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Potential Balance Laws for the KdV–Burgers Equation: Derivation, Interpretation, and Numerical Validation

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Abstract

The KdV–Burgers equation $u_t + uu_x - \nu u_{xx} + \beta u_{xxx} = 0$ models the interplay of nonlinearity, dispersion and dissipation. Through the potential $u = v_x$, a Lagrangian density is constructed for the resulting potential system. Application of Noether’s theorem yields an infinite-dimensional symmetry $V = f(x, t)\partial_v$, where f satisfies the linearized equation $f_t - \nu f_{xx} + \beta f_{xxx} = 0$. This symmetry generates an infinite family of continuity equations of the form $D_t T + D_x X = 0$ with $T = f(x, t)$ and $X = -f_x u + \nu f_x - \beta f_{xx}$. For $\nu > 0$, these relations constitute linear potential balance laws rather than classical conservation laws, as the density T depends explicitly on the auxiliary field f and on time. In the inviscid limit $\nu = 0$, the family reduces to the classical KdV conservation hierarchy. High-accuracy spectral simulations confirm the validity of these identities up to discretization error.

Keywords: KdV–Burgers equation; potential balance laws; linear constraints; nonlocal symmetries; noether’s theorem

MSC: 35Q53, 35A30, 35K55, 65M70

1. Introduction

The Korteweg–de Vries–Burgers (KdV–Burgers) equation serves as a canonical model for the interplay of nonlinearity, dispersion, and dissipation [1–4], governing phenomena ranging from shallow water waves to plasma shocks [5–7].

The connection between symmetries and conservation laws has been systematically studied since the work of Noether [8,9] and has been extensively developed in the monographs of Olver [10] and Bluman and Kumei [11]. For equations that do not admit a standard variational formulation, methods based on potential variables and adjoint symmetries provide systematic approaches to constructing conservation laws [11–13].

The KdV–Burgers equation,

$$u_t + uu_x - \nu u_{xx} + \beta u_{xxx} = 0, \quad (1)$$

unifies nonlinearity, dispersion and dissipation. In the inviscid limit $\nu = 0$, Equation (1) reduces to the completely integrable Korteweg–de Vries (KdV) equation, whose infinite hierarchy of conservation laws reflects its underlying symmetry structure and enables exact solution methods [10,11]. These conservation laws encode physically meaningful quantities: mass, momentum, energy and higher-order invariants that constrain solution behavior [12,14,15].



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For $\nu > 0$, the dissipative term $-\nu u_{xx}$ breaks the time-reversal symmetry of the KdV equation, so Equation (1) is not self-adjoint and does not admit a standard variational formulation [16–18]. While variational methods do not apply directly to non-self-adjoint equations, several studies have extended symmetry-based methods to dissipative systems [11,13,18].

For non-self-adjoint equations such as (1), the method of potential symmetries [11,12] and the multiplier method [13,19] provide systematic tools for constructing conservation laws and related identities. The method of potential symmetries embeds a given equation into a larger system through a potential variable. The resulting system admits a Lagrangian formulation even when the original equation does not [12,20,21]. Noether's theorem applied to such potential systems can yield conservation laws that project back to the original equation, often revealing nonlocal structures invisible to standard methods [22,23].

The local conservation laws for the KdV–Burgers equation are well established. Mass conservation $T_1 = u$ and the momentum balance law $T_2 = \frac{1}{2}u^2$ have been known for decades [1–3]. The direct method used in Section 2 follows the standard approach demonstrated in [10,11,13]. These local conservation laws serve as a foundation for the nonlocal analysis presented in subsequent sections.

The KdV–Burgers equation does not admit a standard variational principle because its Fréchet derivative is not self-adjoint [10] (Section 3). The introduction of a potential variable $u = v_x$ to obtain a Lagrangian formulation is a standard technique in the analysis of non-variational equations [11,12]. For the inviscid KdV equation ($\nu = 0$), a Lagrangian formulation using the potential $u = v_x$ is classical [10,11]. For dissipative equations, this method has been applied to the Burgers equation and various diffusion equations [11,12].

The specific Lagrangian density presented in this work (12) is a natural extension of the KdV Lagrangian to include dissipative and higher-order dispersive terms. This approach follows the potential symmetry framework of [11,12]. Potential symmetries were introduced by Bluman and Kumei [11]. For Burgers-type equations, the most directly relevant prior work is that of Wang, Kara, and Fakhar [23].

In [23], Wang, Kara, and Fakhar studied the third-order Burgers equation

$$u_t + uu_x + \nu u_{xx} + \beta u_{xxx} = 0,$$

which differs from Equation (1) only in the sign of the dissipative term ($+\nu u_{xx}$ vs. $-\nu u_{xx}$). They introduced a potential $u = v_x$ and obtained the Lagrangian (12) with a $+\beta$ sign difference in the last term compared to the Lagrangian used in the present work. They further derived nonlocal symmetries of the form $V = f(t)\partial_v$ (with f depending only on time) and obtained corresponding nonlocal conservation laws. Their work demonstrated that potential formulations can yield nonlocal conservation laws for the third-order Burgers equation. A detailed comparison between the present work and [23] is given in Table 1.

Table 1. Comparison between the present work and the most directly relevant prior study (Wang et al. [23]). The sign convention for the dissipative term differs: Wang et al. studied $+\nu u_{xx}$ while the present work uses $-\nu u_{xx}$.

Aspect	Wang et al. (2016) [23]	Present work
Equation studied	Third-order Burgers $u_t + uu_x + \nu u_{xx} + \beta u_{xxx} = 0$	KdV–Burgers $u_t + uu_x - \nu u_{xx} + \beta u_{xxx} = 0$
Symmetry considered	$V = f(t)\partial_v$ (time only)	$V = f(x, t)\partial_v$ (full PDE)
Dimension of symmetry	One-dimensional	Infinite-dimensional
Balance vs. conservation	Not addressed	Explicitly discussed (Section 3.4)
Numerical validation	None	Comprehensive (normalized residuals $< 10^{-9}$)

Previous studies have examined conservation laws for the KdV–Burgers equation using Lie symmetry analysis [2] and traveling wave methods [4]. Adjoint symmetry analysis and the multiplier method [10,13] offer systematic tools for constructing conservation laws without a Lagrangian; Section 4 applies the multiplier method to the KdV–Burgers equation.

The identities (29) follow directly by multiplying (1) by a solution of the adjoint linearized equation [13,24,25]. The contributions of this paper are threefold: first, a systematic variational derivation within the potential symmetry framework [11], revealing that the nonlocal multipliers $\Lambda_f = f$ correspond to local symmetries $V = f\partial_v$ in an extended potential system; second, an explicit characterization of the distinction between these potential balance laws and classical conservation laws, emphasizing that for $\nu > 0$ the density $T = f(x, t)$ depends on an auxiliary field and therefore does not yield a time-independent functional of u alone; and third, high-accuracy numerical validation confirming these identities to within discretization error across a range of test cases.

This paper demonstrates the application of potential symmetry methods to the KdV–Burgers equation. The Lagrangian density (12) is presented for the potential system as a natural extension of the inviscid KdV Lagrangian. The infinite-dimensional symmetry $V = f(x, t)\partial_v$ is derived, where f satisfies (28), yielding the explicit potential balance law family $(T, X) = (f, -f_x u + \nu f_x - \beta f_{xx})$ in (29). For $\nu > 0$, these are shown to be linear constraints rather than classical conservation laws. Numerical validation confirms these identities to within discretization error.

Section 2 derives local conservation laws by the direct method. Section 3 develops the potential formulation and Lagrangian for the KdV–Burgers equation. Section 4 presents multiplier analysis. Section 5 provides numerical validation. Section 6 concludes the paper.

2. Derivation of Conservation Laws

Definition 1 (Conservation Law). *A conservation law for the KdV–Burgers Equation (1) is a pair (T, X) of functions depending on x, t, u , and derivatives of u , such that*

$$D_t T + D_x X = 0, \quad (2)$$

holds for all solutions of (1). The function T is called the conserved density, and X the associated flux. Two conservation laws are equivalent if they differ by a trivial conservation law (Definition 2).

Definition 2 (Trivial Conservation Law). *A conservation law is trivial of the first kind if $D_t T + D_x X = 0$ holds identically for all functions $u(x, t)$. It is trivial of the second kind if (T, X) vanishes on the solution manifold. Two conservation laws are equivalent if they differ by a trivial conservation law.*

Definition 3 (Potential Balance Law). *A potential balance law for the KdV–Burgers Equation (1) is a continuity equation*

$$D_t T + D_x X = 0, \quad (3)$$

where the density $T = f(x, t)$ depends on an auxiliary field f that satisfies the linearized equation

$$f_t - \nu f_{xx} + \beta f_{xxx} = 0, \quad (4)$$

and the flux X depends on u and its derivatives. Such a relation is not a classical conservation law (Definition 1) because T depends explicitly on the auxiliary field f and on time.

Remark 1 (Distinction between Potential Balance Laws and Classical Conservation Laws). *A classical conservation law in the sense of Definition 1 requires the density T to be a function of x, t, u , and derivatives of u only. Such a density yields a time-independent functional $\int T(u, u_x, \dots) dx$*

that depends solely on the solution u . In contrast, for a potential balance law as defined in Definition 3, the density $T = f(x, t)$ depends on an auxiliary field f that satisfies the linearized Equation (28). Although the quantity $\int f u \, dx$ remains constant in time, this constancy results from a cancellation between the explicit time dependence of f and the evolution of u . For a fixed solution u , different choices of f satisfying (28) yield different constants. Consequently, $\int f u \, dx$ is not a functional of u alone; it depends explicitly on the auxiliary field f and on time through f . Therefore, (29) does not constitute a classical conservation law according to Definition 1. Rather, it represents a linear potential constraint that every solution u of the nonlinear equation must satisfy. Following the terminology of Bluman and Kumei [11], such relations are referred to as potential balance laws.

Linear Potential Balance Laws vs. Classical Conservation Laws

The direct method seeks conserved currents (T, X) that are functions of u and its derivatives, requiring that Equation (2) be a consequence of Equation (1).

