

0132-Avoiding Ascent Sequences

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Abstract

An ascent sequence of length n is a sequence $a_1 a_2 \cdots a_n$ of non-negative integers satisfying $a_1 = 0$ and, for $1 < i \leq n$, $a_i \leq \text{asc}(a_1 a_2 \cdots a_{i-1}) + 1$, where $\text{asc}(a_1 a_2 \cdots a_k)$ denotes the number of ascents in the sequence $a_1 a_2 \cdots a_k$. In this paper, we investigate ascent sequences avoiding the pattern 0132. By refining the structure of such sequences, we derive a generating tree and recurrence relations that characterize their enumeration. These recurrences are then translated into functional equations for the associated generating functions, which are solved using generating-function techniques. As a consequence, we obtain an explicit formula for the generating function that enumerates 0132-avoiding ascent sequences according to length.

Keywords: ascent sequence; pattern avoidance; 0132-avoiding ascent sequence; generating tree; generating function; kernel method

1. Introduction

An *ascent* in a sequence $a_1 a_2 \cdots a_k$ is a position $j \geq 1$ such that $a_j < a_{j+1}$. An *ascent sequence* $a_1 a_2 \cdots a_n$ is a sequence of non-negative integers satisfying $a_1 = 0$ and, for $1 < i \leq n$,

$$a_i \leq \text{asc}(a_1 a_2 \cdots a_{i-1}) + 1,$$

where $\text{asc}(a_1 a_2 \cdots a_k)$ denotes the number of ascents in the prefix $a_1 a_2 \cdots a_k$. Ascent sequences were introduced by Bousquet-Mélou, Claesson, Dukes, and Kitaev [1] in connection with $(2+2)$ -free posets, and their generating function was first obtained in that work. Since then, ascent sequences have become a central object in enumerative combinatorics due to their rich structure and connections with pattern avoidance, posets, and other Catalan families. We refer the reader to [2,3] for further background and developments.

A *pattern* is a word in $\{0, 1, \dots, \ell\}^m$ containing each letter $0, 1, \dots, \ell$ for some $\ell \geq 0$ and $m \geq 1$. Let $a = a_1 a_2 \cdots a_n$ be an ascent sequence and let $\tau = \tau_1 \tau_2 \cdots \tau_m$ be a pattern. We say that a *contains* τ if it has a subsequence $a_{f(1)}, a_{f(2)}, \dots, a_{f(m)}$ with $1 \leq f(1) < \cdots < f(m) \leq n$ that is order-isomorphic to τ , meaning that all pairwise comparisons between entries are preserved. Otherwise, a is said to *avoid* τ .

Pattern avoidance in ascent sequences has been an exceptionally active area of research over the past decade. The systematic study of short patterns was initiated by Duncan and Steingrímsson [4], who classified and conjectured avoidance formulas for several patterns of lengths 3 and 4, establishing deep connections to Catalan families and classical permutation avoidance. Many of these foundational conjectures were subsequently resolved using a variety of bijective and analytic techniques; for instance, Mansour and Shattuck [5] established the exact counts for patterns like 0012, 0021, and 1012 (see Table 1). Concurrently, Liu, Kitaev, and Zhang [6] successfully enumerated classes avoiding specific subsets of length-4 patterns, such as 0101 and 0102, by introducing specialized refined statistics.



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Beyond classical avoidance, the literature has expanded in several distinct directions to map the structural landscape of these sequences. For example, Fu [7] and Yan [8,9] investigated the avoidance of consecutive patterns and barred patterns, revealing close ties to inversion sequences and restricted permutations. Meanwhile, algorithmic and structural frameworks—such as the use of generating trees and the definition of new Wilf-equivalence classes—have been significantly advanced by works such as Callan and Mansour [10], Boliac et al. [11–13], and Baxter and Pudwell [14].

Despite this rich body of work, a comprehensive classification for all patterns of length four remains incomplete. Indeed, determining avoidance formulas for patterns of length four is widely recognized as an exceptionally difficult problem across classical combinatorial structures, including permutations, involutions, and k -ary words (see, e.g., Bóna [15] and Kitaev [3]). For instance, while Regev [16] successfully determined the asymptotic behavior for k -ary words avoiding 0123 via representation theory tools, an explicit formula for the generating function of permutations avoiding the pattern 1324 (and its Wilf-equivalent classes) remains one of the most famous open problems in the discipline.

The pattern 0132 in the context of ascent sequences represents a parallel, notorious gap. Unlike patterns that yield conventional Catalan or binomial distributions, 0132-avoiding sequences generate a highly complex tree structure where the placement of new elements depends non-trivially on multiple tracking statistics of the prefix. Consequently, its explicit enumeration has remained open, and its generating function cannot be derived from any previously known framework.

The main contribution of this paper is to bridge this gap by providing the first complete enumerative and structural analysis of 0132-avoiding ascent sequences. Specifically, we achieve the following:

- **Exact Enumeration:** We establish that the generating function for 0132-avoiding ascent sequences is algebraic of degree three, satisfying a definitive cubic equation (Theorem 1). This provides a sharp contrast to simpler length-4 patterns that typically yield quadratic generating functions.
- **Methodological Framework:** We introduce a refined generating tree framework utilizing multiple catalytic variables to capture the subtle interactions within the prefix ascents. This provides a systematic workflow for solving the resulting complex systems of functional equations, which may be applicable to other unresolved pattern classes. See the last section.

