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Thermal Analysis of the Downstream Spreading of a Planar Power-Law Liquid Jet with Convective Free-Surface Cooling

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Abstract

The two-dimensional thermal liquid jet of a non-Newtonian power-law fluid is investigated under shear-rate-dependent thermal diffusivity, resulting in a one-way coupled nonlinear system governing momentum and thermal transport. Two physically distinct free-surface thermal boundary conditions are examined: adiabatic insulation and convective heat loss. Conservation laws and conserved quantities for the governing system are derived systematically using the multiplier method. By coupling an appropriate conserved vector with an admitted Lie point symmetry, the governing partial differential equations are reduced to a coupled system of ordinary differential equations. Closed-form parametric families of solutions are then obtained for the thermal field. The analysis reveals fundamentally different thermal transport mechanisms across rheological regimes: shear-thinning fluids enhance thermal redistribution and become increasingly sensitive to convective cooling as the Biot number increases, whereas shear-thickening fluids suppress internal thermal transport, promoting greater thermal retention within the jet core and reducing the influence of free-surface cooling. These findings clarify the interplay between rheology, nonlinear thermal diffusion and free-surface cooling and provide new analytical insight into downstream thermal transport in non-Newtonian liquid jets.

Keywords: thermal liquid jet; power-law fluid; convective cooling; free surface flow; Lie point symmetries; conservation laws; parametric solutions

1. Introduction

The complex rheological response in which the effective viscosity depends nonlinearly on the local rate of deformation exhibited by non-Newtonian fluids has vast and far-reaching natural and industrial applications. In particular, this behaviour is typically encountered in polymer melts and solutions, biological fluids and hydrogel-based materials, and even in coatings and paints. An important non-Newtonian fluid model, the power-law or Ostwald-de Waele model, occupies a central position. The model characterises the fluid through a single rheological index n , with $n > 1$ describing shear-thickening fluids and $n < 1$ pseudoplastic fluids, and the Newtonian fluid is recovered for $n = 1$. Power-law fluid models are widely used for their relative simplicity and mathematical tractability but despite this, they are able to capture dominant non-Newtonian features for a broad class of industrially and biologically important fluids. However, important limitations of the power-law model must be acknowledged. At very high or very low shear, many real fluids (and biological fluids or polymers) are characterised by a viscosity plateau which the power-law model does not capture. For biological fluids in particular, the Casson fluid or Carreau–Yasuda model may more accurately describe the fluid flow.



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The first investigations into the boundary-layer equations and analysis for non-Newtonian fluids were carried out by Schowalter [1], Acrivos et al. [2] and Denier and Dabrowski [3]. This, together with the pioneering work on boundary-layer theory applied to jet flows by Schlichting [4], established the mathematical framework for power-law fluids past external surfaces which has subsequently underpinned a large body of research on non-Newtonian boundary layers and jets.

A liquid jet can be described as a collimated stream of fluid impinging on a surface and spreading out over that surface [5]. Watson [6] was the first to consider such a flow and studied the growth of the boundary layer through approximate methods. The analysis of liquid jets for power-law fluids was first investigated in [7] where asymptotic solutions for shear-thinning fluids were found. Gorla [8] further considered the liquid jet of a swirling power-law fluid and obtained important expressions for the free-surface radial velocity and skin friction coefficient. In Magan et al. [9] analytical solutions in parametric form were obtained in the representative range $\frac{1}{2} < n < \infty$ and an analytical expression for the free surface was given. Free-surface flows of non-Newtonian fluids are of particular practical relevance and one application in particular where these flows are important is in the post-nozzle spreading of hydrogel bioinks in extrusion-based biofabrication [10–12]. Hydrogels such as alginate and gelatin methacryloyl exhibit well-characterised power-law pseudoplastic behaviour. Further, the downstream thermal field during jet spreading onto a heated substrate is crucial for the onset and distribution of gelation which is important for cell viability and scaffold fidelity. In these applications, the coupling of thermal effects with non-Newtonian rheology makes a compelling case for the motivation of investigating thermally driven fluid flow for power-law fluids.

In the present analysis, primary concern is given to the two-dimensional liquid jet of a power-law fluid with a shear-rate-dependent temperature field. In [9], the momentum balance equations were solved for power-law fluids and the present work extends this analysis to the thermal problem, incorporating, firstly, a zero heat flux boundary condition at the free surface through the adiabatic boundary condition and, secondly, incorporating convective heat loss at the free surface which is characterised by the Biot number. Despite the extensive literature on non-Newtonian jet flows, the thermal analyses of the planar liquid jet (or radial liquid jet) with convective heat loss or adiabatic boundary conditions have received comparatively little attention. In this analysis, this gap is addressed by deriving analytical solutions in parametric form for the temperature field in the downstream spreading of a power-law liquid jet with an adiabatic boundary condition and a convective Biot number boundary condition at the free surface.

Thermal boundary-layer analyses for power-law fluids have advanced considerably in wall-bounded configurations such as stretching surfaces and wedge flows [13–15], where the thermal layer develops along a solid boundary. The free-surface liquid jet considered here possesses no such wall; instead, the relevant boundary-layer effect is external, with the developing thermal layer in the surrounding medium producing a streamwise-varying heat-transfer coefficient and thus the effective Biot number being employed in the surface condition.

In jet flows, since the mainstream velocity vanishes, all boundary conditions for the velocity components are homogeneous. However, there are invariant structures conserved along the jet and these invariant quantities are essential to complete the solution since the constants of integration cannot be directly obtained from the similarity solution. A detailed comparison of methods to systematically calculate these flow invariants or conserved quantities is outlined in [16]. These methods have been successfully applied to Newtonian and non-Newtonian fluids [9,17–20]. The construction of these invariant structures can be used with Lie point symmetry analysis to derive group-invariant solu-

tions and the seminal work relating conserved vectors to Lie point symmetries was first given in Kara and Mohamed [21]. In the present paper, a similar approach will be used where the derived conserved vectors will be associated with a Lie point symmetry to reduce the coupled system of partial differential equations to a coupled system of ordinary differential equations.

The main contributions of this work are as follows:

- A thermal model for a planar power-law liquid jet is developed in which the thermal diffusivity depends on the local shear rate, coupling the rheology directly to heat transport.
- Both adiabatic and convective free-surface heat loss conditions are incorporated through the Biot number, with the adiabatic case recovered as a limit.
- Conservation laws for the governing system are derived.
- By associating a Lie point symmetry with a conserved vector, group-invariant solutions are obtained, yielding the temperature field in closed parametric form—an exact result not previously available for this problem.
- The interplay between the rheological index and the Biot number is examined for the first time, quantifying how shear-thinning or shear-thickening behaviour modifies free-surface heat loss.

The paper is structured as follows: In Section 2, the thermal liquid jet with a shear-rate-dependent temperature field will be formulated. The conservation laws will be derived in Section 3. In Section 4 the Lie point symmetries will be calculated and in Section 5 an analytical solution in parametric form will be established for the temperature field. The downstream flow behaviour and thermal response will be discussed in Section 6 and the conclusions will be given in Section 7.

2. Nonlinear Formulation of Thermal Planar Jet Flow

In this section, the nonlinear coupled partial differential equations governing the downstream thermal behaviour of a planar liquid jet for a power-law fluid are formulated.

The equations for the balance of mass and Cauchy's equations for the balance of momentum for the steady, two-dimensional flow of an incompressible fluid in dimensional form for a power-law fluid are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \quad (2)$$

and

$$\rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}. \quad (3)$$

where $u = u(x, y)$ and $v = v(x, y)$ are the velocity components in the x - and y -directions, respectively. For a power-law fluid, we consider the stress tensor

$$\tau_{ik} = -p\delta_{ik} + S_{ik}, \quad (4)$$

where the S_{ik} and rate of strain tensor A_{ik} are defined by

$$S_{ik} = b \left| \sqrt{\frac{1}{2} \text{tr}(A^2)} \right|^{n-1} A_{ik}, \quad A_{ik} = \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \quad (5)$$

for the constant b , to describe the rheological response of the fluid.