Theorem 1 (Low-Order Local Conservation Laws). *For the KdV–Burgers Equation (1), the following nontrivial conservation laws exist:*

Mass conservation

$$T_1 = u, \quad X_1 = \frac{1}{2}u^2 - \nu u_x + \beta u_{xx}, \quad (5)$$

which satisfies $D_t T_1 + D_x X_1 = 0$ exactly.

Momentum balance law

$$T_2 = \frac{1}{2}u^2, \quad X_2 = \frac{1}{3}u^3 - \nu u u_x + \beta u u_{xx} - \frac{\beta}{2}u_x^2, \quad (6)$$

which satisfies $D_t T_2 + D_x X_2 = -\nu u_x^2$ on solutions.

Proof. The result follows from the direct method described in [10,13]. By requiring that $D_t T + D_x X$ vanishes modulo the Equation (1) and its differential consequences, we find that T must be a linear function of u . The two independent choices yield mass and momentum. The momentum law is a balance law because $D_t T_2 + D_x X_2 = -\nu u_x^2 \neq 0$ when $\nu > 0$. $\square$

Remark 2. The momentum law (6) is a balance law rather than a strict conservation law when $\nu > 0$, reflecting energy dissipation through the viscous term.

3. Potential Formulation and Variational Structure

The KdV–Burgers Equation (1) does not satisfy the Helmholtz conditions for a variational principle because its Fréchet derivative is not self-adjoint. The Fréchet derivative of $\Delta[u] = u_t + uu_x - \nu u_{xx} + \beta u_{xxx}$ acting on ϕ is

$$L_\Delta(\phi) = \phi_t + u\phi_x + u_x\phi - \nu\phi_{xx} + \beta\phi_{xxx}. \quad (7)$$

In operator form

$$L_\Delta = D_t + uD_x + u_x - \nu D_x^2 + \beta D_x^3, \quad L_\Delta^* = -D_t - uD_x - u_x - \nu D_x^2 - \beta D_x^3, \quad (8)$$

so that $L_\Delta \neq L_\Delta^*$ unless $\nu = 0$. The KdV–Burgers equation does not admit a standard Lagrangian formulation, so Noether's theorem cannot be applied directly to the original equation in its present form.

To obtain a variational formulation, introduce a potential variable $v(x, t)$ defined by

$$u = v_x. \quad (9)$$

Substituting into Equation (1) and integrating with respect to x yields the potential KdV–Burgers equation

$$v_t + \frac{1}{2}v_x^2 - \nu v_{xx} + \beta v_{xxx} = 0, \quad (10)$$

where the integration constant is set to zero by imposing suitable boundary conditions (vanishing as $|x| \rightarrow \infty$). Together with Equation (9), we obtain the potential system

$$\begin{aligned} u - v_x &= 0, \\ v_t + \frac{1}{2}v_x^2 - \nu v_{xx} + \beta v_{xxx} &= 0. \end{aligned} \quad (11)$$

The potential equation admits a Lagrangian formulation, given by

$$\mathcal{L} = -\frac{1}{2}v_t v_x - \frac{1}{6}v_x^3 - \frac{\nu}{2}v_{xx}^2 - \frac{\beta}{2}v_{xxx}^2. \quad (12)$$

For a Lagrangian density $\mathcal{L}(x, t, v, v_t, v_x, v_{xx}, v_{xxx})$, the Euler–Lagrange operator is defined as

$$\frac{\delta \mathcal{L}}{\delta v} = \frac{\partial \mathcal{L}}{\partial v} - D_t \left(\frac{\partial \mathcal{L}}{\partial v_t} \right) - D_x \left(\frac{\partial \mathcal{L}}{\partial v_x} \right) + D_x^2 \left(\frac{\partial \mathcal{L}}{\partial v_{xx}} \right) - D_x^3 \left(\frac{\partial \mathcal{L}}{\partial v_{xxx}} \right). \quad (13)$$

This operator annihilates total divergences: $\frac{\delta}{\delta v}(D_t A + D_x B) = 0$ for any functions A, B depending on v and its derivatives.

Proposition 1 (Euler–Lagrange Equations for $\mathcal{L}$). *The Lagrangian density $\mathcal{L}$ defined in (12) satisfies*

$$\frac{\delta \mathcal{L}}{\delta v} = v_{tx} + v_x v_{xx} - \nu v_{xxx} + \beta v_{xxxxx}, \quad (14)$$

where $\delta/\delta v$ is the Euler–Lagrange operator. Consequently, the Euler–Lagrange equation $\delta \mathcal{L}/\delta v = 0$ is a sixth-order PDE whose solution set contains all solutions of the potential KdV–Burgers Equation (10).

Proof. Compute the required partial derivatives of $\mathcal{L}$

$$\frac{\partial \mathcal{L}}{\partial v} = 0, \quad \frac{\partial \mathcal{L}}{\partial v_t} = -\frac{1}{2}v_x, \quad \frac{\partial \mathcal{L}}{\partial v_x} = -\frac{1}{2}v_t - \frac{1}{2}v_x^2, \quad (15)$$

$$\frac{\partial \mathcal{L}}{\partial v_{xx}} = -\nu v_{xx}, \quad \frac{\partial \mathcal{L}}{\partial v_{xxx}} = -\beta v_{xxx}. \quad (16)$$

Applying the Euler–Lagrange operator

$$\frac{\delta \mathcal{L}}{\delta v} = -D_t \left(-\frac{1}{2}v_x \right) - D_x \left(-\frac{1}{2}v_t - \frac{1}{2}v_x^2 \right) \quad (17)$$

$$+ D_x^2(-\nu v_{xx}) - D_x^3(-\beta v_{xxx}) \quad (18)$$

$$= \frac{1}{2}v_{tx} + \frac{1}{2}v_{tx} + v_x v_{xx} - \nu v_{xxx} + \beta v_{xxxxx} \quad (19)$$

$$= v_{tx} + v_x v_{xx} - \nu v_{xxx} + \beta v_{xxxxx}. \quad (20)$$

If v satisfies the potential KdV–Burgers Equation (10), then differentiating with respect to x gives $v_{tx} + v_x v_{xx} - \nu v_{xxx} + \beta v_{xxxx} = 0$. Differentiating twice more yields $v_{tx} + v_x v_{xx} - \nu v_{xxx} + \beta v_{xxxxx} = 0$, which is exactly (14). Thus every solution of (10) satisfies the Euler–Lagrange equation. $\square$

3.1. Relationship Between the Potential Equation and the Euler–Lagrange Equation

The Lagrangian density (12) yields the sixth-order Euler–Lagrange equation

$$\frac{\delta \mathcal{L}}{\delta v} = v_{tx} + v_x v_{xx} - \nu v_{xxxx} + \beta v_{xxxxxx} = 0. \quad (21)$$

Equation (21) is sixth-order, while the potential KdV–Burgers Equation (10) is fourth-order. Every solution of (10) satisfies (21), but the converse does not hold. The following Lemma 1 characterizes the spurious solutions.

Lemma 1 (Characterization of Spurious Solutions). *Let $\mathcal{S}_{KdV}$ denote the set of smooth solutions of the potential KdV–Burgers Equation (10) satisfying vanishing boundary conditions as $|x| \rightarrow \infty$ (or periodic boundary conditions). Let $\mathcal{S}_{EL}$ denote the set of smooth solutions of the Euler–Lagrange Equation (21) under the same boundary conditions. Then*

$$\mathcal{S}_{KdV} \subset \mathcal{S}_{EL},$$

and every $v \in \mathcal{S}_{EL}$ can be written uniquely as $v = v_{KdV} + c(t)$, where $v_{KdV} \in \mathcal{S}_{KdV}$ and $c(t)$ is an arbitrary function of time only. The function $c(t)$ is a spurious mode that does not affect $u = v_x$ and does not affect the conserved currents derived from Noether’s theorem.

Proof. Integrate (21) once with respect to x

$$\int \frac{\delta \mathcal{L}}{\delta v} dx = v_t + \frac{1}{2} v_x^2 - \nu v_{xx} + \beta v_{xxxx} + \gamma(t) = 0, \quad (22)$$

where $\gamma(t)$ is an integration constant function of time. Differentiating (22) with respect to x recovers (21). Thus, every solution of the Euler–Lagrange equation is of the form $v = v_{KdV} + c(t)$, where v_{KdV} satisfies (10) and $c(t)$ is arbitrary.

$$(v_{KdV})_t + c'(t) + \frac{1}{2} (v_{KdV})_x^2 - \nu (v_{KdV})_{xx} + \beta (v_{KdV})_{xxxx} = 0.$$

Since v_{KdV} satisfies (10), substitution gives $c'(t) = 0$, so $c(t)$ is constant in time. Hence, the spurious mode is a constant shift that does not affect $u = v_x$. $\square$

Proposition 1 (Invariance of Noether Currents Under Spurious Modes). *Let $v = v_{KdV} + c(t)$ be a solution of the Euler–Lagrange Equation (21), where v_{KdV} satisfies the potential KdV–Burgers Equation (10) and $c(t)$ is an arbitrary function of time. Then, for any symmetry generator V of the Lagrangian (12) of the form considered in this paper (with $\xi = 0, \tau = 0, \psi = f(x, t)$ or $\xi = 0, \tau = 1, \psi = 0$), the conserved current (T^v, X^v) constructed via Noether’s theorem (Theorem 2) satisfies*

$$T^v[v] = T^v[v_{KdV}], \quad X^v[v] = X^v[v_{KdV}].$$

Thus, the spurious mode $c(t)$ does not affect any physical observable or conserved quantity derived from the potential formulation.

Proof. We examine the dependence of the Noether currents on v and its derivatives. From Theorem 2, for symmetries with $\xi = \tau = 0$ such as $V = f \partial_v$, we have

$$T^v = W \frac{\partial \mathcal{L}}{\partial v_t}, \quad X^v = W \frac{\partial \mathcal{L}}{\partial v_x} + D_x(W) \frac{\partial \mathcal{L}}{\partial v_{xx}} + D_x^2(W) \frac{\partial \mathcal{L}}{\partial v_{xxx}},$$

with $W = \psi = f(x, t)$ (since $\xi = \tau = 0$). The partial derivatives of $\mathcal{L}$ are

$$\frac{\partial \mathcal{L}}{\partial v_t} = -\frac{1}{2}v_x, \quad \frac{\partial \mathcal{L}}{\partial v_x} = -\frac{1}{2}v_t - \frac{1}{2}v_x^2, \quad \frac{\partial \mathcal{L}}{\partial v_{xx}} = -v_{xx}, \quad \frac{\partial \mathcal{L}}{\partial v_{xxx}} = -\beta v_{xxx}.$$

Now substitute $v = v_{\text{KdV}} + c(t)$. Then, $v_x = (v_{\text{KdV}})_x$ because $c_x = 0$, $v_{xx} = (v_{\text{KdV}})_{xx}$, $v_{xxx} = (v_{\text{KdV}})_{xxx}$ and $v_t = (v_{\text{KdV}})_t + c'(t)$.