Table 1. Enumeration results for ascent sequences avoiding patterns of length four (exact counts or recurrences).

τ	Reference	τ	Reference
0012	[5]	0021	[5]
0101	[4,6], Section 3	0102	[4,6], Section 3
0112	[4,6], Section 3	0120	[4,6]
0121	[4,6]	0122	Section 3
0123	Theorem 3	0132	Theorem 1
1012	[4,5]		

Our core result is formulated as follows.

Theorem 1. *The generating function $f = f(x)$ for the number of ascent sequences of length $n \geq 1$ avoiding the pattern 0132 satisfies*

$$x^2 - x(1 - 3x)f - x(2 - 3x)f^2 + (1 - x)^2 f^3 = 0.$$

The proof of Theorem 1 proceeds in three stages:

- (1) We construct a generating tree for 0132-avoiding ascent sequences;
- (2) We derive a system of recurrence relations for associated generating functions, including f ;
- (3) We solve the resulting functional equations to obtain the cubic equation satisfied by f .

By applying the above theorem and following the standard procedure for obtaining trigonometric solutions to cubic equations, we obtain that the generating function $f = f(x)$ for the number of ascent sequences of length $n \geq 1$ that avoid 0132 is given by

$$\begin{aligned} & \frac{2\sqrt{x(3-11x+9x^2)}}{3(1-x)^2} \cos\left(\frac{\pi}{3} + \frac{1}{3} \arccos\left(\frac{\sqrt{x}(9-7x-27x^2(1-x))}{2\sqrt{3-11x+9x^2}^3}\right)\right) + \frac{x(2-3x)}{3(1-x)^2} \\ &= x + 2x^2 + 5x^3 + 15x^4 + 52x^5 + 200x^6 + 826x^7 + 3583x^8 + 16,096x^9 + 74,215x^{10} \\ &+ 349,133x^{11} + 1,668,933x^{12} + 8,082,765x^{13} + 39,573,601x^{14} + 195,546,506x^{15} + \dots \end{aligned}$$

This paper is organized as follows. In Section 2, we introduce the generating tree framework that serves as the basis for our enumeration. In Section 3, we derive the recurrence relations governing the generating tree and translate them into a system of functional equations for the associated generating functions. In Sections 2.1 and 2.2, we solve this system of equations and use the resulting generating functions to establish the main enumeration theorem (Theorem 1). In Section 3, we illustrate the flexibility of our approach by applying it to several additional pattern classes, namely those avoiding 0123, 0122, 0120, 0102, and 0101, obtaining their corresponding enumerations through analogous generating tree constructions. We conclude with a discussion of possible directions for future research.

2. Proof of Theorem 1

Generating trees have proved to be an effective tool for studying pattern avoidance in many combinatorial structures, particularly in the study of ascent sequences. Their main advantage is that they encode the recursive growth of ascent sequences and often reveal a finite collection of equivalence classes from which recurrence relations and generating functions can be derived. Following [10], we recall the generating-tree framework for pattern-avoiding ascent sequences.

Let $\mathcal{T}(P)$ denote the pattern-avoidance generating tree associated with the set of ascent sequences avoiding every pattern in a given set P . The root of $\mathcal{T}(P)$ is the sequence 0, placed at level 1. The tree is constructed recursively: for each $n \geq 2$, the nodes at level n are precisely the ascent sequences of length n that avoid all patterns in P . If $a = a_1a_2 \cdots a_n$ is such a sequence, then its parent is the sequence $a_1a_2 \cdots a_{n-1}$. To generate the children of a node $a_1a_2 \cdots a_{n-1}$, we consider all possible ascent sequences $a_1a_2 \cdots a_{n-1}a_n$, where

$$0 \leq a_n \leq \text{asc}(a_1a_2 \cdots a_{n-1}) + 1,$$

and retain only those that continue to avoid every pattern in P . Thus, the generating tree records all admissible extensions of pattern-avoiding ascent sequences.

For any node a in $\mathcal{T}(P)$, let $\mathcal{T}(P; a)$ denote the subtree rooted at a and consisting of all its descendants. Given two nodes a and a' in $\mathcal{T}(P)$, we say that the subtrees $\mathcal{T}(P; a)$ and $\mathcal{T}(P; a')$ are isomorphic, denoted by $\mathcal{T}(P; a) \cong \mathcal{T}(P; a')$, if they are isomorphic as plane trees. This induces an equivalence relation $\sim$ on the nodes of $\mathcal{T}(P)$, where $a \sim a'$ if and only if $\mathcal{T}(P; a) \cong \mathcal{T}(P; a')$.

Finally, we define the reduced generating tree $\mathcal{T}[P]$ by replacing each node a of $\mathcal{T}(P)$ with the leftmost node a' occurring at the lowest possible level such that $a \sim a'$.

