

Article

Locally Scale-Invariant Gravity with Conserved Global Charges

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Abstract

We put forward the idea that in addition to invariance under coordinate transformations of general relativity (GR) the gravitational interaction is invariant under arbitrary scale deformations of the metric field, as well as other fields. In addition, we assume that the scaling field has an internal symmetry. Because scale invariance is imposed only on gravity, while the standard model (SM) retains its usual non-Weyl-invariant form, the full framework is not merely a rewriting of GR plus the SM. The global charges that are associated with the internal symmetry could potentially source the gravitational field. Throughout, the framework is treated as a classical generalization of GR on sufficiently sub-Planckian energies; accordingly, any discussion of global charges is to be understood within this effective classical domain. In the case that isotropic deformations are considered, the theory reduces to a Weyl-invariant scalar–tensor (WIST) version of GR. In case that the reference metric is chosen to be Minkowski, the metric factorization takes the standard vierbein form, apart from the additional internal structure assumed here. A few other implications of WIST are considered as well.

Keywords: scale deformations; Weyl invariance; global charges

1. Introduction

It is hard to conceive any physically realistic description of the Universe in the absence of either inertia or gravitation. Inertia is thought to be a universal phenomenon; the Higgs field that is responsible for the generation of particle masses in the electroweak sector is ubiquitous. Likewise, gravitation is everywhere. It is normally identified with the omnipresence of spacetime, the substratum of physical reality, at least down to spacetime singularities, where the very concepts of space and time are believed to break down.

Our current best theory of the (classical) gravitational interaction is general relativity (GR). The latter has successfully passed a wide range of tests in our solar system. GR is one of the simplest of all conceivable metric theories; it obeys the metricity condition, it is torsionless and it is described by a lagrangian linear in the Ricci scalar. GR provides a covariant field theoretical framework that incorporates the principle of relativity applied to non-inertial observers and is realized on curved spacetimes. The curvature in the presence of a given energy-momentum distribution is governed by G , the universal coupling constant of gravitation that converts energy-momentum density units to length/curvature units. This specific coupling between matter and geometry was adopted in GR so as to guarantee consistency with Newton's law of gravitation in the weak field limit, assuming that the locally inferred G is universal.

Naturally, GR is expected to depart from the Newtonian limit in the strong gravity regime, i.e., in physical systems which are dominated by the gravitational interaction while



Academic Editors: Sergei Odintsov
and Stefano Profumo

Received: 15 May 2026

Revised: 30 June 2026

Accepted: 8 July 2026

Published: 12 July 2026

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other interactions play a secondary role, e.g., in black holes or on the largest cosmological scales. However, in contrast to expectations Newtonian gravity does not adequately describe gravitational phenomena on galactic scales and larger, e.g., at the outskirts of galaxies and galaxy clusters, even though gravity is weak on these scales, i.e., the dimensionless gravitational potential is as low as $\Phi \sim 10^{-4}$ or 10^{-3} . A straightforward remedy for this problem (and others in cosmology) involves introducing some form of non-relativistic (NR) invisible matter, i.e., ‘dark matter’ (DM). By far the leading candidate is some yet-to-be-determined beyond-the-standard-model (SM) particle. According to another, less popular possibility primordial black holes may account for the missing non-luminous matter. Other models such as fuzzy DM exist, along with a plethora of modified gravity theories.

As of yet, no DM particle candidate has been found in spite of decades of intensive search. In addition, the status of beyond-the-SM theories of particle physics—our best hope for new DM particles—is unclear with persistent null results coming from the large hadron collider (LHC). In any case, considerable amount of effort is still needed before the existence of a DM particle that adequately accounts for DM phenomena on Hubble down to galactic scales is compellingly established.

Another possibility is that particulate DM simply does not exist, and that on galactic scales or larger gravitation is not strictly described by GR, but rather by a modified version thereof. Indeed, GR has been tested with high precision in the solar system and is strongly supported in compact and dynamical systems such as binary pulsars and gravitational-wave sources. These tests, however, do not by themselves establish that GR coupled only to SM matter accounts for gravitational phenomena on galactic and cosmological scales. It is precisely on those scales that DM and DE are normally introduced, or alternatively that extensions of the gravitational sector could be in play. One possible extension of GR invokes a higher symmetry—Weyl invariance (WI), where G is determined by a scalar field (while the other three fundamental interactions are assumed to have their standard forms, i.e., they are non-WI). In GR this scalar field is replaced by default with a fixed G throughout space and time. However, its local determination by Cavendish-type experiments does not necessarily imply its universality. In a WI version of GR, G can vary in space and time, freely departing from its locally measured value, and simultaneously inducing scalar metric perturbations. From this perspective, there is *a priori* no preferred conformal frame, and whereas the choice $G = \text{const.}$ seems to work impressively well in our solar system it clearly fails on galactic scales and larger. It has been shown, in a phenomenological setting, that fractional variations in G and active gravitational masses at the $O(10^{-3})$ or $O(10^{-4})$ level on galactic and galaxy-cluster scales can model certain DM-like effects without invoking particulate DM, at the reasonable cost of endowing GR with WI [1]. Importantly, the degree of per-system freedom needed at the galactic level is not greater than what is already standard in particulate-DM halo modeling (e.g., NFW or Einasto profiles with fitted parameters per galaxy or galaxy cluster). Another possible application of a WI version of GR is to the interpretation of the ‘Hubble tension’; the apparent statistically significant $\gtrsim 4\sigma$ tension between local and cosmological inference of the Hubble constant, H_0 [2].

On the largest cosmological scales observations require some form of a non-clustering vacuum-like dark energy (DE), which is puzzling in a few respects. First, it is $\gtrsim 120$ orders of magnitude suppressed relative to naive expectations based on our current understanding of particle physics. This might be hinting towards a gravitational/geometric origin for DE rather than a material one. Second, it is found empirically to be comparable in energy density terms to that of the NR matter at the present time although the two species greatly differ in their evolution history. In this sense, there seems to be a fine-tuning of the initial conditions that are associated with DE [3].

Finally, a third implication of endowing the (Weyl) scaling field with an internal symmetry is that the associated conserved charge can contribute an effective negative energy component in the cosmological background equations, enabling a classical non-singular bounce in the radiation era; this possibility has since been explored quantitatively in [4].

The present work focuses on the foundations of a certain generalization of GR that allows for a fully (generally anisotropic) locally scale-invariant generalization of GR by introducing a scale deformation (tensor) field $\Phi_{\mu\nu}$ living on an arbitrary spacetime. When the reference metric is chosen to be Minkowski, the metric factorization takes the standard vierbein form, apart from the additional internal structure postulated here; obviously, this reference-metric representation does not assert that an arbitrary curved spacetime, particularly one with nontrivial global topology, is globally diffeomorphic to Minkowski spacetime. In the case of an isotropic scale invariance on an arbitrary background the theory becomes a WI scalar–tensor (WIST) version of GR, hereafter referred to in abbreviation as ‘WIST’. In addition to local scale invariance we endow $\Phi_{\mu\nu}$ with an internal symmetry with which a global conserved charge is associated. The latter provides an additional source for spacetime curvature. The theory is treated here as a classical extension of GR (therefore applicable only at sub-Planckian curvatures/energies). Clearly, it is not ultraviolet (UV)-complete. Three comments are in order here. First, within the pure gravitational sector and in the absence of internal charges, the Weyl or scale deformation field may amount to a redundant representation; an appropriate Weyl gauge choice recovers ordinary GR. Second, the complete framework considered here is not simply equivalent to standard GR coupled to the SM of particle physics, because local scale invariance is imposed only on gravitation, whereas the SM is retained in its usual non-WI form (unlike in, e.g., [5] where all four fundamental interactions are endowed with Weyl invariance). Consequently, Planck and active gravitational masses may vary relative to inertial masses, and this relative variation cannot be removed by a Weyl transformation of the gravitational variables alone as was demonstrated and utilized in [1,2]. Third, the scale deformation field is additionally endowed with an internal symmetry. The associated conserved charges are WI and provide new classical sources for the gravitational field even within the purely gravitational sector.

Although various variants of WIST have been contemplated over the past fifty years, e.g., [5–15], objections to this framework range between two extremes. On the one hand, it is often claimed that the WIST is ‘GR in a guise’, that extending the symmetry of GR to allow WI does not make it genuinely so, e.g., [16,17]. In particular it is argued that there is classically no preferred conformal frame, so WIST is equivalent to GR. On the other hand, it is sometimes implied that dressing GR with WI brings about a ghost scalar field that inflicts ‘disastrous instabilities’ on the theory, so much so as to disqualify the theory (while the same is not similarly claimed about the ‘equivalent’ GR), e.g., [18]. However, there are also opposing views, e.g., [19–21]. We address these claims separately in the present work. In particular, the Friedmann–Robertson–Walker (FRW) example is used only to illustrate that the appearance of a wrong-sign kinetic term in the WI representation is not, by itself, sufficient evidence for an ordinary propagating ghost or a classical inconsistency. It is not intended as a general stability proof, and no claim is made here that all quantum questions associated with the conformal sector are thereby resolved. Specifically, it is only argued that within the classical domain assumed throughout, the WI representation does not by itself establish a pathology absent in the corresponding GR description. Again, we stress that it is our assumption that WIST is entirely classical to the extent that GR is, and indeed the validity of GR has only been experimentally established on macroscopic down to μm , i.e., mesoscopic scales, and not smaller.

The scope of the present work should be emphasized, once again. It does not provide a complete Hamiltonian quantization of the conformal/internal sector (much like no such quantization exists in the case of GR itself), nor a full phenomenological replacement for DM and DE across all observational probes. Rather, it does formulate the classical locally scale-invariant generalization of classical GR, distinguish Weyl redundancy from internal conserved charges, and show how such charges can enter the classical gravitational equations. Questions of quantum stability, the full dynamical analysis of the general anisotropic tensor sector, and detailed observational tests in the solar system, galaxies, galaxy clusters, the cosmic microwave background (CMB), and large-scale structure (LSS) are important but are clearly beyond the scope of the present work.

The paper is organized as follows. The general anisotropic scale-invariant version of GR is considered in Section 2, followed by its special isotropic case, the WIST theory, in Section 3. The FRW example is discussed in Section 4, followed by Bianchi spacetimes in Section 5, global symmetries in Section 6, and a summary in Section 7. In Appendix A the transformation from a purely gravitational to non-gravitational description of the weak field approximation is described, and in Appendix B the case of SU(2) global symmetry is briefly discussed. Appendix C summarizes the field equations associated with a homogeneous but anisotropic scale-invariant version of GR. Throughout, we adopt a mostly positive signature for the spacetime metric $(-1, 1, 1, 1)$, with the speed of light c and reduced Planck constant $\hbar$ set to unity.