In the present analysis, the work carried out in [9] is now extended to the thermal problem. This is accomplished through the integration of adiabatic and convective cooling boundary conditions characterised by the Biot number.

The thermal energy equation is given by

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = - \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \right), \quad (6)$$

where ρ is a constant density, c_p is the specific heat capacity at a constant pressure and the temperature gradients in the x - and y -directions are given by T_x and T_y , respectively. Further, the heat flux q is described by

$$q = -k \left| \sqrt{\frac{1}{2} \text{tr}(A^2)} \right|^{n-1} \nabla T. \quad (7)$$

Substituting (7) into Equation (6) and using the boundary-layer approximation that the transverse conduction dominates the streamwise conduction, we obtain

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \frac{\partial}{\partial y} \left(\left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial T}{\partial y} \right). \quad (8)$$

It has been shown in [22–24], that the viscosity of complex fluids, like the power-law fluid or non-Newtonian fluids in general, can be temperature- or shear-rate-dependent while traditionally, the thermal diffusivity has been taken as a constant. Such a description is useful when studying pharmaceutical or biological materials [24] which has motivated the choice of the model.

As is evident from Equations (2), (3) and (8), the temperature field is only one-way coupled to the velocity field. The energy equation does not feed back into the momentum equations but is driven by the power-law velocity solution. This is physically appropriate in regimes where the Brinkman number is small and shear-rate-dependent viscosity variations dominate over temperature-induced ones, as in typical hydrogel or polymer jet flows [10–12]. An important observation is that the thermal coupling in the energy equation does not reduce to a linear equation with the one-way coupling but rather it inherits a nonlinear structure through the velocity field. This explicit dependence on the shearing parameter n is what distinguishes it from the thermal Newtonian jet.

The dimensional governing Equations (1)–(3) and (8) can be reduced to the dimensionless equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (9)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left[\left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \right], \quad (10)$$

$$u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \frac{1}{Pr_e} \frac{\partial}{\partial y} \left[\left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial \theta}{\partial y} \right], \quad (11)$$

where the following non-dimensional quantities,

$$\bar{u} = \frac{u}{U}, \quad \bar{v} = \frac{vL}{\delta U}, \quad \bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{\delta}, \quad \theta(x, y) = \frac{T(x, y) - T_\infty}{T_c - T_\infty}, \quad (12)$$

are used. In the above, U , L and δ are the characteristic velocity and lengths in the u , x , y directions respectively. The fluid temperature is given by $T(x, y)$ while T_c is the centreline jet temperature and T_∞ the ambient temperature of the fluid. In the above analysis, the

boundary-layer approximation is applied where the viscous diffusion in the transverse direction dominates that in the streamwise direction. The pressure is then also constant across the boundary layer. Of particular importance is the effective Reynolds number and effective Prandtl number given by

$$Re_e = \frac{\rho U^{2-n} L \delta^{n-1}}{b}, \quad Pr_e = \frac{k}{\rho c_p} \left(\frac{U}{\delta} \right)^n \sqrt{Re_e}. \quad (13)$$

respectively. The effective Prandtl number describes the ratio between the thermal diffusion and momentum diffusion. The nonlinear term $|\frac{\partial u}{\partial y}|^{n-1}$ is indicative of the thermal diffusivity varying non-uniformly across the jet width unlike a Newtonian fluid which varies uniformly and is controlled by the rheological parameter n . The shear-rate-dependent thermal diffusion is a fundamental feature of the present formulation which distinguishes it qualitatively from the Newtonian case.

In a liquid jet, since the streamwise velocity increases from the lower boundary at $y = 0$ toward the free surface at $y = \phi(x)$, the transverse velocity gradient satisfies $\frac{\partial u}{\partial y} > 0$ throughout the physical domain. Hence, for the remainder of the analysis,

$$\left| \frac{\partial u}{\partial y} \right| = \left(\frac{\partial u}{\partial y} \right). \quad (14)$$

The boundary conditions at the lower surface $y = 0$ are given by

$$y = 0 : \quad u(x, 0) = 0, \quad v(x, 0) = 0, \quad \theta(x, 0) = 1, \quad (15)$$

which describes the no-slip condition, the no-penetration condition and the fact that the lower boundary is isothermal and held at the characteristic temperature T_c . In other words, heat is introduced from the lower isothermal boundary and propagates towards the free surface through the thickness of the jet.

For the description of the boundary condition at the free surface $y = \phi(x)$, we consider two cases. For the first case, we consider an adiabatic free surface. Mathematically, this is

$$y = \phi(x) : \quad \frac{\partial \theta}{\partial y} \Big|_{y=\phi(x)} = 0. \quad (16)$$

This condition means that the flux of heat across the free surface is zero. Essentially, heat can diffuse upward but no heat leaves through the surface. A consequence of this is that the heat accumulates or redistributes through the jet.

The second case that we consider is a convective cooling boundary condition or convective heat loss at the free surface,

$$y = \phi(x) : \quad \left(\frac{\partial u}{\partial y} \right)^{n-1} \frac{\partial \theta}{\partial y} = -Bi\theta, \quad (17)$$

where Bi is the variable Biot number. The adiabatic thermal boundary condition can be recovered when $Bi = 0$. A note on how the Biot number is derived in the boundary condition is given below:

The thermal flux is of the form

$$q = -k|u_y|^{n-1} \nabla T, \quad (18)$$

where k is the generalised thermal conductivity and T the temperature. Under the boundary-layer approximation, the conductive flux delivered to the free surface is given by

$$-q_y = -k|u_y|^{n-1}T_y. \quad (19)$$

Now, at the free surface, the convective heat loss to ambient is written as

$$-q_y = h(T_s - T_\infty), \quad (20)$$

where $T_s = T(x, \phi(x))$ is the free-surface temperature, h the convective heat transfer coefficient, and T_∞ the ambient temperature. Using Equations (19) and (20) for a liquid jet, we obtain

$$-ku_y^{n-1}T_y = h(T_s - T_\infty), \quad (21)$$

where h is the heat transfer coefficient. By introducing the non-dimensional quantities in (12) and using the fact that at the free surface, $T_s = T(x, \phi(x))$, we obtain (17) where the dimensionless Biot number is defined by

$$Bi = \frac{h\delta^n}{kU^{n-1}}. \quad (22)$$

In summary, the governing equations under study are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (23)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left[\left(\frac{\partial u}{\partial y} \right)^{n-1} \frac{\partial u}{\partial y} \right], \quad (24)$$

$$u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \frac{1}{Pr_e} \frac{\partial}{\partial y} \left[\left(\frac{\partial u}{\partial y} \right)^{n-1} \frac{\partial \theta}{\partial y} \right], \quad (25)$$

subject to the following boundary conditions at $y = 0$ and $y = \phi(x)$:

$$y = 0 : u(x, 0) = 0, \quad v(x, 0) = 0, \quad \theta(x, 0) = 1, \quad (26)$$

at the lower surface. At the free surface, the two cases are

Case 1:

$$y = \phi(x) : \frac{\partial u}{\partial y}(x, \phi(x)) = 0, \quad \frac{\partial \theta}{\partial y}(x, \phi(x)) = 0. \quad (27)$$

Case 2:

$$y = \phi(x) : \frac{\partial u}{\partial y}(x, \phi(x)) = 0, \quad \left(\frac{\partial u}{\partial y} \right)^{n-1} \frac{\partial \theta}{\partial y} = -Bi\theta. \quad (28)$$

3. Invariant Conservation Structures and Conserved Fluxes

The conservation laws for the coupled system of Equations (23)–(25) subject to the boundary conditions (26)–(28) are now derived. The derivation of conserved vectors and conserved quantities has been systematically compared in Naz et al. [16] and the multiplier method, in particular, has been used to study the conservation laws in jet flows [9,17–20,25–27]. In the subsequent analysis, we will briefly outline the multiplier method for the coupled system of Equations (23)–(25) since the method is well established.