In the next two figures, we illustrate the first levels of the generating trees $\mathcal{T}(\{0132\})$ and $\mathcal{T}[\{0132\}]$. More precisely, Figure 1 presents the generating tree $\mathcal{T}(\{0132\})$ without using the equivalence relation. For instance, the children of 0 are 00 and 01, and the children of 01 are 010, 011, and 012. Thus, in this generating tree, we have the succession rules $0 \rightsquigarrow 00, 01$ and $01 \rightsquigarrow 010, 011, 012$. Figure 2, on the other hand, presents the generating tree $\mathcal{T}[\{0132\}]$ under the equivalence relation. For instance, the children of 0 are $0 \sim 00$ and 01, and the children of 01 are 010, $01 \sim 011$, and 012. Thus, in this generating tree, we have the succession rules $0 \rightsquigarrow 0, 01$ and $01 \rightsquigarrow 010, 01, 012$. Note that this equivalence relation allows us to characterize the succession rules of the generating tree $\mathcal{T}[\{0132\}]$, as stated in Lemma 1.

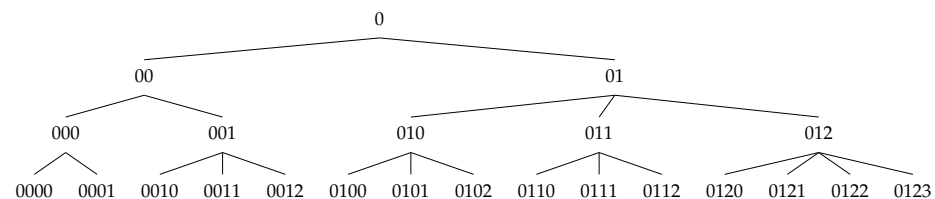


Figure 1. First four levels of $\mathcal{T}(\{0132\})$.

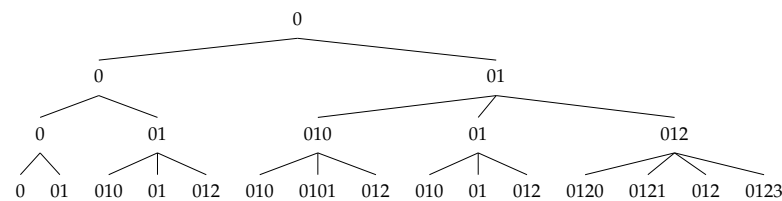


Figure 2. First four levels of $\mathcal{T}[\{0132\}]$.

Now, we are ready to describe the generating tree $\mathcal{T}[0132] = \mathcal{T}[\{0132\}]$.

Lemma 1. The generating tree $\mathcal{T}[0132]$ has root 0 and it satisfies the following succession rules

$$\begin{aligned}
 0 &\rightsquigarrow 0, d_1, \\
 d_j &\rightsquigarrow e_j, d_j, a_{j-1,2}, a_{j-2,3}, \dots, a_{0,j+1}, \quad \text{for } j \geq 1, \\
 e_j &\rightsquigarrow e_j, d_{j+1}, a_{j-1,2}, a_{j-2,3}, \dots, a_{0,j+1}, \quad \text{for } j \geq 1, \\
 a_{j,m} &\rightsquigarrow b_{j,m}, c_{j,m}, a_{j,m}, a_{j,m+1}, a_{j-1,m+2}, \dots, a_{0,m+1+j}, \quad \text{for } m \geq 2, j \geq 0, \\
 b_{j,m} &\rightsquigarrow b_{j,m}, c_{j+1,m}, a_{j+1,m}, a_{j,m+1}, a_{j-1,m+2}, \dots, a_{0,m+1+j}, \quad \text{for } m \geq 2, j \geq 0, \\
 c_{j,m} &\rightsquigarrow b_{j,m}, c_{j,m}, a_{j+1,m}, a_{j,m+1}, a_{j-1,m+2}, \dots, a_{0,m+1+j}, \quad \text{for } m \geq 2, j \geq 0,
 \end{aligned}$$

where $a_{j,m} = (01)^j 01 \dots m$, $b_{j,m} = a_{j,m} 0$, $c_{j,m} = a_{j,m} 1$, $d_j = (01)^j$, and $e_j = d_j 0$ (for a sequence X and an integer d , the constant sequence $XX \dots X$ of d occurrences of X is denoted by X^d).

Proof. Clearly, the children of 0 are 00 and $d_1 = 01$. Since $00b \in A_{0132}(n)$ if and only if $0b \in A_{0132}(n-1)$, it follows that $00 \sim 0$. Therefore, the succession rule $0 \rightsquigarrow 0, d_1$ holds.

Let $j \geq 1$, the children of d_j are $d_j i$ with $i = 0, 1, \dots, j+1$. By definition, we have that $d_j 0 = e_j$, $d_j 1 \sim d_j$, and $d_j i \sim d_{j+1-i} 01 \dots i = a_{j+1-i,i}$ for all $i = 2, 3, \dots, j+1$. Therefore, the succession rule $d_j \rightsquigarrow e_j, d_j, a_{j-1,2}, a_{j-2,3}, \dots, a_{0,j+1}$ holds.