2. Locally Anisotropic Scale-Invariant Theory of Gravitation

GR is described by the covariant Einstein–Hilbert (EH) action

$$\mathcal{I}_{EH} = \int \left(\frac{1}{16\pi G} (R - 2\Lambda) + \mathcal{L}_M \right) \sqrt{-g} d^4x, \quad (1)$$

where R is the Ricci curvature scalar (obtained from the metric field $g_{\mu\nu}$ and its first and second derivatives in the standard way) and Λ is the cosmological constant, respectively, $\mathcal{L}_M$ is the lagrangian density of matter, and g is the metric determinant. The cosmological constant, Λ , can be absorbed in $\mathcal{L}_M$ as a species with vacuum-like properties. Covariance implies that Equation (1) is invariant under the transformation $g_{\mu\nu} \rightarrow g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^\nu}$ (where Greek indices stand for spacetime coordinates and repeated indices are summed over). This symmetry of Equation (1) under change in coordinate basis relies on the commutativity of derivatives.

A natural generalization of covariance (that to the best of our knowledge has not been considered so far in the literature) could be the following replacement everywhere in Equation (1)

$$\begin{aligned} g_{\mu\nu} &\rightarrow \bar{g}_{\mu\nu} \equiv \Phi_\mu^\alpha g_{\alpha\beta} \Phi_\nu^\beta, \\ g^{\mu\nu} &\rightarrow \bar{g}^{\mu\nu} \equiv (\Phi^{-1})^\mu_\alpha g^{\alpha\beta} (\Phi^{-1})^\nu_\beta, \\ \mathcal{L}_M &\rightarrow \bar{\mathcal{L}}_M = \mathcal{L}_M / |\Phi|, \end{aligned} \quad (2)$$

where $\Phi_\mu^\alpha(x)$ is an arbitrary spacetime-dependent real scale deformation tensor of rank two, $|\Phi| \equiv |\det(\Phi_\mu^\alpha)|$, and $\mathcal{L}_M$ denotes the ordinary GR matter lagrangian. This transformation represents a local anisotropic rescaling of the metric field and the lagrangian density (in which Λ is absorbed), thereby providing us with a generalization of the ‘isotropic’ local rescaling, known also as WI, that will be discussed below and in the next section. In this framework gravitation is described by both a tensor field $\Phi_{\mu\nu}$ and the spacetime metric $g_{\mu\nu}$ on which $\Phi_{\mu\nu}$ lives. We assume that $\Phi_{\mu\nu}$ is all-pervading much like the Higgs field is universal.

Naturally, the local (in general anisotropic) rescaling of the various fields is by physical dimension, e.g., the dimensions of $g_{\mu\nu}$, $g^{\mu\nu}$ and $\mathcal{L}_M$ are $length^2$, $length^{-2}$ and $length^{-4}$, respectively. The contravariant coordinates, x^μ , are dimensionless, and the covariant coordinates x_μ are of dimension $length^2$. Specifically, the anisotropic local scale transformations for the metric and tensor fields, as well as the lagrangian density, are

$$\begin{aligned} g_{\mu\nu} &\rightarrow \Omega_\mu^\alpha g_{\alpha\beta} \Omega_\nu^\beta \\ \Phi_\mu^\alpha &\rightarrow \Phi_\mu^\rho (\Omega^{-1})_\rho^\alpha \\ \Phi_\mu^\alpha &\rightarrow (\Omega^{-1})_\rho^\alpha \Phi_\mu^\rho \\ \mathcal{L}_M &\rightarrow \mathcal{L}_M / |\Omega(x)|, \end{aligned} \quad (3)$$

where $|\Omega(x)|$ is the determinant of an arbitrary spacetime-dependent matrix Ω_μ^ν , and leaves $\bar{g}_{\mu\nu}$ and $\bar{\mathcal{L}}_M$ that are defined in Equation (2) invariant. Therefore, if we replace $g_{\mu\nu}$ and $\mathcal{L}_M$ with $\bar{g}_{\mu\nu}$ and $\bar{\mathcal{L}}_M$, respectively, everywhere in Equation (1) then the latter is invariant under Equation (3) in addition to being covariant.

It is useful, once again, to distinguish the pure gravitational construction from the complete theory. If Φ_μ^α is introduced only through the combination $\bar{g}_{\mu\nu}$, carries no independent internal information, and all sectors of the theory are transformed consistently, then the construction may indeed be regarded as a local field redefinition or a redundancy in the description. This conclusion applies to the isolated gravitational sector, as also discussed in related recent Weyl-invariant scalar–tensor and Einstein–Cartan constructions [22–24]. In the framework adopted here, however, only gravitation is endowed with local scale invariance, while the SM of particle physics is retained in its usual non-scale-deformation-invariant form. The relative scaling of Planck and active gravitational masses with respect to inertial masses is therefore physical and cannot be removed by a Weyl transformation of the gravitational sector alone. Independently of this first departure from standard GR coupled to the SM, the scale deformation field may carry internal structure, such as the $U(1)$ or $SU(2)$ sectors discussed below. Conserved quantities associated with these internal symmetries are invariant and may act as genuine classical sources in the gravitational field equations.

In the special case that the scaling matrix is isotropic, $\Phi_\mu^\nu = \phi(x)\delta_\mu^\nu$, i.e., the metric scale deformation is isotropic, the proposed symmetry reduces to a WI version of GR, i.e., the aforementioned WIST theory, in which case the replacement $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = g_{\mu\nu}\phi^2(x)$ and $\mathcal{L}_M \rightarrow \mathcal{L}_M/\phi^4$ is made in Equation (1), where $\phi(x)$ is an arbitrary scalar function. This implies that the theory is invariant under the Weyl transformation

$$\begin{aligned} g_{\mu\nu} &\rightarrow \tilde{g}_{\mu\nu} = g_{\mu\nu}\Omega^2(x) \\ \phi &\rightarrow \tilde{\phi} = \phi\Omega^{-1}(x) \\ \mathcal{L}_M &\rightarrow \tilde{\mathcal{L}}_M = \mathcal{L}_M\Omega^{-4}(x). \end{aligned} \quad (4)$$

This isotropic case will be further explored in the next section.

Setting $G = 3/(8\pi)$ and replacing $\mathcal{L}_M$ with V (the latter contains no derivatives of the various fields), and $g_{\mu\nu}$ with $\bar{g}_{\mu\nu}$ (see Equation (2)) everywhere in Equation (1), we obtain

$$\tilde{\mathcal{I}} = \int \left(\frac{1}{6}(\bar{R} - 2\Lambda) + V \right) \sqrt{-\bar{g}} d^4x. \quad (5)$$

As will be detailed below we assume that V , the source of spacetime curvature, is a potential in $\Phi \equiv \sqrt{\Phi_\alpha^\beta \Phi_\beta^\alpha}$. In particular, we make the assumption that it is an analytic polynomial in

Φ . Consider the transformations described by Equation (3). To first order, an infinitesimal transformation results in

$$\bar{g}_{\mu\nu} = \Omega_\mu^\alpha g_{\alpha\beta} \Omega_\nu^\beta \approx g_{\mu\nu} + \delta\Omega_\mu^\alpha g_{\alpha\nu} + g_{\mu\alpha} \delta\Omega_\nu^\alpha, \quad (6)$$

where we used $\Omega_\mu^\nu \approx \delta_\mu^\nu + \delta\Omega_\mu^\nu$. Similarly,

$$\bar{\Phi}_\mu^\nu = (\Omega^{-1})_\mu^\alpha \Phi_\alpha^\nu \approx (\delta_\mu^\alpha - \delta\Omega_\mu^\alpha) \Phi_\alpha^\nu, \quad (7)$$

from which it follows that

$$\delta\Phi_\mu^\nu \approx -\delta\Omega_\mu^\alpha \Phi_\alpha^\nu. \quad (8)$$

Employing the relation between variations in inverse matrices, $\delta g^{\mu\nu} = -g^{\mu\alpha} \delta g_{\alpha\beta} g^{\beta\nu}$ and using Equation (6) we obtain

$$\delta g^{\mu\nu} \approx -g^{\mu\alpha} \delta\Omega_\alpha^\nu - \delta\Omega_\alpha^\mu g^{\alpha\nu}. \quad (9)$$

In our convention that x^μ are dimensionless (see discussion below Equation (2)), $\int V \sqrt{-g}$ is dimensionless and so $\delta(V \sqrt{-g}) = \frac{(\delta V \sqrt{-g})}{\delta g^{\mu\nu}} \delta g^{\mu\nu} + \frac{(\delta V \sqrt{-g})}{\delta \Phi_\mu^\nu} \delta \Phi_\mu^\nu + \frac{(\delta V \sqrt{-g})}{\delta \Phi_\mu^\alpha} \delta \Phi_\mu^\alpha = 0$ under the transformations Equations (8) and (9). Requiring invariance under arbitrary $\delta\Omega_\mu^\nu$ we obtain

$$S_\mu^\nu = \Phi_\mu^\alpha \frac{\delta V}{\delta \Phi_\alpha^\nu} + \frac{\delta V}{\delta \Phi_\nu^\alpha} \Phi_\mu^\alpha \quad (10)$$

where $S_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{(\delta V \sqrt{-g})}{\delta g^{\mu\nu}}$. This provides a severe constraint on the potential if anisotropic local scale invariance, Equations (1)–(3), is to be a symmetry of the gravitational interaction. We will rederive the analog of this result in the isotropic scaling case and delve into the implications in Sections 3 and 4.

We emphasize that only gravitation is endowed with WI in the proposed framework. All other fundamental interactions, namely the strong and electroweak interactions, are described by the SM of particle physics, with no modifications. Since any realistic physical system is governed by both gravitational and non-gravitational interactions, and since the fundamental interactions are minimally coupled to gravitation, it then follows that this particular framework is clearly not merely a field redefinition of GR and the SM of particle physics. In any case, introducing the global charge that is generally associated with $\Phi_{\mu\nu}$ due to internal symmetry as discussed in Section 6 and in Appendix B renders the theory described here inequivalent to the SM even in purely gravitational systems such as late-time cosmological scales.

Much like the nonvanishing and universality of the vacuum expectation value (VEV) of the Higgs field guarantees that inertia phenomena are ubiquitous, the scale deformation field is here assumed to be all-pervading. In a representation in which a reference metric is chosen, the tensor field Φ_β^α couples to the SM fields through the deformed metric $\bar{g}_{\mu\nu}$. For example, the kinetic term of the Higgs field, $I_{\text{kin}} = \int g^{\mu\nu} (D_\mu H)^\dagger D_\nu H \sqrt{-g} d^4x$, is transformed into $I_{\text{kin}} = \int \bar{g}^{\mu\nu} (D_\mu H)^\dagger D_\nu H \sqrt{-\bar{g}} d^4x$. If the reference metric is chosen to be Minkowski, this takes the form $I_{\text{kin}} = \int (D_\mu H)^\dagger (\Phi^{-1})^\mu_\alpha \eta^{\alpha\beta} (\Phi^{-1})^\nu_\beta D_\nu H |\Phi| d^4x$. Thus the scale deformation tensor couples directly to matter fields in such a representation.