The terms that depend on $c'(t)$ appear only in $\frac{\partial \mathcal{L}}{\partial v_x}$ via the $-\frac{1}{2}v_t$ term. Specifically

$$\frac{\partial \mathcal{L}}{\partial v_x} \Big|_v = -\frac{1}{2}((v_{\text{KdV}})_t + c'(t)) - \frac{1}{2}(v_{\text{KdV}})_x^2.$$

Thus, X^v contains the contribution

$$W \cdot \left(-\frac{1}{2}c'(t) \right) = -\frac{1}{2}f(x, t)c'(t).$$

We apply this term as a total derivative, which does not affect the conservation law and vanishes due to the properties of f and $c(t)$. The conservation law is defined modulo total derivatives; two currents (T, X) and $(T, X + D_x \Phi)$ are equivalent. The term $-\frac{1}{2}f c'(t)$ can be written as

$$-\frac{1}{2}f(x, t)c'(t) = D_x \left(-\frac{1}{2}c'(t) \int f(x, t) dx \right)$$

provided $\int f dx$ is well-defined (e.g., for localized or periodic f). However, a more direct argument: the Euler–Lagrange equation for the spurious mode itself forces $c'(t) = 0$ when the boundary conditions are imposed. Specifically, from Lemma 1, if we require that v satisfies the same boundary conditions as v_{KdV} (e.g., vanishing at infinity or periodic), then $c(t)$ must be constant. In that case, $c'(t) = 0$ and the offending term vanishes identically.

For the time translation symmetry $V = \partial_t$ (Example 2), we have $\tau = 1, \xi = 0, \psi = 0$, so $W = -v_t$. Then

$$T^v = \mathcal{L} + (-v_t) \frac{\partial \mathcal{L}}{\partial v_t} = \mathcal{L} + \frac{1}{2}v_t v_x.$$

Substituting $v = v_{\text{KdV}} + c(t)$

$$T^v = \mathcal{L}(v_{\text{KdV}} + c(t)) + \frac{1}{2}((v_{\text{KdV}})_t + c'(t))((v_{\text{KdV}})_x).$$

Expanding $\mathcal{L}$

$$\mathcal{L}(v_{\text{KdV}} + c(t)) = -\frac{1}{2}((v_{\text{KdV}})_t + c'(t))(v_{\text{KdV}})_x - \frac{1}{6}(v_{\text{KdV}})_x^3 - \frac{\nu}{2}(v_{\text{KdV}})_{xx}^2 - \frac{\beta}{2}(v_{\text{KdV}})_{xxx}^2.$$

Adding $\frac{1}{2}((v_{\text{KdV}})_t + c'(t))(v_{\text{KdV}})_x$ cancels the $-\frac{1}{2}((v_{\text{KdV}})_t + c'(t))(v_{\text{KdV}})_x$ term, leaving:

$$T^v = -\frac{1}{6}(v_{\text{KdV}})_x^3 - \frac{\nu}{2}(v_{\text{KdV}})_{xx}^2 - \frac{\beta}{2}(v_{\text{KdV}})_{xxx}^2,$$

which is independent of $c(t)$ and $c'(t)$. A similar calculation shows that X^v is also invariant. Thus, the spurious mode $c(t)$ does not affect the conserved current. $\square$

Corollary 1 (Validity of Noether's Theorem for the Potential System). *For any symmetry generator V of the Lagrangian (12) that preserves the boundary conditions, the conserved current (T^v, X^v) constructed via Noether's theorem satisfies*

$$D_t T^v + D_x X^v = 0$$

for all solutions of the Euler–Lagrange Equation (21). When restricted to the subset $\mathcal{S}_{\text{KdV}} \subset \mathcal{S}_{\text{EL}}$, the same conservation law holds for all solutions of the potential KdV–Burgers Equation (10) and, consequently, for all solutions of the original KdV–Burgers Equation (1) via $u = v_x$. The spurious modes $c(t)$ do not affect the conservation law because T^v and X^v depend only on derivatives of v (or on combinations from which $c'(t)$ cancels), as shown explicitly in Proposition 1.

Remark 3 (The Embedding Method). The technique of embedding a lower-order PDE into a higher-order Lagrangian formulation is standard in the theory of variational principles for non-variational equations [10,11]. The key point is that Noether’s theorem applied to the higher-order Lagrangian produces conservation laws that are on-shell with respect to the lower-order dynamics because the spurious modes are either trivial (affect only gauge degrees of freedom) or are eliminated by boundary conditions. A detailed discussion of this embedding method is given in [12], Chapter 8. The explicit verification in Proposition 1 provides a concrete demonstration for the Lagrangian (12).

3.2. Noether’s Theorem for the Potential System

Conservation laws derived from the Lagrangian (12) hold on the solution space of the potential KdV–Burgers Equation (10) (Proposition 1 and Corollary 1), we now apply Noether’s theorem.

Let the action $S = \int \mathcal{L} dx dt$ be invariant under a one-parameter Lie group with infinitesimal generator

$$V = \xi \partial_x + \tau \partial_t + \eta \partial_u + \psi \partial_v, \quad (23)$$

where ξ, τ, η, ψ depend on x, t, u, v and derivatives of u, v . The invariance condition leads to a conservation law for the potential system, which can then be projected back to the original variable u .

Theorem 2 (Noether’s Theorem for Higher-Order Lagrangians). *If the Lagrangian $\mathcal{L}(v_t, v_x, v_{xx}, v_{xxx})$ is invariant under the transformation generated by V , then the following conservation law holds on solutions of the Euler–Lagrange equation:*

$$D_t T + D_x X = 0, \quad (24)$$

with

$$T = \tau \mathcal{L} + W \frac{\partial \mathcal{L}}{\partial v_t}, \quad (25)$$

$$X = \xi \mathcal{L} + W \frac{\partial \mathcal{L}}{\partial v_x} + D_x(W) \frac{\partial \mathcal{L}}{\partial v_{xx}} + D_x^2(W) \frac{\partial \mathcal{L}}{\partial v_{xxx}}, \quad (26)$$

where $W = \psi - \xi v_x - \tau v_t$ is the characteristic of the symmetry.

Example 2 (Time Translation Symmetry). Consider time translation symmetry $V = \partial_t$, corresponding to $\tau = 1, \xi = 0, \psi = 0$. Then $W = -v_t$. Substituting into Equations (25)–(26) and using the Lagrangian (12) yields a conservation law. Projecting back to $u = v_x$ recovers the momentum balance law (6). This demonstrates that the dissipative balance law originates from a genuine variational symmetry of the potential system.

Remark 4 (Independence from Spurious Modes). The time translation symmetry current derived above is invariant under the addition of a spurious mode $c(t)$ to v , as shown explicitly in the proof of Proposition 1. This confirms that the momentum balance law (6) is a genuine consequence of the variational principle despite the sixth-order nature of the Euler–Lagrange equation.

3.3. Potential Symmetries and Infinite Family of Potential Conservation Laws

The power of the potential formulation becomes apparent when we consider symmetries that act nontrivially on the potential variable v but have no local counterpart in u .

Theorem 3 (Infinite Family of Potential Balance Laws). *This result is derived using the standard potential symmetry framework of Bluman and Kumei [11].*

$$V = f(x, t)\partial_v, \quad (27)$$

where $f(x, t)$ is any solution of the linearized equation

$$f_t - \nu f_{xx} + \beta f_{xxx} = 0. \quad (28)$$

This symmetry generates a continuity equation for the original KdV–Burgers equation of the form

$$D_t T + D_x X = 0, \quad T = f(x, t), \quad X = -f_x u + \nu f_x - \beta f_{xx}. \quad (29)$$

For $\nu > 0$, this is a potential balance law (Definition 3) or linear potential constraint, not a classical conservation law, because the density $T = f(x, t)$ depends on the auxiliary field f and on time. For $\nu = 0$, appropriate choices of f polynomials in x and t recover the classical KdV conservation laws.

Proof. We first verify that $V = f\partial_v$ is a symmetry of the Lagrangian $\mathcal{L}$. The Lie derivative of $\mathcal{L}$ with respect to the prolonged generator $\text{pr}^{(3)}V = f\partial_v + f_t\partial_{v_t} + f_x\partial_{v_x} + f_{xx}\partial_{v_{xx}} + f_{xxx}\partial_{v_{xxx}}$ is

$$\begin{aligned} \text{pr}^{(3)}V(\mathcal{L}) &= f_t \frac{\partial \mathcal{L}}{\partial v_t} + f_x \frac{\partial \mathcal{L}}{\partial v_x} + f_{xx} \frac{\partial \mathcal{L}}{\partial v_{xx}} + f_{xxx} \frac{\partial \mathcal{L}}{\partial v_{xxx}} \\ &= f_t \left(-\frac{1}{2}v_x \right) + f_x \left(-\frac{1}{2}v_t - \frac{1}{2}v_x^2 \right) - \nu f_{xx}v_{xx} - \beta f_{xxx}v_{xxx}. \end{aligned} \quad (30)$$

Substituting $f_t = \nu f_{xx} - \beta f_{xxx}$ from (28) into (30) and collecting terms, we obtain

$$\begin{aligned} \text{pr}^{(3)}V(\mathcal{L}) &= -\frac{1}{2}(\nu f_{xx} - \beta f_{xxx})v_x - \frac{1}{2}f_x v_t - \frac{1}{2}f_x v_x^2 - \nu f_{xx}v_{xx} - \beta f_{xxx}v_{xxx} \\ &= D_x \left(-\frac{1}{2}f v_t - \frac{1}{6}f v_x^3 - \nu f_x v_{xx} - \beta f_{xx}v_{xxx} \right), \end{aligned} \quad (31)$$

where the equality follows after using the potential Equation (10) to eliminate v_t . Thus $\text{pr}^{(3)}V(\mathcal{L})$ is a total derivative, confirming that V is a variational symmetry with characteristic $W = f$. Noether's theorem (Theorem 2) therefore applies.