We search for multipliers of the form

$$\Lambda_1 = \Lambda_1(x, y, u, v), \quad \Lambda_2 = \Lambda_2(x, y, u, v), \quad \Lambda_3 = \Lambda_3(x, y, u, v). \quad (29)$$

The multipliers must satisfy the relation

$$\Lambda_1(u_x + v_y) + \Lambda_2(uu_x + vu_y - [(u_y)^{n-1}u_y]_y) + \Lambda_3\left(u\theta_x + v\theta_y - \frac{1}{Pr_e}[(u_y)^{n-1}\theta_y]_y\right) = D_x T^1 + D_y T^2, \quad (30)$$

where $T = (T^1, T^2)$ are the conserved vector components and D_x and D_y are the total derivative operators defined by

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v} + \theta_x \frac{\partial}{\partial \theta} + u_{xx} \frac{\partial}{\partial u_x} + v_{xx} \frac{\partial}{\partial v_x} + \theta_{xx} \frac{\partial}{\partial \theta_x} + u_{xy} \frac{\partial}{\partial u_y} + v_{xy} \frac{\partial}{\partial v_y} + \theta_{xy} \frac{\partial}{\partial \theta_y} + \dots, \quad (31)$$

$$D_y = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial v} + \theta_y \frac{\partial}{\partial \theta} + u_{yy} \frac{\partial}{\partial u_y} + v_{yy} \frac{\partial}{\partial v_y} + \theta_{yy} \frac{\partial}{\partial \theta_y} + u_{yx} \frac{\partial}{\partial u_x} + v_{yx} \frac{\partial}{\partial v_x} + \theta_{yx} \frac{\partial}{\partial \theta_x} + \dots \quad (32)$$

To obtain the determining equations and to eliminate the divergence of expression (30), the Euler operator is applied to (30); that is,

$$E_u \left[\Lambda_1(u_x + v_y) + \Lambda_2(uu_x + vu_y - [(u_y)^{n-1}u_y]_y) + \Lambda_3\left(u\theta_x + v\theta_y - \frac{1}{Pr_e}[(u_y)^{n-1}\theta_y]_y\right) \right] = 0, \quad (33)$$

$$E_v \left[\Lambda_1(u_x + v_y) + \Lambda_2(uu_x + vu_y - [(u_y)^{n-1}u_y]_y) + \Lambda_3\left(u\theta_x + v\theta_y - \frac{1}{Pr_e}[(u_y)^{n-1}\theta_y]_y\right) \right] = 0, \quad (34)$$

$$E_\theta \left[\Lambda_1(u_x + v_y) + \Lambda_2(uu_x + vu_y - [(u_y)^{n-1}u_y]_y) + \Lambda_3\left(u\theta_x + v\theta_y - \frac{1}{Pr_e}[(u_y)^{n-1}\theta_y]_y\right) \right] = 0, \quad (35)$$

where the Euler operators are defined by

$$E_u = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_y^2 \frac{\partial}{\partial u_{yy}} + D_x D_y \frac{\partial}{\partial u_{xy}} - \dots, \quad (36)$$

$$E_v = \frac{\partial}{\partial v} - D_x \frac{\partial}{\partial v_x} - D_y \frac{\partial}{\partial v_y} + D_x^2 \frac{\partial}{\partial v_{xx}} + D_y^2 \frac{\partial}{\partial v_{yy}} + D_x D_y \frac{\partial}{\partial v_{xy}} - \dots, \quad (37)$$

$$E_\theta = \frac{\partial}{\partial \theta} - D_x \frac{\partial}{\partial \theta_x} - D_y \frac{\partial}{\partial \theta_y} + D_x^2 \frac{\partial}{\partial \theta_{xx}} + D_y^2 \frac{\partial}{\partial \theta_{yy}} + D_x D_y \frac{\partial}{\partial \theta_{xy}} - \dots \quad (38)$$

and the total derivative operators D_x and D_y are given in (31) and (32).

Expanding Equations (36)–(38) results in an overdetermined system of equations which can be separated into derivatives and derivative powers of u, v and θ . As a result of solving these equations, we obtain the multipliers

$$\Lambda_1 = c_1, \quad \Lambda_2 = c_1 u + c_2, \quad \Lambda_3 = c_1 \theta + c_3. \quad (39)$$

By substituting the multipliers (39) into Equation (30), we are able to write the coupled system of partial differential Equations (23)–(25) in conserved form to obtain the conserved vectors

$$T^1 = u, \quad T^2 = v. \quad (40)$$

$$T^1 = u^2, \quad T^2 = uv + (u_y)^n, \quad (41)$$

$$T^1 = u\theta, \quad T^2 = v\theta - \frac{1}{Pr_e}(u_y)^{n-1}\theta_y. \quad (42)$$

By setting $n = 1$ we can recover the conserved vectors for a thermal Newtonian liquid jet. The conserved vectors for $n = 1$ for the velocity components found for a Newtonian jet were obtained in [25] and for the power-law liquid jet in [9]. The results have been extended to obtain the conserved vectors for the thermal equation with a shear-rate-dependent temperature field.

In order to derive the invariant structures of the flow or the conserved quantities, the expression

$$D_x T^1 + D_y T^2 = \frac{\partial T^1}{\partial x}(x, y) + \frac{\partial T^2}{\partial y}(x, y), \quad (43)$$

must be satisfied. In order to choose an appropriate conserved vector, the integral of the flux component must vanish.

The free surface of the flow is a deformable fluid boundary that is not constrained by a wall but is in contact with other fluid-like air. This boundary is not rigidly prescribed but the free surface is a surface that can move and evolve with the flow. Important conditions that apply at the free surface are the kinematic surface condition and a dynamic condition, where the former condition ensures that a particle on the fluid surface remains on the fluid surface while the latter condition balances the stress in the liquid with the external stresses. To obtain the conserved quantities for the liquid jet, it is necessary to calculate the y -component of velocity on the free surface.

From the definition of the material time derivative evaluated at the free surface $y = \phi(x)$, we find that the relation

$$v(x, \phi(x)) = u(x, \phi(x)) \frac{d\phi(x)}{dx}, \quad (44)$$

for the normal velocity at the free surface holds.

For conserved vectors (40), using relation (43), the boundary conditions (26) and the kinematic surface condition (44), we obtain the conserved quantity

$$J_1 = \int_0^{\phi(x)} u(x, y) dy = \text{constant independent of } x, \quad (45)$$

which represents a conservation of momentum flux along the jet. This has been established for the Newtonian jet in Naz et al. [25] and in [9] for the power-law fluid. Considering now the conserved vectors (42) with relation (43), we obtain

$$\int_0^{\phi(x)} \frac{\partial}{\partial x}(u\theta) dy + \int_0^{\phi(x)} \frac{\partial}{\partial y} \left(v\theta - \frac{1}{Pr_e} (u_y)^{n-1} \theta_y \right) dy = 0. \quad (46)$$

Using the formula for differentiation under an integral sign [28] with the kinematic condition (44) and boundary conditions (26) at $y = \phi(x)$, we obtain the invariant heat flux along the jet:

$$J_2 = 2\rho c_p \int_0^{\phi(x)} u(x, y) \theta(x, y) dy = \text{constant independent of } x. \quad (47)$$

The conserved quantities for the thermal Newtonian jet, $n = 1$, are also given by Equations (45) and (47). Similar results have been derived in [18,27].

4. Lie Point Symmetries and Associated Lie Point Symmetries

The Lie point symmetry method has been previously applied to jet flow problems in [9,17,18,20,26,29]. Further, the seminal work by Kara and Mohamed [21] allows one to construct a group-invariant solution through the association of a conserved vector with the Lie point symmetry. This approach allows one to systematically reduce the number of

independent variables. Further reading on the theory of Lie point symmetries to differential equations can be found in [30].