Let $j \geq 1$, the children of e_j are $e_j i$ with $i = 0, 1, \dots, j+1$. By definition, we have that $e_j 0 = e_j$, $e_j 1 \sim d_{j+1}$, and $e_j i \sim d_{j+1-i} 01 \dots i = a_{j+1-i,i}$ for all $i = 2, 3, \dots, j+1$. Therefore, the succession rule $e_j \rightsquigarrow e_j, d_{j+1}, a_{j-1,2}, a_{j-2,3}, \dots, a_{0,j+1}$ holds.

Now, let $j \geq 0$ and $m \geq 2$, the children of $a_{j,m}$ are $a_{j,m} i$ with $i = 0, 1, \dots, m+j+1$. By definition, we have that $a_{j,m} 0 = b_{j,m}$, $a_{j,m} 1 = c_{j,m}$, $a_{j,m} i$ with $i = 2, 3, \dots, m-1$ contains

0132, $a_{j,m}m \sim a_{j,m}$, and $a_{j,m}(m+1+i) \sim a_{j-i,m+1+i}$ for all $i = 0, 1, \dots, j$. Therefore, the succession rule $a_{j,m} \rightsquigarrow b_{j,m}, c_{j,m}, a_{j,m}, a_{j,m+1}, a_{j-1,m+2}, \dots, a_{0,m+1+j}$ holds.

Similarly, all the other succession rules hold. $\square$

Let $A_v(x)$ be the generating function for the number of nodes at level n in the subtree of $\mathcal{T}[P]$ rooted at v and containing all its descendants, where the root stays at level 1; that is,

$$A_v(x) = \sum_{n \geq 1} (\text{number of nodes at level } n \text{ in the subtree of } \mathcal{T}[P] \text{ rooted at } v) x^n.$$

Define $A_{j,m}(x) = A_{a_{j,m}}(x)$, $B_{j,m}(x) = A_{b_{j,m}}(x)$, $C_{j,m}(x) = A_{c_{j,m}}(x)$, $D_j(x) = A_{d_j}(x)$, and $E_j(x) = A_{e_j}(x)$. Thus, by Lemma 1, we have

$$\begin{aligned} A_0(x) &= x + xA_0(x) + xD_1(x), \end{aligned} \quad (1)$$

$$\begin{aligned} D_j(x) &= x + xE_j(x) + xD_j(x) + x \sum_{i=2}^{j+1} A_{j+1-i,i}(x), \quad \text{for } j \geq 1, \end{aligned} \quad (2)$$

$$\begin{aligned} E_j(x) &= x + xE_j(x) + xD_{j+1}(x) + x \sum_{i=2}^{j+1} A_{j+1-i,i}(x), \quad \text{for } j \geq 1, \end{aligned} \quad (3)$$

$$\begin{aligned} A_{j,m}(x) &= x + xB_{j,m}(x) + xC_{j,m}(x) + xA_{j,m}(x) + x \sum_{i=0}^j A_{j-i,m+1+i}(x), \quad \text{for } m \geq 2, j \geq 0, \end{aligned} \quad (4)$$

$$\begin{aligned} B_{j,m}(x) &= x + xB_{j,m}(x) + xC_{j+1,m}(x) + xA_{j+1,m} + x \sum_{i=0}^j A_{j-i,m+1+i}(x), \quad \text{for } m \geq 2, j \geq 0, \end{aligned} \quad (5)$$

$$\begin{aligned} C_{j,m}(x) &= x + xB_{j,m}(x) + xC_{j,m}(x) + xA_{j+1,m}(x) + x \sum_{i=0}^j A_{j-i,m+1+i}(x), \quad \text{for } m \geq 2, j \geq 0. \end{aligned} \quad (6)$$

Define $L(v, u) = \sum_{m \geq 2} \sum_{i \geq 0} L_{i,m} v^i u^{m-2}$ with $L \in \{A, B, C\}$ and $L(v) = \sum_{i \geq 1} L_i(x) v^{i-1}$ with $L \in \{D, E\}$. Here, we used either one or two catalytic variables. By multiplying (4)–(6) by $v^j u^{m-2}$ and summing over $m \geq 2$ and $j \geq 0$, and by multiplying (2)–(3) by v^{j-1} and summing over $j \geq 1$, we obtain