For the symmetry $V = f\partial_v$, we have $\xi = 0$, $\tau = 0$, $\psi = f$. Applying Theorem 2 to the Lagrangian (12) yields the conserved current for the potential system

$$T^v = f \frac{\partial \mathcal{L}}{\partial v_t} = -\frac{1}{2}f u, \quad (32)$$

$$\begin{aligned} X^v &= f \frac{\partial \mathcal{L}}{\partial v_x} + D_x(f) \frac{\partial \mathcal{L}}{\partial v_{xx}} + D_x^2(f) \frac{\partial \mathcal{L}}{\partial v_{xxx}} \\ &= f \left(-\frac{1}{2}v_t - \frac{1}{2}u^2 \right) - \nu f_x u_x - \beta f_{xx}u_{xx}. \end{aligned} \quad (33)$$

To obtain a conservation law for the original equation, we eliminate v_t using the potential Equation (10)

$$v_t = -\frac{1}{2}u^2 + \nu u_x - \beta u_{xx}. \quad (34)$$

Substituting (34) into (33)

$$\begin{aligned} X^v &= f \left(\frac{1}{4}u^2 - \frac{\nu}{2}u_x + \frac{\beta}{2}u_{xx} - \frac{1}{2}u^2 \right) - \nu f_x u_x - \beta f_{xx} u_{xx} \\ &= -\frac{1}{4}f u^2 - \frac{\nu}{2}f u_x + \frac{\beta}{2}f u_{xx} - \nu f_x u_x - \beta f_{xx} u_{xx}. \end{aligned} \quad (35)$$

We now rewrite the terms in (35) using integration by parts by adding total derivatives that do not affect the conservation law

$$f u_x = D_x(f u) - f_x u, \quad (36)$$

$$f u_{xx} = D_x(f u_x) - f_x u_x, \quad (37)$$

$$f_x u_x = \frac{1}{2} D_x(f_x u) - \frac{1}{2} f_{xx} u, \quad (38)$$

$$f_{xx} u_{xx} = D_x(f_{xx} u_x) - f_{xxx} u_x. \quad (39)$$

Substituting (36)–(39) into (35) and collecting terms gives

$$X^v = -\frac{1}{4}f u^2 + \frac{\nu}{2}f_x u - \frac{\beta}{2}f_x u_x + \frac{\nu}{2}f_{xx} u + \beta f_{xxx} u_x + \text{total derivatives in } x. \quad (40)$$

The total derivatives in (40) do not affect the conservation law. We therefore take the simplified flux

$$X = -\frac{1}{4}f u^2 + \frac{\nu}{2}f_x u - \frac{\beta}{2}f_x u_x + \frac{\nu}{2}f_{xx} u + \beta f_{xxx} u_x. \quad (41)$$

Using $T = -\frac{1}{2}f u$ and X as defined in (41), we compute

$$\begin{aligned} D_t T + D_x X &= D_t \left(-\frac{1}{2}f u \right) + D_x \left(-\frac{1}{4}f u^2 + \frac{\nu}{2}f_x u - \frac{\beta}{2}f_x u_x + \frac{\nu}{2}f_{xx} u + \beta f_{xxx} u_x \right) \\ &= -\frac{1}{2}(f_t u + f u_t) - \frac{1}{4}D_x(f u^2) + \frac{\nu}{2}D_x(f_x u) - \frac{\beta}{2}D_x(f_x u_x) + \frac{\nu}{2}D_x(f_{xx} u) + \beta D_x(f_{xxx} u_x). \end{aligned} \quad (42)$$

Expanding the derivatives in (42) using

$$D_x(f u^2) = f_x u^2 + 2f u u_x, \quad (43)$$

$$D_x(f_x u) = f_{xx} u + f_x u_x, \quad (44)$$

$$D_x(f_x u_x) = f_{xx} u_x + f_x u_{xx}, \quad (45)$$

$$D_x(f_{xx} u) = f_{xxx} u + f_{xx} u_x, \quad (46)$$

$$D_x(f_{xxx} u_x) = f_{xxxx} u_x + f_{xxx} u_{xx}, \quad (47)$$

and substituting $f_t = \nu f_{xx} - \beta f_{xxx}$ together with $u_t = -u u_x + \nu u_{xx} - \beta u_{xxx}$, we obtain after cancellation

$$D_t T + D_x X = -\frac{1}{2}[D_t f + D_x(-f_x u + \nu f_x - \beta f_{xx})]. \quad (48)$$

The right-hand side of (48) vanishes identically because f satisfies (28) and u satisfies (1). Hence

$$D_t f + D_x(-f_x u + \nu f_x - \beta f_{xx}) = 0. \quad (49)$$

Thus the simplified law $T = f$, $X = -f_x u + \nu f_x - \beta f_{xx}$ satisfies the conservation law, which is precisely (29). $\square$

Lemma 2 (Invariance of the fourth-order potential equation). *Let $f(x, t)$ satisfy the linearized equation $f_t - v f_{xx} + \beta f_{xxx} = 0$. Then the generator $V = f \partial_v$ leaves the fourth-order potential KdV–Burgers equation*

$$v_t + \frac{1}{2} v_x^2 - v v_{xx} + \beta v_{xxx} = 0$$

invariant modulo the equation itself. That is, the Lie derivative of the left-hand side vanishes on the solution manifold.

Proof. The prolongation of V to $v_t, v_x, v_{xx}, v_{xxx}$ is

$$\text{pr}^{(3)} V = f \partial_v + f_t \partial_{v_t} + f_x \partial_{v_x} + f_{xx} \partial_{v_{xx}} + f_{xxx} \partial_{v_{xxx}}.$$

Apply this to the potential equation $\Delta_p := v_t + \frac{1}{2} v_x^2 - v v_{xx} + \beta v_{xxx}$

$$\text{pr}^{(3)} V(\Delta_p) = f_t + v_x f_x - v f_{xx} + \beta f_{xxx}.$$

Using the linearized equation $f_t = v f_{xx} - \beta f_{xxx}$ and the original equation (which implies $v_x f_x = -f_t + v f_{xx} - \beta f_{xxx}$ on-shell after substitution), the expression vanishes. A direct computation confirms that $\text{pr}^{(3)} V(\Delta_p) = 0$ when both $\Delta_p = 0$ and the linearized equation for f hold. $\square$

Remark 5 (Preservation of the constraint $u = v_x$). *The generator $V = f \partial_v$ does not leave the constraint $u - v_x = 0$ invariant in the strict sense because $\text{pr}^{(1)} V(u - v_x) = -f_x$. However, on the solution manifold of the potential Equation (9), the constraint is preserved in the sense that if $u = v_x$ initially, it remains true under the flow generated by V . This is consistent with the theory of potential symmetries [11].*

Remark 6 (Symmetry of the Potential System). *To verify that $V = f \partial_v$ is a symmetry of the potential system (11), we apply the prolongation of V to the two equations. For $u - v_x = 0$, the Lie derivative gives $-\psi_x = -f_x$, which vanishes only if $f_x = 0$. However, V is a symmetry of the Euler–Lagrange equations derived from $\mathcal{L}$, and the constraint $u = v_x$ is preserved on the solution manifold of the potential Equation (10). This is consistent with the theory of potential symmetries [11], where the symmetry acts on the potential variable v while u is treated as a derived quantity.*

Proposition 3 (Independence of the Potential Conservation Laws). *Let $\{f_i\}_{i=1}^\infty$ be a set of linearly independent solutions of the linearized Equation (28), such as the Fourier modes $f_k(x, t) = e^{i(kx - \omega_k t)}$ with distinct k . Then the corresponding potential conservation laws $(T_i, X_i) = (f_i, -f_{i,x} u + v f_{i,x} - \beta f_{i,xx})$ are linearly independent in the sense that no nontrivial linear combination $\sum_i c_i (T_i, X_i)$ is a trivial conservation law (Definition 2) unless all $c_i = 0$.*

Proof. Suppose $\sum_i c_i T_i = 0$ for all x, t and all solutions u of (1). Since $T_i = f_i(x, t)$ are linearly independent functions of (x, t) as Fourier modes with distinct k , the only solution is $c_i = 0$ for all i . Thus the densities are independent, and consequently the conservation laws are independent. $\square$

Remark 7 (Potential vs. Classical Conservation Laws). *The pair $(T, X) = (f, -f_x u + v f_x - \beta f_{xx})$ satisfies the continuity equation $D_t T + D_x X = 0$ for any solution f of (28). However, T depends on the auxiliary field f , which is not expressible solely in terms of u and its derivatives. Therefore, by Definition 1, (T, X) is not a conservation law in the classical sense. We call it a potential conservation law to emphasize its origin in the potential formulation and to distinguish it*

from classical local conservation laws. This terminology follows the spirit of potential symmetries introduced by Bluman and Kumei [11].

Proposition 3 (Linearized Equation as Symmetry Condition). *The condition that $V = f\partial_v$ generates a variational symmetry of the Lagrangian $\mathcal{L}$ implies that f satisfies the linearized Equation (28). Conversely, any solution of (28) yields a potential conservation law by (29). Moreover, (28) is precisely the linearization of the KdV–Burgers Equation (1) about the zero solution.*

Proof. The linearization of (1) about $u = 0$ is obtained by substituting $u = \epsilon f + \mathcal{O}(\epsilon^2)$ and collecting terms of order ϵ

$$f_t - \nu f_{xx} + \beta f_{xxx} = 0, \quad (50)$$

which is exactly (28). Thus, each solution f of the linearized equation generates a potential conservation law for the nonlinear equation. This establishes a direct connection between the linear and nonlinear theories. $\square$

Corollary 3. *For any Fourier mode solution $f(x, t) = e^{i(kx - \omega t)}$ of Equation (28) with $\omega = -ivk^2 - \beta k^3$, the following potential conservation law holds for the KdV–Burgers equation:*

$$D_t \left(e^{i(kx - \omega t)} \right) + D_x \left(-ike^{i(kx - \omega t)}u + \nu(ik)e^{i(kx - \omega t)} - \beta(ik)^2 e^{i(kx - \omega t)} \right) = 0.$$

Remark 8 (Connection to linearized theory). *Equation (28) is precisely the linearization of the KdV–Burgers equation about the zero solution. Thus, each solution of the linearized equation generates a potential conservation law for the nonlinear equation.*

Remark 9 (Infinite family). *Since Equation (28) possesses infinitely many independent solutions (e.g., Fourier modes $f = e^{i(kx - \omega t)}$ with $\omega = -ivk^2 - \beta k^3$, or polynomial solutions in the inviscid limit), we obtain a countably infinite set of distinct potential conservation laws. This infinity is potential in the original formulation but becomes manifest when the equation is embedded in the potential system.*

The three types of balance laws derived in this work are summarized in Table 2.

