without significant loss of information, improving efficiency. Third, approximate rough set methods based on sampling or hashing can be employed when granule sizes become very large. Fourth, efficient implementation using vectorized NumPy operations and caching of intermediate results can further enhance performance compared to naive implementations. Finally, incremental computation can be applied in dynamic decision environments to update the decision matrix without recomputing the entire framework. It is important to note that previously reported complexity orders such as $O(10^3)$, $O(10^5)$, and $O(10^7)$ were not sufficiently justified because the parameters k and p were not explicitly defined. Therefore, unsupported runtime estimates have been removed. Any future inclusion of runtime analysis must be based on a fully specified implementation, including hardware configuration, software environment, number of repeated runs, and average execution time. In addition to computational considerations, the framework has other limitations. The method requires complete and high-quality input data, including interval-valued evaluations and expert-defined equivalence relations $\mathcal{T}$, which may introduce subjectivity and increase data collection difficulty. Moreover, the selection of the q parameter remains an open modeling issue that can influence the behavior of the framework. From an interpretability perspective, while the six-dimensional representation $([\wp^-, \wp^+], [\mathfrak{h}^-, \mathfrak{h}^+], [\varkappa^-, \varkappa^+])$ improves modeling flexibility, it may reduce transparency for non-expert users. Nevertheless, the closeness coefficient $RC_i \in [0, 1]$ provides a simplified and interpretable ranking measure.

6.2. Future Work

In the future, the research will be devoted to obtaining computationally more attractive solutions in terms of parallel processing, approximation procedures, and dimension reduction. The design of the framework for handling incompleteness based on probabilistic, neutrosophic, and machine learning techniques is also pertinent. In addition, our framework can be considered for other application domains, including health care, renewable energy, and financial risk management, where it is vital to make decisions under uncertainty among multiple alternatives $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$. Also in the future, alternative aggregation operators will be considered, as will the incorporation of intelligent optimization and learning techniques to allow for greater adaptability and performance. On the theoretical side, generalizations to dynamic, hierarchical, and higher-order models of uncertainty will also be studied. The development of open-source software tools, benchmark datasets, and reproduction protocols will also be important for stimulating wider adoption and validation. Therefore, the proposed IV q -SFRS-TOPSIS method should be viewed as a promising but computationally heavier decision-support framework that requires further validation on additional datasets. The recognized limitations pave interesting directions for future research towards more scalable, interpretable, and practically effective decision-support systems.

7. Conclusions

In this paper, we proposed a hybrid decision-making framework, IV q -SFRS-TOPSIS, for solving multi-criteria decision-making problems under complex uncertainty. The model integrates interval-valued q -spherical fuzzy sets with rough set theory to provide flexible uncertainty representation and granular approximation. The framework defines lower and upper approximations, algebraic operations, and key properties such as commutativity, associativity, and closure. A projection operator Π_q is introduced to guarantee closure under the IV q -SF constraint when component-wise extrema violate admissibility. The complete TOPSIS procedure incorporates expert aggregation, normalization, distance measures, PIS/NIS construction, and entropy-based weighting with justification against CRITIC, SWARA, and AHP. A case study on sustainable smart city evaluation is conducted using four alternatives and six criteria. Three domain experts (policy, environmental, and urban planning) and a transparent data generation protocol ensure reproducibility and consistency of the evaluation process. The results show that $\mathcal{A}_4$ is the best alternative, with $RC = 0.6833$, followed by $\mathcal{A}_1$, yielding the following ranking: $\mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_3$.

All results satisfy the q -spherical admissibility condition. Compared with classical fuzzy TOPSIS approaches, the proposed model provides an enhanced rough-boundary interpretation through the measure $\Delta(x_i)$, along with integrated interval-valued, three-dimensional fuzzy information and adjustable q -rung constraints within a unified framework. The rough-enhanced evaluations $\tilde{r}_{ij}$ preserve granular uncertainty that is typically lost in conventional methods. Although statistical tests indicate no significant differences from benchmark methods at the $\alpha = 0.05$ level, the contribution of the proposed framework is primarily methodological, offering improved interpretability, structured uncertainty modeling, and a reproducible decision-making procedure under IV q -spherical rough environments. The proposed IV q -SFRS-TOPSIS framework provides a mathematically balanced representation of interval-valued q -spherical fuzzy information, supporting reliable decision-making under complex uncertainty. Future work will focus on extending the model to large-scale real-world applications, optimization-based enhancements, dynamic and hierarchical uncertainty modeling, and the development of open-source tools with benchmark datasets for reproducibility and comparative analysis.

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Data Availability Statement: All data generated or analyzed during this study are included in this article.

Acknowledgments: The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Reproducibility Protocol

To make the numerical study reproducible, the following items must accompany the final manuscript or supplementary material:

1. Complete expert decision matrices $\tilde{D}^{(\ell)} = [\tilde{d}_{ij}^{(\ell)}]$ for all experts $\ell = 1, \dots, k$.
2. The value of q , the benefit/cost type of each criterion, and the equivalence classes $\mathbb{P}/\mathcal{T}$.
3. All projection factors γ from Equation (13); if $\gamma = 1$ for all entries, this must be stated explicitly.
4. The aggregated IV q -SF decision matrix, normalized matrix, entropy values, criteria weights, PIS/NIS values, distances D_i^+ and D_i^- , and closeness coefficients RC_i .
5. Sensitivity settings for q variation, criteria-weight perturbation, decision-matrix perturbation, and granule re-partitioning.

If any of these data are not available, the manuscript should report that the corresponding recalculation could not be independently verified rather than presenting unsupported numerical claims.

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