$$\begin{aligned}
A_0(x) &= x + xA_0(x) + xD_1(x), \\
A(v, u) &= \sum_{m \geq 2} \sum_{j \geq 1} A_{j,m}(x) v^j u^{m-2} \\
&= \sum_{m \geq 2} \sum_{j \geq 0} x v^j u^{m-2} + x \sum_{m \geq 2} \sum_{j \geq 0} B_{j,m}(x) v^j u^{m-2} \\
&\quad + x \sum_{m \geq 2} \sum_{j \geq 0} C_{j,m}(x) v^j u^{m-2} + x \sum_{m \geq 2} \sum_{j \geq 0} A_{j,m}(x) v^j u^{m-2} \\
&\quad + x \sum_{m \geq 2} \sum_{j \geq 0} \sum_{i=0}^j A_{j-i, m+1+i}(x) v^j u^{m-2} \\
&= \frac{x}{(1-v)(1-u)} + xB(v, u) + xC(v, u) + xA(v, u) \\
&\quad + x \sum_{m \geq 2} \sum_{i \geq 0} \sum_{j \geq 0} A_{j, m+1+i}(x) v^{j+i} u^{m-2} \\
&= \frac{x}{(1-v)(1-u)} + xB(v, u) + xC(v, u) + xA(v, u) \\
&\quad + x \sum_{i \geq 0} \sum_{m \geq i+3} \sum_{j \geq 0} A_{j,m}(x) v^{j+i} u^{m-i-3} \\
&= \frac{x}{(1-v)(1-u)} + xB(v, u) + xC(v, u) + xA(v, u) \\
&\quad + x \sum_{m \geq 3} \sum_{i=0}^{m-3} \sum_{j \geq 0} A_{j,m}(x) v^{j+i} u^{m-i-3} \\
&= \frac{x}{(1-v)(1-u)} + xB(v, u) + xC(v, u) + xA(v, u) \\
&\quad + \frac{x}{v-u} \sum_{m \geq 3} \sum_{j \geq 0} A_{j,m}(x) (v^{j+m-2} - v^j u^{m-2}) \\
&= \frac{x}{(1-v)(1-u)} + xB(v, u) + xC(v, u) + xA(v, u) + \frac{x}{v-u} (A(v, v) - A(v, u)).
\end{aligned} \tag{7}$$

and similarly,

$$\begin{aligned}
B(v, u) &= \frac{x}{(1-v)(1-u)} + xB(v, u) + \frac{x}{v} (C(v, u) - C(0, u)) + \frac{x}{v} (A(v, u) - A(0, u)) \\
&\quad + \frac{x}{v-u} (A(v, v) - A(v, u)), \\
C(v, u) &= \frac{x}{(1-v)(1-u)} + xB(v, u) + xC(v, u) + \frac{x}{v} (A(v, u) - A(0, u)) \\
&\quad + \frac{x}{v-u} (A(v, v) - A(v, u)), \\
D(v) &= \frac{x}{1-v} + xE(v) + xD(v) + xA(v, v), \\
E(v) &= \frac{x}{1-v} + xE(v) + \frac{x}{v} (D(v) - D(0)) + xA(v, v).
\end{aligned} \tag{8}$$

In order to solve this system of equations, we assume the following:

$$L(v, u) = \frac{1}{1-u} \tilde{L}(v), \quad L \in \{A, B, C\}. \tag{9}$$

This assumption is motivated by the computation of the first terms of the generating functions A , B , and C .

The strategy to solve (7) and (8) is as follows. First, we solve (7) and (8) under the assumption of (9). Then, we verify whether the resulting solution satisfies both (7)–(9). If it does, the obtained solution is indeed a solution to (7) and (8).

2.1. Solving (7) and (8) Under the Assumption of (9)

By (7)–(9), we have

$$\begin{aligned}
A_0(x) &= x + xA_0(x) + xD_1(x), \\
\tilde{A}(v) &= \frac{x}{1-v} + x\tilde{B}(v) + x\tilde{C}(v) + x\tilde{A}(v) + \frac{x}{1-v} \tilde{A}(v), \\
\tilde{B}(v) &= \frac{x}{1-v} + x\tilde{B}(v) + \frac{x}{v} (\tilde{C}(v) - \tilde{C}(0)) + \frac{x}{v} (\tilde{A}(v) - \tilde{A}(0)) + \frac{x}{1-v} \tilde{A}(v), \\
\tilde{C}(v) &= \frac{x}{1-v} + x\tilde{B}(v) + x\tilde{C}(v) + \frac{x}{v} (\tilde{A}(v) - \tilde{A}(0)) + \frac{x}{1-v} \tilde{A}(v), \\
D(v) &= \frac{x}{1-v} + xE(v) + xD(v) + \frac{x}{1-v} \tilde{A}(v), \\
E(v) &= \frac{x}{1-v} + xE(v) + \frac{x}{v} (D(v) - D(0)) + \frac{x}{1-v} \tilde{A}(v).
\end{aligned} \tag{10}$$

Solving the third and fourth equations of (10) for $\tilde{B}(v)$ and $\tilde{C}(v)$, and then substituting at the second equation of (10), we obtain

$$K(x, v)\tilde{A}(v) = -(1-v)(vx - 2v - x)x^2\tilde{A}(0) + v(1-v)x^2\tilde{C}(0) - v^2x, \quad (11)$$

where $K(x, v) = x^3 + 3(1-x)x^2v - (3x^2 - 3x + 1)(1-x)v^2 + (1-x)^3v^3$. This type of equation can be solved using the kernel method (see [17]). For the equation $K(x, v) = 0$, there are three roots $v_j = v_j(x)$, $j = 0, 1, 2$, where

$$\begin{aligned} v_0(x) &= 1 - x - 3x^2 - 9x^3 - 30x^4 + \dots, \\ v_1(x) &= -x\sqrt{x} + \frac{3}{2}x^2 - \frac{25}{8}x^2\sqrt{x} + 5x^3 - \frac{1167}{128}x^3\sqrt{x} + 16x^4 - \frac{30,729}{1024}x^4\sqrt{x} + \dots, \\ v_2(x) &= x\sqrt{x} + \frac{3}{2}x^2 + \frac{25}{8}x^2\sqrt{x} + 5x^3 + \frac{1167}{128}x^3\sqrt{x} + 16x^4 + \frac{30,729}{1024}x^4\sqrt{x} + \dots. \end{aligned}$$