This should not be confused with the statement that Φ_μ^α is literally the standard vierbein e^a_μ . Indeed, the construction is tetrad-like in the special sense that a metric is represented through a product of a reference metric and deformation fields. However, the usual vierbein carries one spacetime index and one internal Lorentz-frame index, whereas

Φ^α_μ is introduced here as a spacetime scale deformation tensor. In the special case in which the reference metric is Minkowski, the index α may be interpreted as an internal Lorentz-frame index and Φ^α_μ reduces, apart from the additional internal structure postulated in the present work, to the standard vierbein e^a_μ [25–27]. The present construction is nevertheless more general, since $g_{\alpha\beta}$ may be an arbitrary non-Minkowski reference metric, in which case Φ^α_μ is a scale deformation tensor relating two general metrics rather than an ordinary vierbein.

3. WIST Theory

In this section we focus on the isotropic case, i.e., $\Phi^\nu_\mu = \delta^\nu_\mu \phi(x)$, where $\phi(x)$ is a scale deformation (scalar) field. Although it was assumed in Equation (2) that Φ^ν_μ is real (in order for the new metric field, $\bar{g}_{\mu\nu}$, to be real) it is clear that at least in the special case considered in this section, i.e., $\Phi^\nu_\mu = \delta^\nu_\mu \phi(x)$, promoting $\phi(x)$ to a complex scalar field renders $\bar{g}_{\mu\nu}$ real. Since the scalar field is assumed to be complex, it has a U(1) internal symmetry. The more general case of an arbitrary internal symmetry is briefly discussed in Section 6. Gravitation is then fundamentally endowed with WI, i.e., an isotropic local scale invariance, within a WIST theory, and standard GR is recovered in a particular conformal frame, where $\phi = \text{constant}$. However, this conformal equivalence does not exhaust the present framework once the scale field is treated as genuine additional field content and endowed with an internal symmetry: in that case, conserved quantities may remain even after a Weyl gauge choice, and the theory need not collapse to standard GR.

3.1. WIST Action and Basic Interpretation

Rather than using the more general results from the previous section we derive them directly in the isotropic case. The WIST theory readily follows from locally rescaling the spacetime metric $g_{\mu\nu} \rightarrow \phi\phi^* g_{\mu\nu}$ everywhere in Equation (1). The resulting theory is then described by the following action

$$\begin{aligned}\mathcal{I}_{WIST} &= \int \left(\frac{1}{6} |\phi|^2 R - \phi^* \square \phi + V(|\phi|, \{\Psi\}) \right) \sqrt{-g} d^4x \\ &= \int \left(\frac{1}{6} |\phi|^2 R + \phi_\mu \phi^{*\mu} + V(|\phi|, \{\Psi\}) \right) \sqrt{-g} d^4x,\end{aligned}\quad (11)$$

where $\phi_\mu \equiv \frac{\partial \phi}{\partial x^\mu}$, V that replaces $\mathcal{L}_M$ is now allowed to explicitly depend on $|\phi|$ but not on its derivatives, and the second equality follows from integration by parts of the kinetic term associated with the scalar field. All other fields (including $g_{\mu\nu}$) are collectively denoted by $\{\Psi\}$. The potential/source term $V(|\phi|, \{\Psi\})$ is not an arbitrary scalar potential. A generic term such as $m^2 |\phi|^2$ with a fixed mass parameter would indeed break WI. In the present construction the allowed dependence of V on $|\phi|$ and on the other fields is constrained by the requirement that the full source term has the appropriate Weyl weight. This is the isotropic analog of the constraint derived in Equation (10), and is made explicit in the discussion leading to Equation (18) below. We emphasize that in the new formulation, Equation (11), gravitation is sourced by V rather than by ‘matter’, $\mathcal{L}_M$. In general, GR is consistent with observations within the solar system and so on these scales V can be replaced by $\mathcal{L}_M$, i.e., ‘ordinary’ matter. However, on larger scales, where both DM and DE are required by observations we favor the form of Equation (11) which does not imply the existence of DM particles. In addition, in Equation (11), geometry and matter are not presented as two different entities. Rather, the existence of ϕ in both the source, V , and the kinetic term reflects the viewpoint that they are actually intertwined. They are ‘unified’ in the sense that WI of Equation (11), a symmetry that is not recognized by GR, depends on both the scalar and metric fields.

3.2. Origin of the Wrong-Sign Term and WI

With V being explicitly ϕ -dependent, Equation (11) is now a scalar–tensor theory of the Bergmann–Wagoner type [28,29]. It was first obtained by Deser [6] and later in [12] with $\mathcal{L}_M$ [referred to as V in Equation (11)] that does not necessarily depend on ϕ . In [6] ϕ was assumed to be real. A similar procedure to the replacement $g_{\mu\nu} \rightarrow \phi\phi^*g_{\mu\nu}$ that transforms Equations (1)–(11) was employed by Chamseddine and Mukhanov in their Mimetic Gravity [30] but in the latter case the scalar field was an irrotational velocity potential φ , i.e., $g_{\mu\nu} \rightarrow \varphi_{,\mu}\varphi^{,\mu}g_{\mu\nu}$.

Formally, the kinetic term associated with the scalar field that appears in Equation (11), $\mathcal{L}_\phi \equiv \phi_{,\mu}\phi^{*,\mu}$ or $\mathcal{L}_\phi \equiv -\phi^*\square\phi$, can be viewed as a new source of the gravitational interaction. This term is completely ignored in GR where ϕ is set to a constant value, $\phi_0 \equiv \sqrt{\frac{3}{8\pi G}}$. Invariance of Equation (11) implies that $V \rightarrow |\phi|^{-4}V$ under $g_{\mu\nu} \rightarrow |\phi|^2g_{\mu\nu}$. This simple derivation also underscores the origin of the ‘ghost’ scalar field whose kinetic term, $\mathcal{L}_\phi \equiv \phi_{,\mu}\phi^{*,\mu}$, appears with the ‘wrong’ sign in Equation (11); it simply stems from locally stretching/squeezing the spacetime metric, or equivalently stretching/squeezing the yardstick with which distances are measured, and in our case it is the Planck length $l_p \propto \sqrt{G}$ in vacuum, or $\propto (G\rho)^{-1/2}$ where ρ is the energy density in non-vacuum configurations. This observation is crucial for evaluation of the significance of the ghost field. We see that ϕ is tightly related to $g_{\mu\nu}$ (merely a local stretch of the spacetime metric), and as $g_{\mu\nu}$ is treated classically so should be ϕ and V ; the ghostly nature of ϕ derives solely from the specific form in which $g_{\mu\nu}$ and its derivatives appear in the EH action. Below we illustrate, through the FRW example, that the mere presence of the wrong-sign term does not by itself imply an ordinary classical runaway of the standard ghost type. Although this does not count as a general proof of stability it provides a useful guidance into interpretation within the WI classical framework. Instabilities are considered disastrous in the quantum regime because the Hamiltonian is unbounded from below.

Since Equation (11) was obtained from Equation (1) by effecting the transformation $g_{\mu\nu} \rightarrow \phi\phi^*g_{\mu\nu}$, invariance of the former is guaranteed insofar $|\phi|^2g_{\mu\nu}$ as invariant, i.e.,

$$\begin{aligned}\phi &\rightarrow \phi/\Omega \\ g_{\mu\nu} &\rightarrow \Omega^2g_{\mu\nu},\end{aligned}\tag{12}$$

for an arbitrary $\Omega(x) > 0$, i.e., Equation (4). Equation (12) is the Weyl transformation of metric and scalar fields, respectively. It is thanks to this freedom that $\phi = \phi_0/\Omega(x)$ is arbitrary, and so although it is spacetime-dependent it is not dynamical in the sense that it is not determined by a dynamical differential equation which has an attractor solution, that, e.g., drives the kinetic term, $\mathcal{L}_\phi \equiv \phi_{,\mu}\phi^{*,\mu}$, to arbitrary negative values, as is the case with generic ghost fields. Section 4 instead shows that, after fixing a static conformal representation, the scale field reproduces the same constrained FRW evolution as the usual scale factor and does not introduce an additional classical runaway absent from GR.

In fact, the presence of the kinetic term in $\mathcal{L}_\phi \equiv \phi_{,\mu}\phi^{*,\mu}$ is expected once G is promoted to a field. More specifically, since the curvature scalar depends on derivatives of the metric field it transforms inhomogeneously under Equation (12)

$$R \rightarrow \Omega^{-2}\left(R - 6\frac{\square\Omega}{\Omega}\right),\tag{13}$$

where $\square$ is the d’Alembertian. Invariance of Equation (11) under Equation (12) then requires the presence of the kinetic term $\mathcal{L}_\phi \equiv \phi_{,\mu}\phi^{*,\mu}$ that transforms inhomogeneously and guarantees the mutual cancelation of derivatives of Ω (provided that the appropriate integration by parts has been carried out). The prefactor $\frac{1}{6}$ in front of the curvature coupling

term in Equation (11) guarantees that the inhomogeneous term on the right hand side of Equation (13) is compensated by a similar inhomogeneous term from the transformation of the kinetic term, $\phi_\mu^* \phi^\mu$. In other words, under the transformation in Equation (12) the combination of the first two terms in Equation (11) transforms as $\frac{1}{6}|\phi|^2 R - \phi \square \phi^* \rightarrow \Omega^{-4}(\frac{1}{6}|\phi|^2 R - \phi \square \phi^*)$, i.e., it has the appropriate well-defined conformal weight of $length^{-4}$, such that its product with the volume element in Equation (11) is WI.

In the special case that ϕ is a real field and V is independent of ϕ , Equation (11) can be cast in the form of a Brans–Dicke (BD) theory [31] $\mathcal{I}_{BD} = \int \left(\Phi R - \omega_{BD} \frac{\Phi_\mu \Phi^\mu}{\Phi} + \mathcal{L}_M \right) \sqrt{-g} d^4x$ with Φ (not to be confused with the norm of $\Phi_{\mu\nu}$ that is defined just below Equation (5)) and ω_{BD} being the BD scalar field and dimensionless BD parameter, respectively, where $\Phi \equiv \phi^2/6$ and $\omega_{BD} \equiv -3/2$. Whereas a tight lower limit, $\omega_{BD} > 40,000$, has been reported in [32], it should be stressed that it was obtained from satellite observations directly exploring the impact of varying G within our own solar system, and not beyond. In addition, in the (generally non-WI) BD framework the source term $\mathcal{L}_M$ does not in general depend on ϕ while in WIST the potential V does.

3.3. Wrong-Sign Kinetic Term, Quantization, and Why This Is Not an Ordinary Ghost

Equation (11) contains a wrong-sign kinetic term for the Weyl compensator, and this is sometimes taken in the literature to indicate a fatal ghost instability. In the WI isotropic case considered here, however, that default conclusion does not readily follow. At the classical level, the sign by itself does not establish an ordinary propagating negative energy mode; rather, its interpretation is related to the gravitational conformal sector. This point is clarified in the following subsection and made more explicit in the comparison with the more general non-minimally coupled scalar theory below.

