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Stress–Strength Reliability Inference with Upper Record Ranked Set Sampling Under the Scale-Parameter Exponential Family

Yinuo Dong ¹, Yihan Ding ² and Wenhao Gui ^{1,*} ¹ School of Mathematics and Statistics, Beijing Jiaotong University, Beijing 100044, China; 25121504@bjtu.edu.cn² School of Science, Beijing University of Civil Engineering and Architecture, Beijing 102616, China; 23398057@bjtu.edu.cn

* Correspondence: whgui@bjtu.edu.cn

Abstract

In reliability engineering and life-testing experiments, exact measurements can be costly or destructive, whereas ranking information and upper record observations are often easier to obtain. Upper record ranked set sampling (RRSS) provides an efficient design for such settings, and the scale-parameter exponential family (SPEF) offers a unified framework covering several common lifetime distributions. Motivated by these advantages, this paper investigates point and interval estimation of the stress–strength reliability parameter $\mathcal{R}$ under upper RRSS and SPEF assumptions. Explicit expressions for the maximum likelihood estimator and the uniformly minimum variance unbiased estimator of $\mathcal{R}$ are derived, and their performance is evaluated. To address inference limitations in small samples, Bayesian point and highest posterior density (HPD) interval estimators are developed via Monte Carlo simulation under conjugate gamma priors. Several frequentist confidence intervals are also constructed, including an exact interval, an approximate MLE-based interval, and three parametric bootstrap intervals. Simulation studies demonstrate that the Bayesian HPD intervals attain nominal coverage with competitive expected lengths in small samples. Finally, the practical utility and model adequacy of the proposed methodologies are demonstrated through an analysis of real lifetime data from steel specimens.

Keywords: record ranked set sampling; stress–strength model; uniformly minimum variance unbiased estimation; reliability estimation; bootstrap confidence intervals; Bayesian inference; highest posterior density credible interval

1. Introduction

Stress–strength analysis is a well-established topic in the statistical literature. The reliability parameter $\mathcal{R}$ is defined as $\Pr(X < Y)$, i.e., the probability that the applied stress X does not exceed the available strength Y . In engineering practice, stress–strength models are widely used in material reliability evaluation and design, as well as in product life testing. For example, in the aerospace industry, life tests are routinely performed on aircraft components to ensure operational safety. During service, a product may be exposed to external stresses such as pressure, wind, humidity, and temperature, while its inherent strength allows it to resist these loads. Because of this interaction between stress and strength, such models have broad applications, especially in reliability assessment.

Research on stress–strength reliability has progressed from basic models to more complex systems under diverse probabilistic frameworks. Ref. [1] analyzed consecutive



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k -out-of- n :G systems. Ref. [2] extended the framework to multivariate settings with dependent components and multi-component stresses, and Ref. [3] studied inference for two-parameter exponential stress–strength models. Whereas traditional analyses rely on binary system states (working or failed), a more realistic approach may define multiple performance levels according to the magnitude of $Y - X$, thereby moving toward multi-state system modeling.

Later work has broadened the distributional assumptions. Refs. [4] and [5] considered the Lindley and Power Lindley distributions, respectively, using maximum likelihood, bootstrap, and Bayesian methods when exact distributional theory is intractable. Ref. [6] subsequently provided a generalized framework for distributions in general exponential form.

The multi-component stress–strength model has also been studied under various distributions. Ref. [7] obtained results for the Chen distribution, whereas Ref. [8] applied classical and Bayesian approaches to the Topp–Leone distribution via approximations and Gibbs sampling. Ref. [9] introduced a system-based perspective, using system signatures to express system reliability and optimize configurations with exponential components. Overall, the literature shows a trend from classical estimation toward incomplete-data likelihoods, resampling methods, and Bayesian computation for increasingly complex systems and distributional settings.

A family of probability distributions $\{F(\cdot; \theta) : \theta \in \Theta\}$ with probability density functions (PDFs) $\{f(\cdot; \theta)\}$ is said to belong to the scale-parameter exponential family (SPEF) if its density admits the canonical representation

$$f(z; \theta) = h(z) g(\theta) \exp\{-\omega(\theta) T(z)\}, \quad z \in \mathcal{Z}, \quad (1)$$

where the support $\mathcal{Z} \subseteq \mathbb{R}$ is independent of the parameter $\theta \in \Theta$. Here, $T : \mathcal{Z} \rightarrow \mathbb{R}$ is a given measurable function, $\omega(\theta)$ is a real-valued parameter depending on θ , $h(z)$ is a non-negative baseline function independent of θ , and $g(\theta)$ is a normalizing factor chosen such that $\int_{\mathcal{Z}} f(z; \theta) dz = 1$; the non-negativity of $h(z)$ ensures that $f(z; \theta)$ defines a valid probability density, since the exponential term is strictly positive and the factor $g(\theta)$ does not depend on z . The normalization condition implies that

$$g(\theta) = \left[\int_{\mathcal{Z}} h(z) \exp\{-\omega(\theta) T(z)\} dz \right]^{-1}, \quad (2)$$

and, defining $A(\theta) = -\log g(\theta) = \log \int_{\mathcal{Z}} h(z) \exp\{-\omega(\theta) T(z)\} dz$, the density can equivalently be written in exponential family form as

$$f(z; \theta) = h(z) \exp\{-\omega(\theta) T(z) - A(\theta)\}. \quad (3)$$

The corresponding cumulative distribution function (CDF) is given by

$$F(z; \theta) = \int_{z_0}^z h(y) \exp\{-\omega(\theta) T(y) - A(\theta)\} dy, \quad (4)$$

where z_0 denotes the lower endpoint of $\mathcal{Z}$. Further simplification is possible under the additional assumptions that $\mathcal{Z} = [0, \infty)$, $T(z)$ is strictly increasing and continuously differentiable on $[0, \infty)$ with $T(0) = 0$ and $T(z) \geq 0$ for all $z \geq 0$, the baseline function satisfies $h(z) = T'(z)$, and the parameter satisfies $\omega(\theta) > 0$.

Under these conditions, the normalizing constant simplifies to $g(\theta) = \omega(\theta)$, the density reduces to

$$f(z; \theta) = \omega(\theta) T'(z) \exp\{-\omega(\theta) T(z)\}, \quad z \geq 0, \quad (5)$$

and the cumulative distribution function admits the closed-form expression

$$F(z; \theta) = 1 - \exp\{-\omega(\theta) T(z)\}, \quad z \geq 0. \quad (6)$$

Examples of continuous scale-parameter exponential family distributions with explicit parameterization are summarized in Table 1.

Table 1. Examples of continuous scale-parameter exponential family distributions with explicit parameterization.

Distribution	$T(z)$	$T'(z)$	$\omega(\theta)$	Support
Exponential ($\theta > 0$)	z	1	$1/\theta$	$z \geq 0$
Weibull ($k > 0, \theta > 0$)	z^k	kz^{k-1}	$1/\theta^k$	$z \geq 0$
Rayleigh ($\theta > 0$)	$z^2/2$	z	$1/\theta^2$	$z \geq 0$
Gamma ($\alpha = 1, \theta > 0$)	z	1	$1/\theta$	$z \geq 0$
Lomax (Pareto II) ($\alpha > 0$)	$\ln(1+z)$	$(1+z)^{-1}$	α	$z \geq 0$

Ranked set sampling (RSS) offers an efficient alternative to simple random sampling (SRS) by incorporating ranking information into the sampling design, thereby improving estimator precision while reducing measurement cost. As reviewed by [10], both balanced and unbalanced RSS schemes outperform SRS in a wide range of settings, particularly when ranking can be performed with reasonable accuracy. However, inference under RSS can be sensitive to ranking errors, and classical maximum likelihood estimation may lose efficiency in such cases. To address this issue, Ref. [11] investigated several robust estimation methods and showed that goodness-of-fit-based procedures perform better under imperfect ranking. Moreover, Ref. [12] demonstrated that rank-based sampling schemes can be applied effectively to complex distributional models, including skewed populations, where order statistics and Fisher information play a key role in efficient inference. Recent stress–strength studies further confirm the usefulness of RSS-type designs: Ref. [13] developed median-ranked set sampling estimators for Lomax and inverse Lomax models, while Ref. [14] showed that RSS and extreme RSS improve power Lindley reliability estimation under uncensored and right-censored settings.

Ref. [15] introduced a sampling scheme called record ranked set sampling (RRSS), which extends the traditional ranked set sampling framework. In this scheme, only the final record from each sequence, denoted by $R_{i,i}$, is retained for subsequent analysis. Consequently, the dataset $\mathbf{R} = (R_{1,1}, R_{2,2}, \dots, R_{n,n})^\top$ constitutes a record ranked set sample of size n . The sampling procedure can be illustrated by the following schematic representation in Figure 1.

$$\begin{array}{ll}
 1 : R_{(1)1} & \rightarrow R_{1,1} = R_{(1)1} \\
 2 : R_{(1)2} \ R_{(2)2} & \rightarrow R_{2,2} = R_{(2)2} \\
 \vdots & \vdots \\
 n : R_{(1)n} \ R_{(2)n} \ \cdots \ R_{(n)n} & \rightarrow R_{n,n} = R_{(n)n}
 \end{array}$$

Figure 1. Schematic illustration of the record ranked set sampling (RRSS) procedure.

With $F(\cdot; \theta)$ and $f(\cdot; \theta)$ denoting the population CDF and PDF, the joint density of $\mathbf{R}$ under upper RRSS of size n is obtained from the marginal density of ordinary records; see [16]:

$$f_{\mathbf{R}}(\mathbf{r}; \theta) = \prod_{i=1}^n \frac{\{-\ln(1 - F(r_{i,i}; \theta))\}^{i-1}}{(i-1)!} f(r_{i,i}; \theta), \quad \theta \in \Theta. \quad (7)$$

Existing studies have employed RRSS to enhance stress–strength reliability estimation for exponential distributions. Ref. [17] developed basic point and interval estimators under upper RRSS. Ref. [18] used prior information to construct shrinkage and preliminary test estimators, showing that these estimators uniformly dominate the maximum likelihood estimator in terms of asymptotic efficiency. Further extensions cover record-based designs and other exponential-type models. Ref. [19] compared likelihood and Bayesian procedures for stress–strength reliability under lower record RRSS, whereas Ref. [20] proposed MLE and Bayesian estimators for linear exponential distributions in one- and multi-cycle RRSS settings and reported efficiency gains relative to simple random sampling. More recent record-based reliability work has combined likelihood, bootstrap, and Bayesian procedures for specific lifetime models. For instance, Ref. [21] studied generalized Rayleigh stress–strength reliability under lower RRSS with Bayesian and bootstrap intervals, and Ref. [22] developed classical and Bayesian inference based on lower record values from inverse Lomax models. Recent reliability work has also emphasized the practical importance of distribution-specific Bayesian estimation techniques; for example, Ref. [23] studied Rayleigh stress–strength reliability for series systems with cold standby redundancy. These studies motivate a more general treatment that preserves the efficiency of record-based sampling while avoiding restriction to a single lifetime model.

The present study focuses on estimating the stress–strength reliability parameter $\mathcal{R} = \Pr(X < Y)$ within the scale-parameter exponential family (SPEF), a widely used model class in reliability analysis and life testing. Methodologically, we adopt an upper RRSS scheme with possibly unequal sample sizes n and m , which can improve estimation efficiency when data collection is limited or costly. Unlike earlier RRSS stress–strength studies that mainly focus on a specific distribution or a narrower set of estimators, the present work develops a unified inferential framework for SPEF models.

The main contributions of this paper are summarized as follows. First, closed-form MLE and UMVUE expressions are derived for $\mathcal{R}$ under upper RRSS. Second, exact, approximate, and parametric bootstrap confidence intervals are constructed and compared. Third, a Bayesian procedure with independent conjugate gamma priors for $\omega(\theta_1)$ and $\omega(\theta_2)$ is developed, leading to tractable posterior sampling and HPD credible intervals. Fourth, simulation studies and a steel specimen lifetime application are used to assess finite-sample behavior and practical applicability.

The remainder of this paper is organized as follows. Section 2 derives the MLE and UMVUE of $\mathcal{R}$ and compares them with respect to bias, variance, and MSE. Section 3 presents several frequentist confidence intervals for $\mathcal{R}$. Section 4 develops Bayesian point and HPD interval estimation under independent conjugate gamma priors and examines the finite-sample behavior of $\hat{\mathcal{R}}_{\text{Bayes}}$. Section 5 compares the coverage probabilities and expected lengths of the frequentist and Bayesian intervals via simulation. Section 6 applies the proposed methods to a real dataset. Section 7 concludes.