The Lie point symmetry generator is given by

$$\begin{aligned} X = & \zeta^1(x, y, u, v, \theta) \frac{\partial}{\partial x} + \zeta^2(x, y, u, v, \theta) \frac{\partial}{\partial y} \\ & + \eta^1(x, y, u, v, \theta) \frac{\partial}{\partial u} + \eta^2(x, y, u, v, \theta) \frac{\partial}{\partial v} \\ & + \eta^3(x, y, u, v, \theta) \frac{\partial}{\partial \theta}. \end{aligned} \quad (48)$$

The determining equations are derived by solving the invariance conditions

$$X^{[2]}(F_i)|_{F_1=F_2=F_3} = 0, \quad i = 1, 2, 3 \quad (49)$$

where F_1, F_2 and F_3 are given by governing Equations (23), (24) and (25) respectively. The generator is prolonged to the second order because the highest derivatives that occur in the system are of order two. The second prolongation is defined by

$$\begin{aligned} X^{[2]} = & X + \zeta_x^1 \frac{\partial}{\partial u_x} + \zeta_x^2 \frac{\partial}{\partial v_x} + \zeta_x^3 \frac{\partial}{\partial \theta_x} + \zeta_y^1 \frac{\partial}{\partial u_y} + \zeta_y^2 \frac{\partial}{\partial v_y} + \zeta_y^3 \frac{\partial}{\partial \theta_y} \\ & + \zeta_{xx}^1 \frac{\partial}{\partial u_{xx}} + \zeta_{xy}^1 \frac{\partial}{\partial u_{xy}} + \zeta_{yy}^1 \frac{\partial}{\partial u_{yy}} + \zeta_{xx}^2 \frac{\partial}{\partial v_{xx}} + \zeta_{xy}^2 \frac{\partial}{\partial v_{xy}} + \zeta_{yy}^2 \frac{\partial}{\partial v_{yy}} \\ & + \zeta_{xx}^3 \frac{\partial}{\partial \theta_{xx}} + \zeta_{xy}^3 \frac{\partial}{\partial \theta_{xy}} + \zeta_{yy}^3 \frac{\partial}{\partial \theta_{yy}}, \end{aligned} \quad (50)$$

where the extended infinitesimals are defined by

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\zeta^j), \quad \zeta_{ij}^\alpha = D_j(\zeta_i^\alpha) - u_{ik}^\alpha D_j(\zeta^k), \quad (51)$$

where the total derivative operators are defined in (31) and (32).

A set of overdetermined determining equations is obtained by expanding the invariance criterion (49). These equations can be separated in derivatives and derivative powers of u, v and θ to obtain the explicit infinitesimals

$$\begin{aligned} \zeta^1 &= [(2-n)c_4 + (n+1)c_5]x + c_1, \\ \zeta^2 &= c_5y + c_2, \\ \eta^1 &= c_4u, \\ \eta^2 &= [(n-1)c_4 - nc_5]v, \\ \eta^3 &= c_3 + c_6\theta. \end{aligned} \quad (52)$$

and therefore the Lie generator is given by

$$\begin{aligned} X = & ([(2-n)c_4 + (n+1)c_5]x + c_1) \frac{\partial}{\partial x} + (c_5y + c_2) \frac{\partial}{\partial y} \\ & + c_4u \frac{\partial}{\partial u} + ([(n-1)c_4 - nc_5]v) \frac{\partial}{\partial v} + (c_3 + c_6\theta) \frac{\partial}{\partial \theta}. \end{aligned} \quad (53)$$

To determine the Lie point symmetry (53) which is associated with conserved vectors (40) and (42), we make use of the relation

$$X(T^j) + T^j D_l(\zeta^l) - T^l D_l(\zeta^j) = 0, \quad j = 1, 2. \quad (54)$$

Since the conserved vectors depend on at most the first derivatives, the generator in (54) is prolonged up to the first order and the explicit first-order prolongation is given by

$$\begin{aligned} X^{[1]} = X &+ [(n-1)c_4 - (n+1)c_5]u_x \frac{\partial}{\partial u_x} + (c_4 - c_5)u_y \frac{\partial}{\partial u_y} \\ &+ [(2n-3)c_4 - (2n+1)c_5]v_x \frac{\partial}{\partial v_x} + [(n-1)c_4 - (n+1)c_5]v_y \frac{\partial}{\partial v_y} \\ &+ [c_6 - (2-n)c_4 - (n+1)c_5]\theta_x \frac{\partial}{\partial \theta_x} + (c_6 - c_5)\theta_y \frac{\partial}{\partial \theta_y}. \end{aligned} \quad (55)$$

For conserved vector (40), using relation (54) up to the first prolongation (55) and the fact that the total derivative operators are defined in (31) and (32), we obtain the relation between the constants $c_4 = -c_5$. Similarly, it is found that for conserved vector (42), for Equation (54) to be satisfied, the relations $c_4 = -c_5$ (as above) and $c_6 = 0$ and $c_3 = 0$ must hold. Thus, the Lie point symmetry associated with conserved vectors (40) and (42) is given by

$$X = [c_1 + (1-2n)c_4x] \frac{\partial}{\partial x} + (c_2 - c_4y) \frac{\partial}{\partial y} + c_4u \frac{\partial}{\partial u} - c_4(2n-1)v \frac{\partial}{\partial v}. \quad (56)$$

5. Group-Invariant Solutions and Parametric Solutions

In this section, the group-invariant solution to the coupled system of partial differential equations is derived using the associated Lie point symmetry (56) and the one-way coupled system of partial differential is reduced to a system of ordinary differential equations where the nonlinear structure of the flow is introduced through the velocity field. The system of equations is then solved making use of the fact that these equations are autonomous which allows for analytical solutions in parametric form to be found.

In the present study, the previously developed analytical framework for downstream power-law liquid jet dynamics is extended [9] by incorporating thermal transport into the model through shear-rate-dependent thermal diffusivity. While the earlier work in [9] established exact and parametric solutions for the momentum field, thereby characterising the velocity structure and free-surface evolution of the jet, the thermal behaviour remained unresolved. Here, we address this gap by deriving and analysing the associated energy equation under two physically distinct free-surface thermal boundary conditions: adiabatic insulation, representing negligible heat exchange with the surrounding environment, and convective heat loss at the free surface, which models realistic thermal loss from the jet surface to the ambient medium. This extension fundamentally broadens the model from purely hydrodynamic to thermo-fluidic behaviour, allowing us to quantify how surface heat transfer mechanisms influence temperature evolution, thermal boundary-layer structure, and coupled transport characteristics in power-law liquid jets. By comparing insulated and convectively cooled regimes, the work reveals distinct classes of thermal solutions, and provides a first-principles framework relevant to applications where downstream temperature regulation is critical.

5.1. Group-Invariant Solution

The equations

$$X(u - U(x, y))|_{u=U(x, y)} = 0, \quad (57)$$

$$X(v - V(x, y))|_{v=V(x, y)} = 0, \quad (58)$$

$$X(\theta - \Theta(x, y))|_{\theta=\Theta(x, y)} = 0, \quad (59)$$

must be satisfied for the group-invariant solutions $u = U(x, y)$, $v = V(x, y)$ and $\theta = \Theta(x, y)$ generated by the Lie point symmetry (56). The group-invariant solutions for Equations (57) and (58) are derived in detail in [9] and are found to be

$$u(x, y) = x^{-\frac{1}{2n-1}} F(\xi), \quad v(x, y) = x^{-1} H(\xi), \quad \xi = \frac{y}{x^{\frac{1}{2n-1}}}. \quad (60)$$

In a similar fashion, we can solve Equation (59) for the temperature θ by solving the differential equations of characteristic curves

$$\frac{dx}{(1-2n)c_4x} = \frac{dy}{(c_2 - c_4y)} = \frac{d\theta}{0}, \quad (61)$$

to give the invariant solution

$$\theta(x, y) = G(\xi), \quad (62)$$

where the similarity variable ξ is defined in (60).

By substituting group-invariant solutions (60) and (62) into the coupled system of Equations (23)–(25), we obtain the following set of coupled ordinary differential equations:

$$\frac{dH}{d\xi} - \frac{1}{(2n-1)} \left(\xi \frac{dF}{d\xi} + F \right) = 0, \quad (63)$$

$$n \left(\frac{dF}{d\xi} \right)^{n-1} \frac{d^2F}{d\xi^2} - H(\xi) \frac{dF}{d\xi} + \frac{1}{(2n-1)} \left(\xi \frac{dF}{d\xi} + F \right) F = 0, \quad (64)$$

$$\left[\frac{1}{(1-2n)} \xi F(\xi) + H(\xi) \right] \frac{dG}{d\xi} - \frac{1}{Pr_e} \left[(n-1) \left(\frac{dF}{d\xi} \right)^{n-2} \frac{d^2F}{d\xi^2} \frac{dG}{d\xi} + \left(\frac{dF}{d\xi} \right)^{n-1} \frac{d^2G}{d\xi^2} \right] = 0, \quad (65)$$

The corresponding boundary conditions at the lower surface $y = 0$ can thus be written as

$$y = 0 : F(0) = 0, \quad H(0) = 0, \quad G(0) = 1. \quad (66)$$

For the boundary conditions at the free surface, we consider first the adiabatic insulation boundary condition and secondly the convective heat loss boundary condition at the free surface.