In an ordinary non-gravitational field theory, a wrong-sign kinetic term is often taken as evidence for pathology. In the WIST case, however, this inference is not warranted. The field ϕ in the WI representation does not play the role of an ordinary unconstrained matter degree of freedom (DOF) with negative energy; rather, the sign reflects the indefinite kinetic structure already familiar from gravity. This is seen most simply in minisuperspace, where the FRW scale factor also appears with a wrong-sign kinetic term (and the scale factor plays the role of a scalar field living in static FRW-like spacetime), yet this is not interpreted as establishing a propagating ghost field in standard cosmology. The same standard applies here as well; at the classical level, the wrong sign by itself does not indicate a fatal instability. This limited aspect should not be confused with a claim to have resolved all quantization issues of the gravitational conformal sector. However, those issues are already familiar at the level of gravity itself and are not specific to the present representation.

The fact that FRW scale factor a enters the reduced action in minisuperspace with a wrong-sign kinetic term, yet one does not ordinarily conclude from this that standard FRW cosmology is plagued by a propagating ghost, is because a is part of a constrained gravitational system rather than an ordinary unconstrained field theory. Likewise here, rewriting the gravitational sector in WI form does not create a new pathological DOF; it merely makes explicit a sign structure that is already implicit in the GR-based gravitational action. In particular, as shown in Section 4, the FRW action can be rewritten directly in WIST form, with the same apparent sign issue appearing in a well-known gravitational setting.

At the classical level this is sufficient for the purposes of the present work. WI implies that ϕ is not dynamically driven by an independent second-order field equation in the special case $\xi = 1/6$; rather, it reflects the freedom to choose the local scale. The usual classical runaway argument associated with non-Weyl ghost models therefore does not apply here.

In the quantum regime additional considerations apply, but the conclusion is still subtler than the standard ghost diagnosis. Canonical quantization of cosmology already quantizes variables with this same wrong-sign character. In the Wheeler–DeWitt approach, for example, the wave function of the Universe depends on variables such as a and matter fields, and the kinetic operator on minisuperspace is again negative rather than positive definite. And despite the wrong sign, one still quantizes the gravitational DOF. What is being quantized is not an ordinary negative energy particle field, but a constrained gravitational variable. The negative sign reflects the indefinite signature of superspace, not by itself the presence of a propagating ghost particle.

This does not mean that all quantum issues are resolved. In Euclidean approaches to quantum gravity, including Hartle–Hawking-type constructions, the same sign issue is related to the well-known conformal-factor problem; the Euclideanized gravitational action is not bounded below along the conformal direction. But this is a problem of defining the gravitational path integral, not a clear demonstration that the theory contains a standard quantum field theory (QFT) ghost with catastrophic instability. Put alternatively, the wrong sign is real, but its interpretation is gravitational and constrained, not that of an ordinary matter ghost. Likewise, the conserved internal charges are treated here as classical integration constants or classical Noether charges. Their negative effective contribution in the reduced cosmological equations is not interpreted as a quantum ghost instability; it is precisely the classical source term that enables the non-singular bounce discussed in [4]. A full constrained Hamiltonian analysis and a quantum stability analysis of the combined Weyl/internal sector would be valuable, but they are beyond the scope of the present classical work.

Within this realm, the appearance of a wrong-sign kinetic term in Equation (11) should not be taken as evidence for a fatal instability of the sort familiar from unconstrained ghost fields in standard QFT. Rather, it is interpreted as a manifestation of the indefinite kinetic structure of the gravitational conformal sector. Clearly, this is not meant as a proof of quantum stability or as a resolution of the Euclidean conformal-factor ‘problem’, and indeed a long-standing and well-recognized problem that frequently reappears in WI formulations. Classically, we maintain this is not a problem.

3.4. *Physical Departures from Standard GR Coupled to the Standard Model*

This section re-states the two points at which the present framework physically departs from the standard combination of GR and the SM. At the level of the pure gravitational sector, promoting the fixed Planck scale to a spacetime-dependent scalar field ϕ , or more generally to a tensorial scale deformation field Φ^μ_ν , does not by itself enlarge the physical content; in the absence of internal charges, the Weyl freedom can be gauge-fixed back to an ordinary GR description. The complete framework considered here is nevertheless not merely a reformulation of standard GR coupled to the SM, because the gravitational sector is locally scale-invariant whereas the SM remains non-WI. Thus, Planck and active gravitational masses transform with the scale field while inertial masses do not. The internal-symmetry sector then provides a second and independent extension, via additional internal DOFs and their conserved global charges.

In this sense, the present construction is conceptually no different from other familiar model-building choices in fundamental physics. One may formulate Yang–Mills (YM) theories with different internal groups, such as $SU(2)$ or $SU(3)$, and the resulting theories simply possess different field content and different numbers of physical DOF. Nature then decides which choice, if any, is realized. Likewise here, the point is not that GR “should” contain the extra sector introduced in the present work, but that one may consistently pos-

tulate such a sector and explore its consequences. The question of necessity is observational; the question of consistency and phenomenology is theoretical.

This internal sector should be distinguished from local WI, which is a local spacetime redundancy associated with rescalings of the metric and scale field. By contrast, the $U(1)$ or $SU(2)$ structures discussed here refer to an internal organization of the additional scale DOF. Once this internal sector is present, conserved quantities may arise that are not removed by fixing the local Weyl gauge. Thus, even when one works in an EF description, the theory need not reduce to standard GR, because the extra sector can still carry physical information.

This point is explicit in the cosmological reduction. In the $SU(2)$ case discussed in Appendix B, the scalar multiplet is written in terms of a modulus ρ and angular variables on the internal space. After integrating the corresponding EL equations once, one obtains conserved integration constants, and the reduced kinetic term becomes equivalent to replacing φ by ρ together with an additional contribution $-c_\theta^2/\rho^2$ to the effective potential. The constants c_χ and c_ϕ drop out of the final reduced expression, whereas c_θ remains and acts as a conserved global charge associated with the relevant $U(1)$ subgroup. This is not a gauge artifact of conformal frame choice, but a genuine remnant of the internal-symmetry sector.

The physical significance of this extra sector is that it can modify the cosmological dynamics in a way impossible in standard GR cosmology with ordinary matter. In particular, the term proportional to $-c_\theta^2/\rho^2$ competes with the radiation term at sufficiently small ρ and can generate a non-singular bounce. The bounce is therefore one concrete example of what the additional internal-charge sector can naturally accomplish.

The correct comparison therefore has two levels. In the isolated gravitational sector, and in the absence of independent internal charges, the scale field may be removed by a Weyl gauge choice and ordinary GR is recovered. Once this sector is coupled to the non-WI SM, however, the relative scaling of Planck and active gravitational masses with respect to inertial masses becomes physical, and the resulting theory is not equivalent to standard GR coupled to the SM. Finally, when the scale sector is endowed with a nontrivial internal symmetry, conserved charges survive the Weyl gauge choice and further enlarge the solution space, including in purely gravitational systems. Whether either departure is realized in nature is ultimately an observational question.

3.5. Why φ Is Non-Dynamical in the WI Case

It is constructive at this point to explore a slightly more general case of a dynamical scalar field which is non-minimally coupled to spacetime curvature

$$\mathcal{I}_{ST} = \int \left(\xi |\phi|^2 R + \phi_\mu^* \phi^\mu + V(|\phi|) \right) \sqrt{-g} d^4x, \quad (14)$$

where ξ is an arbitrary dimensionless coupling parameter, thereby generalizing Equation (11), i.e., the theory is in general not WI, unless $\xi = \frac{1}{6}$.

The field equations derived from variation in Equation (14) with respect to the metric and scalar field are, respectively,

$$2\xi |\phi|^2 G_{\mu\nu} = S_{M,\mu\nu} + \Theta_{\mu\nu}, \quad (15)$$

and

$$\xi \phi R - \square \phi + \frac{\partial V}{\partial \phi} = 0, \quad (16)$$

where

$$3\Theta_{\mu\nu} \equiv 6\zeta\phi^*\phi_{;\mu;\nu} + (6\zeta - 3)\phi_\mu^*\phi_\nu - 6\zeta g_{\mu\nu} \left[\phi^*\square\phi - \left(\frac{1}{4\zeta} - 1 \right) \phi_\alpha^*\phi^\alpha \right] + c.c. \quad (17)$$

Here and throughout $S_{M,\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}V)}{\delta g^{\mu\nu}}$ is the analog of the energy-momentum tensor. Equation (15) is a generalization of Einstein equations with $2\zeta|\phi|^2$ replacing $1/(16\pi G)$, and $\Theta_{\mu\nu}$ is an effective contribution to the energy-momentum tensor essentially due to gradients of G and active gravitational masses; the latter appear in the source term, $\mathcal{L}_M$. Multiplying Equation (16) by ϕ^* , adding the result to its complex conjugate and to the trace of Equation (15) results in

$$\phi \frac{\partial V}{\partial \phi} + \phi^* \frac{\partial V}{\partial \phi^*} = S_M + (1 - 6\zeta)(\phi^*\square\phi + \phi_\alpha^*\phi^\alpha + c.c.), \quad (18)$$

where S_M is the trace of $S_{M,\mu\nu}$. Together, Equations (15) and (18) provide a system of dynamical equations for the ten metric components and the complex scalar field. In the BD form (ϕ is real and $\mathcal{L}_M$, that is here replaced by V , is independent of ϕ) Equation (18) reduces to the well-known form $(3 + 2\omega_{BD})\square\Phi = T_M$, where ω_{BD} is the dimensionless BD parameter. It follows from Equation (18) that for any $\zeta \neq \frac{1}{6}$ (equivalently $\omega_{BD} \neq -3/2$) the scalar field is dynamically determined by the matter distribution via a differential equation, which thereby embodies the Machianity of BD theory of gravity.

In contrast, in the special case $\zeta = \frac{1}{6}$ (equivalently $\omega_{BD} = -3/2$ [5–15]) Equation (14) reduces to the WIST form of Equations (11) and (18) to a constraint that only determines the dependence of V on the scalar field (as in the more general Equation (10)). In this special case, ϕ is indeed non-dynamical; it can be set to any desired function irrespective of the matter distribution, and Machianity is lost. Although ϕ still appears in Equation (15) we now have only ten differential equations for ten metric components and a scalar field—this under-determined system simply reflects the freedom to locally rescale the fields in the special case $\zeta = \frac{1}{6}$, merely a manifestation of WI, Equation (12) [5–15]. The latter implies invariance of Equation (11) under local rescaling of fields such that $V \rightarrow |\phi|^{-4}V$. This requires in particular that, e.g.,

$$\begin{aligned} \psi &\rightarrow \Omega^{-\frac{3}{2}}\psi \\ A_\mu &\rightarrow A_\mu, \end{aligned} \quad (19)$$

where ψ and A_μ are Dirac and vector fields, respectively, in addition to the transformation of the scalar and metric fields described in Equation (12)—each field is rescaled by its mass/length dimension. In the special case $\zeta = \frac{1}{6}$, Equation (18) could be derived solely from the requirement that $V\sqrt{-g}$ is WI, as was done in the previous section in the more general case, in Equation (10).