2. Point Estimation

Let Z follow a scale-parameter exponential family (SPEF) distribution with parameter θ . Under the assumptions above, the model admits a simplified (canonical) form, denoted

by $Z \sim \text{SPEF}(\theta)$. Here the normalizing constant becomes $g(\theta) = \omega(\theta)$, and the PDF and CDF are

$$f(z; \theta) = \omega(\theta) T'(z) \exp\{-\omega(\theta) T(z)\}, \quad z \geq 0, \quad (8)$$

and

$$F(z; \theta) = 1 - \exp\{-\omega(\theta) T(z)\}, \quad z \geq 0. \quad (9)$$

Let $X \sim \text{SPEF}(\theta_1)$ and $Y \sim \text{SPEF}(\theta_2)$ be independent. The stress–strength reliability parameter $\mathcal{R} = \Pr(X < Y)$ admits a closed-form expression under the regularity conditions $\omega(\theta) > 0$, $T(0) = 0$, and $T(+\infty) = +\infty$; the derivation is given in Appendix A (A1).

$$\mathcal{R} = \frac{\omega(\theta_1)}{\omega(\theta_1) + \omega(\theta_2)}. \quad (10)$$

Let $\mathbf{r} = (r_{1,1}, r_{2,2}, \dots, r_{n,n})^\top$ and $\mathbf{s} = (s_{1,1}, s_{2,2}, \dots, s_{m,m})^\top$ denote observations from $\mathbf{R} = (R_{1,1}, R_{2,2}, \dots, R_{n,n})^\top$ and $\mathbf{S} = (S_{1,1}, S_{2,2}, \dots, S_{m,m})^\top$, where $\mathbf{R}$ and $\mathbf{S}$ are independent upper RRSS of sizes n and m from $\text{SPEF}(\theta_1)$ and $\text{SPEF}(\theta_2)$, respectively.

2.1. Maximum Likelihood Estimation

First, we find the maximum likelihood estimation (MLE) of $\omega(\theta_1)$ and $\omega(\theta_2)$. By substituting (8) and (9) into (7), the likelihood functions follow

$$\begin{aligned} L_1(\theta_1; \mathbf{r}) &= \prod_{i=1}^n f(r_{i,i}) \\ &= \frac{\omega^N(\theta_1) \prod_{i=1}^n T^{i-1}(r_{i,i}) T'(r_{i,i})}{\prod_{i=1}^n (i-1)!} \exp\left\{-\omega(\theta_1) \sum_{i=1}^n T(r_{i,i})\right\}, \quad N = \frac{n(n+1)}{2}. \end{aligned} \quad (11)$$

$$\begin{aligned} L_2(\theta_2; \mathbf{s}) &= \prod_{j=1}^m f(s_{j,j}) \\ &= \frac{\omega^M(\theta_2) \prod_{j=1}^m T^{j-1}(s_{j,j}) T'(s_{j,j})}{\prod_{j=1}^m (j-1)!} \exp\left\{-\omega(\theta_2) \sum_{j=1}^m T(s_{j,j})\right\}, \quad M = \frac{m(m+1)}{2}. \end{aligned} \quad (12)$$

Then, the MLE of the parameters $\omega(\theta_1)$ and $\omega(\theta_2)$ can be readily given by

$$\omega(\hat{\theta}_{1\text{ML}}) = \frac{N}{\sum_{i=1}^n T(R_{i,i})}, \quad N = \frac{n(n+1)}{2}. \quad (13)$$

$$\omega(\hat{\theta}_{2\text{ML}}) = \frac{M}{\sum_{j=1}^m T(S_{j,j})}, \quad M = \frac{m(m+1)}{2}. \quad (14)$$

with $\omega(\theta) > 0$, $T(+\infty) = +\infty$, $T(0) = 0$, respectively. Using the invariance property of the MLE, we find the MLE of $\mathcal{R}$ as

$$\hat{\mathcal{R}}_{\text{ML}} = \frac{\omega(\hat{\theta}_{1\text{ML}})}{\omega(\hat{\theta}_{1\text{ML}}) + \omega(\hat{\theta}_{2\text{ML}})} = \left(1 + \frac{M \sum_{i=1}^n T(R_{i,i})}{N \sum_{j=1}^m T(S_{j,j})}\right)^{-1}, \quad (15)$$

From [16], $T(r_{i,i})$ has a gamma distribution. Throughout this paper, $\text{Ga}(a, b)$ denotes a gamma distribution with shape parameter a and scale-parameter b , whose density is proportional to $x^{a-1} \exp(-x/b)$ for $x > 0$. Under this convention, $T(r_{i,i}) \sim \text{Ga}(i, 1/\omega(\theta))$ with PDF

$$f(T(r_{i,i}); \theta) = \frac{[\omega(\theta) T(r_{i,i})]^{i-1} [\omega(\theta)] \exp\{-\omega(\theta) T(r_{i,i})\}}{(i-1)!}, \quad r_{i,i} > 0, \quad (16)$$

As mentioned earlier, $T(r_{i,i})$ are independent random variables. Therefore

$$\sum_{i=1}^n T(R_{i,i}) \sim \text{Ga}(N, 1/\omega(\theta_1)), \quad (17)$$

$$\sum_{j=1}^m T(S_{j,j}) \sim \text{Ga}(M, 1/\omega(\theta_2)), \quad (18)$$

Consequently, from (15), we have

$$\hat{R}_{\text{ML}} \stackrel{d}{=} \left(1 + \frac{1 - \mathcal{R}}{\mathcal{R}} F_{2N, 2M} \right)^{-1}, \quad (19)$$

where $\stackrel{d}{=}$ denotes equality in distribution and $F_{2N, 2M}$ represents the F -distribution with $2N$ and $2M$ degrees of freedom.

Using (19), the bias, variance, and mean squared error (MSE) of $\hat{R}_{\text{ML}}$ are evaluated numerically for selected pairs (n, m) ; Figure 2 reports the results.

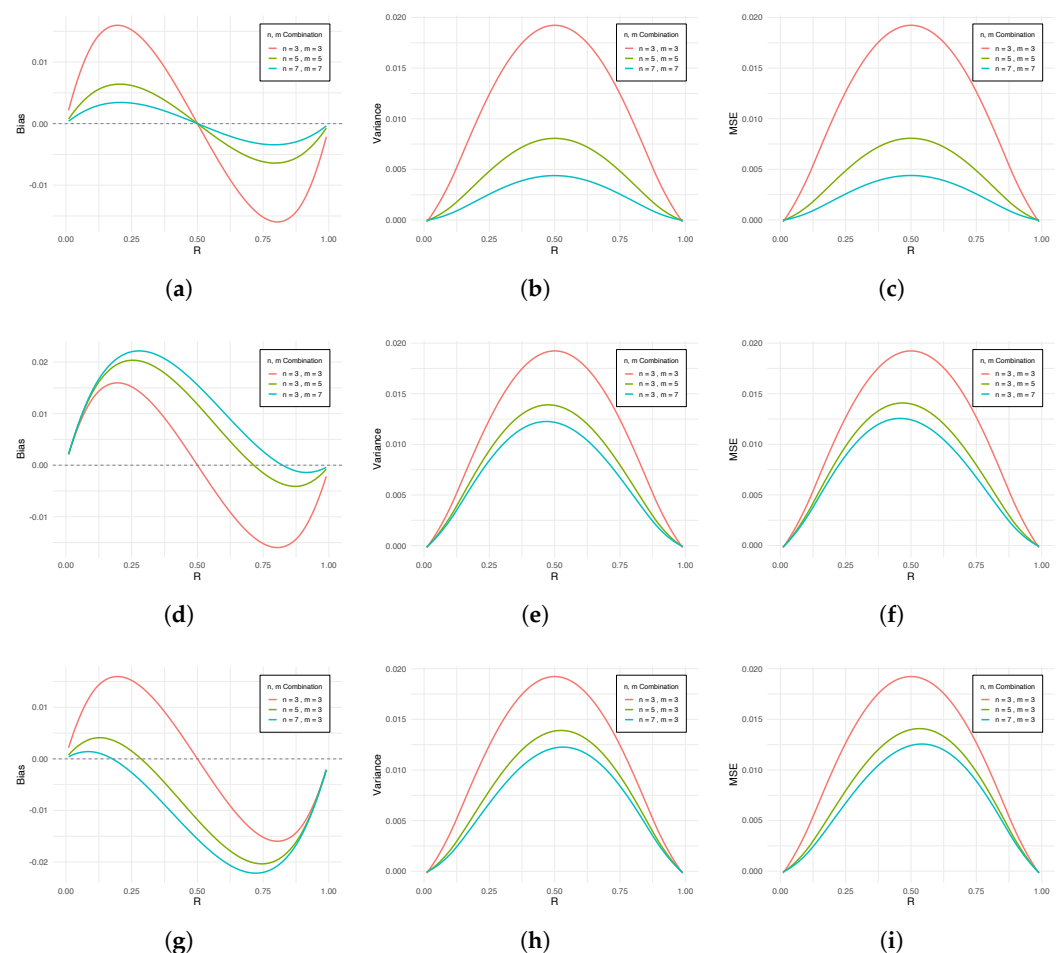


Figure 2. (a) Bias of $\hat{R}_{\text{ML}}$ under balanced (n, m) ; (b) Variance of $\hat{R}_{\text{ML}}$ under balanced (n, m) ; (c) MSE of $\hat{R}_{\text{ML}}$ under balanced (n, m) ; (d) Bias under fixed n with increasing m ; (e) Variance under fixed n with increasing m ; (f) MSE under fixed n with increasing m ; (g) Bias under fixed m with increasing n ; (h) Variance under fixed m with increasing n ; (i) MSE under fixed m with increasing n .

Figure 2 yields the following observations:

- When n and m increase simultaneously with $n = m$, the bias of $\hat{R}_{\text{ML}}$ remains approximately symmetric about $\mathcal{R} = 0.5$ and decreases rapidly with sample size. In this balanced setting, both variance and MSE exhibit a symmetric unimodal pattern,

attaining their maximum near the center and approaching zero near the boundaries, suggesting stable finite-sample behavior and near-unbiasedness.

- When n is fixed and m increases, the symmetry of the bias curve is lost and a pronounced right-skewed pattern emerges, with larger bias for high $\mathcal{R}$. Although variance and MSE decrease monotonically as m increases, the directional bias persists, reflecting unequal contributions from the two samples under an unbalanced design.
- When m is fixed and n increases, the bias becomes left-skewed with larger deviations for small $\mathcal{R}$. As in the previous case, increasing n reduces variance and MSE; however, the asymmetric bias induced by sample imbalance is not entirely eliminated.

2.2. Uniformly Minimum Variance Unbiased Estimation

Theorem 1. Let $n, m \geq 2$ and suppose that $X \sim \text{SPEF}(\theta_1)$ and $Y \sim \text{SPEF}(\theta_2)$ are independent. Then the uniformly minimum variance unbiased estimator (UMVUE) of the stress–strength reliability $\mathcal{R} = \Pr(X < Y)$ exists and admits the following explicit form:

$$\hat{\mathcal{R}}_{\text{UMVU}} = \begin{cases} 1 - H_{n,m}(\hat{R}_{\text{ML}}), & \text{if } \hat{R}_{\text{ML}} \leq \frac{N}{N+M}, \\ H_{m,n}(1 - \hat{R}_{\text{ML}}), & \text{if } \hat{R}_{\text{ML}} \geq \frac{N}{N+M}, \end{cases} \quad (20)$$

where $N = n(n+1)/2$, $M = m(m+1)/2$, and $\hat{R}_{\text{ML}}$ denotes the maximum likelihood estimator of $\mathcal{R}$. The function $H_{n,m}(\cdot)$ is defined as

$$H_{n,m}(t) = \sum_{d=0}^{N-1} \frac{\binom{N-1}{d}}{\binom{d+M-1}{d}} \left(-\frac{M}{N} \frac{t}{1-t} \right)^d. \quad (21)$$

Remark 1. When $\hat{R}_{\text{ML}} = N/(N+M)$ exactly, both branches of (20) yield the same value, and either expression may be used.

Proof. When $n = m = 1$, an unbiased estimator with finite variance does not exist; hence, we restrict attention to the case $n, m \geq 2$.

If the transformation $T(\cdot)$ is strictly increasing on the supports of X and Y , then the indicator $I(T(R_{1,1}) < T(S_{1,1}))$ provides an unbiased estimator of $\Pr(X < Y)$. If $T(\cdot)$ is strictly decreasing, the unbiasedness is preserved by reversing the inequality.