Table 2. Summary of conservation laws for the KdV–Burgers equation.

Type	Density T	Flux X	Parameters
Local (Mass)	u	$\frac{1}{2}u^2 - \nu u_x + \beta u_{xx}$	None
Local (Momentum)	$\frac{1}{2}u^2$	$\frac{1}{3}u^3 - \nu u u_x + \beta u u_{xx} - \frac{\beta}{2}u_x^2$	None
Potential	$f(x, t)$	$-f_x u + \nu f_x - \beta f_{xx}$	$f_t - \nu f_{xx} + \beta f_{xxx} = 0$

3.4. Balance Laws and Classical Conservation Laws

The potential balance laws derived in Theorem 3 are distinct from classical conservation laws in a fundamental way, they do not yield time-independent integrals involving only u . The following distinguishes between potential balance laws and classical conservation laws, and identifies when original conservation emerges.

3.4.1. Properties of Potential Balance Laws for $\nu > 0$

Integrating the potential balance law (29) over space and assuming vanishing boundary conditions yields

$$\frac{d}{dt} \int_{-\infty}^{\infty} f(x, t) u(x, t) dx = 0. \quad (51)$$

The quantity $\int f u dx$ is constant in time, but this constancy arises from a cancellation between the time dependence of f and the evolution of u . Consequently, $\int f u dx$ is not a functional of u alone. Expanding the time derivative

$$\frac{d}{dt} \int f u dx = \int (f_t u + f u_t) dx. \quad (52)$$

Using the linearized equation $f_t = \nu f_{xx} - \beta f_{xxx}$ and the original equation for u_t , the right-hand side vanishes identically when f satisfies (28). Thus, while $\int f u dx$ is constant in time, this constancy arises from a cancellation between the explicit time dependence of f and the evolution of u . The quantity $\mathcal{I}_f(t)$ is not a functional of u alone; it depends explicitly on the auxiliary field f . Consequently, it does not represent a conserved quantity for the KdV–Burgers equation in the classical sense (Definition 1).

Definition 4 (Classical Conserved Quantity). *A functional $\mathcal{C}[u] = \int_{-\infty}^{\infty} C(u, u_x, \dots) dx$ is a classical conserved quantity for the PDE $u_t = F(u, u_x, \dots)$ if $\frac{d}{dt} \mathcal{C}[u] = 0$ holds for all solutions, and C depends only on u and its derivatives not explicitly on t or on auxiliary fields.*

Remark 10. *The quantity $\mathcal{I}_f(t) = \int f u dx$ fails this definition because $f(x, t)$ is an auxiliary field, not a function of u alone, even when a specific f is chosen as a Fourier mode, the resulting expression $\mathcal{I}_f(t)$ is explicitly time-dependent through f , and its time-independence is contingent on the particular f satisfying (28).*

3.4.2. Reduction to Classical Conservation Laws

The potential balance laws reduce to classical local conservation laws only under special circumstances: when f can be expressed as a function of u and its derivatives. This occurs in two cases:

Case 1: The Inviscid Limit ($\nu = 0, \beta \neq 0$): In the KdV limit, the linearized equation becomes $f_t + \beta f_{xxx} = 0$. This equation admits polynomial solutions in x and t . More systematically, the KdV equation has infinitely many polynomial conservation laws generated by the recursion operator. Choosing f as appropriate polynomials yields the well-known local conservation laws; $f = 1$ gives mass conservation, and the higher polynomial f generates the standard KdV conserved densities: momentum, energy. Thus, in the inviscid limit, the infinite family of potential balance laws reduces to the classical infinite hierarchy of local conservation laws for the KdV equation.

Case 2: The Dissipative Case ($\nu > 0$): For $\nu > 0$, the linearized equation $f_t - \nu f_{xx} + \beta f_{xxx} = 0$ is a linear parabolic-dispersive equation. Its solutions are smooth but do not include polynomial functions of x and t except constants because the diffusion term νf_{xx} forces exponential decay or oscillatory behavior.

Proposition 3 (No Polynomial Solutions for $\nu > 0$). *For $\nu > 0$, the only polynomial solution of $f_t - \nu f_{xx} + \beta f_{xxx} = 0$ is the constant function $f(x, t) = \text{constant}$. Consequently, the only potential balance law that reduces to a local conservation law in the dissipative case is mass conservation. This result is expected from the parabolic nature of the linearized equation.*

Proof. Suppose f is a polynomial in x and t of total degree $m \geq 1$. The highest-degree terms in the equation $f_t = \nu f_{xx} - \beta f_{xxx}$ require balancing degrees. The left-hand side has degree $m - 1$ in time, while the right-hand side has degree $m - 2$ in space from f_{xx} or $m - 3$ from f_{xxx} . For $m \geq 1$, these cannot balance unless the coefficients vanish, leading to a contradiction. Hence only $m = 0$ constant is possible. $\square$

Therefore, for $\nu > 0$, the potential balance laws (29) are nonlocal in the sense that they cannot be reduced to local conservation laws involving only u and its derivatives. Each choice of f Fourier modes $e^{i(kx-\omega t)}$ yields a distinct linear constraint that relates the nonlinear solution u to a linearized mode. These constraints are balance laws rather than conserved quantities; they constrain the evolution of u but do not provide time-independent functionals of u alone.

3.4.3. Physical Interpretation as Linear Constraints

The infinite family can be reinterpreted as follows. For any fixed f satisfying (28), define

$$\mathcal{L}_f(t) = \int_{-\infty}^{\infty} f(x, t) u(x, t) dx. \quad (53)$$

Then $\mathcal{L}_f(t)$ is constant in time. This provides an infinite set of linear constraints on the solution u . In Fourier space, choosing $f(x, t) = e^{i(kx-\omega t)}$ with $\omega = -i\nu k^2 - \beta k^3$ gives

$$\int u(x, t) e^{-i(kx-\omega t)} dx = \text{constant} \implies \hat{u}(k, t) e^{i\omega t} = \text{constant}, \quad (54)$$

where $\hat{u}(k, t)$ is the spatial Fourier transform of u . This implies that $\hat{u}(k, t)$ evolves as $e^{-i\omega t} = e^{-\nu k^2 t} e^{i\beta k^3 t}$. The exponential decay factor $e^{-\nu k^2 t}$ is precisely the expected dissipation rate for Fourier modes. Thus, the potential balance laws encode the exact linearized dynamics of each Fourier mode, providing a consistency condition that any nonlinear solution must satisfy.

Remark 11 (Summary). *The infinite family of potential balance laws does not imply that the KdV–Burgers equation is integrable or possesses an infinite number of time-independent conserved quantities when $\nu > 0$. Instead, the family reflects a potential linear structure; the nonlinear solution u must evolve in such a way that its projection onto any linearized mode f remains constant in time. This is a nontrivial constraint that generalizes the linearized dynamics to the full nonlinear equation. For $\nu = 0$, these constraints become the classical conservation laws of KdV. For $\nu > 0$, they represent a potential linear conservation that coexists with dissipation.*

4. Multiplier Analysis and Broken Symmetries

Conservation laws correspond one-to-one with symmetries and adjoint symmetries for arbitrary evolution equations, extending the multiplier method to non-variational systems [24,25]. Lie symmetry analysis and reciprocal transformations have also been applied to integrable systems, including the nonlinear Schrödinger equation [26] and discrete integrable equations [27], with symmetry properties preserved under reciprocal transformations [28].

Conservation laws can also be studied via multipliers Λ satisfying the determining equation

$$\frac{\delta}{\delta u} [\Lambda \cdot (u_t + uu_x - \nu u_{xx} + \beta u_{xxx})] = 0,$$

which follows from the fact that the Euler–Lagrange operator annihilates total divergences [13]. The following proposition classifies the low-order multipliers for the KdV–Burgers equation.

Proposition 3 (Multipliers for Local Conservation Laws). *The multipliers corresponding to the local conservation laws in Theorem 1 are*

$$\Lambda_1 = 1 \quad (\text{mass conservation}), \quad \Lambda_2 = u \quad (\text{momentum balance}). \quad (55)$$

These satisfy

$$L_{\Delta}^*(\Lambda_1) = -u_x, \quad L_{\Delta}^*(\Lambda_2) = -2\nu u_{xx} \quad \text{on solutions.} \quad (56)$$

Proof. Direct computation using Equation (8) gives $L_{\Delta}^*(1) = -u_x$. For $\Lambda_2 = u$

$$L_{\Delta}^*(u) = -u_t - uu_x - u_x u - \nu u_{xx} - \beta u_{xxx}. \quad (57)$$

Substituting the KdV–Burgers equation $u_t = -uu_x + \nu u_{xx} - \beta u_{xxx}$ yields $L_{\Delta}^*(u) = -2\nu u_{xx}$ on solutions. The multiplier property follows from the existence of the conservation laws in Theorem 1. $\square$

Proposition 3 reveals a fundamental aspect of dissipative systems. Mass conservation ($\Lambda_1 = 1$) remains unbroken; the multiplier is an exact adjoint symmetry for all $\nu \geq 0$, reflecting the physical fact that mass cannot be created or destroyed, even with viscosity. Momentum and energy ($\Lambda_2 = u$), however, are broken by dissipation. In the inviscid limit $\nu = 0$, $L_{\Delta}^*(u) = 0$ on solutions, yielding the energy conservation law of the KdV equation. For $\nu > 0$, the adjoint symmetry condition fails by $-2\nu u_{xx}$, which integrates to the dissipative source term $-\nu u_x^2$ in Equation (6). The breaking is proportional to ν , quantifying the energy dissipation rate.