3.6. Conformal Modulus, the $U(1)$ Phase, and the General Internal-Symmetry Sector

It is useful to separate the local Weyl DOF from the genuinely physical internal sector. In the special isotropic WI case $\zeta = 1/6$, the modulus of the scalar field is not an independently propagating DOF in the usual sense. Indeed, given the local Weyl symmetry

$$\phi \rightarrow \phi/\Omega(x), \quad g_{\mu\nu} \rightarrow \Omega^2(x)g_{\mu\nu}, \quad (20)$$

we have

$$\phi(x) = \phi_0/\Omega(x), \quad (21)$$

with arbitrary positive $\Omega(x)$. Thus, whereas ϕ is generally spacetime-dependent, its modulus is not determined by a differential equation with ghost dynamics. In this sense the conformal modulus is a gauge choice associated with local scale symmetry, rather than an ordinary propagating matter ghost field. The appearance of a wrong-sign kinetic term in this parametrization therefore does not by itself signal any fatal classical instability; rather, it reflects the familiar conformal-sector structure already present in Weyl-related formulations of gravity, including the classical FRW minisuperspace reformulation discussed above.

The situation changes once the scale field carries a nontrivial internal symmetry. The simplest example is that of the complex scalar field that corresponds to a global $U(1)$ internal symmetry. Specifically, in case that

$$\phi(x) = \rho(x)e^{i\alpha(x)}, \quad (22)$$

the phase α does not appear in the curvature coupling term proportional to R , nor in a potential $V(|\phi|)$ that depends only on the modulus. It enters only through the kinetic term, whose α -dependent part is

$$S_\alpha \propto \int \rho^2 \alpha_{,\mu} \alpha^{,\mu} \sqrt{-g} d^4x. \quad (23)$$

Varying with respect to α and integrating by parts gives

$$\nabla_\mu (\rho^2 \partial^\mu \alpha) = 0, \quad (24)$$

or equivalently

$$\partial_\mu (\sqrt{-g} \rho^2 \partial^\mu \alpha) = 0. \quad (25)$$

Hence the $U(1)$ phase defines a conserved current,

$$J^\mu = \rho^2 \partial^\mu \alpha, \quad \nabla_\mu J^\mu = 0, \quad (26)$$

or, in density form,

$$j^\mu = \sqrt{-g} \rho^2 \partial^\mu \alpha, \quad \partial_\mu j^\mu = 0. \quad (27)$$

Since $\rho^2 \sqrt{-g}$ is WI, this conserved current density is itself WI.

This observation extends beyond $U(1)$. More generally, the additional DOFs are encapsulated by an internal group, such as $SU(2)$, $SU(N)$, or $O(N)$. In that case, the single phase α is replaced by an internal field $U(x)$, and one may schematically write

$$\Phi(x) = \rho(x) U(x). \quad (28)$$

The Weyl/radial part ρ again belongs to the conformal sector, whereas the genuinely new local dynamics resides in the internal orientation field $U(x)$. The corresponding kinetic term reduces to a sigma-model-type structure on the group manifold,

$$(\partial\Phi)^2 \sim (\partial\rho)^2 + \rho^2 \mathcal{G}_{ab}(\pi) \partial_\mu \pi^a \partial^\mu \pi^b, \quad (29)$$

where $\pi^a(x)$ are local coordinates on the internal manifold and $\mathcal{G}_{ab}$ is its induced metric. In the abelian $U(1)$ case this reduces to the single-phase equation above; in the non-abelian case one obtains a coupled set of equations for the internal coordinates together with the corresponding Noether currents associated with the generators of the symmetry group.

In summary, the conformal modulus belongs to the Weyl gauge sector, whereas the internal orientation fields yield conserved currents and charges that remain physical even after the Weyl modulus has been gauge-fixed.

3.7. Physical Interpretation: Masses, Frame Choice, and Relation to BD

Any mass terms that appear in V are by definition active gravitational masses, which need not be equivalent to inertial or passive gravitational masses. While the three types of mass are not necessarily equivalent [33], the notion of a passive gravitational mass in any theory of gravity that satisfies the equivalence principle is vacuous. If the ratio of the latter two mass types is a universal constant then the equivalence principle is satisfied—an assumption that we indeed make in the present work. Again, we emphasize that the other three fundamental interactions are described by the lagrangian of the SM of particle physics, $\mathcal{L}_{SM}$, where all masses are inertial, whether generated via the Higgs mechanism in the electroweak sector or via the explicitly broken chiral symmetry in quantum chromodynamics (QCD).

Since $\phi(x) = \Omega^{-1}(x)\phi_0$, where ϕ_0 is the fixed GR value, then for any given value such a choice $\Theta_{\mu\nu}$ can be calculated according to Equation (17) and used in Equation (15) to solve for the corresponding $g_{\mu\nu}$. In practice, however, it is easier to start from any known solution of the Einstein equations, ϕ_0 and $g_{\mu\nu}$, and transform it to $\Omega^{-1}(x)\phi_0$ and $\Omega^2(x)g_{\mu\nu}$ with arbitrary $\Omega(x)$ by virtue of WI. We emphasize, once again, that this latter procedure is only possible assuming that there are no global charges associated with the field.

In the theory described by Equation (11), the scalar field ϕ determines not only G but also active gravitational masses. Only pure radiation $S_{rad} = 0$ is consistent with the theory described by Equation (14) with $\xi = \frac{1}{6}$ unless V explicitly depends on $|\phi|$. In the case of perfect fluid $S = V$ [34], and with equation of state (EOS) w_M the trace is $S = -\rho_M(1 - 3w_M)$. It then follows from Equation (18) that $V \propto |\phi|^{1-3w_M}$ in case that $\xi = \frac{1}{6}$, i.e., it is linear and quartic in $|\phi|$ in cases of NR- and vacuum-like terms, respectively, and is independent of $|\phi|$ in case of pure radiation. Linearity of V in $|\phi|$ in the case of vanishing EOS, $w_M = 0$, suggests that active gravitational masses are regulated by $|\phi|$. Not only that the same ϕ determines both Planck mass and active gravitational masses is a necessary condition for the consistency of non-radiation sources with this WI theory, it is also a conceptually natural “conclusion” as the concept of active gravitational mass is meaningless unless it couples to G , and in this sense it seems natural that both quantities are determined by the same scalar field. This clearly does not have necessarily to be the case in general (as in, e.g., BD theory), but it is a nice property of the WI model described by Equation (11).

We emphasize that in the original BD proposal [31] the matter lagrangian, $\mathcal{L}_M$, does not explicitly depend on the scalar field. Notably, Brans and Dicke required $\omega_{BD} > 0$ to guarantee the positivity of the Hamiltonian in their original proposal [31], a fact that was emphasized and reinterpreted in [35]. The instability of BD theories with $\omega_{BD} < 0$ was further emphasized in [36], but as we argued above, the non-positive kinetic term of the scalar field is not an issue in the special case $\omega_{BD} = -3/2$ and $V = V(|\phi|)$ due to WI. The observational lower limit $\omega_{BD} \geq 40,000$ reported in [32] implies that the BD field Φ is very nearly a constant, essentially reducing the theory to GR. But, again, this conclusion applies to the BD theory, where $\mathcal{L}_M$ is independent of the scalar field, and in addition (perhaps

not less important and relevant to astrophysics and cosmology) this tight limit has been obtained in our solar system and clearly does not readily apply on galactic scales or beyond.

4. The FRW Spacetime in WIST Representation

In the present section we cast the FRW action in the form of Equation (11) in a particular conformal frame where spacetime is static. The redshifting Universe may then be represented, in this conformal frame, not as an expansion of the spatial metric but as a temporal evolution of the mass/length standards used to describe observations. Specifically, the Big Bang is formally the divergence of the Planck yardsticks at $t = 0$. In other words, the divergence of curvature in the standard description is replaced by divergence of a scalar field, the Planck length. This construction should be understood as a change in representation, not as a proof of geodesic completeness or frame-invariant singularity resolution.

The FRW action in its minisuperspace form

$$\mathcal{I}_{FRW} = \int [-a'^2 + Ka^2 - \Lambda a^4/3 + a^4 \mathcal{L}_M(a)] \sqrt{-\gamma} d^4x, \quad (30)$$

is obtained from Equation (1) in units where $3/(8\pi G) \equiv 1$ and assuming that the metric is $g_{\mu\nu} = a^2 \gamma_{\mu\nu}$, where the two metrics $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ are conformally related, and $\gamma_{\mu\nu} \equiv \text{diag}(-1, \frac{1}{1-Kr^2}, r^2, r^2 \sin^2 \theta)$ is the metric in the comoving frame, $\gamma \equiv \det(\gamma_{\mu\nu})$, and after the term proportional to the corresponding curvature scalar $a^4 R = 6a^4(a''/a + K)/a^2$ is integrated by parts. Here, K is the spatial curvature parameter, $a(\eta)$ is the purely time-dependent scale factor, a prime denotes derivatives with respect to η , where $d\eta \equiv dt/a$ and $d^4x = d\eta dr d\theta d\phi$.

Since null geodesics are blind to conformal metric transformations we expect light to still be redshifted in the WIST description with the spatially static metric $\gamma_{\mu\nu}$ now replacing the standard expanding space metric $g_{\mu\nu}$. This is explained by the fact that $-d\eta^2 = -dt^2/a^2$, and so the lapse function, γ_{00} , is a^{-2} if the metric $\gamma_{\mu\nu}$ is presented in cosmic time, t , coordinates, rather than conformal η . Incoming photon wavelengths are not stretched by space expansion in this description (space is static) but rather by the temporally evolving gravitational potential, $\gamma_{00}(t)$, which is $\propto a^{-2}(\eta)$, which as we see below would be $\propto \phi^{-2}(\eta)$. In other words, cosmological redshift is (indirectly) due to the growing (decreasing) Planck mass (length). A more direct way to explain redshift is by the fact that in the comoving frame, described by $\gamma_{\mu\nu}$, Planck masses M_0 are effectively transformed to $M = a(\eta)M_0$, i.e., the Planck length monotonically decreases with time. With respect to this contracting yardstick space seems to expand. Of course, only dimensionless observables are directly measurable. The static-space/varying-mass description is therefore not meant to introduce a new observable redshift in addition to the usual one, but rather to provide an equivalent conformal representation of the same dimensionless ratios.