From the likelihood functions in (11) and (12), it follows that $\left(\sum_{i=1}^n T(R_{i,i}), \sum_{j=1}^m T(S_{j,j}) \right)^\top$ is a complete sufficient statistic for $(\omega(\theta_1), \omega(\theta_2))$. Applying the Rao–Blackwell Theorem together with the Lehmann–Scheffé Theorem yields the UMVUE of $\mathcal{R}$ as

$$\hat{\mathcal{R}}_{\text{UMVU}} = \Pr \left\{ T(R_{1,1}) < T(S_{1,1}) \mid \sum_{i=1}^n T(R_{i,i}), \sum_{j=1}^m T(S_{j,j}) \right\}. \quad (22)$$

To evaluate the probability in (22), we substitute the pivot

$$w_0 = \frac{M}{N} \cdot \frac{\sum_{i=1}^n T(R_{i,i})}{\sum_{j=1}^m T(S_{j,j})} = \frac{M}{N} \cdot \frac{1 - \hat{R}_{\text{ML}}}{\hat{R}_{\text{ML}}} \quad (23)$$

into the CDF of the ancillary statistic W derived in Appendix A (A3). Setting $t = \hat{R}_{\text{ML}}$, the CDF in (A3) evaluated at w_0 gives

$$\begin{aligned}
 F_W(w_0) &= 1 - \sum_{d=0}^{N-1} \frac{\binom{N-1}{d}}{\binom{d+M-1}{d}} (-w_0)^d \\
 &= 1 - \sum_{d=0}^{N-1} \frac{\binom{N-1}{d}}{\binom{d+M-1}{d}} \left(-\frac{M}{N} \cdot \frac{1-t}{t} \right)^d = H_{n,m}(t),
 \end{aligned} \tag{24}$$

where the function $H_{n,m}(\cdot)$ is as defined in (21), with the factor $(-1)^d$ absorbed naturally into the binomial expansion.

Equivalently, this probability can be written as

$$\hat{R}_{\text{UMVU}} = \Pr \left\{ W < \frac{\sum_{j=1}^m T(S_{j,j})}{\sum_{i=1}^n T(R_{i,i})} \right\}, \tag{25}$$

where

$$W = \frac{T(R_{1,1})}{T(S_{1,1})} \cdot \frac{\sum_{j=1}^m T(S_{j,j})}{\sum_{i=1}^n T(R_{i,i})} \stackrel{d}{=} \frac{U_1}{U_2}, \tag{26}$$

where $U_1 \sim \text{Beta}(1, N-1)$ and $U_2 \sim \text{Beta}(1, M-1)$ are independent. The ratio U_1/U_2 is well-defined almost surely since $U_2 > 0$ a.s., and its CDF is derived explicitly in Appendix A (A3).

W is an ancillary statistic. By Basu's Lemma, W is independent of the complete sufficient statistic. Evaluating the above probability yields the stated expression for $\hat{R}_{\text{UMVU}}$. $\square$

Figure 3 leads to the following observations:

- When $n = m$ increases simultaneously, the variance and MSE of $\hat{R}_{\text{UMVU}}$ remain approximately symmetric about $\mathcal{R} = 0.5$ and display a smooth unimodal pattern. Both error measures decrease substantially over the entire range of $\mathcal{R}$ as sample size increases, indicating improved estimation precision under balanced sampling.
- When n is fixed and m increases, the variance and MSE curves remain nearly symmetric about $\mathcal{R} = 0.5$ and show no apparent skewness. Increasing m leads to a uniform reduction in both measures across all values of $\mathcal{R}$.
- When m is fixed and n increases, variance and MSE decrease steadily while their overall shapes remain symmetric and well behaved, suggesting that the UMVUE maintains stable error characteristics even under unbalanced sample configurations.

A comparison between the MLE and the UMVUE reveals notable differences in their finite-sample behavior. While both estimators benefit from increasing sample sizes in terms of reduced variance and MSE, the UMVUE exhibits more stable and symmetric error structures across the entire range of $\mathcal{R}$. In particular, the variance and MSE of $\hat{R}_{\text{UMVU}}$ remain approximately symmetric about $\mathcal{R} = 0.5$ even under unbalanced sample configurations, indicating robustness to sample size imbalance. In contrast, the MLE is more sensitive to unequal sample sizes, as evidenced by skewed bias patterns when $n \neq m$, which may adversely affect its finite-sample performance. Overall, although the MLE is computationally simpler and asymptotically efficient, the UMVUE demonstrates superior stability and reliability in finite samples, especially for small or unbalanced designs. Section 4 provides a complementary Bayesian analysis based on conjugate gamma priors, whose finite-sample behavior is examined in Figure 4.

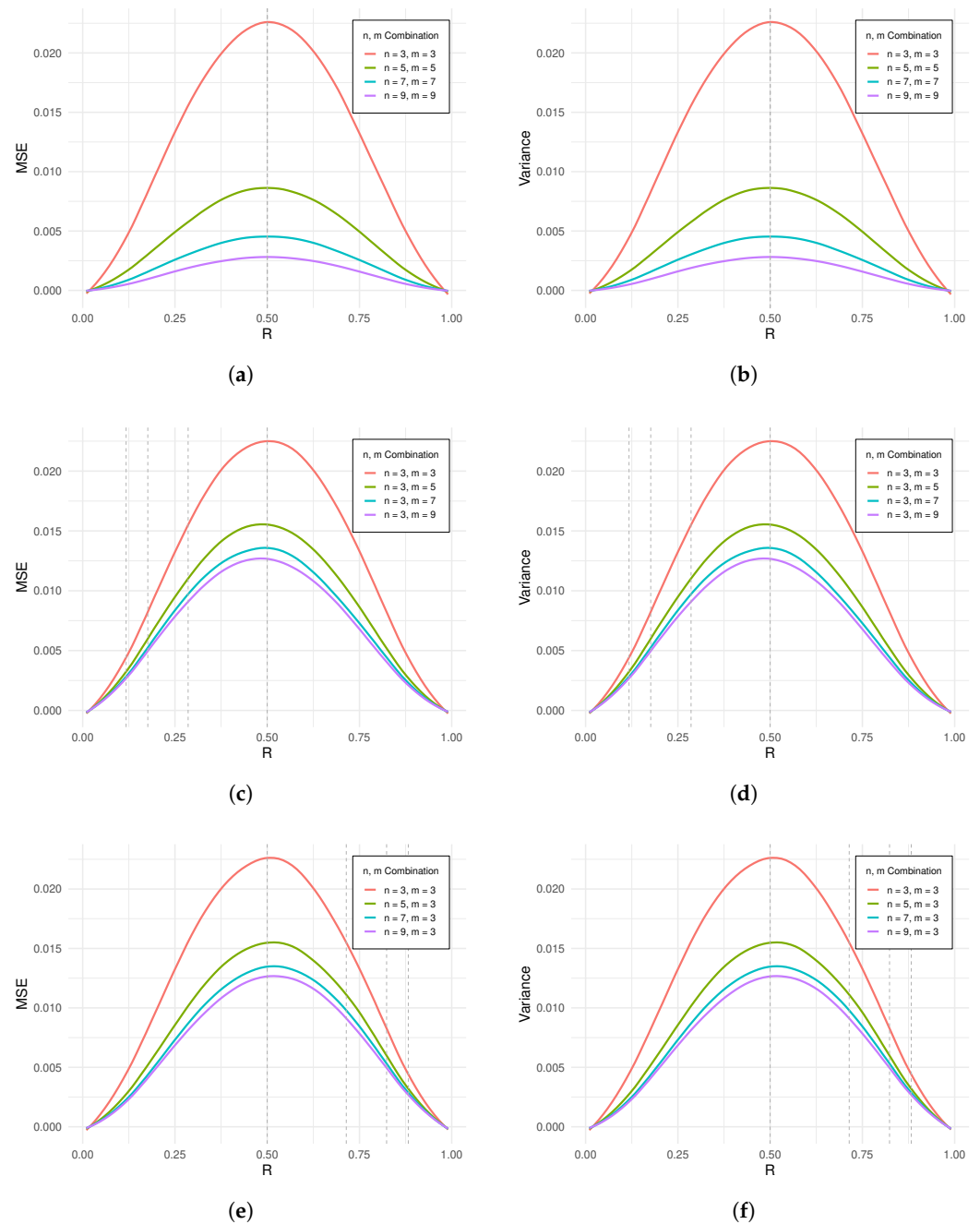


Figure 3. (a) MSE of $\hat{R}_{UMVU}$ under balanced (n, m) ; (b) Variance of $\hat{R}_{UMVU}$ under balanced (n, m) ; (c) MSE under fixed n with increasing m ; (d) Variance under fixed n with increasing m ; (e) MSE under fixed m with increasing n ; (f) Variance under fixed m with increasing n .

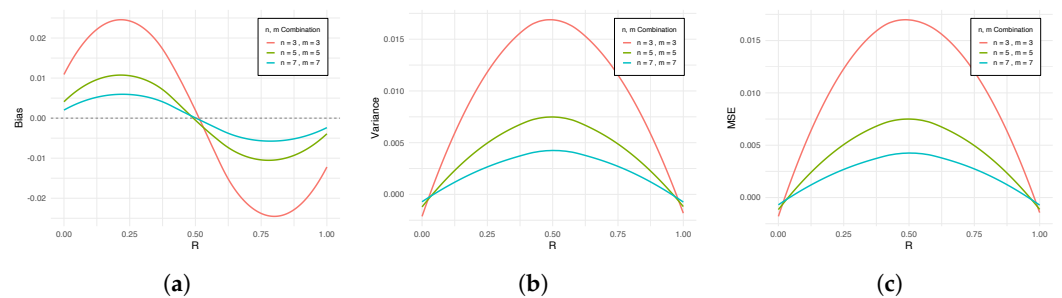


Figure 4. Cont.

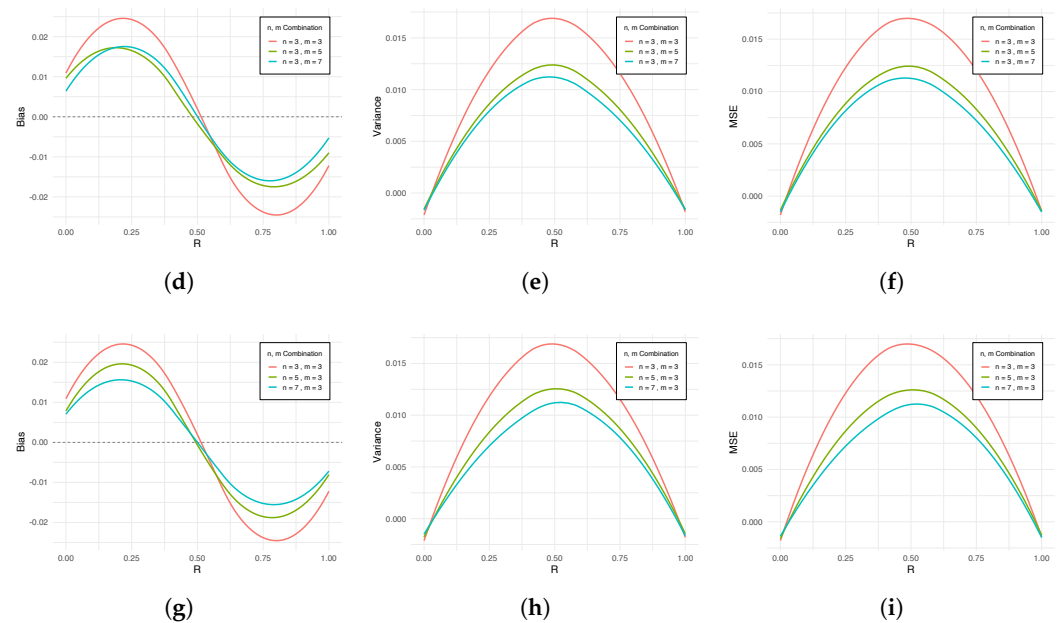


Figure 4. (a) Bias of $\hat{\mathcal{R}}_{\text{Bayes}}$ under balanced (n, m) ; (b) Variance of $\hat{\mathcal{R}}_{\text{Bayes}}$ under balanced (n, m) ; (c) MSE of $\hat{\mathcal{R}}_{\text{Bayes}}$ under balanced (n, m) ; (d) Bias under fixed n with increasing m ; (e) Variance under fixed n with increasing m ; (f) MSE under fixed n with increasing m ; (g) Bias under fixed m with increasing n ; (h) Variance under fixed m with increasing n ; (i) MSE under fixed m with increasing n .