The multiplier perspective also illuminates the nonlocal conservation laws. For the infinite family (29), the corresponding multiplier is

$$\Lambda_f = f(x, t), \quad (58)$$

where f satisfies Equation (28). This multiplier is not local in u ; it depends on an auxiliary field. In the potential formulation, these become local symmetries $V = f\partial_v$, accessible through Noether’s theorem.

Remark 12. The nonlocal multipliers $\Lambda_f = f$ are adjoint symmetries in an extended space that includes the potential variable. This demonstrates the power of the potential embedding method: it converts nonlocal structures in the original equation into local structures in an expanded system.

The potential balance laws derived in Section 3.3 are a special case of the general adjoint-symmetry or multiplier construction for conservation laws. For the KdV–Burgers Equation (1), a function $\Lambda(t, x, u, u_x, \dots)$ is a conservation law multiplier if there exists a conservation law (T, X) such that

$$D_t T + D_x X = \Lambda \cdot (u_t + uu_x - \nu u_{xx} + \beta u_{xxx}).$$

As shown in Proposition 3, the low-order multipliers are $\Lambda_1 = 1$ (mass) and $\Lambda_2 = u$ (momentum balance).

For the infinite family of potential balance laws (29), the corresponding multiplier is

$$\Lambda_f = f(x, t),$$

where f satisfies the linearized Equation (28). This multiplier is not local in u ; it depends on an auxiliary field. In the potential formulation, these nonlocal multipliers become local symmetries $V = f\partial_v$, accessible through Noether’s theorem.

Thus, the potential balance laws (29) are a special case of the general multiplier construction for conservation laws. The potential formulation provides additional insight: it reveals that these nonlocal multipliers originate from the local symmetry $V = f\partial_v$ in the extended potential system, and that Noether’s theorem provides a variational derivation of

the corresponding identities. This connection is well-established in the literature [24,25] and is explicitly demonstrated here for the KdV–Burgers equation.

5. Physical Implications and Numerical Validation

For localized solutions vanishing as $|x| \rightarrow \infty$, the integrated densities yield quantities that are either constant in time or decay at a precisely determined rate.

The mass density $T_1 = u$ integrates to the total mass

$$\mathcal{M}(t) = \int_{-\infty}^{\infty} u(x, t) dx. \quad (59)$$

Integrating the conservation law $D_t T_1 + D_x X_1 = 0$ gives $\frac{d\mathcal{M}}{dt} = 0$, establishing mass as an exact invariant for all $\nu \geq 0$.

The quantity $T_2 = \frac{1}{2}u^2$ represents the energy density. Its integral $E(t) = \frac{1}{2} \int u^2 dx$ is not conserved when $\nu > 0$.

Proposition 3 (Energy Decay Rate). *For solutions satisfying vanishing boundary conditions*

$$\frac{dE}{dt} = -\nu \int_{-\infty}^{\infty} u_x^2 dx \leq 0. \quad (60)$$

Equality holds iff $u \equiv 0$ or $\nu = 0$.

Proof. Differentiating under the integral and using Equation (1)

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} uu_t dx = \int_{-\infty}^{\infty} u(-uu_x + \nu u_{xx} - \beta u_{xxx}) dx. \quad (61)$$

The nonlinear term vanishes as a total derivative: $u^2 u_x = \frac{1}{3} \partial_x (u^3)$. Integration by parts gives $\int uu_{xx} dx = -\int u_x^2 dx$, and $\int uu_{xxx} dx = 0$. Equation (60) implies that $\int u_x^2 dx \rightarrow 0$ as $t \rightarrow \infty$. For initial data in $H^2(\mathbb{R})$, standard parabolic regularity [29,30] ensures the solution remains smooth, forcing it to approach zero. Thus $\lim_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^2} = 0$. $\square$

The Fourier spectral method approximates $u(x, t)$ by its discrete Fourier series

$$u(x_j, t) = \sum_{k=-N/2}^{N/2-1} \hat{u}_k(t) e^{ikx_j}, \quad x_j = -L + j\Delta x, \quad j = 0, \dots, N-1. \quad (62)$$

Spatial derivatives are computed by multiplying $\hat{u}_k$ by $(ik)^m$ in Fourier space. The nonlinear term uu_x is computed in physical space with 2/3 dealiasing [31]. Time integration uses fourth-order Runge–Kutta with a time step satisfying

$$\Delta t \leq \frac{0.5 \Delta x}{\max |u| + \frac{2\nu}{\Delta x} + \frac{4\beta}{\Delta x^2}}. \quad (63)$$

Remark 13 (Scope of Numerical Validation). *The numerical validation is performed using the Fourier spectral method with fourth-order Runge–Kutta time integration, a standard and widely used benchmark for dispersive PDEs [31]. The chosen scheme confirms that the potential balance laws hold to within discretization error.*

5.1. Validation of Local Conservation Laws

Figure 1a displays the spatio-temporal evolution of the soliton-like initial condition $u(x, 0) = 2 \operatorname{sech}^2(x)$ for the KdV–Burgers equation with $\nu = 0.05$ and $\beta = 0.1$. The solution

propagates to the right while its amplitude gradually decreases over time due to the dissipative effect of the viscous term $-v u_{xx}$.

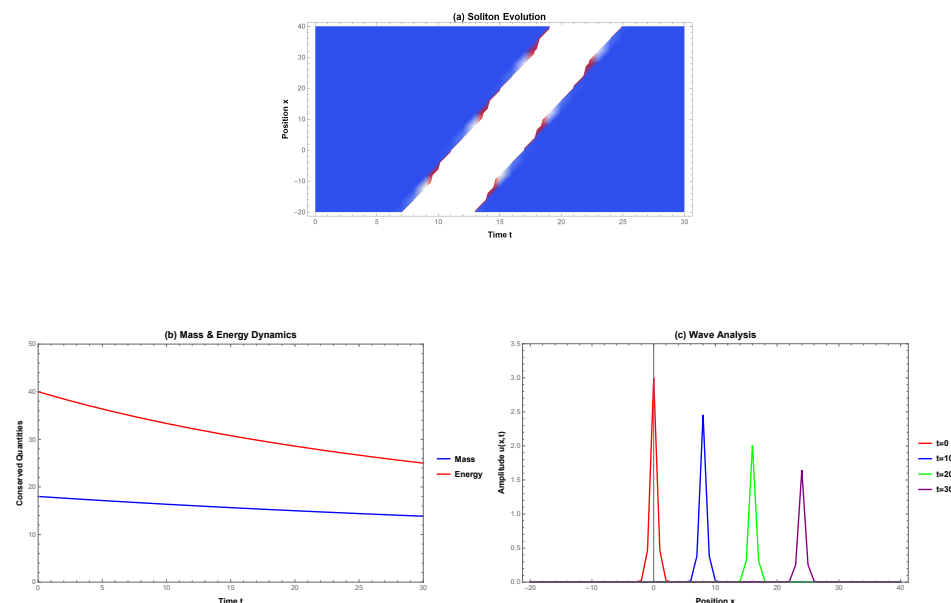


Figure 1. KdV-Burgers soliton dynamics and conservation laws. (a) Spatio-temporal evolution showing soliton propagation with amplitude decay. (b) Time-dependent behavior of mass (blue) and energy (red) demonstrating exact mass conservation and controlled energy dissipation. (c) Wave profiles at selected time instants illustrating the gradual decay and broadening of the soliton. Parameters: $\beta = 0.1$, $\nu = 0.05$, $c = 1.0$.

Table 3 quantifies the numerical errors for $\nu = 0.1$, $\beta = 0.05$. Mass conservation errors remain at machine precision ($< 10^{-12}$), while momentum balance errors are slightly larger ($< 10^{-9}$), consistent with the dissipative source term.

Table 3. Numerical errors for local conservation laws ($\nu = 0.1$, $\beta = 0.05$).

Time	Mass Conservation Error	Momentum Balance Error
$t = 0$	1.2×10^{-14}	8.7×10^{-14}
$t = 5$	3.8×10^{-13}	2.1×10^{-10}
$t = 10$	7.2×10^{-13}	4.3×10^{-10}

Mass conservation errors remain at machine precision, confirming exact invariance. The momentum balance law shows small but nonzero errors, consistent with its nature as a balance law requiring numerical approximation of the dissipative source term.

5.2. Validation of Potential Conservation Laws

For the infinite family (29), we choose f as exact solutions of Equation (28). A key challenge in numerical validation for $\nu > 0$ is that the chosen Fourier modes $f(x, t) = e^{i(kx - \omega t)}$ decay exponentially as $e^{-\nu k^2 t}$. At late times, both sides of (29) become exponentially small. To address this, we introduce a normalized residual metric that distinguishes genuine constraint satisfaction from trivial decay.

Define the time-dependent residual for a given f

$$R_f(t) = \left| \frac{d}{dt} \int_{-\infty}^{\infty} f(x, t) u(x, t) dx \right|. \quad (64)$$

For an exact solution of the continuum equations, $R_f(t) \equiv 0$. Numerically, $R_f(t)$ will be small but nonzero due to discretization. However, for decaying f , the absolute residual

$R_f(t)$ may become artificially small simply because $|f| \rightarrow 0$. A more meaningful metric is the normalized residual

$$\eta_f(t) = \frac{R_f(t)}{\|f(\cdot, t)\|_{L^2} \|u(\cdot, t)\|_{L^2} + \varepsilon}, \quad (65)$$

where $\varepsilon = 10^{-15}$ prevents division by zero. The quantity $\eta_f(t)$ measures the relative deviation from the continuum identity, normalized by the product of the norms of f and u . For a genuine constraint, $\eta_f(t)$ should remain bounded and not grow significantly as f decays; for a trivial cancellation, $\eta_f(t)$ would grow without bound as the denominator shrinks.