Case 1: Adiabatic insulation

$$y = \phi(x) : \frac{dF}{d\xi}(k) = 0, \quad \frac{dG}{d\xi}(k) = 0, \quad (67)$$

where the quantity k is derived using similarity variable (60). That is, at the free surface, $y = \phi(x)$, and so

$$\xi = \frac{\phi(x)}{x^{\frac{1}{2n-1}}} = k. \quad (68)$$

Case 2: Convective heat loss

$$y = \phi(x) : \frac{dF}{d\xi}(k) = 0, \quad -x^{-1} \left(\frac{dF(k)}{d\xi} \right)^{n-1} \frac{dG}{d\xi} = BiG(k). \quad (69)$$

From this boundary condition we can conclude that the constant Biot convective boundary condition is not invariant under this similarity scaling. To retain the self-similarity thermal structure under the convective heat exchange, we consider a downstream varying effective Biot number, $Bi(x) = \frac{Bi_0}{x}$. This does not alter the boundary-layer governing equations but modifies the thermal boundary condition at the free surface. The choice of the Biot

number reflects a spatially evolving heat transfer coefficient. We can justify the choice of the effective Biot number since this number changes along the jet for a clear physical reason. The Biot number is defined, in this case, as $Bi = \frac{h\delta^n}{kU^{n-1}}$, where δ is the transverse length scale, which we hold fixed. So when the Biot number changes, it is because the heat transfer coefficient h changes, not the geometry. The coefficient h is not the same everywhere along the free surface. As the jet moves downstream, the thermal boundary layer in the surrounding medium gets thicker which subsequently slows the rate of heat loss. As a result, the heat transfer coefficient becomes smaller further along the jet, so h depends on position, $h = h(x)$. Thus, the Biot number now depends on x as well. For the Biot number to stay constant, h would have to be the same all along the surface but that rarely happens when a surface is cooled as it moves. This kind of position-dependent cooling occurs in several real processes that involve power-law fluids, such as melt and fibre spinning, the casting of polymer films and coatings, and 3D bioprinting, where a shear-thinning bioink is cooled as it is laid down and sets by gelling. In all of these, the fluid follows an Ostwald-de Waele (power-law) model, and the heat lost at the surface varies along the flow. Thus, the convective heat loss boundary condition at the thermal free surface becomes

$$y = \phi(x) : -\left(\frac{dF}{d\xi}\right)^{n-1} \frac{dG}{d\xi} = Bi_0 G(k). \quad (70)$$

5.2. Parametric Solutions

The parametric solutions to the momentum equations have been established in [9]. The thermally coupled system with a shear-rate-dependent thermal field under distinct boundary conditions admits a rich non-trivial analytical structure which is the focus of the present analysis.

Although the momentum field governing the downstream spreading of the power-law liquid jet was derived in our previous work, those hydrodynamic results form the essential mechanical foundation for the present thermal analysis and are therefore restated here in concise form. Specifically, the previously obtained similarity transformations, invariant reductions, and parametric solutions for the velocity components are reintroduced not as repeated contributions, but as the required governing structure upon which the energy equation depends through convective transport and shear-dependent diffusion. In this context, the known momentum solution provides the velocity and shear-rate distributions that directly determine the thermal field, enabling closure of the energy problem under both adiabatic and convective free-surface boundary conditions. Restating these hydrodynamic results ensures continuity and completeness while positioning the velocity field as a precursor to the study's principal novelty: new temperature solutions, thermal conserved quantities, and the effects of surface heat exchange. The results are summarised in Table 1.

The analysis is restricted to the representative range $\frac{1}{2} < n < \infty$ only. It is shown in [9] that physically reasonable solutions do not exist in the range $0 < n < \frac{1}{2}$ or at $n = \frac{1}{2}$.

Table 1. Summary of results derived in [9].

Description	Parametric Expression
Invariant streamwise velocity profile ($0 \leq R \leq 1$, $0 < F(\xi) < F(k)$)	$F(\xi) = \left[\frac{n+1}{3n(2n-1)} \right]^{\frac{1}{2n-1}} \left[\frac{J_1}{I_1(n)} \right]^{\frac{n+1}{2n-1}} R$
Invariant derivative form ($0 \leq R \leq 1$)	$\frac{dF}{d\xi} = \left[\frac{n+1}{3n(2n-1)} \right]^{\frac{2}{(2n-1)}} \left[\frac{J_1}{I_1(n)} \right]^{\frac{3}{2n-1}} [1 - R^3]^{\frac{1}{n+1}}$

Table 1. Cont.

Description	Parametric Expression
Invariant transverse velocity function ($0 \leq R \leq 1$)	$H(\xi) = \frac{1}{2n-1} \left[\frac{J_1}{I_1(n)} \right] G \int_0^R \frac{dr}{(1-r^3)^{\frac{1}{n+1}}}$
Similarity variable ($0 \leq \xi \leq k$)	$\xi = \left[\frac{3n(2n-1)}{n+1} \right]^{\frac{1}{2n-1}} \left[\frac{J_1}{I_1(n)} \right]^{\frac{n-2}{2n-1}} \int_0^R \frac{dr}{(1-r^3)^{\frac{1}{n+1}}}$
Dimensional streamwise velocity field	$u(x, y) = x^{-\frac{1}{2n-1}} \left[\frac{n+1}{3n(2n-1)} \right]^{\frac{1}{2n-1}} \left[\frac{J_1}{I_1(n)} \right]^{\frac{n+1}{2n-1}} R$
Dimensional transverse velocity field	$v(x, y) = x^{-1} \frac{1}{2n-1} \left[\frac{J_1}{I_1(n)} \right] R \int_0^R \frac{dr}{(1-r^3)^{\frac{1}{n+1}}}$
Physical coordinate transformation	$y = x^{\frac{1}{2n-1}} \left[\frac{3n(2n-1)}{n+1} \right]^{\frac{1}{2n-1}} \left[\frac{J_1}{I_1(n)} \right]^{\frac{n-2}{2n-1}} \int_0^R \frac{dr}{(1-r^3)^{\frac{1}{n+1}}}$
Momentum normalisation integral	$I_1(n) = \int_0^1 \frac{R dR}{(1-R^3)^{\frac{1}{n+1}}}$
Free-surface thickness integral	$I_2(n) = \int_0^1 \frac{dr}{(1-r^3)^{\frac{1}{n+1}}}$
Free-surface boundary location ($k = \xi_{\max}$)	$k = \xi_{\max} = \left[\frac{3n(2n-1)}{n+1} \right]^{\frac{1}{2n-1}} \left[\frac{J_1}{I_1(n)} \right]^{\frac{n-2}{2n-1}} I_2(n)$
Free-surface profile ($y = \phi(x)$ at $\xi = k$)	$\phi(x) = \left[\frac{3n(2n-1)}{n+1} \right]^{\frac{1}{2n-1}} \left[\frac{J_1}{I_1(n)} \right]^{\frac{n-2}{2n-1}} I_2(n) x^{\frac{1}{2n-1}}$

We will now derive the expressions that determine the convective and shear-dependent transport mechanisms under adiabatic and convective cooling boundary conditions.