3. Interval Estimation

This section develops frequentist confidence intervals for $\mathcal{R}$; Bayesian credible intervals under the conjugate posterior specification are presented in Section 4.

3.1. Maximum Likelihood Confidence Interval

Theorem 2. Based on the pivotal representation in (19), the $100(1 - \alpha)\%$ maximum likelihood confidence interval for $\mathcal{R}$ is given by

$$\Pr\left\{\left(1 + \frac{1 - \hat{\mathcal{R}}_{\text{ML}}}{\hat{\mathcal{R}}_{\text{ML}}} \left\{F_{2N, 2M}\left(\frac{\alpha}{2}\right)\right\}^{-1}\right)^{-1} \leq \mathcal{R} \leq \left(1 + \frac{1 - \hat{\mathcal{R}}_{\text{ML}}}{\hat{\mathcal{R}}_{\text{ML}}} \left\{F_{2N, 2M}\left(1 - \frac{\alpha}{2}\right)\right\}^{-1}\right)^{-1}\right\} = 1 - \alpha, \quad (27)$$

where $F_{2N, 2M}(\gamma)$ denotes the γ th quantile of the F distribution with $2N$ and $2M$ degrees of freedom.

Proof. According to (19), the statistic

$$\frac{1 - \hat{\mathcal{R}}_{\text{ML}}}{\hat{\mathcal{R}}_{\text{ML}}} \cdot \frac{\mathcal{R}}{1 - \mathcal{R}} \quad (28)$$

constitutes a pivotal quantity following an $F_{2N, 2M}$ distribution. By inverting the corresponding probability inequalities with respect to $\mathcal{R}$, the confidence limits in (27) are obtained. $\square$

3.2. Approximate Confidence Interval

Theorem 3. Assume $N, M \rightarrow \infty$ such that $M/N \rightarrow p \in (0, \infty)$, where N and M are the numbers of record values. Let $\hat{\theta}_{1\text{ML}}$ and $\hat{\theta}_{2\text{ML}}$ be the maximum likelihood estimators of θ_1 and θ_2 , respectively. From (19), an approximate $100(1 - \alpha)\%$ confidence interval for $\mathcal{R}$ is given by

$$(\hat{\mathcal{R}}_{\text{ML}} - z_{1-\alpha/2} \hat{W}, \hat{\mathcal{R}}_{\text{ML}} + z_{1-\alpha/2} \hat{W}), \quad (29)$$

where z_γ is the γ -th quantile of the standard normal distribution, and

$$\hat{W} = \frac{\omega(\hat{\theta}_{1ML}) \omega(\hat{\theta}_{2ML})}{[\omega(\hat{\theta}_{1ML}) + \omega(\hat{\theta}_{2ML})]^2} \sqrt{\frac{1}{N} + \frac{1}{M}}. \quad (30)$$

Proof. We know that as $n, m \rightarrow \infty$, then $(\hat{\theta}_1 - \theta_1) \xrightarrow{d} N(0, \sigma_1^2)$ and $(\hat{\theta}_2 - \theta_2) \xrightarrow{d} N(0, \sigma_2^2)$, where $\sigma_1^2 = -1/E\left(\frac{\partial^2}{\partial \theta_1^2} \log L_1(\theta_1; \mathbf{R})\right)$ and $\sigma_2^2 = -1/E\left(\frac{\partial^2}{\partial \theta_2^2} \log L_2(\theta_2; \mathbf{S})\right)$ with L_1 and L_2 given in (11) and (12), respectively.

$$I(\theta_1) = -N \left[\frac{\omega''(\theta_1) \omega(\theta_1) - (\omega'(\theta_1))^2}{\omega^2(\theta_1)} \right] - \omega''(\theta_1) E \left[\sum_{i=1}^n T(r_{i,i}) \right], \quad (31)$$

Before computing $I(\theta_1)$, we establish the required moment. From (16), and under the shape-scale convention for $\text{Ga}(a, b)$, $T(R_{i,i}) \sim \text{Ga}(i, 1/\omega(\theta_1))$, so $E[T(R_{i,i})] = i/\omega(\theta_1)$. Summing over $i = 1, \dots, n$ and using $N = n(n+1)/2$ gives

$$E \left[\sum_{i=1}^n T(R_{i,i}) \right] = \frac{1}{\omega(\theta_1)} \sum_{i=1}^n i = \frac{N}{\omega(\theta_1)}. \quad (32)$$

Substituting (32) into the Fisher information expression (31) then yields

$$I(\theta_1) = -N \frac{\omega''(\theta_1) \omega(\theta_1) - [\omega'(\theta_1)]^2}{\omega^2(\theta_1)} - \omega''(\theta_1) \cdot \frac{N}{\omega(\theta_1)}, \quad (33)$$

from which $\sigma_1^2 = \omega^2(\theta_1)/[N(\omega'(\theta_1))^2]$ follows directly. An analogous calculation applies to $I(\theta_2)$ with n, N, θ_1 replaced by m, M, θ_2 .

$$I(\theta_2) = -M \left[\frac{\omega''(\theta_2) \omega(\theta_2) - (\omega'(\theta_2))^2}{\omega^2(\theta_2)} \right] - \omega''(\theta_2) E \left[\sum_{j=1}^m T(s_{j,j}) \right]. \quad (34)$$

It shows that $\sigma_1^2 = \frac{\omega^2(\theta_1)}{N(\omega'(\theta_1))^2}$ and $\sigma_2^2 = \frac{\omega^2(\theta_2)}{M(\omega'(\theta_2))^2}$. We further assume that n and m grow large such that $M/N \rightarrow p$, where $p \in (0, \infty)$,

$$\sqrt{N}(\hat{\theta}_{1ML} - \theta_1) \xrightarrow{d} N\left(0, \frac{\omega^2(\theta_1)}{(\omega'(\theta_1))^2}\right), \quad (35)$$

$$\sqrt{M}(\hat{\theta}_{2ML} - \theta_2) \xrightarrow{d} N\left(0, \frac{\omega^2(\theta_2)}{(\omega'(\theta_2))^2}\right), \quad (36)$$

$$\sqrt{N}[\omega(\hat{\theta}_{1ML}) - \omega(\theta_1)] \xrightarrow{d} N(0, \omega^2(\theta_1)), \quad (37)$$

$$\sqrt{M}[\omega(\hat{\theta}_{2ML}) - \omega(\theta_2)] \xrightarrow{d} N(0, \omega^2(\theta_2)). \quad (38)$$

Since the two samples are independent, the corresponding estimators are asymptotically independent. Rescaling the second component by $\sqrt{N}$ and using $M/N \rightarrow p$, the joint convergence holds as

$$\sqrt{N} \begin{pmatrix} \omega(\hat{\theta}_{1ML}) - \omega(\theta_1) \\ \omega(\hat{\theta}_{2ML}) - \omega(\theta_2) \end{pmatrix} \xrightarrow{d} N_2 \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma \right), \quad (39)$$

where

$$\Sigma = \begin{pmatrix} \omega^2(\theta_1) & 0 \\ 0 & \frac{\omega^2(\theta_2)}{p} \end{pmatrix}. \quad (40)$$

Define $G(t_1, t_2) = t_1 / (t_1 + t_2)$. Note that

$$\mathcal{R} = \frac{\omega(\theta_1)}{\omega(\theta_1) + \omega(\theta_2)} = G(\omega(\theta_1), \omega(\theta_2)), \quad (41)$$

$$\hat{R}_{ML} = \frac{\omega(\hat{\theta}_{1ML})}{\omega(\hat{\theta}_{1ML}) + \omega(\hat{\theta}_{2ML})} = G(\omega(\hat{\theta}_{1ML}), \omega(\hat{\theta}_{2ML})). \quad (42)$$

The gradient of G is:

$$\nabla G(t_1, t_2) = \begin{pmatrix} \frac{\partial G}{\partial t_1} \\ \frac{\partial G}{\partial t_2} \end{pmatrix} = \frac{1}{(t_1 + t_2)^2} \begin{pmatrix} t_2 \\ -t_1 \end{pmatrix}. \quad (43)$$

Applying the multivariate delta method to G at $(\omega(\theta_1), \omega(\theta_2))$ yields

$$\sqrt{N}(\hat{R}_{ML} - R) \xrightarrow{d} N(0, \nabla G(\omega(\theta_1), \omega(\theta_2))^\top \Sigma \nabla G(\omega(\theta_1), \omega(\theta_2))). \quad (44)$$

Therefore, the asymptotic variance of $\sqrt{N}(\hat{R}_{ML} - R)$ is

$$\begin{aligned} \tau^2 &= \nabla G(\omega(\theta_1), \omega(\theta_2))^\top \Sigma \nabla G(\omega(\theta_1), \omega(\theta_2)) \\ &= \frac{\omega^2(\theta_1)\omega^2(\theta_2)}{[\omega(\theta_1) + \omega(\theta_2)]^4} \left(1 + \frac{1}{p}\right). \end{aligned} \quad (45)$$

Equivalently, the approximate variance of $\hat{R}_{ML}$ itself is

$$W^2 = \frac{\omega^2(\theta_1)\omega^2(\theta_2)}{[\omega(\theta_1) + \omega(\theta_2)]^4} \left(\frac{1}{N} + \frac{1}{M}\right). \quad (46)$$

Finally, substituting θ_1 and θ_2 with their respective MLEs $\hat{\theta}_{1ML}$ and $\hat{\theta}_{2ML}$ in W yields the estimated standard error $\hat{W}$; consequently, by Slutsky's Theorem

$$\frac{\hat{R}_{ML} - R}{\hat{W}} \xrightarrow{d} N(0, 1). \quad (47)$$

The confidence interval (29) follows directly. $\square$

3.3. Bootstrap Confidence Interval

The bootstrap procedures below rely on the totals $\sum_{i=1}^n T(R_{i,i})$ and $\sum_{j=1}^m T(S_{j,j})$. Because $\sum_{i=1}^n T(R_{i,i}) \sim \text{Ga}(N, 1/\omega(\theta_1))$ and $\sum_{j=1}^m T(S_{j,j}) \sim \text{Ga}(M, 1/\omega(\theta_2))$, parametric bootstrap resampling provides a natural basis for interval estimation. Accordingly, Algorithms 1–3 construct parametric bootstrap confidence intervals for $\mathcal{R}$.

Algorithm 1 Percentile bootstrap confidence interval

```

1: Input: Independent observed samples
    $\mathbf{T}(\mathbf{r}) = (T(r_{1,1}), T(r_{2,2}), \dots, T(r_{n,n}))^\top$ ,  $\mathbf{T}(\mathbf{s}) = (T(s_{1,1}), T(s_{2,2}), \dots, T(s_{m,m}))^\top$ 
2: Compute  $\omega(\hat{\theta}_{1ML})$ ,  $\omega(\hat{\theta}_{2ML})$ , and  $\hat{R}_{ML}$  from (13), (14), and (15), respectively.
3: for  $b = 1$  to  $B$  do
4:   for  $i = 1$  to  $n$  do
5:     Generate  $T_{i1}^* \sim \text{Ga}(i, 1/\omega(\hat{\theta}_{1ML}))$ 
6:   end for
7:   for  $j = 1$  to  $m$  do
8:     Generate  $T_{j2}^* \sim \text{Ga}(j, 1/\omega(\hat{\theta}_{2ML}))$ 
9:   end for
10:  Compute  $\omega(\hat{\theta}_{1ML}^*)$  and  $\omega(\hat{\theta}_{2ML}^*)$ 
11:  Obtain the bootstrap estimator  $\hat{\mathcal{R}}_{(b)ML}^*$ 
12: end for

```

Algorithm 1 *Cont.*

-
- 13: Let $\hat{F}$ denote the empirical cumulative distribution function of $\{\hat{\mathcal{R}}_{(b)ML}^*\}_{b=1}^B$.
 14: Construct the $100(1 - \alpha)\%$ percentile bootstrap confidence interval for $\mathcal{R}$ as
 $(\hat{F}^{-1}(\frac{\alpha}{2}), \hat{F}^{-1}(1 - \frac{\alpha}{2}))$
 15: **Output:** Percentile bootstrap confidence interval for $\mathcal{R}$
-

Algorithm 2 Bootstrap- t confidence interval (based on asymptotic variance)