We thus see that the ratio between either the Planck or active gravitational mass to the inertial mass is time-dependent on cosmological scales where the distribution of matter is assumed to be homogeneous and isotropic and Equation (30) applies. We stress that this is generally the case in general spacetimes and matter distributions since the inertial mass is independent of the scalar field. It is only due to the specific symmetries of the cosmological model making it amenable to a particular combination of Weyl (e.g., $g_{\mu\nu} \rightarrow \gamma_{\mu\nu} = g_{\mu\nu}/a^2$) and coordinate (i.e., $dt \rightarrow d\eta = dt/a$) transformations that, e.g., the effective ratio of inertial-to-Planck mass is fixed at the background level. However, in the presence of small perturbations of the scalar field the Planck and active gravitational mass are perturbed while inertial masses are not. This difference is critical for certain applications. For example, it allows DM-like perturbative effects to be modeled in the present framework by scalar field perturbations, essentially by $\lesssim 10^{-3} - 10^{-4}$ fractional variations in G on galactic and

galaxy-cluster scales, as explored phenomenologically in [1,4]. Otherwise they could be gauged away and DM phenomena cannot be explained this way.

As emphasized above, only the effective ratio of inertial-to-Planck mass is fixed at the background level in the sense that inferentially both scale linearly with $a(\eta)$ although fundamentally only the Planck and active gravitational masses do. Specifically, since the time coordinate rescales ($dt \rightarrow d\eta$) so does the frequency of emitted light. If the mass of the electron that appears in the Rydberg formula is to be inferred from the redshifted light on a static spacetime it is then concluded that its inertial masses evolve $\propto a(\eta)$. In that sense the ratio of the Planck and (the effective) inertial masses is universally fixed.

It can be readily verified that the Euler–Lagrange (EL) equation for the scale factor $a(\eta)$ that extremizes the action $\mathcal{I}_{FRW}$ is the acceleration equation, whose first integral is indeed the Friedmann equation $\mathcal{H}^2 + K = a^2\rho_M + \frac{\Lambda a^2}{3} + \text{const.}/a^2$, where $\mathcal{H} \equiv a'/a$ is the conformal Hubble function, and it is assumed that $\mathcal{L}_M(a)$ is a power-law in a , much like ρ_M is a power-law in standard cosmology. Here, $\mathcal{L}_M(a)$ accounts for all sources (e.g., dust, radiation, etc.) other than spatial curvature and cosmological constant (which are characterized by effective equations of state $-1/3$ and -1 , respectively). Completing the derivation requires that the integration constant is related to the energy density of radiation $\rho_r = \text{const.}/a^4$. Since $\rho_M > 0$ it is clear that the kinetic and potential terms in Equation (30) have the ‘wrong’ relative sign (unless $K > 0$ or $\Lambda < 0$), yet the GR-based FRW model is not considered to be ‘plagued with ghosts’ or ‘disastrous instabilities’ at the classical level. On the contrary, FRW is the backbone of the remarkably successful standard cosmological model. This is similar to how in minisuperspace the Friedmann equation is obtained as the Hamiltonian constraint by retaining the lapse function before fixing the time gauge and varying with respect to that lapse. Minisuperspace-like variation similarly results in the acceleration equation, e.g., [37], which contains no independent information. Equivalently, in the gauge-fixed conformal-time action above, the scale-factor equation may be integrated once, with the integration constant identified with the radiation contribution.

Redefining the scale factor $a \rightarrow \phi$, the FRW action is reformulated as a WIST theory with a non-positive kinetic term, defined on a static background

$$\mathcal{I}_{FRW} = \int \left(\frac{1}{6} R \phi^2 - \phi'^2 - \lambda \phi^4 + \tilde{\mathcal{L}}_M(\phi) \right) \sqrt{-\gamma} d^4x. \quad (31)$$

Here, $\tilde{\mathcal{L}}_M = a^4 \mathcal{L}_M$, $R = 6K$ and $\lambda \equiv \Lambda/3$. The single DOF of standard FRW spacetime, $a(\eta)$, is here replaced by the scalar field $\phi(\eta)$. Clearly, renaming the scale factor, $a \rightarrow \phi$, in going from the general relativistic Equation (30) to its WIST presentation, Equation (31), does not introduce any new dynamics; in particular, no unacceptable new instabilities that are not already present in the FRW spacetime arise due to this field redefinition. The relative ‘wrong’ sign between the kinetic and potential term is simply manifested in that the integration of the Friedmann equation results in $a(\eta)$ [or equivalently $\phi(\eta)$] that evolves monotonically with η rather than having oscillatory behavior.

As Equations (30) and (31) are equivalent any perturbation that is either $a(\eta)$ of Equation (30) or $\phi(\eta)$ of Equation (31) will have precisely the same dynamics. Exactly as in the standard cosmological model δa is never considered as a dynamical DOF, so should be the case with $\delta \phi$. This is so since $g_{\mu\nu} = a^2 \gamma_{\mu\nu}$, and therefore any perturbation in a can be absorbed in $\delta \gamma_{\mu\nu}$, a perturbation of $\gamma_{\mu\nu}$. Considering scalar metric perturbations only—these are the Newtonian potentials. Therefore, invoking WI implies that any perturbation in G induces scalar metric perturbations [2], and vice versa—any excessive gravitational potential (not accounted for by ordinary SM matter) could be explained by corresponding perturbations in G [1].

As we just saw, local isotropic scaling has sufficient freedom to allow for reformulation of the (background) FRW model on a static background. However, this is not generally the case with Bianchi Universes. The latter are homogeneous but not isotropic. As such, they may have up to three different expansion rates along the three principal axes for which a corresponding number of functional DOFs are required. For that purpose, the more general anisotropic local rescaling freedom described in Section 2 is required. In addition, at the perturbation level and going beyond the cosmological domain, vector and tensor perturbation modes cannot be accounted for by a single function ϕ , and for that purpose the more general theory described in Section 2 is required as well, as was already mentioned above.

5. Bianchi Spacetimes in Anisotropic Scale-Invariant Representation

In this section we generalize the construction of Section 4 to homogeneous but anisotropic cosmologies of Bianchi type. Although such models are observationally disfavored as global descriptions of the Universe, they provide a useful testbed for understanding anisotropic gravitational collapse, shear-dominated epochs, and intermediate stages of structure formation. In particular, they allow us to illustrate how anisotropic spacetime dynamics can be reinterpreted, within the locally anisotropic scale deformation framework described in Section 2, as evolution of direction-dependent G and active gravitational masses rather than spacetime geometry itself.

For concreteness we focus on the type-I Bianchi model, whose line element in comoving coordinates is

$$ds^2 = -dt^2 + \sum_{i=1}^3 a_i^2(t) (dx^i)^2, \quad (32)$$

where the a_i s are the scale factors along three principal axes on that homogeneous but anisotropic spacetime. Introducing the mean (volume) scale factor

$$a(t) \equiv (a_1 a_2 a_3)^{1/3}, \quad (33)$$

and anisotropy variables $\beta_i(t)$ satisfying $\sum_i \beta_i = 0$, we write

$$a_i(t) = a(t) e^{\beta_i(t)}. \quad (34)$$

Defining conformal time by

$$d\eta \equiv \frac{dt}{a(t)}, \quad (35)$$

the metric becomes

$$ds^2 = a^2(\eta) \left[-d\eta^2 + \sum_{i=1}^3 e^{2\beta_i(\eta)} (dx^i)^2 \right]. \quad (36)$$

Unlike in the isotropic FRW case, the metric inside the brackets still carries explicit time dependence through the shear variables $\beta_i(\eta)$. However, within the locally anisotropic scale-invariant theory of Section 2, this time dependence need not be attributed to spacetime geometry. To make this explicit, we introduce a fixed, time-independent homogeneous background metric $\gamma_{\mu\nu}$ appropriate to the given Bianchi type, and transfer all time dependence into a scale deformation tensor $\Phi_\mu{}^\nu(\eta)$. Specifically, the scale deformation tensor is defined by

$$\Phi_\mu{}^\nu(\eta) = \phi(\eta) \Sigma_\mu{}^\nu(\eta), \quad (37)$$

with

$$\phi(\eta) \equiv a(\eta), \quad (38)$$

and

$$\Sigma_\mu{}^\nu(\eta) = \text{diag}(1, e^{\beta_1(\eta)}, e^{\beta_2(\eta)}, e^{\beta_3(\eta)}), \quad \det \Sigma_\mu{}^\nu = 1. \quad (39)$$

With these definitions, the spacetime metric can be written identically as

$$g_{\mu\nu}(\eta) = \Phi_\mu{}^\alpha(\eta) \gamma_{\alpha\beta} \Phi_\nu{}^\beta(\eta), \quad (40)$$

where $\gamma_{\mu\nu}$ is time-independent by construction. All cosmological evolution, including anisotropic expansion and shear, is thus encoded in the tensor $\Phi_\mu{}^\nu$ rather than in the background spacetime geometry. In the anisotropic theory, $\Phi_\mu{}^\nu$ can be viewed as a direction-dependent local ruler (in a four-dimensional sense), equivalently a ‘mass tensor’ that regulates active gravitational masses and effective couplings anisotropically in spacetime.

This construction is the anisotropic generalization of the FRW reformulation of Section 4. There, isotropic expansion was reinterpreted as evolution of a scalar mass scale ϕ on a static background. Here, anisotropic expansion and shear are reinterpreted as evolution of the tensorial mass scale $\Phi_\mu{}^\nu = \phi \Sigma_\mu{}^\nu$, while spacetime itself remains static and isotropic in comoving coordinates.

Analogous to the isotropic case described in Equation (30) the EH action specialized to the Bianchi I metric can be written, after integration by parts and in units $3/(8\pi G) = 1$, as

$$\mathcal{I}_{BI} = \int \left[-a'^2 + a^2 \sum_i \beta_i'^2 - \lambda a^4 + a^4 \mathcal{L}_M \right] \sqrt{-\gamma} d^4x, \quad (41)$$

where primes denote derivatives with respect to conformal time and $\lambda \equiv \Lambda/3$.

Rewriting this action in terms of ϕ and $\Sigma_\mu{}^\nu$ yields

$$\mathcal{I}_{BI} = \int \left[-\phi'^2 + \phi^2 \text{Tr}[(\Sigma^{-1}\Sigma')^2] - \lambda \phi^4 + \tilde{\mathcal{L}}_M(\phi, \Sigma) \right] \sqrt{-\gamma} d^4x, \quad (42)$$

where $\text{Tr}[(\Sigma^{-1}\Sigma')^2]$ reduces exactly to $\sum_i \beta_i'^2$ for the diagonal $\Sigma_\mu{}^\nu$ defined above, and $\tilde{\mathcal{L}}_M \equiv a^4 \mathcal{L}_M$.

No explicit time dependence appears in $\gamma_{\mu\nu}$, reflecting the fact that the background spacetime is static (when the time coordinate is η), while all gravitational dynamics reside in $\Phi_\mu{}^\nu$. As in the FRW case, the non-positive kinetic term of ϕ does not signal an instability; ϕ and $\Sigma_\mu{}^\nu$ satisfy exactly the same constraint structure as the scale factor and shear variables in standard Bianchi cosmology. No additional DOFs or new instabilities are introduced. For completeness, a minisuperspace derivation of the EL equations (equivalent to the Einstein equations) that correspond to Equation (42) is provided in Appendix C.