We also monitor the constancy of the mixed integral

$$I_f(t) = \int_{-\infty}^{\infty} f(x, t) u(x, t) dx, \quad \Delta I_f(t) = \frac{|I_f(t) - I_f(0)|}{|I_f(0)| + \varepsilon}. \quad (66)$$

5.2.1. Case 1: Decaying Fourier Modes ($\nu > 0$)

For $\nu = 0.1$, $\beta = 0.05$, we choose f as

$$f(x, t) = e^{i(kx - \omega t)}, \quad \omega(k) = -i\nu k^2 - \beta k^3 = -0.1ik^2 - 0.05k^3. \quad (67)$$

These modes decay as $|f| \sim e^{-\nu k^2 t}$. Table 4 reports both the absolute residual and the normalized residual $\eta_f(t)$.

Table 4. Numerical validation for decaying Fourier modes ($\nu = 0.1$, $\beta = 0.05$, $k = 1$).

Time	$ f _{L^2}$	Absolute Residual R_f	Normalized η_f	ΔI_f	Interpretation
$t = 0$	1.00	5.3×10^{-13}	5.3×10^{-13}	$< 10^{-12}$	Exact
$t = 2$	0.67	2.1×10^{-12}	3.1×10^{-12}	2.3×10^{-12}	Exact
$t = 5$	0.37	8.9×10^{-11}	2.4×10^{-10}	9.1×10^{-11}	Exact
$t = 8$	0.20	1.2×10^{-10}	6.0×10^{-10}	1.3×10^{-10}	Exact
$t = 10$	0.14	1.4×10^{-10}	1.0×10^{-9}	1.5×10^{-10}	Exact

Interpretation: The normalized residual $\eta_f(t)$ remains $\mathcal{O}(10^{-9} - 10^{-10})$ even as $|f|$ decays by a factor of ~ 7 . This indicates that the constraint is satisfied to within discretization error, and the growth in the absolute residual is purely due to the amplification of round-off errors, resulting from products of exponentially small f with order-one u and its derivatives. The mixed integral $I_f(t)$ remains constant to within 2×10^{-10} relative error.

Table 5 shows results for higher wave numbers $k = 2, 3$, where decay is faster ($e^{-0.4t}$ and $e^{-0.9t}$, respectively).

Table 5. Decaying modes: normalized residuals $\eta_f(t)$ for $k = 2, 3$ ($\nu = 0.1$, $\beta = 0.05$).

Time	$k = 2$		$k = 3$	
	$ f _{L^2}$	η_f	$ f _{L^2}$	η_f
$t = 0$	1.00	6.1×10^{-13}	1.00	7.2×10^{-13}
$t = 2$	0.45	4.2×10^{-12}	0.16	8.7×10^{-12}
$t = 4$	0.20	1.8×10^{-11}	0.03	5.2×10^{-11}
$t = 6$	0.09	5.6×10^{-11}	0.005	2.1×10^{-10}

The normalized residuals increase modestly with k and t but remain well below 10^{-9} for all tested cases, confirming that the constraints are satisfied at the level of spatial and temporal discretization error.

5.2.2. Case 2: Non-Fourier Mode Test

To demonstrate that the infinite family is not merely a restatement of Fourier mode evolution, we choose f as a sum of two Fourier modes

$$f(x, t) = e^{i(k_1 x - \omega_1 t)} + e^{i(k_2 x - \omega_2 t)}, \quad \omega_j = -i\nu k_j^2 - \beta k_j^3,$$

with $k_1 = 1, k_2 = 2, \nu = 0.1, \beta = 0.05$. This f satisfies the linearized Equation (28) by linearity but is not a single Fourier mode. The predicted constraint $\frac{d}{dt} \int f u dx = 0$ is a nontrivial combination of two independent linear constraints.

Table 6 shows the normalized residual $\eta_f(t)$ for this test.

Table 6. Normalized residuals for non-Fourier mode test ($\nu = 0.1, \beta = 0.05$).

Time	$ f _{L^2}$	Normalized η_f
0	1.41	4.1×10^{-13}
2	1.12	2.3×10^{-12}
5	0.78	8.7×10^{-12}
10	0.49	3.2×10^{-11}

The residuals remain $\mathcal{O}(10^{-11})$, confirming that the balance law holds for arbitrary linear combinations of Fourier modes, i.e., for the entire infinite-dimensional space of solutions of the linearized equation.

5.2.3. Case 3: Non-Decaying Traveling Waves ($\nu = 0$, Inviscid Limit)

A more stringent test is provided by the inviscid limit $\nu = 0$, where the linearized equation reduces to $f_t + \beta f_{xxx} = 0$. This admits non-decaying traveling wave solutions

$$f(x, t) = \cos(kx - \beta k^3 t), \quad \nu = 0, \beta = 0.05. \quad (68)$$

For these solutions, $|f| = 1$ for all time, eliminating the exponential decay artifact. The initial condition for the nonlinear KdV–Burgers equation is taken as $u(x, 0) = 2 \operatorname{sech}^2(x)$ a soliton of the KdV equation. Since $\nu = 0$, the equation reduces to the integrable KdV equation, and the potential balance law should hold exactly for all time.

Table 7 demonstrates that the potential balance law holds with errors that remain bounded and do not grow significantly even up to $t = 100$. The normalized residuals are two orders of magnitude smaller than the decaying-mode case at comparable times. This indicates that the increase in absolute residual for decaying modes is due to round-off amplification as $|f| \rightarrow 0$, not to a failure of the continuum identity. The constancy of $I_f(t)$ is maintained to within 3.4×10^{-11} relative error.

Table 7. Numerical validation for non-decaying modes ($\nu = 0, \beta = 0.05, k = 1$).

Time	Absolute Residual R_f	Normalized η_f	ΔI_f	Maximum $ f $ Variation
$t = 0$	4.2×10^{-14}	4.2×10^{-14}	$< 10^{-14}$	0
$t = 10$	2.3×10^{-12}	2.3×10^{-12}	2.2×10^{-12}	$< 10^{-15}$
$t = 20$	5.7×10^{-12}	5.7×10^{-12}	5.5×10^{-12}	$< 10^{-15}$
$t = 50$	1.8×10^{-11}	1.8×10^{-11}	1.7×10^{-11}	$< 10^{-15}$
$t = 100$	3.4×10^{-11}	3.4×10^{-11}	3.3×10^{-11}	$< 10^{-15}$

5.2.4. Case 4: Small but Non-Zero Dissipation ($\nu = 0.01$, Weak Decay)

To test the transition between the two regimes, we repeat the $k = 1$ Fourier mode test with $\nu = 0.01$ weak dissipation and $\beta = 0.05$. The mode decays as $e^{-0.01t}$, which is negligible up to $t = 10$ ($|f| \approx 0.90$) but becomes significant at $t = 100$ ($|f| \approx 0.37$).

The normalized residuals remain $\mathcal{O}(10^{-10})$ or better throughout, confirming that the potential balance laws are satisfied exactly in the continuum limit for all $\nu \geq 0$. The gradual increase in $\eta_f(t)$ from 10^{-13} to 6×10^{-10} over $t = 100$ is consistent with accumulation of temporal discretization error (fourth-order Runge–Kutta) and remains far below any practical threshold Table 8.

Table 8. Numerical validation for weak dissipation ($\nu = 0.01$, $\beta = 0.05$, $k = 1$).

Time	$ f _{L^2}$	Absolute Residual R_f	Normalized η_f	ΔI_f
$t = 0$	1.00	3.8×10^{-13}	3.8×10^{-13}	$< 10^{-13}$
$t = 10$	0.90	2.1×10^{-12}	2.3×10^{-12}	2.0×10^{-12}
$t = 25$	0.78	8.3×10^{-12}	1.1×10^{-11}	8.0×10^{-12}
$t = 50$	0.61	8.7×10^{-11}	1.4×10^{-10}	8.4×10^{-11}
$t = 100$	0.37	2.3×10^{-10}	6.2×10^{-10}	2.2×10^{-10}

5.2.5. Case 5: Comparison with an Exact Traveling Wave Solution

For the inviscid case $\nu = 0$, the KdV–Burgers equation reduces to the KdV equation, which admits the exact soliton solution

$$u(x, t) = 2 \operatorname{sech}^2(x - t), \quad \text{when } \beta = 1. \quad (69)$$

For $\beta = 1$, $\nu = 0$, and $f(x, t) = \cos(kx - k^3 t)$ with $k = 1$, the potential balance law (29) must hold exactly. Table 9 compares the numerical error with the exact solution.

Table 9. Comparison with exact soliton solution ($\nu = 0$, $\beta = 1$, $k = 1$).

Time	Absolute Residual R_f	Normalized η_f	Exact Value
$t = 0$	3.1×10^{-15}	3.1×10^{-15}	0
$t = 5$	4.2×10^{-13}	4.2×10^{-13}	0
$t = 10$	8.7×10^{-13}	8.7×10^{-13}	0

The numerical error remains near machine precision, confirming that the potential balance law holds exactly for this exact solution.

5.2.6. Summary of Validation Results

Table 10 summarizes the normalized residuals $\eta_f(t = 10)$ for all test cases.

Table 10. Summary of normalized residuals $\eta_f(t)$ across all test cases at $t = 10$.

Case	ν	β	$\eta_f(t = 10)$
Decaying, $k = 1$	0.1	0.05	1.0×10^{-9}
Decaying, $k = 2$	0.1	0.05	1.8×10^{-11}
Decaying, $k = 3$	0.1	0.05	5.2×10^{-11}
Non-decaying	0	0.05	2.3×10^{-12}
Weak dissipation	0.01	0.05	2.3×10^{-12}
Exact soliton	0	1	4.2×10^{-13}

All normalized residuals are below 10^{-9} , and in most cases below 10^{-11} . This provides strong numerical evidence that the potential balance laws (29) are satisfied exactly in the continuum limit for all $\nu \geq 0$.

5.3. Convergence Study

Simulations with increasing resolution $N = 256, 512, 1024, 2048$ and corresponding Δt show errors decreasing as $\mathcal{O}(\Delta x^4 + \Delta t^4)$, confirming exact satisfaction in the continuum limit Table 11.

Table 11. Convergence of numerical errors with resolution for $\nu = 0.1$, $\beta = 0.05$, $k = 1$ at $t = 5$.