-
- 1: **Input:** Independent observed samples
 $\mathbf{T}(\mathbf{r}) = (T(r_{1,1}), \dots, T(r_{n,n}))^\top$, $\mathbf{T}(\mathbf{s}) = (T(s_{1,1}), \dots, T(s_{m,m}))^\top$
 2: Compute $\omega(\hat{\theta}_{1ML})$, $\omega(\hat{\theta}_{2ML})$, and $\hat{R}_{ML}$, and compute $\hat{W}$ by replacing $\omega(\theta_1)$ and $\omega(\theta_2)$ with their MLEs.
 3: **for** $b = 1$ to B **do**
 4: **for** $i = 1$ to n **do**
 5: Generate $T_{i1}^* \sim \text{Ga}(i, 1/\omega(\hat{\theta}_{1ML}))$
 6: **end for**
 7: **for** $j = 1$ to m **do**
 8: Generate $T_{j2}^* \sim \text{Ga}(j, 1/\omega(\hat{\theta}_{2ML}))$
 9: **end for**
 10: Compute $\omega(\hat{\theta}_{1ML}^*)$ and $\omega(\hat{\theta}_{2ML}^*)$
 11: Obtain $\hat{\mathcal{R}}_{(b)ML}^*$ and $\hat{W}_{(b)}^*$
 12: **end for**
 13: Construct the studentized statistics

$$z_{(b)}^* = \frac{\hat{\mathcal{R}}_{(b)ML}^* - \hat{R}_{ML}}{\hat{W}_{(b)}^*}, \quad b = 1, \dots, B$$

 14: Construct the $100(1 - \alpha)\%$ bootstrap- t confidence interval for $\mathcal{R}$ as
 $(\hat{R}_{ML} - z_{1-\frac{\alpha}{2}}^* \hat{W}, \hat{R}_{ML} - z_{\frac{\alpha}{2}}^* \hat{W})$
 15: **Output:** Bootstrap- t confidence interval for $\mathcal{R}$
-

Algorithm 3 Double bootstrap- t confidence interval (based on bootstrap standard error)

-
- Require:** Independent observed samples $\mathbf{T}(\mathbf{r}) = (T(r_{1,1}), \dots, T(r_{n,n}))^\top$, $\mathbf{T}(\mathbf{s}) = (T(s_{1,1}), \dots, T(s_{m,m}))^\top$
 1: Compute $\omega(\hat{\theta}_{1ML})$, $\omega(\hat{\theta}_{2ML})$, and $\hat{R}_{ML}$.
 2: **for** $b = 1$ to B **do**
 3: **for** $i = 1$ to n **do**
 4: Generate $T_{i1}^* \sim \text{Ga}(i, 1/\omega(\hat{\theta}_{1ML}))$
 5: **end for**
 6: **for** $j = 1$ to m **do**
 7: Generate $T_{j2}^* \sim \text{Ga}(j, 1/\omega(\hat{\theta}_{2ML}))$
 8: **end for**
 9: Compute outer-loop MLEs $\omega(\hat{\theta}_{(b)1ML}^*)$ and $\omega(\hat{\theta}_{(b)2ML}^*)$, and obtain $\hat{R}_{(b)ML}^*$.
 10: **for** $b' = 1$ to B' **do** ▷ Inner bootstrap
 11: **for** $i = 1$ to n **do**
 12: Generate $T_{i1}^{**} \sim \text{Ga}(i, 1/\omega(\hat{\theta}_{(b)1ML}^*))$ ▷ resample from outer-loop estimates
 13: **end for**
 14: **for** $j = 1$ to m **do**
 15: Generate $T_{j2}^{**} \sim \text{Ga}(j, 1/\omega(\hat{\theta}_{(b)2ML}^*))$
 16: **end for**
 17: Compute $\hat{R}_{(b')ML}^{**}$ from $\{T_{i1}^{**}\}$ and $\{T_{j2}^{**}\}$.
 18: **end for**
 19: Estimate the bootstrap standard error of the outer estimator:

$$\hat{\sigma}(\hat{R}_{(b)ML}^*) = \sqrt{\frac{1}{B' - 1} \sum_{b'=1}^{B'} (\hat{R}_{(b')ML}^{**} - \bar{R}^{**})^2}, \quad \bar{R}^{**} = \frac{1}{B'} \sum_{b'=1}^{B'} \hat{R}_{(b')ML}^{**}.$$

- 20: **end for**
 21: Construct studentized statistics

$$t_{(b)}^* = \frac{\hat{R}_{(b)ML}^* - \hat{R}_{ML}}{\hat{\sigma}(\hat{R}_{(b)ML}^*)}, \quad b = 1, \dots, B.$$

- 22: Construct the $100(1 - \alpha)\%$ double bootstrap- t interval:

$$(\hat{R}_{ML} - t_{1-\alpha/2}^* \hat{\sigma}, \hat{R}_{ML} - t_{\alpha/2}^* \hat{\sigma}).$$

- Ensure:** Double bootstrap- t confidence interval for $\mathcal{R}$.
-

Within the inner bootstrap loop, resampling uses the estimates $\omega(\hat{\theta}_{(b)1ML}^*)$ and $\omega(\hat{\theta}_{(b)2ML}^*)$ from the b th outer bootstrap iteration, so $\hat{\sigma}(\hat{R}_{(b)ML}^*)$ estimates the standard error of the outer bootstrap replicate, conditional on the observed data.

4. Bayesian Inference

4.1. Prior and Posterior Distributions

From (11) and (12), the upper RRSS likelihood for samples $\mathbf{r}$ and $\mathbf{s}$ from $\text{SPEF}(\theta_1)$ and $\text{SPEF}(\theta_2)$ depends on the natural parameters $\omega(\theta_1)$ and $\omega(\theta_2)$. For convenience, define the sufficient statistics

$$U = \sum_{i=1}^n T(r_{i,i}), \quad V = \sum_{j=1}^m T(s_{j,j}), \quad (48)$$

with $N = n(n+1)/2$ and $M = m(m+1)/2$. Then the joint likelihood function is

$$L(\omega(\theta_1), \omega(\theta_2) \mid \mathbf{r}, \mathbf{s}) \propto \omega(\theta_1)^N \exp\{-\omega(\theta_1)U\} \omega(\theta_2)^M \exp\{-\omega(\theta_2)V\}. \quad (49)$$

Independent gamma priors are assigned to $\omega(\theta_1)$ and $\omega(\theta_2)$ because the kernel of (49) coincides with that of a gamma density, which ensures conjugacy and facilitates posterior computation. Under the shape-scale convention for $\text{Ga}(a, b)$ stated in Section 2, we take

$$\omega(\theta_1) \sim \text{Ga}(a_1, 1/b_1), \quad \pi(\omega(\theta_1)) \propto \omega(\theta_1)^{a_1-1} \exp\{-b_1\omega(\theta_1)\}, \quad (50)$$

$$\omega(\theta_2) \sim \text{Ga}(a_2, 1/b_2), \quad \pi(\omega(\theta_2)) \propto \omega(\theta_2)^{a_2-1} \exp\{-b_2\omega(\theta_2)\}, \quad (51)$$

where $a_1, b_1, a_2, b_2 > 0$ are hyperparameters and $1/b_j$ is the scale-parameter in the gamma notation. If historical data or engineering knowledge are available, these hyperparameters can be chosen to match prior beliefs about the means and variances of the natural parameters $\omega(\theta_1)$ and $\omega(\theta_2)$. For example, under the above shape-scale convention, $E\{\omega(\theta_j)\} = a_j/b_j$ and $\text{Var}\{\omega(\theta_j)\} = a_j/b_j^2$.

In the simulation study and real-data illustration below, little reliable prior information is assumed to be available. We therefore adopt $a_1 = b_1 = a_2 = b_2 = 10^{-4}$ as a weakly informative baseline. Assigning the same small hyperparameters to both populations keeps the prior specification symmetric with respect to stress and strength and avoids favoring either $\omega(\theta_1)$ or $\omega(\theta_2)$ before observing the data. Under this choice, the posterior is driven mainly by the RRSS likelihood while retaining the computational convenience of conjugacy.

The joint posterior is proportional to the likelihood times the priors:

$$\pi(\omega(\theta_1), \omega(\theta_2) \mid \mathbf{r}, \mathbf{s}) \propto L(\omega(\theta_1), \omega(\theta_2) \mid \mathbf{r}, \mathbf{s}) \pi(\omega(\theta_1)) \pi(\omega(\theta_2)). \quad (52)$$

Substituting (49)–(51), and separating the variables, we obtain

$$\pi(\omega(\theta_1), \omega(\theta_2) \mid \mathbf{r}, \mathbf{s}) \propto [\omega(\theta_1)^{N+a_1-1} \exp\{-\omega(\theta_1)(U+b_1)\}] \times [\omega(\theta_2)^{M+a_2-1} \exp\{-\omega(\theta_2)(V+b_2)\}]. \quad (53)$$

It follows that the posterior distributions of $\omega(\theta_1)$ and $\omega(\theta_2)$ remain independent gamma distributions:

$$\omega(\theta_1) \mid \mathbf{r} \sim \text{Ga}\left(N + a_1, \frac{1}{U + b_1}\right), \quad (54)$$

$$\omega(\theta_2) \mid \mathbf{s} \sim \text{Ga}\left(M + a_2, \frac{1}{V + b_2}\right). \quad (55)$$

4.2. Monte Carlo Approximation

From (10), $\mathcal{R} = \frac{\omega(\theta_1)}{\omega(\theta_1) + \omega(\theta_2)}$. Although (54) and (55) are explicit, $\mathcal{R}$ is a strongly nonlinear function of $(\omega(\theta_1), \omega(\theta_2))$, so its posterior has no simple closed form. Monte Carlo simulation is therefore used to approximate the posterior of $\mathcal{R}$ and to obtain Bayesian estimates.

Algorithm 4 summarizes the computation.

Algorithm 4 Bayesian point and HPD interval estimation via Monte Carlo simulation

- 1: **Input:** Independent observed samples
 $\mathbf{r} = (r_{1,1}, r_{2,2}, \dots, r_{n,n})^\top$, $\mathbf{s} = (s_{1,1}, s_{2,2}, \dots, s_{m,m})^\top$
 prior hyperparameters (a_1, b_1, a_2, b_2) ; Monte Carlo size K ; significance level α
 - 2: Compute $U = \sum_{i=1}^n T(r_{i,i})$ and $V = \sum_{j=1}^m T(s_{j,j})$, with $N = n(n+1)/2$ and $M = m(m+1)/2$ as in (48)
 - 3: **for** $k = 1$ to K **do**
 - 4: Generate $\omega^{(k)}(\theta_1) \sim \text{Ga}(N + a_1, \frac{1}{U + b_1})$ and $\omega^{(k)}(\theta_2) \sim \text{Ga}(M + a_2, \frac{1}{V + b_2})$ independently
 - 5: Set $\mathcal{R}^{(k)} = \omega^{(k)}(\theta_1) / [\omega^{(k)}(\theta_1) + \omega^{(k)}(\theta_2)]$ as in (56)
 - 6: **end for**
 - 7: Compute the Bayesian point estimate $\hat{\mathcal{R}}_{\text{Bayes}} = K^{-1} \sum_{k=1}^K \mathcal{R}^{(k)}$ under squared error loss (SEL), as in (57)
 - 8: Sort $\{\mathcal{R}^{(k)}\}_{k=1}^K$ in ascending order
 - 9: Select the shortest interval containing $\lceil K(1 - \alpha) \rceil$ of the sorted draws to form the $100(1 - \alpha)\%$ highest posterior density (HPD) credible interval
 - 10: **Output:** $\hat{\mathcal{R}}_{\text{Bayes}}$ and the HPD credible interval for $\mathcal{R}$
-

$$\mathcal{R}^{(k)} = \frac{\omega^{(k)}(\theta_1)}{\omega^{(k)}(\theta_1) + \omega^{(k)}(\theta_2)}, \quad k = 1, \dots, K, \quad (56)$$

$$\hat{\mathcal{R}}_{\text{Bayes}} = \frac{1}{K} \sum_{k=1}^K \mathcal{R}^{(k)}. \quad (57)$$

With closed-form gamma posteriors in (54) and (55), Algorithm 4 draws independently from the marginal posteriors and yields a Monte Carlo approximation to the posterior distribution of $\mathcal{R}$.

4.3. Finite-Sample Performance of the Bayesian Estimator

A supplementary Monte Carlo study evaluates the finite-sample performance of $\hat{\mathcal{R}}_{\text{Bayes}}$ under exponential RRSS, independently of the classical development in Section 2. The simulation design parallels that of Figure 2: for each (n, m) , bias, variance, and MSE are computed on a grid of $\mathcal{R}$ and smoothed with `loess`. We set $n_{\text{sim}} = 800$ replications, $K = 3000$ posterior draws per replication, and $a_1 = b_1 = a_2 = b_2 = 10^{-4}$.