Hence, by substituting $v = v_1$ and $v = v_2$ into (11), and then solving for $\tilde{A}(0)$ and $\tilde{C}(0)$, we obtain

$$\begin{aligned} \tilde{A}(0) &= -\frac{v_1v_2}{x^2(1-v_1)(1-v_2)}, \\ \tilde{C}(0) &= \frac{2v_1v_2(1-x) + (v_1+v_2)x}{x^2(1-v_1)(1-v_2)}. \end{aligned}$$

Thus, by the second, the third, and the fourth equations of (10), we obtain

$$\begin{aligned} \tilde{A}(v) &= \frac{x(v-v_1)(v-v_2)}{K(x, v)(1-v_1)(1-v_2)}, \\ \tilde{B}(v) &= \frac{\tilde{a}(v)}{xK(x, v)(1-v_1)(1-v_2)}, \\ \tilde{C}(v) &= \frac{\tilde{b}(v)}{xK(x, v)(1-v_1)(1-v_2)}. \end{aligned}$$

where

$$\begin{aligned} \tilde{a}(v) &= x(v_1v_2(1-x) + x) \\ &\quad - (1-x)(v_1v_2(1-2x)(1-x) + x(1-2x)(v_1+v_2) + 2x^2)(1-v) \\ &\quad + (1-x)^2(v_2(x-1) - x)(v_1(x-1) - x)(1-v)^2, \\ \tilde{b}(v) &= x(v_1v_2(x-1) - (v_1+v_2)x + x) - (v_1v_2(1-x)(1-x+x^2) \\ &\quad + x^2(v_1+v_2)(1-x) + x^2(x-2))(1-v) - (1-x)(v_1v_2(1-x)^2 - x^2)(1-v)^2. \end{aligned}$$

By solving the sixth equation of (10) for $E(v)$, and substituting its expression into the fifth equation of (10) with $v = \frac{x^2}{(1-x)^2}$, we obtain

$$D(0) = \frac{x^2 - v_1v_2(1-x)^2}{x(1-v_1)(1-v_2)(1-x)^2}.$$

2.2. A Solution for (7) and (8)

Therefore, by taking $L(v, u) = \frac{1}{1-u}\tilde{L}(v)$ for all $L \in \{A, B, C\}$ and the expression of $D(0)$, we see that the second, third, fourth, fifth, and sixth equations of (10) hold. Hence, by the first equation of (10), we solved the system (10), which implies the following result.

Theorem 2. The generating function $\sum_{n \geq 1} A_{0132}(n)x^n$ is given by

$$A_0(x) = \frac{x}{1-x} + \frac{x}{1-x} D(0) = \frac{x}{1-x} + \frac{x^2 - v_1 v_2 (1-x)^2}{(1-v_1)(1-v_2)(1-x)^3}.$$

Note that by the definitions of the roots v_0, v_1, v_2 , we have $v_0 v_1 v_2 = -\frac{x^3}{(1-x)^3}$ and $v_0 + v_1 + v_2 = \frac{1-3x+3x^2}{(1-x)^2}$. So, by Theorem 2, we have

$$A_0(x) = \frac{x}{1-x} \left(1 + \frac{x}{v_0(1-x)^2 - x^2} \right).$$

Note that v_0 satisfies $K(x, v_0) = 0$. Hence, by finding v_0 in terms of $A_0(x)$ and then substituting into the equation $K(x, v_0) = 0$, we obtain the result in Theorem 1.

3. Further Results and Conclusions

In this section, we present several applications of the generating-tree framework developed in the proof of Theorem 1. In particular, we derive explicit formulas for the generating functions that count ascent sequences avoiding a specific pattern of length four, illustrating the utility and versatility of our recursive approach.

As a first consequence, we find an explicit formula for the generating function for the number of ascent sequences of length n avoiding the pattern 0101. One can show that the generating tree $\mathcal{T}[0101]$ has root 0 and satisfies the succession rules

$$01 \cdots m \rightsquigarrow 0, 01, 012, \dots, 01 \cdots m(m+1)$$

for all $m \geq 0$. Thus, the corresponding weights satisfy the recurrence relation

$$A_{01 \cdots m}(x) = x + x \sum_{j=0}^{m+1} A_{01 \cdots j}(x).$$

To solve this, we define the bivariate generating function $A(x; u) = \sum_{m \geq 0} A_{01 \cdots m}(x) u^m$. Multiplying the recurrence by u^m and summing over $m \geq 0$ yields the functional equation

$$A(x; u) = \frac{x}{1-u} + \frac{x}{1-u} A(x; u) + \frac{x}{u} (A(x; u) - A(x; 0)).$$

This type of linear functional equation can be solved systematically using the kernel method (see, e.g., [17]). By setting $u = \frac{1-\sqrt{1-4x}}{2}$ to cancel the kernel coefficient, we obtain

$$A(x; 0) = \frac{1 - \sqrt{1-4x}}{2x} - 1,$$

which matches the known results established in [4,6].