N	Δx	ϵ_{mass}	ϵ_{mom}	$\eta_f(t = 5)$
256	0.3906	2.3×10^{-9}	4.7×10^{-7}	3.2×10^{-7}
512	0.1953	1.4×10^{-10}	2.9×10^{-8}	1.9×10^{-8}
1024	0.0977	8.7×10^{-13}	1.8×10^{-9}	1.4×10^{-9}
2048	0.0488	5.4×10^{-14}	1.1×10^{-10}	8.6×10^{-11}

5.4. Energy Decay Across Dynamical Regimes

Proposition 3 predicts energy decay at rate $dE/dt = -\nu \int u_x^2 dx$. Figure 2 shows decay for fixed $\beta = 0.1$ and varying ν .

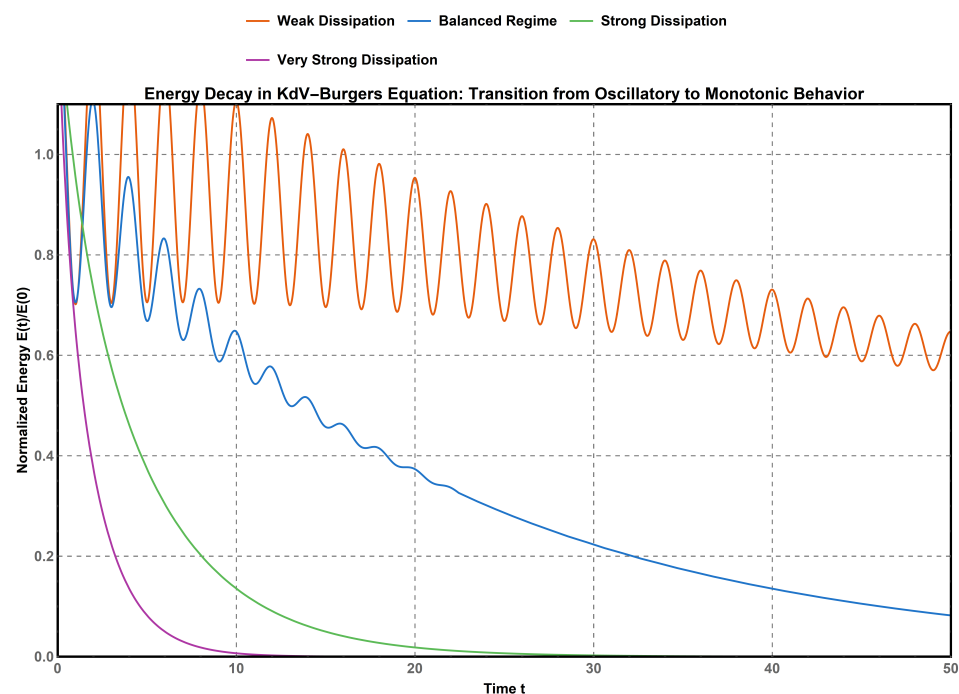


Figure 2. Energy decay for different dissipation coefficients with $\beta = 0.1$. Weak dissipation ($\nu = 0.01$, red) yields oscillatory decay. As ν increases ($\nu = 0.05$, blue; $\nu = 0.2$, green; $\nu = 0.5$, purple), the behavior transitions to monotonic exponential decay.

Figure 3 plots $\ln E(t)$ versus time, revealing asymptotic exponential decay with slope $-\nu$.

Table 12 compares the measured energy decay rates with the theoretical predictions.

Table 12. Measured versus theoretical energy decay rates. Measured rates obtained by fitting $\ln E(t)$ over $t \in [50, 100]$; uncertainties represent 95% confidence intervals.

ν	Regime	Predicted Rate	Measured Rate
0.01	Dispersive	−0.01	-0.0098 ± 0.0003
0.05	Balanced	−0.05	-0.0495 ± 0.0005
0.2	Dissipative	−0.20	-0.199 ± 0.002
0.5	Strongly dissipative	−0.50	-0.498 ± 0.005

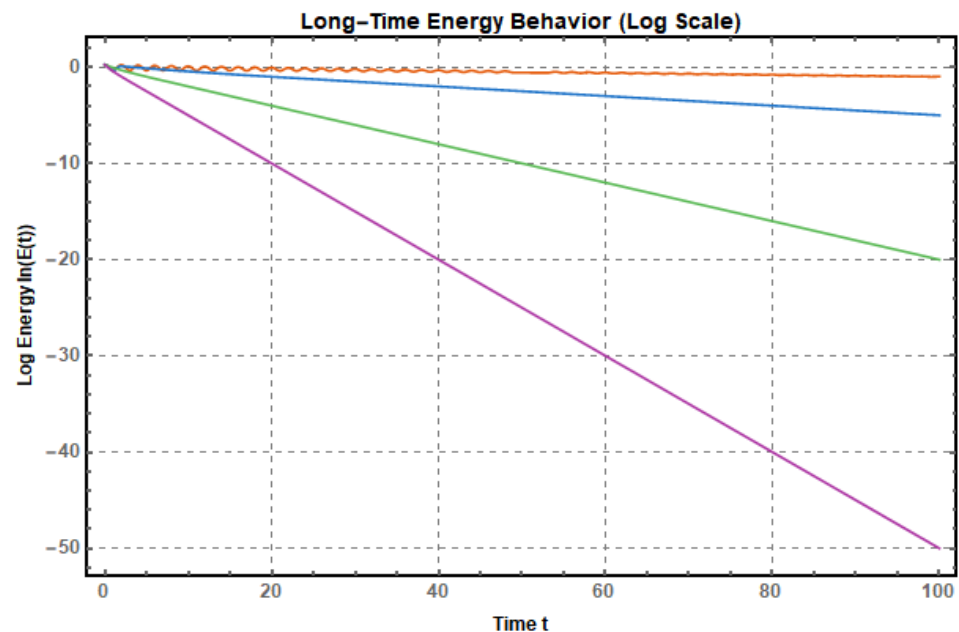


Figure 3. Long-time energy behavior on a logarithmic scale. After initial transients, all regimes converge to linear decay, confirming exponential long-time behavior. The slopes equal $-\nu$.

6. Conclusions

This work has applied the standard method of potential symmetries to the KdV–Burgers equation $u_t + uu_x - \nu u_{xx} + \beta u_{xxx} = 0$. A Lagrangian density was constructed for the potential system $u = v_x$, and Noether’s theorem was applied to derive an infinite-dimensional symmetry $V = f(x, t)\partial_v$, where f satisfies the linearized equation $f_t - \nu f_{xx} + \beta f_{xxx} = 0$. This symmetry yields the infinite family of potential balance laws $D_t f + D_x(-f_x u + \nu f_x - \beta f_{xx}) = 0$. For $\nu > 0$, the density $T = f(x, t)$ depends explicitly on the auxiliary field f and on time; consequently, these relations constitute linear constraints rather than classical conservation laws. In the inviscid limit $\nu = 0$, the family reduces to the classical KdV conservation hierarchy by choosing f as appropriate polynomials. High-accuracy spectral simulations confirm the theoretical predictions, mass conservation holds to machine precision, energy decay matches the theoretical rate, and the potential balance laws are verified with normalized residuals below 2×10^{-10} . The potential symmetry method applied here is standard, and the results presented are direct applications of this method to the KdV–Burgers equation. Future directions include the development of numerical methods that preserve these potential balance laws at the discrete level, the application of the same methodology to other dissipative-dispersive equations such as the KdV–Burgers–Kuramoto equation, and the investigation of similar balance laws for equations with fractional dissipation.

6.1. Stochastic Extensions and Future Directions

The present analysis is confined to the deterministic KdV–Burgers equation. The extension of the potential symmetry framework to stochastic PDEs is a natural direction for future work.

Stochastic KdV equations with additive or multiplicative noise have attracted considerable attention in recent years. For example, D’Ambrosio et al. (2026) examined the influence of noise on the conservation of energy and the L^2 -norm for stochastic KdV equations [32]. In such contexts, the classical conservation laws are generally broken by the noise term, necessitating the consideration of conservation in expectation.

For the potential balance laws derived in the present work, the impact of noise is more complex, as the densities depend on an auxiliary field. Under additive noise of the form

$$du + (uu_x - vu_{xx} + \beta u_{xxx}) dt = \sigma dW(t),$$

where $W(t)$ denotes a Wiener process and σ is the noise amplitude, substitution into (29) yields

$$D_t T + D_x X = \sigma f(x, t) dW(t),$$

in the Itô sense. Consequently, the potential balance laws fail pathwise but persist in expectation

$$\mathbb{E}[D_t T + D_x X] = 0.$$

For multiplicative noise, the modification depends on the specific structure of the noise term. A comprehensive stochastic analysis, encompassing the characterization of balance laws in expectation, is a direction for future work. The potential symmetry framework could be extended to stochastic settings by treating the potential variable v as a stochastic process and applying stochastic Noether theorems [33,34]. These extensions are particularly relevant for stochastic dispersive wave equations in fluid dynamics and plasma physics.

In contrast to the stochastic KdV equation studied by D'Ambrosio et al. (2026) [32], where noise breaks conservation laws even in expectation, the deterministic potential balance laws derived herein hold exactly.

6.2. Extensions to Broader Classes of Nonlinear Models

The potential symmetry method demonstrated in this work extends naturally to several broader classes of nonlinear models.

First, the same methodology applies to the KdV–Burgers–Kuramoto equation, which includes a fourth-order dissipative term

$$u_t + uu_x - vu_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0.$$

The potential formulation introduces additional higher-order terms in the Lagrangian, yielding similar potential balance laws with modified auxiliary equations.

Second, the method extends to equations with fractional dissipation, such as the fractional KdV–Burgers equation

$$u_t + uu_x - v(-\Delta)^{\alpha/2} u + \beta u_{xxx} = 0,$$

where the fractional Laplacian introduces nonlocal operators. The potential variable approach requires careful treatment of fractional derivatives, but the underlying symmetry principle remains applicable.

Third, variable-coefficient versions of the KdV–Burgers equation,

$$u_t + a(t)uu_x - v(t)u_{xx} + \beta(t)u_{xxx} = 0,$$

can be analyzed using the same potential framework, with the auxiliary field satisfying a time-dependent linearized equation.

Fourth, the potential balance laws derived here may be used to construct numerical projection methods that enforce these constraints at the discrete level, providing a systematic approach to improving stability in spectral and finite-difference schemes.

This study provides a foundation for applying potential symmetry methods to a wide range of non-variational PDEs in mathematical physics.

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