Figure 4 shows that $\hat{\mathcal{R}}_{\text{Bayes}}$ exhibits smoother bias and MSE curves than $\hat{R}_{\text{ML}}$ under unbalanced sample sizes (cf. Figure 2); the HPD intervals also maintain coverage close to the nominal level in small samples, as reported in Section 5.

5. Simulation Study

A Monte Carlo simulation study was conducted in R version 4.2.2 (31 October 2022) to compare the finite-sample performance of the frequentist intervals in Section 3 and the Bayesian HPD intervals in Section 4. All combinations were considered for $n, m \in \{3, 5, 7\}$, $\mathcal{R} \in \{0.1, 0.3, 0.5, 0.95, 0.99\}$, and $\alpha \in \{0.05, 0.1\}$. For each setting, 1000 independent pairs $(\mathbf{r}, \mathbf{s})$ were generated from $\text{Exp}(\theta_1)$ and $\text{Exp}(\theta_2)$ via Algorithm 5, with $\theta_2 = \theta_1 \mathcal{R} / (1 - \mathcal{R})$ and $\theta_1 = 1$. Bootstrap runs used $B = 200$ and $B' = 25$. For each Monte Carlo replication, we computed the MLCI, ACI, CI.Perc, Boot_1 - t , Boot_2 - t , and HPD interval from Algorithm 4 with $K = 3000$ and $a_1 = b_1 = a_2 = b_2 = 10^{-4}$.

Algorithm 5 Upper RRSS record generation

```

1: Input: Desired number of upper records  $n$  and distribution parameter  $\theta$ 
2: Initialize a vector  $\mathbf{R} = (R_{1,1}, R_{2,2}, \dots, R_{n,n})^\top$ 
3: for  $i = 1$  to  $n$  do
4:   Set counter  $c = 0$  and initial maximum  $M = -\infty$ 
5:   while  $c < i$  do
6:     Generate a random observation  $x$  from the specified SPEF( $\theta$ ) distribution with parameter  $\theta$ 
7:     if  $x > M$  then
8:       Update  $M = x$ 
9:       Increment  $c = c + 1$ 
10:    end if
11:  end while
12:  Set  $R_{i,i} = M$ 
13: end for
14: Output: Upper record vector  $\mathbf{R} = (R_{1,1}, R_{2,2}, \dots, R_{n,n})^\top$ 

```

Upper record-ranked set samples were generated by Algorithm 5.

For each parameter configuration, empirical coverage probability (C.P.) and expected length (E.L.) were recorded. Tables 2 and 3 report results for $\alpha = 0.05$ and $\alpha = 0.10$; the HPD column refers to Algorithm 4.

Table 2. C.P. and E.L. of $\mathcal{R}$ for $\alpha = 0.05$.

R	n	m	MLCI		ACI		CI.Perc		Boot _{1-t}		Boot _{2-t}		HPD	
			C.P.	E.L.	C.P.	E.L.	C.P.	E.L.	C.P.	E.L.	C.P.	E.L.	C.P.	E.L.
0.10	3	3	0.952	0.243	0.903	0.214	0.943	0.235	0.939	0.242	0.968	0.281	0.936	0.224
0.10	3	5	0.936	0.175	0.931	0.185	0.918	0.212	0.939	0.208	0.959	0.219	0.944	0.183
0.10	3	7	0.928	0.156	0.936	0.174	0.931	0.198	0.917	0.200	0.969	0.200	0.940	0.165
0.10	5	3	0.941	0.216	0.899	0.173	0.915	0.170	0.948	0.172	0.962	0.228	0.930	0.181
0.10	5	5	0.947	0.139	0.925	0.131	0.944	0.137	0.938	0.137	0.955	0.152	0.951	0.134
0.10	5	7	0.948	0.116	0.937	0.116	0.940	0.121	0.937	0.121	0.957	0.127	0.942	0.117
0.10	7	3	0.926	0.206	0.890	0.157	0.902	0.152	0.914	0.153	0.961	0.210	0.949	0.170
0.10	7	5	0.947	0.124	0.922	0.113	0.952	0.113	0.934	0.112	0.944	0.129	0.947	0.117
0.10	7	7	0.950	0.098	0.937	0.095	0.943	0.096	0.939	0.097	0.952	0.105	0.943	0.097
0.30	3	3	0.947	0.448	0.897	0.453	0.952	0.436	0.941	0.438	0.967	0.569	0.927	0.437
0.30	3	5	0.935	0.365	0.911	0.390	0.908	0.389	0.933	0.387	0.968	0.458	0.937	0.369
0.30	3	7	0.929	0.337	0.923	0.368	0.907	0.371	0.906	0.369	0.953	0.424	0.924	0.344
0.30	5	3	0.937	0.398	0.908	0.379	0.908	0.354	0.933	0.360	0.959	0.473	0.950	0.376
0.30	5	5	0.949	0.293	0.929	0.294	0.953	0.286	0.949	0.287	0.958	0.332	0.935	0.288
0.30	5	7	0.949	0.255	0.933	0.261	0.939	0.256	0.945	0.257	0.953	0.284	0.943	0.256
0.30	7	3	0.925	0.379	0.897	0.350	0.923	0.331	0.908	0.330	0.958	0.440	0.935	0.354
0.30	7	5	0.946	0.263	0.928	0.258	0.940	0.251	0.944	0.249	0.949	0.281	0.932	0.256
0.30	7	7	0.950	0.217	0.938	0.218	0.931	0.211	0.947	0.212	0.961	0.236	0.932	0.215
0.50	3	3	0.950	0.501	0.892	0.522	0.931	0.491	0.936	0.490	0.955	0.639	0.909	0.492
0.50	3	5	0.937	0.432	0.900	0.446	0.939	0.424	0.921	0.423	0.960	0.537	0.930	0.429
0.50	3	7	0.926	0.407	0.905	0.418	0.915	0.399	0.926	0.400	0.958	0.500	0.923	0.403
0.50	5	3	0.937	0.432	0.910	0.447	0.935	0.424	0.943	0.424	0.968	0.539	0.938	0.429
0.50	5	5	0.949	0.339	0.924	0.346	0.935	0.331	0.945	0.332	0.952	0.383	0.925	0.335
0.50	5	7	0.945	0.300	0.930	0.305	0.933	0.294	0.928	0.293	0.957	0.335	0.938	0.298
0.50	7	3	0.928	0.408	0.905	0.418	0.922	0.399	0.933	0.398	0.960	0.500	0.931	0.405
0.50	7	5	0.944	0.300	0.932	0.305	0.944	0.294	0.930	0.293	0.953	0.335	0.931	0.298
0.50	7	7	0.953	0.254	0.939	0.257	0.947	0.248	0.938	0.247	0.947	0.276	0.942	0.252
0.95	3	3	0.950	0.143	0.899	0.118	0.947	0.136	0.943	0.140	0.965	0.166	0.948	0.130
0.95	3	5	0.938	0.126	0.898	0.093	0.928	0.095	0.928	0.097	0.944	0.129	0.958	0.105
0.95	3	7	0.929	0.119	0.891	0.084	0.922	0.085	0.909	0.083	0.950	0.118	0.940	0.095
0.95	5	3	0.937	0.097	0.931	0.102	0.929	0.123	0.930	0.123	0.953	0.123	0.932	0.098
0.95	5	5	0.949	0.077	0.932	0.071	0.950	0.074	0.938	0.076	0.947	0.082	0.957	0.073
0.95	5	7	0.950	0.068	0.926	0.060	0.952	0.063	0.946	0.060	0.945	0.071	0.947	0.063
0.95	7	3	0.926	0.085	0.939	0.095	0.924	0.118	0.920	0.120	0.957	0.107	0.958	0.092
0.95	7	5	0.945	0.063	0.936	0.062	0.939	0.065	0.941	0.066	0.941	0.063	0.930	0.061
0.95	7	7	0.950	0.053	0.936	0.051	0.927	0.052	0.951	0.051	0.937	0.055	0.939	0.052
0.99	3	3	0.952	0.033	0.899	0.024	0.934	0.032	0.939	0.032	0.955	0.037	0.959	0.030
0.99	3	5	0.934	0.029	0.890	0.019	0.929	0.021	0.923	0.021	0.927	0.028	0.952	0.023
0.99	3	7	0.919	0.028	0.882	0.017	0.922	0.017	0.915	0.017	0.958	0.025	0.953	0.021
0.99	5	3	0.935	0.021	0.928	0.022	0.948	0.027	0.925	0.028	0.942	0.026	0.943	0.022
0.99	5	5	0.949	0.016	0.930	0.015	0.933	0.016	0.952	0.016	0.948	0.017	0.939	0.016
0.99	5	7	0.943	0.014	0.928	0.012	0.924	0.013	0.943	0.013	0.936	0.014	0.959	0.013
0.99	7	3	0.929	0.018	0.945	0.020	0.922	0.026	0.912	0.026	0.942	0.023	0.950	0.020
0.99	7	5	0.949	0.013	0.939	0.013	0.945	0.014	0.923	0.014	0.932	0.014	0.945	0.013
0.99	7	7	0.949	0.011	0.942	0.010	0.950	0.011	0.929	0.010	0.947	0.011	0.942	0.011

Note: MLCI: Maximum Likelihood Confidence Interval; ACI: Approximate Confidence Interval; CI.Perc: Percentile Bootstrap CI (Algorithm 1); Boot_{1-t}: Bootstrap- t CI with $\hat{W}$ (Algorithm 2); Boot_{2-t}: Double Bootstrap- t CI with SE (Algorithm 3); HPD: Bayesian Highest Posterior Density Credible Interval (Algorithm 4).

Table 3. C.P. and E.L. of $\mathcal{R}$ for $\alpha = 0.10$.