Consider the energy equation

$$\left[\frac{1}{(1-2n)} \xi F(\xi) + H(\xi) \right] \frac{dG}{d\xi} - \frac{1}{Pr_e} \left[(n-1) \left(\frac{dF}{d\xi} \right)^{n-2} \frac{d^2 F}{d\xi^2} \frac{dG}{d\xi} + \left(\frac{dF}{d\xi} \right)^{n-1} \frac{d^2 G}{d\xi^2} \right] = 0, \quad (71)$$

which will be solved subject to the following boundary conditions:

$$y = 0 : F(0) = 0, \quad H(0) = 0, \quad G(0) = 1. \quad (72)$$

Case 1: Adiabatic insulation

$$y = \phi(x) : \frac{dF}{d\xi}(k) = 0, \quad \frac{dG}{d\xi}(k) = 0, \quad (73)$$

Case 2: Convective heat loss

$$y = \phi(x) : \frac{dF}{d\zeta}(k) = 0, \quad -\left(\frac{dF}{d\zeta}\right)^{n-1} \frac{dG}{d\zeta} = Bi_0 G(k). \quad (74)$$

Using the results for $H(\zeta)$ from Table 1, ordinary differential Equation (71) reduces to

$$\frac{d}{d\zeta} \left[\frac{1}{Pr_e} \left(\frac{dF}{d\zeta} \right)^{n-1} \frac{dG}{d\zeta} \right] = 0, \quad (75)$$

which, upon integration with respect to ζ and using the boundary condition $\frac{dG(k)}{dk} = 0$, describes a thermally insulated free surface. That is,

$$\theta(x, y) = G(\zeta) = \text{constant}. \quad (76)$$

From a physical perspective, a temperature gradient does not develop, meaning that there is no heating or cooling through the boundary layer. This does suggest that the adiabatic insulation boundary condition may not be strong enough to construct a meaningful thermal field.

To derive a non-trivial thermal boundary layer, we consider the convective cooling boundary condition at the free surface.

From Equation (75), we can integrate with respect to ζ to obtain

$$\frac{dG}{d\zeta} = A \left(\frac{dF}{d\zeta} \right)^{1-n}, \quad (77)$$

where A is a constant of integration. Using the convective heat loss boundary condition at the free surface, we find that the constant of integration A is given by

$$A = -Bi_0 G(k), \quad (78)$$

and so the ordinary differential equation becomes

$$\frac{dG}{d\zeta} = -Bi_0 G(k) \left(\frac{dF}{d\zeta} \right)^{1-n}. \quad (79)$$

Since $\zeta = k$ at the free surface, $G(\zeta) = G(k)$ at this point. Now, integrating Equation (79) from ζ to k , so that we may infer information about $G(\zeta)$ for an arbitrary ζ , yields

$$G(\zeta) = G(k) \left[1 + Bi_0 \int_{\zeta}^k \left(\frac{dF(s)}{ds} \right)^{1-n} ds \right]. \quad (80)$$

The solution for the temperature field depends on $G(k)$ which can only be determined from conserved quantity (47). In terms of the invariant solutions, this conserved quantity is written as

$$I_2 = 2\rho c_p \int_0^k F(\zeta) G(\zeta) d\zeta. \quad (81)$$

The value of $G(k)$ can be found using conserved quantity (81), and with the form of $G(\zeta)$ in (80), the conserved quantity is written as

$$I_2 = 2\rho c_p G(k) \int_0^k F(\zeta) \left[1 + Bi_0 \int_{\zeta}^k \left(\frac{dF(s)}{ds} \right) ds \right] d\zeta. \quad (82)$$

To evaluate the thermal conserved quantity, we may directly substitute the similarity profile $F(\zeta)$ into the integral. We, however, recast this in terms of the derivative of F . This is done to avoid potential endpoint singularities associated with the thermal parameter R

which can obscure convergence properties even though the integral is well defined. Further, from a physical perspective, since the temperature field is governed by the shear rate, the rheological influence on temperature transport arises explicitly through local deformation rates rather than solely through velocity magnitude and so expressing the solution in terms of F' more directly reflects the thermal coupling.

Since

$$\int_0^k F(\xi) d\xi = \int_0^{F(k)} F(\xi) \frac{d\xi}{dF} dF = \int_0^{F(k)} \frac{F}{F'} dF, \quad (83)$$

conserved quantity (82) can be written as

$$J_2 = 2\rho c_p G(k) \int_0^{F(k)} \frac{F}{F'} \left[1 + Bi_0 \int_F^{F(k)} \left(\frac{dp}{dq} \right)^{-n} dp \right] dF, \quad (84)$$

which can be solved for $G(k)$ to give

$$G(k) = \left(\frac{J_2}{2\rho c_p} \right) \left[\int_0^{F(k)} \frac{F}{F'} \left[1 + Bi_0 \int_F^{F(k)} \left(\frac{dp}{dq} \right)^{-n} dp \right] dF \right]^{-1}. \quad (85)$$

In order to obtain the results in Table 1, the transformation

$$R(\xi) = \frac{F(\xi)}{F(k)}, \quad (86)$$

was used and it was found that $F(k)$ is given by

$$F(k) = \left[\frac{n+1}{3n(2n-1)} \right]^{\frac{1}{2n-1}} \left[\frac{J_1}{I_1(n)} \right]^{\frac{n+1}{2n-1}}. \quad (87)$$

Using these results with the definitions of F' and $I_1(n)$ in Table 1, the value of $G(k)$ is

$$G(k) = \left[\frac{J_2 J_1}{2\rho c_p (I_1(n))^2} \right] \left[\frac{3n(2n-1)}{n+1} \right]^{-\frac{4}{(n+1)(2n-1)}} \left[1 + Bi_0 \int_0^{F(k)} \left(\frac{dp}{dq} \right)^{-n} dp \right]^{-1}. \quad (88)$$

We can thus complete the solution for the temperature field using (62). Thus,

$$\begin{aligned} \theta(x, y) = & \left[\frac{J_2 J_1}{2\rho c_p (I_1(n))^2} \right] \left[\frac{3n(2n-1)}{n+1} \right]^{-\frac{4}{(n+1)(2n-1)}} \\ & \times \left[1 + Bi_0 \int_0^{F(k)} \left(\frac{dp}{dq} \right)^{-n} dp \right]^{-1} \left[1 + Bi_0 \int_{\xi}^k \left(\frac{dF(s)}{ds} \right)^{1-n} ds \right]. \end{aligned} \quad (89)$$

In the above equation, the last two terms of the solution are important. The first of the last two terms ensures that the thermal field satisfies the global energy balance across the jet and allows for $G(k)$ to be determined, while the last term describes the spatial distribution of the temperature profile.

6. Downstream Flow Evolution and Thermal Response

The behaviour of the fluid downstream will now be investigated. The key features that are considered include the temperature profiles for the adiabatic and convective heat loss boundary conditions, and the heat flux at the surface substrate $y = 0$ and at the free surface $y = \phi(x)$. Further, the mean temperature across both is also established using the parametric solutions. Finally, the sensitivity of the flow to the shearing parameter n , effective Prandtl number and Biot number is also discussed.

At the substrate $y = 0$, the heat flux from the substrate into the fluid is measured and can be approximated by

$$q_{\text{substrate}} \propto -(F'(0))^{n-1} G'(0). \quad (90)$$

The heat loss at the free surface can be approximated by

$$q_{\text{surface}} \propto B_i G(k), \quad (91)$$

where $G(k)$ is the temperature at the free surface.

In Figures 1–3 the effects of the Biot number and the rheological index n on the normalised temperature profiles from the jet centreline to the free surface are discussed.