As another application, we derive an explicit formula for the generating function counting ascent sequences of length n that avoid the pattern 0102. It can be shown that the generating tree $\mathcal{T}[0102]$ has root 0 and obeys the following succession rules:

$$\begin{aligned} 01 \cdots m &\rightsquigarrow 010, 0121, \dots, 01 \cdots (m-1)m(m-1), 01 \cdots m, 01 \cdots (m+1), \\ 01 \cdots (m-1)m(m-1) &\rightsquigarrow 010, 0121, \dots, 01 \cdots (m-1)m(m-1), 01 \cdots (m-1)m(m-1), \end{aligned}$$

for all $m \geq 0$. Translating these rules into a system of recurrence relations and converting them into a functional equation allows us to apply the kernel method once more. This

shows that the generating function for the number of ascent sequences of length n avoiding 0102 is given by

$$\frac{x(1-2x)}{(1-x)(1-3x)},$$

which matches the known results established in [4,6].

Similarly, we can study the pattern 0112. It can be shown that the generating tree $\mathcal{T}[0112]$ has root 0 and obeys the following succession rules:

$$\begin{aligned} 01 \cdots m &\rightsquigarrow 01 \cdots m, 011, 0122, \dots, 01 \cdots (m-1)mm, 01 \cdots (m+1), \\ 01 \cdots (m-1)mm &\rightsquigarrow 01 \cdots (m-1)mm, 011, 0122, \dots, 01 \cdots (m-1)mm. \end{aligned}$$

From these rules, it is straightforward to see that the generating trees $\mathcal{T}[0102]$ and $\mathcal{T}[0112]$ are isomorphic, immediately implying that the two corresponding generating functions are equal.

Also, we can study the pattern 0123. It can be shown that the generating tree $\mathcal{T}[0123]$ has root 0 and obeys the following succession rules:

$$\begin{aligned} 0 &\rightsquigarrow 0, 01, \\ (01)^m &\rightsquigarrow (01)^m 0, (01)^m, 012, (01)^2 3, \dots, (01)^m(m+1), \\ (01)^m(m+1) &\rightsquigarrow (01)^m(m+1), (01)^m(m+1), 012, (01)^2 3, \dots, (01)^m(m+1), \\ (01)^m 0 &\rightsquigarrow (01)^m 0, (01)^{m+1}, 012, (01)^2 3, \dots, (01)^m(m+1), \end{aligned}$$

for all $m \geq 1$. Translating these rules into a system of recurrence relations, we obtain

$$\begin{aligned} A_0(x) &= x + xA_0(x) + xA_{01}(x), \\ A_{(01)^m}(x) &= x + xA_{(01)^m 0}(x) + xA_{(01)^m}(x) + x \sum_{j=1}^m A_{(01)^j(j+1)}(x), \\ A_{(01)^m(m+1)}(x) &= x + 2xA_{(01)^m(m+1)}(x) + x \sum_{j=1}^m A_{(01)^j(j+1)}(x), \\ A_{(01)^m 0}(x) &= x + xA_{(01)^m 0}(x) + xA_{(01)^{m+1}}(x) + x \sum_{j=1}^m A_{(01)^j(j+1)}(x), \end{aligned}$$

for all $m \geq 1$. Translating these recurrence relations to functional equations allows us to apply the kernel method once more. This shows that

$$\begin{aligned} A_{(01)^m}(x) &= \frac{x(1-2x)^m}{(1-3x)^{m-1}(1-5x+6x^2-x^3)}, \\ A_{(01)^m 0}(x) &= \frac{x(1-3x+x^2)(1-2x)^m}{(1-3x)^m(1-5x+6x^2-x^3)}, \\ A_{(01)^m(m+1)}(x) &= \frac{x(1-2x)^{m-1}}{(1-3x)^m}, \end{aligned}$$

for all $m \geq 1$. Hence, we have the following theorem.

Theorem 3. *The generating function for the number of ascent sequences of length n avoiding 0123 is given by*

$$\frac{x(1-3x+x^2)}{1-5x+6x^2-x^3}.$$

Similarly, we can study the pattern 0122. It can be shown that the generating tree $\mathcal{T}[0122]$ has root 0 and obeys the following succession rules:

$$\begin{aligned} 0 &\rightsquigarrow 0, 01, \\ (01)^m &\rightsquigarrow (01)^m 0, (01)^m, (01)^m, (01)^m 3, (01)^m 4, \dots, (01)^m (m+1), \\ (01)^m 0 &\rightsquigarrow (01)^m 0, (01)^{m+1}, (01)^m, (01)^m 3, (01)^m 4, \dots, (01)^m (m+1), \\ (01)^m j &\rightsquigarrow (01)^m 0, (01)^m, (01)^{m-1}, (01)^{m-1} 3, (01)^{m-1} 4, \dots, (01)^{m-1} (j-1), \\ &\quad (01)^m j, (01)^m (j+1), \dots, (01)^m (m+1), \end{aligned}$$

for all $m \geq 1$ and $3 \leq j \leq m+1$. Translating these rules into a system of recurrence relations, we obtain