<i>R</i>	<i>n</i>	<i>m</i>	<i>MLCI</i>		<i>ACI</i>		<i>CI.Perc</i>		<i>Boot₁-t</i>		<i>Boot₂-t</i>		<i>HPD</i>	
			C.P.	E.L.	C.P.	E.L.	C.P.	E.L.	C.P.	E.L.	C.P.	E.L.	C.P.	E.L.
0.10	3	3	0.904	0.199	0.865	0.182	0.884	0.196	0.879	0.194	0.920	0.225	0.870	0.180
0.10	3	5	0.883	0.145	0.887	0.155	0.886	0.176	0.860	0.172	0.926	0.181	0.888	0.153
0.10	3	7	0.881	0.131	0.889	0.146	0.867	0.164	0.874	0.165	0.920	0.162	0.889	0.138
0.10	5	3	0.890	0.176	0.859	0.145	0.889	0.147	0.857	0.141	0.906	0.180	0.896	0.152
0.10	5	5	0.901	0.115	0.888	0.111	0.876	0.112	0.890	0.114	0.909	0.126	0.883	0.109
0.10	5	7	0.901	0.096	0.898	0.098	0.893	0.101	0.883	0.101	0.901	0.106	0.876	0.096
0.10	7	3	0.876	0.167	0.846	0.131	0.861	0.128	0.883	0.132	0.892	0.165	0.896	0.139
0.10	7	5	0.901	0.103	0.889	0.096	0.887	0.095	0.908	0.095	0.916	0.108	0.874	0.096
0.10	7	7	0.898	0.082	0.892	0.080	0.879	0.081	0.904	0.081	0.894	0.084	0.897	0.080
0.30	3	3	0.895	0.379	0.845	0.380	0.899	0.371	0.885	0.377	0.914	0.448	0.855	0.365
0.30	3	5	0.887	0.310	0.866	0.327	0.880	0.329	0.899	0.331	0.935	0.379	0.865	0.315
0.30	3	7	0.875	0.287	0.874	0.308	0.854	0.315	0.886	0.313	0.896	0.345	0.867	0.294
0.30	5	3	0.885	0.336	0.856	0.316	0.878	0.306	0.872	0.306	0.913	0.366	0.874	0.315
0.30	5	5	0.898	0.247	0.877	0.247	0.897	0.243	0.910	0.246	0.902	0.272	0.890	0.243
0.30	5	7	0.897	0.214	0.885	0.219	0.878	0.217	0.878	0.218	0.898	0.231	0.885	0.214
0.30	7	3	0.878	0.319	0.857	0.295	0.885	0.282	0.876	0.282	0.910	0.351	0.873	0.297
0.30	7	5	0.890	0.220	0.887	0.216	0.871	0.211	0.892	0.211	0.912	0.233	0.881	0.215
0.30	7	7	0.897	0.182	0.892	0.182	0.894	0.180	0.900	0.179	0.910	0.192	0.873	0.180
0.50	3	3	0.897	0.428	0.840	0.438	0.881	0.421	0.879	0.419	0.907	0.515	0.850	0.420
0.50	3	5	0.889	0.368	0.865	0.375	0.893	0.363	0.873	0.362	0.915	0.433	0.870	0.365
0.50	3	7	0.878	0.346	0.867	0.352	0.866	0.342	0.863	0.340	0.918	0.404	0.857	0.343
0.50	5	3	0.889	0.368	0.862	0.375	0.884	0.362	0.868	0.361	0.917	0.429	0.863	0.364
0.50	5	5	0.895	0.287	0.876	0.290	0.885	0.283	0.882	0.283	0.898	0.319	0.881	0.284
0.50	5	7	0.894	0.254	0.871	0.256	0.878	0.250	0.899	0.251	0.912	0.275	0.867	0.252
0.50	7	3	0.880	0.347	0.858	0.351	0.898	0.343	0.874	0.341	0.899	0.403	0.863	0.343
0.50	7	5	0.898	0.254	0.885	0.256	0.881	0.250	0.889	0.250	0.914	0.275	0.877	0.252
0.50	7	7	0.896	0.214	0.883	0.215	0.901	0.211	0.884	0.211	0.892	0.226	0.876	0.213
0.95	3	3	0.901	0.115	0.872	0.101	0.894	0.113	0.905	0.112	0.921	0.130	0.895	0.102
0.95	3	5	0.887	0.101	0.861	0.078	0.866	0.080	0.875	0.079	0.899	0.099	0.860	0.084
0.95	3	7	0.886	0.095	0.846	0.070	0.885	0.072	0.873	0.070	0.903	0.092	0.882	0.075
0.95	5	3	0.890	0.081	0.896	0.085	0.893	0.099	0.885	0.097	0.926	0.098	0.884	0.082
0.95	5	5	0.901	0.063	0.890	0.060	0.901	0.062	0.897	0.062	0.886	0.065	0.885	0.060
0.95	5	7	0.900	0.055	0.881	0.050	0.878	0.051	0.889	0.051	0.905	0.057	0.907	0.051
0.95	7	3	0.878	0.071	0.898	0.080	0.864	0.093	0.874	0.095	0.914	0.088	0.864	0.076
0.95	7	5	0.903	0.052	0.896	0.052	0.892	0.054	0.897	0.054	0.882	0.056	0.882	0.052
0.95	7	7	0.893	0.044	0.895	0.042	0.874	0.043	0.878	0.043	0.883	0.045	0.893	0.043
0.99	3	3	0.900	0.026	0.863	0.022	0.905	0.026	0.887	0.025	0.898	0.029	0.908	0.023
0.99	3	5	0.891	0.022	0.865	0.016	0.873	0.017	0.864	0.017	0.897	0.021	0.878	0.018
0.99	3	7	0.884	0.021	0.857	0.015	0.884	0.014	0.878	0.014	0.879	0.019	0.888	0.017
0.99	5	3	0.892	0.017	0.897	0.018	0.867	0.013	0.871	0.022	0.904	0.021	0.893	0.018
0.99	5	5	0.901	0.013	0.892	0.012	0.896	0.016	0.897	0.013	0.908	0.014	0.905	0.013
0.99	5	7	0.894	0.012	0.886	0.010	0.913	0.010	0.89	0.010	0.897	0.011	0.901	0.011
0.99	7	3	0.880	0.015	0.909	0.017	0.868	0.021	0.882	0.020	0.908	0.019	0.897	0.016
0.99	7	5	0.900	0.011	0.897	0.011	0.894	0.011	0.891	0.011	0.883	0.011	0.892	0.011
0.99	7	7	0.901	0.009	0.895	0.009	0.871	0.009	0.875	0.009	0.890	0.009	0.895	0.009

Note: MLCI: Maximum Likelihood Confidence Interval; ACI: Approximate Confidence Interval; CI.Perc: Percentile Bootstrap CI (Algorithm 1); Boot₁-t: Bootstrap-t CI with $\hat{W}$ (Algorithm 2); Boot₂-t: Double Bootstrap-t CI with SE (Algorithm 3); HPD: Bayesian Highest Posterior Density Credible Interval (Algorithm 4).

Tables 2 and 3 support the following comparisons among interval estimators:

- MLCI and bootstrap-*t* intervals achieve coverage close to the nominal level in most settings, yielding satisfactory finite-sample performance.
- ACI frequently exhibits undercoverage in small samples, particularly when $\mathcal{R}$ is near 0 or 1, although its intervals tend to be shorter.
- CI.Perc provides stable coverage with moderate expected length and improves as *n* and *m* increase.
- Among bootstrap methods, Boot₁-*t* offers a favorable balance between coverage and expected length; Boot₂-*t* is more conservative and may produce wider intervals.
- HPD intervals maintain coverage close to the nominal level in small samples. At $\mathcal{R} = 0.10$ with *n* = *m* = 3 and $\alpha = 0.05$, HPD yields C.P. = 0.936 and E.L. = 0.224, comparable to MLCI (C.P. = 0.952, E.L. = 0.243) and exceeding that of ACI (C.P. = 0.903). Near the upper boundary (e.g., $\mathcal{R} = 0.95$), HPD coverage remains satisfactory with expected lengths comparable to, or shorter than, those of bootstrap-*t* intervals.
- Relative to bootstrap-*t* intervals, HPD intervals exhibit more stable coverage under sparse record configurations and avoid the occasional excess width of Boot₂-*t*, rendering them a practically useful Bayesian alternative in small-sample RRSS designs.

- Increasing n or m reduces expected interval length and stabilizes coverage for all methods considered.

6. Real Data Analysis

The proposed methods are illustrated using steel specimen lifetimes at different stress levels from [24]. Ref. [23] recently analyzed this dataset for stress–strength reliability of series systems with cold standby redundancy under a Rayleigh model, employing both maximum likelihood and Bayesian estimation.

Because the Rayleigh distribution belongs to the SPEF, it serves as a representative case for the RRSS procedures developed here. To further examine model adequacy and generalizability, we compared the Rayleigh model with several common lifetime distributions, including the exponential, Weibull, and Lomax models. Table 4 reports Kolmogorov–Smirnov (K-S) statistics, descriptive K-S p -values, AIC, and BIC for the two stress levels used in the stress–strength analysis. Maximum-likelihood scale estimates under the Rayleigh model at stress levels 34.5 and 33.5 are $\hat{\sigma} = 298.88$ and $\hat{\sigma} = 503.76$, respectively. The K-S tests yield $D = 0.164$ (p -value = 0.599) and $D = 0.179$ (p -value = 0.487), so the Rayleigh specification is not rejected at conventional significance levels.

Table 4. Goodness-of-fit comparison for candidate lifetime distributions in the steel specimen data.

Stress	Model	Parameter Estimates	K-S D	K-S p	AIC	BIC
34.5	Exponential	rate = 0.002727	0.317	0.027	278.18	279.18
34.5	Rayleigh	$\sigma = 298.877$	0.164	0.599	267.02	268.02
34.5	Weibull	shape = 1.889; scale = 415.932	0.151	0.695	268.89	270.88
34.5	Lomax	$\alpha = 1.70 \times 10^{10}$; $\lambda = 6.23 \times 10^{12}$	0.317	0.027	280.18	282.17
33.5	Exponential	rate = 0.001694	0.230	0.208	297.23	298.23
33.5	Rayleigh	$\sigma = 503.756$	0.179	0.487	292.39	293.39
33.5	Weibull	shape = 1.620; scale = 664.816	0.118	0.913	292.48	294.47
33.5	Lomax	$\alpha = 2.49 \times 10^6$; $\lambda = 1.47 \times 10^9$	0.230	0.208	299.23	301.22

Although the two-parameter Weibull distribution gives slightly larger K-S p -values, the Rayleigh model has the smallest AIC and BIC values for both stress levels and belongs directly to the one-parameter SPEF structure used in the proposed RRSS inference. It is therefore adopted as the primary model for illustrating the methodology.

The Rayleigh CDF and PDF with scale σ are

$$f(x; \sigma) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x \geq 0, \quad (58)$$

$$F(x; \sigma) = 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x \geq 0. \quad (59)$$

This member of the scale-parameter exponential family corresponds to $\omega(\sigma) = 1/\sigma^2$ and $T(x) = x^2/2$, as in Table 1.

Let $\mathbf{R} = (R_{1,1}, R_{2,2}, \dots, R_{n,n})^\top$ and $\mathbf{S} = (S_{1,1}, S_{2,2}, \dots, S_{m,m})^\top$ be two independent sets of upper record values drawn from Rayleigh distributions with scale parameters σ_1 and σ_2 , respectively. The MLEs for the scale parameters are:

$$\hat{\sigma}_{1\text{ML}} = \sqrt{\frac{\sum_{i=1}^n R_{i,i}^2}{2N}}, \quad \hat{\sigma}_{2\text{ML}} = \sqrt{\frac{\sum_{j=1}^m S_{j,j}^2}{2M}}, \quad (60)$$

where $N = \frac{n(n+1)}{2}$ and $M = \frac{m(m+1)}{2}$. With $\hat{\omega}_{jML} = 1/\hat{\sigma}_{jML}^2$, the MLE for $\mathcal{R} = \Pr(X < Y)$ in (10) is

$$\hat{R}_{ML} = \frac{\hat{\omega}_{1ML}}{\hat{\omega}_{1ML} + \hat{\omega}_{2ML}} = \frac{\hat{\sigma}_{2ML}^2}{\hat{\sigma}_{1ML}^2 + \hat{\sigma}_{2ML}^2} = \left(1 + \frac{M}{N} \cdot \frac{\sum_{i=1}^n R_{i,i}^2}{\sum_{j=1}^m S_{j,j}^2} \right)^{-1}. \quad (61)$$

This formulation exploits the distributional properties $\sum_{i=1}^n R_{i,i}^2 \sim \text{Ga}(N, 2\sigma_1^2)$ and $\sum_{j=1}^m S_{j,j}^2 \sim \text{Ga}(M, 2\sigma_2^2)$ under the same shape-scale convention.

Figure 5 is consistent with the K-S test results, showing close agreement between sample and fitted Rayleigh quantiles at both stress levels.

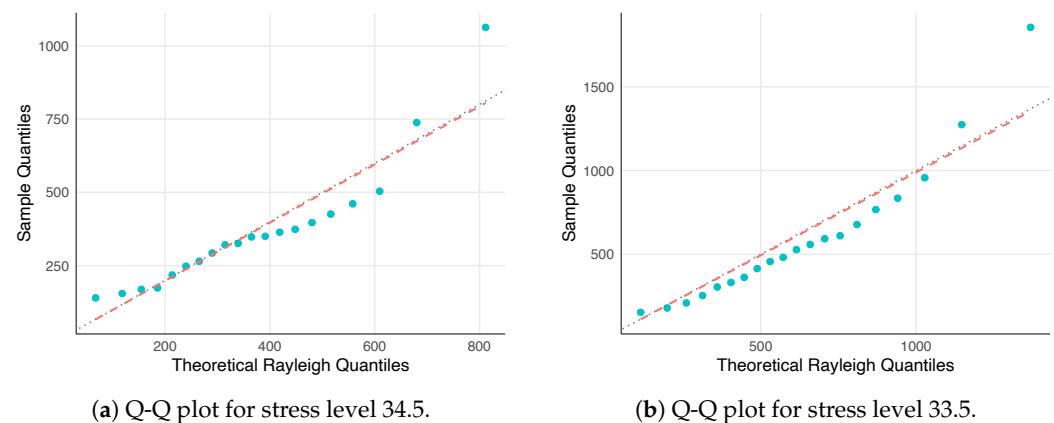


Figure 5. Q-Q plots for the steel specimen lifetime data against the fitted Rayleigh distribution at two stress levels.

The empirical upper record values from the dataset are used to implement the estimation procedures in Sections 2–4. For each (n, m) , MLE, UMVUE, and Bayes point estimates, together with the MLCI, ACI, percentile bootstrap, bootstrap- t , and HPD intervals, are computed from the same record sequences. Bayesian intervals in Table 5 use Algorithm 4 with $K = 5000$ and $a_1 = b_1 = a_2 = b_2 = 10^{-4}$. This Monte Carlo size is sufficient for stable numerical HPD interval estimates in the table.