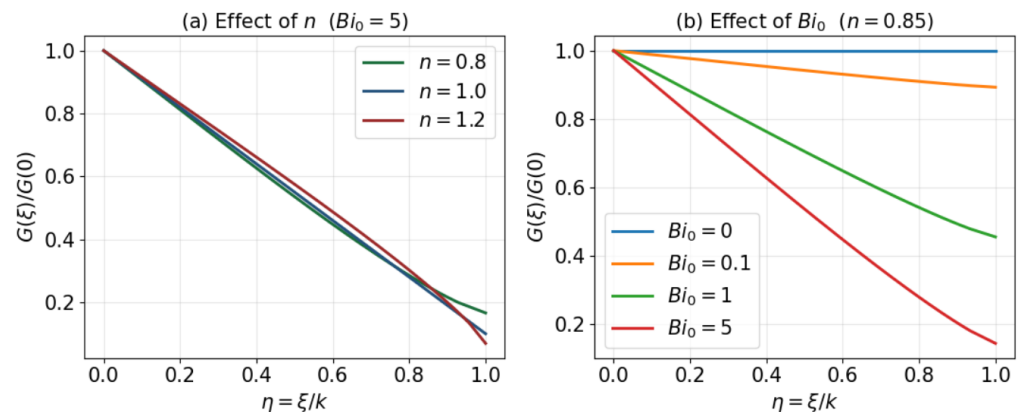


Figure 1. Cross-stream temperature profile normalised by its centreline value, $G(\xi)/G(0)$, against the scaled coordinate $\eta = \xi/k$ from the jet centreline ($\eta = 0$) to the free surface ($\eta = 1$). (a) Variation with the power-law index n at fixed $Bi_0 = 5$. (b) Variation with the Biot number Bi_0 at fixed $n = 0.85$ (shear-thinning).

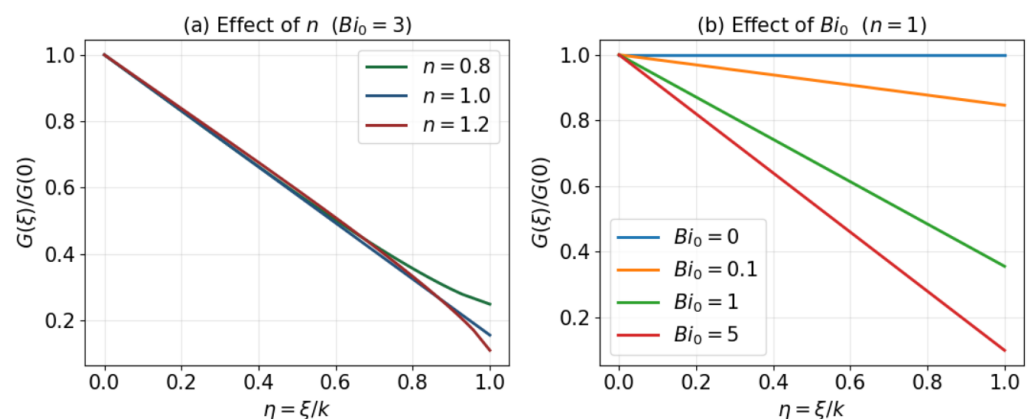


Figure 2. Cross-stream temperature profile normalised by its centreline value, $G(\xi)/G(0)$, against the scaled coordinate $\eta = \xi/k$ from the jet centreline ($\eta = 0$) to the free surface ($\eta = 1$). (a) Variation with the power-law index n at fixed $Bi_0 = 5$. (b) Variation with the Biot number Bi_0 at fixed $n = 1$ (Newtonian).

The Biot number is a dimensionless number in heat transfer that measures the competition between heat conduction inside the jet in this case and the heat convection at the free surface. It can be interpreted as the ratio between the internal temperature gradients and the external heat transfer.

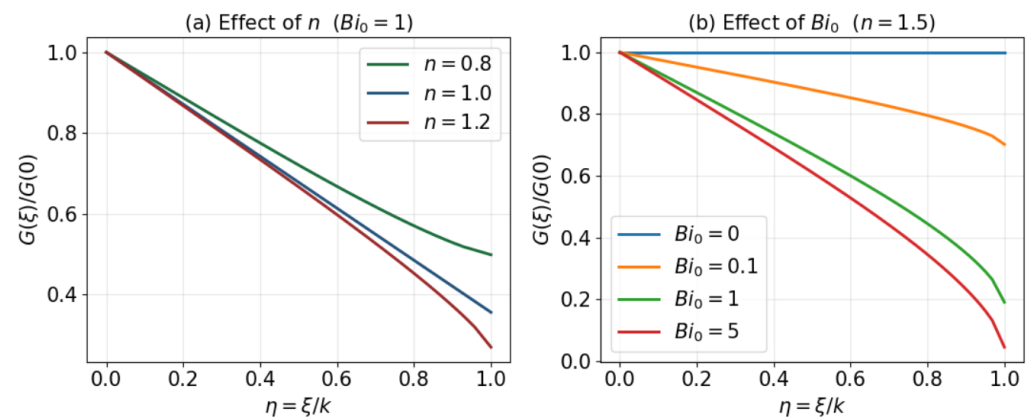


Figure 3. Cross-stream temperature profile normalised by its centreline value, $G(\xi)/G(0)$, against the scaled coordinate $\eta = \xi/k$ from the jet centreline ($\eta = 0$) to the free surface ($\eta = 1$). (a) Variation with the power-law index n at fixed $Bi_0 = 5$. (b) Variation with the Biot number Bi_0 at fixed $n = 1.5$ (shear-thickening).

From the above figures, it is evident that for shear-thinning fluids for small Biot numbers, the convection at the surface is weak while the conduction inside the jet is more efficient. This results in greater thermal retention at lower Biot numbers. For Newtonian and shear-thickening fluids, the decrease in cross-stream temperature is more pronounced. To explain this more clearly, we must understand the coupling between heat transport and surface heat loss. The Biot number does not by itself generate the heat loss. Instead it determines how efficiently heat is extracted at the surface. Essentially, we want to investigate how much thermal energy is available near the free surface for the Biot condition to remove and this depends on the rheological index n .

For shear-thinning fluids, heat is transported through the jet more efficiently and so the heat that is removed from the free surface is quickly and continually replenished. Hence, the temperature remains better retained and distributed through the jet. For Newtonian fluids this replenishment is more moderate but less effective than shear-thinning fluids. For shear-thickening fluids, thermal transport in the jet is less effective and by increasing the Biot number, the free surface loses heat quickly but the interior of the jet cannot replenish that heat as quickly. As a consequence, the free surface cools rapidly.

We now investigate the case of fixing the Biot number while varying the shearing index n . At lower Biot numbers, the shear-thinning temperature profile sits above the Newtonian and shear-thickening profiles and this is because the internal spreading of heat dominates surface cooling and so heat spreads through the jet more easily. This therefore means that the temperature will remain higher over a larger portion of the jet. For higher Biot numbers the opposite effect occurs. The shear-thickening profile sits above the shear-thinning profile, indicating that shear-thinning fluids lose heat more effectively because more heat is being supplied to the surface while for shear-thickening fluids, heat is not supplied to the free surface as easily and so more heat is retained in the jet.

In Figure 4, the temperature that is retained at the free surface is plotted for shear-thinning, Newtonian and shear-thickening fluids across a range of Biot numbers.

The heat retention at the free surface for all fluids exhibits a monotonic decrease in thermal retention as the Biot number increases. Evidently, shear-thinning fluids retain the most heat at the free surface while shear-thickening fluids the least. The free-surface temperature is a function of the Biot number and using the boundary condition at the free surface implies that the temperature there depends on how strongly the surface removes heat through the effective Biot number and how much heat is transported to the surface through the rheological index n . For shear-thinning fluids, heat is better distributed through

the jet, allowing for easier thermal transport from a warmer jet core to the interface. Shear-thickening fluids on the other hand are characterised by weaker internal transport due to the higher viscosity and so heat at the surface cannot be replaced as effectively from the jet core. Thus, the temperature at the free surface will be lower.