$$\begin{aligned} A_0(x) &= x + xA_0(x) + xA_{01}(x), \\ A_{(01)^m}(x) &= x + xA_{(01)^m 0}(x) + 2xA_{(01)^m}(x) + x \sum_{j=3}^{m+1} A_{(01)^m j}(x), \\ A_{(01)^m 0}(x) &= x + xA_{(01)^m 0}(x) + xA_{(01)^{m+1}}(x) + xA_{(01)^m}(x) + x \sum_{j=3}^m A_{(01)^m j}(x), \\ A_{(01)^m j}(x) &= x + xA_{(01)^m 0}(x) + xA_{(01)^m}(x) + xA_{(01)^{m-1}}(x) \\ &\quad + x \sum_{i=3}^{j-1} A_{(01)^{m-1} i}(x) + x \sum_{i=j}^{m+1} A_{(01)^{m-1} i}(x), \end{aligned}$$

for all $m \geq 1$ and $3 \leq j \leq m+1$. Define $A(u) = \sum_{m \geq 1} A_{(01)^m}(x) u^{m-1}$, $B(u) = \sum_{m \geq 1} A_{(01)^m 0}(x) u^{m-1}$, and $C(u, v) = \sum_{m \geq 2} \sum_{j=3}^{m+1} A_{(01)^m j}(x) u^{m-2} v^{m+1-j}$. Now, the above recurrence relations can be written as

$$\begin{aligned} A(u) &= \frac{x}{1-u} + xB(u) + 2xA(u) + xuC(u, 1), \\ B(u) &= \frac{x}{1-u} + xB(u) + xA(u) + \frac{x}{u}(A(u) - A(0)) + xuC(u, 1), \\ C(u, v) &= \frac{x}{(1-u)(1-uv)} + \frac{x}{u(1-v)}(B(u) - B(uv)) + \frac{x}{u(1-v)}(A(u) - A(uv)) \\ &\quad + \frac{x}{1-v}(A(u) - vA(uv)) + \frac{xu}{1-v}(C(u, 1) - vC(u, v)) \\ &\quad + \frac{x}{1-v}(C(u, v) - vC(uv, 1)). \end{aligned}$$

Here, we were unable to solve this system to derive an explicit generating function for the number of ascent sequences of length n avoiding the pattern 0122.

Another direction is to consider the set of ascent sequences of length n that avoid both 0132 and another pattern τ . For instance, it can be shown that the generating tree $\mathcal{T}[0132, 0123]$ has root 0 and obeys the following succession rules:

$$\begin{aligned}
0 &\rightsquigarrow 0, 01, \\
01 &\rightsquigarrow 010, 01, 012, \\
010 &\rightsquigarrow 010, a_2, 012, \\
012 &\rightsquigarrow 012, 012, 012, \\
a_m &\rightsquigarrow b_m, a_m, \underbrace{012, \dots, 012}_{m \text{ times}}, \\
b_m &\rightsquigarrow b_m, a_{m+1}, \underbrace{012, \dots, 012}_{m \text{ times}},
\end{aligned}$$

where $a_m = (01)^m$ and $b_m = (01)^m 0$. Translating these rules into a system of recurrence relations and converting them into a functional equation allows us to apply the kernel method once more. This shows that the generating function for the number of ascent sequences of length n avoiding both 0132 and 0123 is given by

$$\frac{x(1 - 5x + 7x^2 - x^3)}{(1 - 3x)(1 - 2x)^2}.$$

In another example, it can be shown that the generating tree $\mathcal{T}[0132, 0012]$ has root 0 and obeys the following succession rules:

$$\begin{aligned}
0 &\rightsquigarrow 00, 01, \\
00 &\rightsquigarrow 00, 00, \\
01 &\rightsquigarrow 010, 011, 012, \\
010 &\rightsquigarrow 010, 00, 010, \\
011 &\rightsquigarrow 010, 011, 011, \\
012 &\rightsquigarrow 0120, 0121, 0122, 012, \\
0120 &\rightsquigarrow 0120, 00, 010, 010, \\
0121 &\rightsquigarrow 0120, 0121, 011, 011, \\
0122 &\rightsquigarrow 0120, 0121, 0122, 011.
\end{aligned}$$

Translating these rules into a system of recurrence relations and converting them into a functional equation allows us to show that the generating function for the number of ascent sequences of length n avoiding both 0132 and 0123 is given by

$$\frac{x(1 - 5x + 9x^2 - 5x^3 + x^4)}{(1 - x)(1 - 2x)^3}.$$

In conclusion, the methodology presented above—constructing the underlying generating tree, translating its succession rules into a system of recurrence relations, deriving the corresponding functional equations, and systematically solving them using tools such as the kernel method—provides a powerful framework for obtaining explicit formulas for pattern-avoiding ascent sequences. We have demonstrated the effectiveness of this approach by resolving the non-trivial avoidance case of 0132, along with several other cases considered in this work. A limitation of the method is that, in its current form, it does not immediately lead to a universal procedure for deriving closed formulas for all pattern-avoidance classes.

We anticipate that this approach is robust enough to be applied to many of the pattern sets discussed in the introduction, although a complete characterization remains open. As a natural direction for future research, we therefore propose the following conjecture: it should be possible to obtain explicit generating functions for ascent sequences avoiding any given single pattern τ of length four. More generally, one may also investigate the simultaneous avoidance of arbitrary sets of length-four patterns.

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