Figure 6 displays the posterior density of $\mathcal{R}$ for $(n, m) = (5, 5)$ only as a representative graphical illustration; the complete numerical analysis for all considered (n, m) combinations is reported in Table 5. The pair $(5, 5)$ is chosen because it is a balanced and moderate record configuration: it gives equal effective record information from the two samples, avoids the instability of the sparsest setting $(3, 3)$, and does not rely on the largest record size $(7, 7)$. Thus, it provides a clear middle-case visualization of the posterior distribution while retaining the small-sample character of upper RRSS. The posterior density in Figure 6 uses $K = 100,000$ draws only to reduce visual Monte Carlo roughness and obtain a smoother plotted density; the numerical interval results remain those reported in Table 5.

On this dataset, the MLE, UMVUE, and Bayes point estimates are generally close for a fixed record configuration; for $(n, m) = (5, 5)$, they are 0.620, 0.624, and 0.617, consistent with the posterior concentration in Figure 6. Different choices of (n, m) affect the effective record information through $N = n(n+1)/2$ and $M = m(m+1)/2$. Balanced settings tend to produce mutually consistent point estimates, whereas strongly unbalanced settings can shift the estimated reliability because one sample contributes more record information than the other. For interval estimation, HPD, MLCI, ACI, and parametric bootstrap intervals are now centered consistently around the corresponding point estimates. The wider bootstrap intervals observed in sparse or unbalanced rows reflect the additional resampling variability of the record-based design rather than a systematic displacement of the interval center.

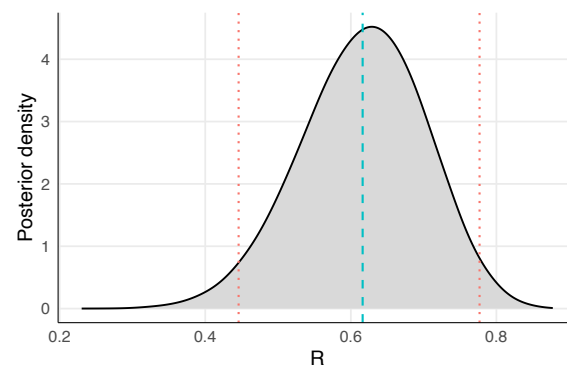


Figure 6. Monte Carlo approximation to the posterior density of $\mathcal{R}$ for the steel specimen data with $n = m = 5$, $K = 100,000$, and hyperparameters $a_1 = b_1 = a_2 = b_2 = 10^{-4}$. The dashed and dotted vertical lines indicate the posterior mean and HPD limits, respectively.

Table 5. Point and interval estimates of stress–strength reliability $\mathcal{R}$ for the steel specimen data.

Row	n	m	MLE	UMVUE	Bayes	HPD	MLCI	ACI	CI.Perc	Boot ₁ - t	Boot ₂ - t
1	3	3	0.898	0.911	0.883	(0.759, 0.976)	(0.728, 0.966)	(0.794, 1.000)	(0.767, 0.972)	(0.725, 0.985)	(0.695, 0.988)
2	3	5	0.687	0.686	0.667	(0.461, 0.855)	(0.425, 0.841)	(0.483, 0.890)	(0.470, 0.853)	(0.386, 0.840)	(0.410, 0.849)
3	3	7	0.610	0.600	0.590	(0.373, 0.786)	(0.354, 0.774)	(0.400, 0.820)	(0.420, 0.801)	(0.313, 0.772)	(0.408, 0.748)
4	5	3	0.951	0.959	0.949	(0.902, 0.989)	(0.890, 0.983)	(0.908, 0.995)	(0.888, 0.978)	(0.899, 0.986)	(0.880, 0.982)
5	5	5	0.620	0.624	0.617	(0.445, 0.779)	(0.440, 0.772)	(0.451, 0.789)	(0.460, 0.788)	(0.433, 0.778)	(0.371, 0.785)
6	5	7	0.768	0.770	0.761	(0.647, 0.869)	(0.629, 0.858)	(0.656, 0.880)	(0.658, 0.868)	(0.640, 0.866)	(0.601, 0.862)
7	7	3	0.507	0.524	0.521	(0.304, 0.729)	(0.320, 0.746)	(0.286, 0.727)	(0.297, 0.693)	(0.313, 0.803)	(0.203, 0.825)
8	7	5	0.734	0.742	0.733	(0.605, 0.848)	(0.601, 0.843)	(0.612, 0.856)	(0.587, 0.827)	(0.584, 0.843)	(0.602, 0.867)
9	7	7	0.746	0.749	0.743	(0.641, 0.840)	(0.633, 0.833)	(0.646, 0.845)	(0.664, 0.836)	(0.627, 0.835)	(0.621, 0.844)

Note: For each (n, m) combination, all point and interval estimates in the table are based on the same empirical upper record samples.

7. Conclusions

Point and interval estimation of the stress–strength reliability parameter $\mathcal{R}$ under upper RRSS and the scale-parameter exponential family (SPEF) was studied. Closed-form MLE and UMVUE formulas were obtained and compared by simulation: both improve as sample size increases, but the UMVUE maintains a more symmetric error profile about $\mathcal{R} = 0.5$ even when $n \neq m$, whereas the MLE exhibits greater sensitivity to imbalance and skewed finite-sample bias. A Bayesian analysis with independent conjugate gamma priors yields tractable posteriors for the natural parameters; the Bayes estimator under squared error loss displays comparatively smooth bias and MSE curves, especially under unbalanced record sizes. For interval estimation, an exact MLE-based interval, a normal approximation, three parametric bootstrap procedures, and HPD credible intervals based on Monte Carlo posterior draws were examined. Simulations indicate that MLCI and bootstrap- t intervals generally achieve coverage close to the nominal level, while ACI may exhibit undercoverage near the boundaries in small samples. The percentile bootstrap interval (CI.Perc) exhibits stable coverage with moderate expected length, and Boot₁- t offers a favorable balance between coverage and precision. HPD credible intervals achieve coverage close to the nominal level with competitive expected lengths, particularly when $\mathcal{R}$ is near 0 or 1. The steel specimen application demonstrates close agreement among the point estimators and confirms the practical value of the unified framework.

Several extensions are worth pursuing. First, the current analysis assumes perfect ranking; inference under imperfect ranking or ranking errors would be important for practical RRSS applications. Second, adaptive and multi-cycle RRSS designs may improve efficiency when more record information can be collected sequentially. Third, the framework could be extended to dependent stress–strength models, robust or semiparametric SPEF-type models,

and Bayesian procedures under alternative loss functions or formally elicited informative priors. These directions would further broaden the applicability of record-based stress–strength reliability inference.

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Appendix A. Derivation of $\mathcal{R}$ and the CDF of W

Let $X \sim \text{SPEF}(\theta_1)$ and $Y \sim \text{SPEF}(\theta_2)$ be two independent random variables. The stress–strength reliability parameter $\mathcal{R} = \Pr(X < Y)$ can then be derived as follows:

$$\mathcal{R} = \Pr(X < Y)$$

Using the transformation $U = Y - X$, $V = X$

$$\begin{aligned} &= \int_0^{+\infty} f_u(u) du \\ &= \int_0^{+\infty} \int_0^{+\infty} f_{u,v}(u, v) dv du \\ &= \int_0^{+\infty} \int_0^{+\infty} \omega(\theta_1) \exp\{-\omega(\theta_1)T(v)\} T'(v) \omega(\theta_2) \exp\{-\omega(\theta_2)T(u+v)\} T'(u+v) dv du \end{aligned}$$

Using the transformation $\alpha = \exp\{-\omega(\theta_2)T(u+v)\}$

$$\begin{aligned} &= \int_0^{+\infty} \left(\int_{\exp\{-\omega(\theta_2)T(v)\}}^{\exp\{-\omega(\theta_2)T(+\infty)\}} d\alpha \right) - \omega(\theta_1) \exp\{-\omega(\theta_1)T(v)\} T'(v) dv \\ &= \int_0^{+\infty} [\exp\{-\omega(\theta_2)T(+\infty)\} - \exp\{-\omega(\theta_2)T(v)\}] (-\omega(\theta_1)) \exp\{-\omega(\theta_1)T(v)\} T'(v) dv \end{aligned}$$

Using the transformation $\beta = \exp\{-\omega(\theta_1)T(v)\}$ (A1)

$$\begin{aligned} &= \int_{\exp\{-\omega(\theta_1)T(0)\}}^{\exp\{-\omega(\theta_1)T(+\infty)\}} \exp\{-\omega(\theta_2)T(+\infty)\} d\beta \\ &\quad - \int_0^{+\infty} (-\omega(\theta_1)) \exp[-\omega(\theta_2) - \omega(\theta_1)]T(v) T'(v) dv \end{aligned}$$

Using the transformation $\gamma = \exp\{-[\omega(\theta_2) - \omega(\theta_1)]T(v)\}$

$$\begin{aligned} &= \int_{\exp\{-\omega(\theta_1)T(0)\}}^{\exp\{-\omega(\theta_1)T(+\infty)\}} \exp\{-\omega(\theta_2)T(+\infty)\} d\beta \\ &\quad - \int_{\exp\{-[\omega(\theta_1) - \omega(\theta_2)]T(0)\}}^{\exp\{-[\omega(\theta_1) - \omega(\theta_2)]T(+\infty)\}} \frac{-\omega(\theta_1)}{-\omega(\theta_1) - \omega(\theta_2)} d\gamma \\ &= \frac{\omega(\theta_1)}{\omega(\theta_1) + \omega(\theta_2)} \exp\{-[\omega(\theta_1) - \omega(\theta_2)]T(0)\} \\ &\quad + \frac{\omega(\theta_2)}{\omega(\theta_1) + \omega(\theta_2)} \exp\{-[\omega(\theta_1) - \omega(\theta_2)]T(+\infty)\} + \exp\{-\omega(\theta_1)T(0) - \omega(\theta_2)T(+\infty)\} \end{aligned}$$

Applying the boundary conditions $T(0) = 0$ and $T(+\infty) = +\infty$, together with $\omega(\theta_1), \omega(\theta_2) > 0$, we simplify each term as follows:

- First term: $\exp\{-[\omega(\theta_1) - \omega(\theta_2)]T(0)\} = \exp\{0\} = 1$, so the first term reduces to $\frac{\omega(\theta_1)}{\omega(\theta_1) + \omega(\theta_2)}$.

- Second term: $\exp\{-\omega(\theta_1) - \omega(\theta_2)\} T(+\infty) = \exp\{-\infty\} = 0$, so the second term vanishes.
- Third term: $\exp\{-\omega(\theta_1) T(0) - \omega(\theta_2) T(+\infty)\} = \exp\{0 - \infty\} = 0$, so the third term vanishes.

Therefore

$$R = \frac{\omega(\theta_1)}{\omega(\theta_1) + \omega(\theta_2)}, \quad (\text{A2})$$

which establishes (10).

The CDF of the ancillary statistic $W = \frac{T(R_{1,1}) \sum_{j=1}^m T(S_{j,j})}{T(S_{1,1}) \sum_{i=1}^n T(R_{i,i})} \stackrel{d}{=} \frac{\text{Beta}(1, N-1)}{\text{Beta}(1, M-1)}$ is derived as follows:

$$\begin{aligned} F_W(w) &= P\left(\frac{U}{V} \leq w\right) \\ &= P(U \leq wV) \\ &= \int_{-\infty}^{+\infty} P(U \leq wv \mid V = v) f_V(v) dv \\ &= \int_0^1 P(U \leq wv \mid V = v) f_V(v) dv \\ &= \int_0^1 F_U(wv) f_V(v) dv \\ &= \int_0^1 [1 - (1 - wv)^{N-1}] (M-1)(1-v)^{M-2} dv \\ &= \int_0^1 (M-1)(1-v)^{M-2} dv - \int_0^1 (1-wv)^{N-1} (M-1)(1-v)^{M-2} dv \\ &= -(1-v)^{M-1} \Big|_0^1 - (M-1) \int_0^1 (1-wv)^{N-1} (1-v)^{M-2} dv \\ &= 1 - (M-1) \int_0^1 \sum_{d=0}^{N-1} \binom{N-1}{d} (-wv)^d (1-v)^{M-2} dv \\ &= 1 - (M-1) \sum_{d=0}^{N-1} \binom{N-1}{d} (-w)^d \int_0^1 v^d (1-v)^{M-2} dv \\ &= 1 - (M-1) \sum_{d=0}^{N-1} \binom{N-1}{d} (-w)^d \frac{1}{\binom{d+M-1}{d}} \cdot \frac{1}{M-1} \\ &= 1 - \sum_{d=0}^{N-1} \frac{\binom{N-1}{d}}{\binom{d+M-1}{d}} (-w)^d \end{aligned} \quad (\text{A3})$$

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