This reformulation clarifies the physical interpretation of anisotropic gravitational collapse. What is conventionally described as anisotropic spacetime contraction or shear growth can equivalently be described as the growth of anisotropy in the mass (or Planck) tensor, while spacetime itself remains static and isotropic in comoving coordinates. This perspective may be useful for describing anisotropic collapse of proto-structures and shear-dominated phases.

6. Global Symmetry

As reviewed in Section 3 one of the often-made claims about WIST is that the procedure of ‘conformalizing’ GR does not really result in modified physics. Specifically, it is argued that the conformal frame can always be so chosen so as to recover GR from Equation (11) by setting ϕ to a constant. Irrespective of the validity of this claim in light of the discussion towards the end of Section 2, it is clear that if a global symmetry is associated with the tensor field Φ_μ^ν then the presence of a global conserved charge definitely modifies the theory.

A widespread expectation in UV-complete quantum gravity is that *exact* global symmetries are not allowed (see, e.g., [38,39]). The proposed framework, however, is explicitly formulated as a classical (sub-Planckian) generalization of GR, and therefore does not attempt to address (or satisfy) such UV-completeness constraints. Within this classical domain, conserved global charges can be legitimately considered.

Specifically, the field Φ_μ^ν itself can be generalized to carry the indices of an internal (non-coordinate) symmetry, i.e., $\Phi_\mu^\nu \rightarrow \Phi_\mu^\nu = (\Phi_I^J)_\mu^\nu$ where boldface letters indicate that the tensor field carries internal indices (capital Latin letters) of a (typically unitary) group. The metric on (internal) group space is denoted $\mathcal{G}_{IJ}$, such that, e.g., $\Phi_\mu^\alpha \Phi_\alpha^\nu$ is a shorthand notation for $(\Phi_I^K)_\mu^\alpha \mathcal{G}_{KL} (\Phi^{IL})_\alpha^\nu$, and it is understood that contracted capital Latin letters are summed over. Equation (2) is then generalized to

$$\begin{aligned} g_{\mu\nu} &\rightarrow \bar{g}_{\mu\nu} \equiv \Phi_\mu^\alpha g_{\alpha\beta} \Phi_\nu^\beta, \\ g^{\mu\nu} &\rightarrow \bar{g}^{\mu\nu} \equiv (\Phi^{-1})^\mu_\alpha g^{\alpha\beta} (\Phi^{-1})^\nu_\beta, \\ \mathcal{L}_M &\rightarrow \bar{\mathcal{L}}_M = \mathcal{L}_M / |\Phi|, \end{aligned} \quad (43)$$

Here, it is important to distinguish between the isotropic and fully anisotropic case. The decomposition discussed above, in which the field is effectively characterized by a single modulus together with internal angular variables, corresponds only to the isotropic decomposition $\Phi_\mu^\nu(x) = \phi(x) \delta_\mu^\nu$ of the more general tensorial theory. In that special case the spacetime scale sector carries only one external DOF, namely the scalar modulus $|\phi|$, while the nontrivial structure resides in the internal group space.

In the fully anisotropic theory, however, Φ_μ^ν is itself a general spacetime tensor field and need not factorize into a single scalar modulus times an internal orientation. Rather, each spacetime component may carry its own internal structure. Clearly, the single-modulus construction should be viewed only as a particular isotropic ansatz, applicable only in the isotropic case, but not as the most general form of the theory.

The discussion in this section is meant to provide a framework to equipping cosmological Bianchi models with internal charges absent from the corresponding GR-based models, much like it has been done in the simpler case of the FRW model in [4]. While anisotropic Bianchi models are essentially ruled out observationally as global cosmological models they are still useful toy models for anisotropic spherical collapse and in that context it is desirable to have a non-singular (perhaps even bouncing) description.

When expanded, Equation (43) is $g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} \equiv (\Phi_I^K)_\mu^\alpha g_{\alpha\beta} \mathcal{G}_{KL} (\Phi^{LI})_\nu^\beta$. In the special case that the internal symmetry is described by a unitary group (whose elements are U_{IJ}) then no stretch and contraction of the fundamental units takes place ($|U| = 1$), i.e., no local scale invariance, but rather only a global charge is added to the theory. In this case, derivatives of the new metric field generically result in quadratic kinetic terms of phases that function as ‘cyclic coordinates’ on the group manifold, thereby resulting in conserved charges.

We stress that the internal symmetry is assumed to apply only to the field Φ_μ^ν , and not to any other fields. There is a conserved global charge that is associated with this global symmetry that physically alters the field equations of gravitation. The presence of this

charge modifies the kinetic term associated with the field, as is demonstrated in, e.g., the SU(2) case in Appendix B. An example of this within a WIST-based bouncing cosmological model is described in [4].

7. Summary

Symmetry plays a central role in our current understanding of the fundamental interactions. The SM of particle physics is based on a local $SU(3) \times SU(2) \times U(1)$ gauge symmetry, and GR is covariant. In the present work, the idea that GR accommodates a broader symmetry, anisotropic local scale invariance (of which WI is just a special case), is explored first. The underlying idea is that not only that the gravitational interaction (and by induction also all the other interactions which take place in the spacetime arena) does not depend on the reference frame of the observer (as is manifested by covariance of, e.g., the Einstein and geodesic equations) it also does not depend on (*a priori* arbitrary) choices of units, i.e., yardsticks, e.g., the Planck length scale, l_P . The latter is the natural yardstick in vacuum. In the non-vacuum case the natural meter stick would be the dynamical gravitational time scale, $(G\rho_M)^{-1/2} \propto l_P^{-1} \rho_M^{-1/2}$, where G is the Newton constant and ρ_M is the energy density of matter. The standard unit choice adopted in GR is that both G and particle masses (active gravitational masses included) are fixed constants.

The Newton constant, G , has been measured in our solar system at a reasonable precision, and is purportedly also deduced (at the few percent precision) from Big Bang nucleosynthesis (BBN) via observational inference of light element abundances, etc. However, all these estimates of G are based on the premise that all other universal constants are truly constants, as is clearly manifested in the BD program. On another front, Newtonian gravity fails to account for a few observed phenomena on galactic scales and beyond in spite of the fact that the gravitational pull is weak in these systems and Newtonian gravity is expected, at least naively, to provide a reasonably accurate description. The standard solution to this problem involves an assumption about the existence of a non-luminous DM substance, presumably exotic electrically neutral particles. Another way, albeit less popular, to tackle the problem has involved modifying the Newtonian dynamics on sufficiently large scales. Both remedies assume that G is a fixed universal constant across space and time; in the absence of compelling direct evidence for the universality of G this has to be assumed.

However, positing local (generally anisotropic) scale invariance as an additional symmetry of (what is usually considered GR) opens up new possibilities. Promoting G to a tensor (Section 2) or scalar (Section 3) scale deformation field necessitates that active gravitational masses are regulated by the same fields as well. Clearly, it is assumed here that (anisotropic or isotropic) local scale invariance applies only within the domain of validity of GR. This is of course not the case on microscopic scales. While GR is assumed here to be locally scale-invariant, the SM of particle physics is assumed to be as is, i.e., non-scale-invariant, with no modifications. The effective energy and pressure contained in spatio-temporal variations in the scalar field, i.e., G and active gravitational masses, then provide an additional (non-particulate) source for spacetime curvature, thereby accounting for, e.g., perturbative ‘DM phenomena’ with no recourse to DM. In the present work a few key aspects have been explored of the underlying locally scale-invariant generalizations of GR. In addition to local scale invariance, the ‘scaling’ fields (ϕ or $\Phi_{\mu\nu}$) have been equipped with internal symmetry and generally nonvanishing global charge.

It should be emphasized that whereas the case of isotropic local rescaling has long been explored, its generalization, i.e., the anisotropic case that is discussed in Section 2, has not been considered in the literature in full generality (apart from tetrad-like constructions in Minkowski reference-background formulations) to the best of our knowledge.

We also reiterate that the present work is classical in the same sense in which the GR examples used throughout are classical. The internal conserved charges are treated as classical Noether charges or integration constants. Their role in the homogeneous cosmological case is to provide an effective negative energy source term capable of producing a bounce, not to define a quantized negative-norm particle sector. The quantum theory of this sector, if any, requires a separate constrained Hamiltonian analysis.

A notable property of ϕ (or $\Phi_{\mu\nu}$ for that matter) is that it is formally a ‘ghost’ field, i.e., its kinetic and potential terms have the ‘wrong’ relative sign. However, this property by itself does not disqualify the theory because the problem with ghost-afflicted theories is that the field configuration dynamically runs away from its ground state, or equilibrium, by ever-lowering the kinetic energy while increasing the potential term. However, as was explicitly shown in Section 3, and as is equally clear from the underlying WI, the spacetime-dependent ϕ in the case of WIST, or Φ_{μ}^{ν} in the anisotropic case, is non-dynamical; it is not determined by an independent dynamical EL equation in the unfixed WI formulation. Consequently, physical systems described by locally scale-invariant theories of gravity are not doomed to runaway from stable field configurations. We also illustrated this conclusion with the redshifting Universe in Section 4. The FRW spacetime can be recast in the canonical WIST form on a static background. Cosmological redshift is then explained not by space expansion but rather by contraction of our fundamental yardsticks, e.g., the Planck ‘constant’, which are (directly and indirectly) regulated by ϕ . Whereas ϕ can be locally rescaled along with a corresponding rescaling of $g_{\mu\nu}$, adopting a specific conformal frame in which the metric is static implies that ϕ satisfies exactly the same Friedmann equation which is normally satisfied by a , the scale factor that describes space expansion in the standard cosmological model. More general cosmological models, e.g., Bianchi-type models, that are described by non-isotropic expansion rates are in general described by three scale factors. This family of models can be easily described by the anisotropic locally scale-invariant theory.

Funding: This research has been supported by the Joan and Irwin Jacobs donor-advised fund at the JCF (San Diego, CA, USA).

Data Availability Statement: The original contributions presented in the study are included in the article. Further inquiries can be directed to the author.

Acknowledgments: Anonymous referees are gratefully acknowledged for their useful and constructive comments.

Conflicts of Interest: The author declares no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

Appendix A. The Weak Gravitational Field Limit as an Inertial Force

The classical trajectories followed by particles in GR are described by geodesics

$$\ddot{x}^{\mu} + \Gamma_{\rho\sigma}^{\mu} \dot{x}^{\rho} \dot{x}^{\sigma} = 0, \quad (\text{A1})$$

where $\Gamma_{\rho\sigma}^{\mu}$ is the Christoffel symbol that is calculated from the metric field and its derivatives in the usual way, and $\dot{x}^{\mu} \equiv \frac{dx^{\mu}}{d\lambda}$, with λ an affine parameter. Equation (A1) is subject to the constraint $g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} = c$ where c equals -1 or 0 in case of massive or massless particles, respectively. In the presence of additional non-gravitational forces, F^{μ} , Equation (A1) generalizes to

$$\ddot{x}^{\mu} + \Gamma_{\rho\sigma}^{\mu} \dot{x}^{\rho} \dot{x}^{\sigma} = \frac{F^{\mu}}{m}, \quad (\text{A2})$$

in the case of a massive particle of mass m . Now, the Christoffel symbol transforms inhomogeneously under Weyl transformations (Equation (4))

$$\Gamma_{\rho\sigma}^{\mu} \rightarrow \tilde{\Gamma}_{\rho\sigma}^{\mu} = \Gamma_{\rho\sigma}^{\mu} + \Omega^{-1}(\delta_{\rho}^{\mu}\Omega_{\sigma} + \delta_{\sigma}^{\mu}\Omega_{\rho} - g_{\rho\sigma}g^{\mu\delta}\Omega_{\delta}). \quad (\text{A3})$$

Therefore, in the case $F^{\mu} = 0$, Equation (A2) is rewritten in the new conformal frame as

$$\dot{x}^{\mu} + \tilde{\Gamma}_{\rho\sigma}^{\mu}\dot{x}^{\rho}\dot{x}^{\sigma} = \frac{2\Omega_{\sigma}}{\Omega}\dot{x}^{\sigma}\dot{x}^{\mu} - cg^{\mu\nu}\Omega_{\nu}/\Omega, \quad (\text{A4})$$

or equivalently

$$\dot{x}^{\mu} + \tilde{\Gamma}_{\rho\sigma}^{\mu}\dot{x}^{\rho}\dot{x}^{\sigma} = -\frac{2\phi_{\sigma}}{\phi}\dot{x}^{\sigma}\dot{x}^{\mu} + cg^{\mu\nu}\phi_{\nu}/\phi. \quad (\text{A5})$$

We see that what is considered as a pure gravitational force in one conformal frame is interpreted as not pure gravitational force in other frames. A striking example is the case where spacetime is described by the Minkowski metric in the new frame, and in cartesian coordinates then $\tilde{\Gamma}_{\rho\sigma}^{\mu}$ vanishes. In this particular case Equation (A5) describes a purely non-gravitational interaction. The particles (massive or massless) follow exactly the same geodesics (that describe, e.g., gravitational clustering of massive particles or light bending of massless photons) in the two conformal frames, but while in the first frame it is due to spacetime curvature in the other frame it is due to an inertial ‘force’, $F \equiv -2\dot{x}^{\mu}\dot{x}^{\nu}\phi_{\nu}/\phi + c\eta^{\mu\nu}\phi_{\nu}/\phi$, where $\eta^{\mu\nu}$ is the Minkowski metric.

For example, the line element that describes a weakly perturbed Minkowski spacetime in Cartesian coordinates and in case that stress is negligible, and neglecting vector and tensor modes, is $ds^2 = -(1 + 2\varphi)dt^2 + (1 - 2\varphi)dx^i dx^i$. Here $\varphi(x)$ is the gravitational potential. Applying a Weyl transformation $\Omega = 1 + \varphi$ the metric in the new conformal frame becomes purely Minkowski, and $\tilde{\Gamma}_{\rho\sigma}^{\mu}$ vanishes in Cartesian coordinates. However, the r.h.s of Equation (A5) does not vanish now. The entire kinematics of test particles is now due to the fact that $\phi_0 \rightarrow \tilde{\phi} \approx \phi_0[1 - \varphi(x)]$. In other words, the standard picture according to which light bending and gravitational clustering are due to spacetime curvature is replaced by spacetime dependence of G and active gravitational masses on Minkowski background. In the presence of vector and/or tensor modes the single functional DOF, ϕ , added by the WIST theory is insufficient for entirely flattening the spacetime geometry and the more general locally scale-invariant theory that is described in Section 2 is required. The tensor field $\Phi_{\alpha\beta}$ can be used to reformulate GR on a Minkowski spacetime, in which case it reduces to the vierbein field if internal DOFs are ignored. The generalization of Equation (A5) in that case is straightforward.

Appendix B. The Case of SU(2) Global Symmetry

As an example for possibilities opened up by introducing internal symmetries associated with the gravitational interaction, we consider the case that the scalar field ϕ is a SU(2) multiplet. Elements of this group are represented by

$$\phi = \rho \begin{pmatrix} \cos(\theta)e^{i\varphi} & \sin(\theta)e^{i\chi} \\ -\sin(\theta)e^{-i\chi} & \cos(\theta)e^{-i\varphi} \end{pmatrix}, \quad (\text{A6})$$

where ρ is the modulus of the field and θ , φ and χ are three phases. The line element on this group manifold is $ds_{\phi}^2 = G_{IJ}d\phi^I d\phi^J$, where contracted capital Latin indices are summed over. Since this is a unitary group then $ds_{\phi}^2 = \frac{1}{2}\text{Tr}(d\phi^{\dagger}d\phi) = -\frac{1}{2}\text{Tr}(\phi^{-1}d\phi)^2 = d\rho^2 + \rho^2[\cos^2(\theta)d\varphi^2 + \sin^2(\theta)d\chi^2 + d\theta^2]$. The lagrangian of the kinetic term, $\mathcal{L}_{kin} = G_{IJ}\dot{\phi}^I\dot{\phi}^J$ (which is a straightforward generalization of the kinetic term in Equation (11)), is therefore

$\mathcal{L}_{kin} = -\dot{\rho}^2 - \rho^2[\cos^2(\theta)\dot{\phi}^2 + \sin^2(\theta)\dot{\chi}^2 + \dot{\theta}^2]$. The EL equations associated with χ and φ that are obtained from variations in $\mathcal{L}_{kin}$ can be each integrated once to give

$$\begin{aligned}\dot{\chi} &= \frac{c_\chi}{\rho^2 \sin^2 \theta} \\ \dot{\phi} &= \frac{c_\varphi}{\rho^2 \cos^2 \theta'}\end{aligned}\quad (\text{A7})$$

where c_χ and c_φ are integration constants. Employing these in the field equation for θ and integrating once we obtain

$$\rho^4 \dot{\theta}^2 = -\frac{c_\chi^2}{\sin^2 \theta} - \frac{c_\varphi^2}{\cos^2 \theta} + c_\theta^2, \quad (\text{A8})$$

where c_θ is another integration constant. Employing all these in the kinetic term $\phi_\mu \phi^\mu = -\dot{\rho}^2 - \rho^2[\cos^2(\theta)\dot{\phi}^2 + \sin^2(\theta)\dot{\chi}^2 + \dot{\theta}^2]$ we obtain $-\dot{\rho}^2 - \frac{c_\theta^2}{\rho^2}$, and both integration constants c_χ and c_φ vanish from our final expression. All this amounts to replacing ϕ with ρ everywhere in Equation (11) in addition to introducing a negative contribution to the potential, $-\frac{c_\theta^2}{\rho^2}$.

In the cosmological context this new term competes with the radiation term at sufficiently small ρ . This term, obtained due to the SU(2) symmetry of the scalar field (a symmetry which is not recognized by GR) rather than having to recourse to exotic forms of matter with negative energy density, is responsible to a bounce, where the cosmological expansion rate momentarily halts. As expected, this is exactly the result obtained in [4] in case of U(1) symmetry. The constant c_θ is simply a global charge associated with the U(1) subgroup, which is the relevant symmetry in a ‘central’ potential, i.e., a potential which is independent on the phase, and depends only on ρ . In cosmology the term $\propto c_\theta$ serves as an effective stiff matter with a negative energy density, thereby replacing the initial Big Bang singularity with a Big Bounce [4].

Appendix C. Type-I Bianchi-like Anisotropic Evolution

In this appendix a minisuperspace-style derivation of the EL equations is provided for homogeneous and anisotropic mass tensors living on a Minkowski spacetime. The derivation very closely follows the standard minisuperspace Type-I Bianchi procedure and is included here for completeness.

Consider Minkowski spacetime with an explicit lapse function $N(\eta)$, with infinitesimal spacetime interval

$$ds^2 = -N^2(\eta) d\eta^2 + \sum_{i=1}^3 (dx^i)^2. \quad (\text{A9})$$

Effecting the transformation $d\eta \rightarrow N(\eta)d\eta$ in Equation (42) results in

$$\mathcal{I}_{BI} = \int \left[-\frac{\phi'^2}{N} + \frac{\phi^2}{N} \sum_i \beta_i'^2 - N \lambda \phi^4 + N \phi^4 \mathcal{L}_M(\phi, \Sigma; \psi) \right] \sqrt{-\gamma} d^4 x, \quad (\text{A10})$$

with $\lambda \equiv \Lambda/3$ and where ψ collectively denotes matter DOFs and $\tilde{\mathcal{L}}_M$ is replaced by $\mathcal{L}_M$ for notational simplicity. In Σ -notation this is equivalently

$$\mathcal{I}_{BI} = \int \left[-\frac{\phi'^2}{N} + \frac{\phi^2}{N} \text{Tr}[(\Sigma^{-1} \Sigma')^2] - N \lambda \phi^4 + N \phi^4 \mathcal{L}_M(\phi, \Sigma; \psi) \right] \sqrt{-\gamma} d^4 x. \quad (\text{A11})$$

The crucial point is that the lapse N is retained so that the Hamiltonian constraint arises as an EL equation.

Varying the action with respect to N results in the constraint

$$\frac{\phi'^2}{N^2} - \frac{\phi^2}{N^2} \sum_i \beta_i'^2 - \lambda \phi^4 + \phi^4 \mathcal{L}_M = 0, \quad (\text{A12})$$

Equation (A12) is the type-I Bianchi Friedmann-like (Hamiltonian) constraint, with the shear contribution entering through $\sum_i \beta_i'^2$.

Varying Equation (A10) with respect to β_i results in

$$\frac{d}{d\eta} \left(\frac{\phi^2}{N} \beta_i' \right) = \frac{N\phi^4}{2} \frac{\partial \mathcal{L}_M}{\partial \beta_i}. \quad (\text{A13})$$

For isotropic matter (no anisotropic stress), $\partial \mathcal{L}_M / \partial \beta_i = 0$, and one obtains the standard free-shear law

$$\beta_i' \propto \frac{N}{\phi^2}. \quad (\text{A14})$$

Varying Equation (A10) with respect to ϕ yields

$$\left(\frac{\phi'}{N} \right)' + \frac{\phi}{N} \sum_i \beta_i'^2 - 2N\lambda\phi^3 + \frac{N}{2} \left[4\phi^3 \mathcal{L}_M + \phi^4 \frac{\partial \mathcal{L}_M}{\partial \phi} \right] = 0. \quad (\text{A15})$$

Once the EL Equations (A12), (A13) and (A15) are obtained, a time gauge should be imposed, $N(\eta) = 1$ in our case.

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