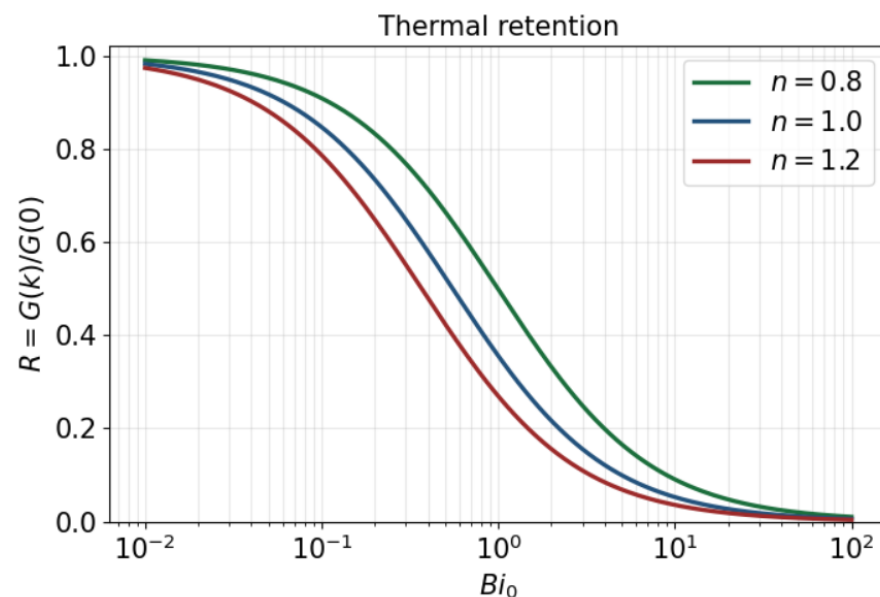


Figure 4. The fraction of the centreline temperature surviving at the free surface, versus Bi_0 . Retention decays monotonically as surface cooling strengthens, with shear-thinning fluids retaining the most heat and shear-thickening fluids the least.

The heat loss at the free surface is now investigated in Figure 5. The balance of the conductive flux arriving at the free surface and the convective loss to ambient are given by (91). It is clear that as the Biot number becomes very large, convection becomes so efficient that the maximum possible heat loss is determined entirely by how fast the thermal energy diffuses through the jet. This ceiling is rheologically driven.

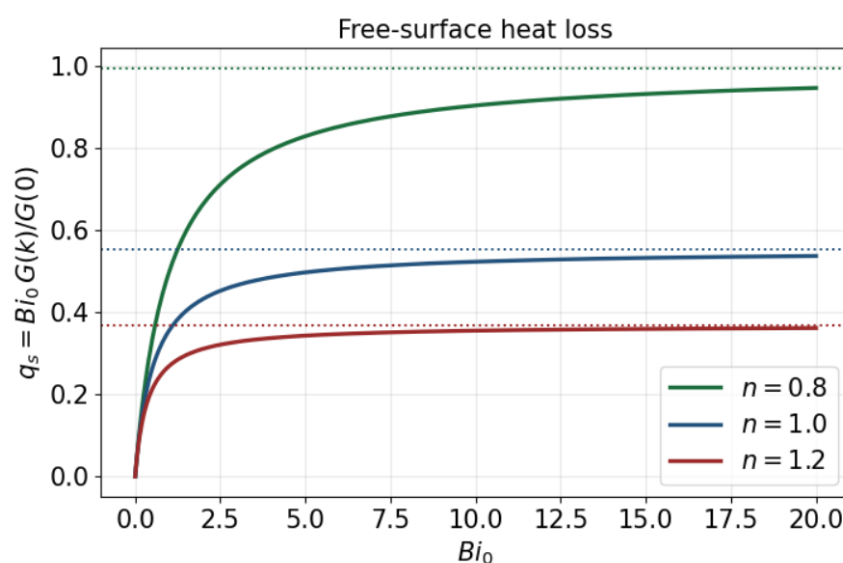


Figure 5. Dimensionless free-surface heat loss $q_s = Bi_0 G(k)/G(0)$ versus Bi_0 . At small Bi_0 the loss grows linearly ($q_s \approx Bi_0$, surface-coefficient-limited); as $Bi_0 \rightarrow \infty$, it saturates at a finite, rheology-set ceiling, where the loss becomes internal-conduction-limited: the jet sheds heat no faster than the nonlinear conductivity delivers it to the surface. Shear-thinning fluids reach the highest ceiling.

In order to understand where heat transfer is the strongest and how effectively heat penetrates the jet, we can study the effects of the shearing parameter n and Biot number on the thermal gradients.

In Figure 6a, the cross-stream temperature gradient across the jet is plotted for a fixed Biot number from the centreline of the jet to the free surface. Shear-thinning fluids have large gradients near the substrate indicating that heat is concentrated near the surface and because this large gradient remains significant across the thickness of the jet, thermal penetration is deeper for shear-thinning fluids than it is for shear-thickening or Newtonian fluids. There is a change in the magnitude of the gradient closer to the surface. The thermal gradient exhibits markedly different behaviour for shear-thinning and shear-thickening fluids. In the shear-thinning regime ($n < 1$), the gradient decreases rapidly towards the free surface, indicating that the temperature field becomes increasingly uniform. This behaviour is attributed to the enhanced effective thermal diffusivity near the free surface, which promotes the redistribution of heat throughout the fluid and therefore requires a smaller temperature gradient to sustain heat transfer. Consequently, although heat is transported efficiently, the local thermal gradient is reduced.

In contrast, shear-thickening fluids ($n > 1$) exhibit an increasing thermal gradient near the free surface. The reduced effective thermal diffusivity inhibits the redistribution of heat within the jet, resulting in a larger temperature difference between the fluid interior and the cooled free surface. This produces steeper temperature gradients, indicating greater thermal resistance and stronger localisation of heat transfer near the surface. These observations are consistent with the earlier heat-retention analysis, where shear-thinning fluids were found to redistribute heat more efficiently throughout the jet, while shear-thickening fluids retained larger internal temperature differences, leading to steeper gradients near the free surface.

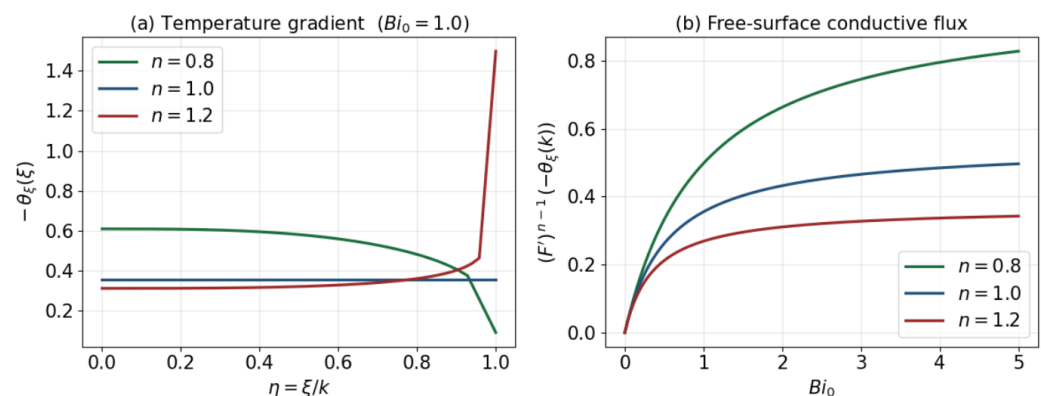


Figure 6. (a) Cross-stream temperature gradient $\theta'_\xi(\xi)$ across the jet at fixed $Bi_0 = 1$, from the centreline ($\eta = 0$) to free surface ($\eta = 1$): for shear-thinning fluids the gradient is largest in the interior and falls toward zero at the surface, the Newtonian gradient is uniform, and for shear-thickening fluids it rises steeply toward the surface. (b) Free-surface conductive flux $(F')^{n-1}(-\theta'_\xi(k))$ versus Bi_0 . It grows linearly at small Bi_0 and saturates at the rheology-set ceiling as cooling becomes internal-conduction-limited.

The free-surface conductive flux follows the same qualitative trend as the convective heat loss because they are coupled through the interfacial energy balance. Increasing the Biot number strengthens the surface cooling and increases conductive supply of heat to the surface, but further increases of the Biot number will produce diminishing changes because the rate of heat removal is limited by the rate at which heat can be transported from the jet interior to the free surface.

To study and identify the cooling intensity required to achieve a prescribed level of thermal retention, a parametric retention response map in the (n, Bi_0) –plane is plotted in Figure 7. This map provides a global view of the coupled influence of rheology and convective cooling. The map confirms that as the Biot number increases, retention decreases, but it also shows how this decrease is rheology-dependent. As expected, greater thermal retention is shown for low Biot numbers and strongly shear-thinning fluids compared with shear-thickening fluids and even Newtonian fluids. It is further evident that the flattening of the contours at large Biot numbers reflects the onset of an internally transport-limited regime where cooling strength produces diminishing returns.

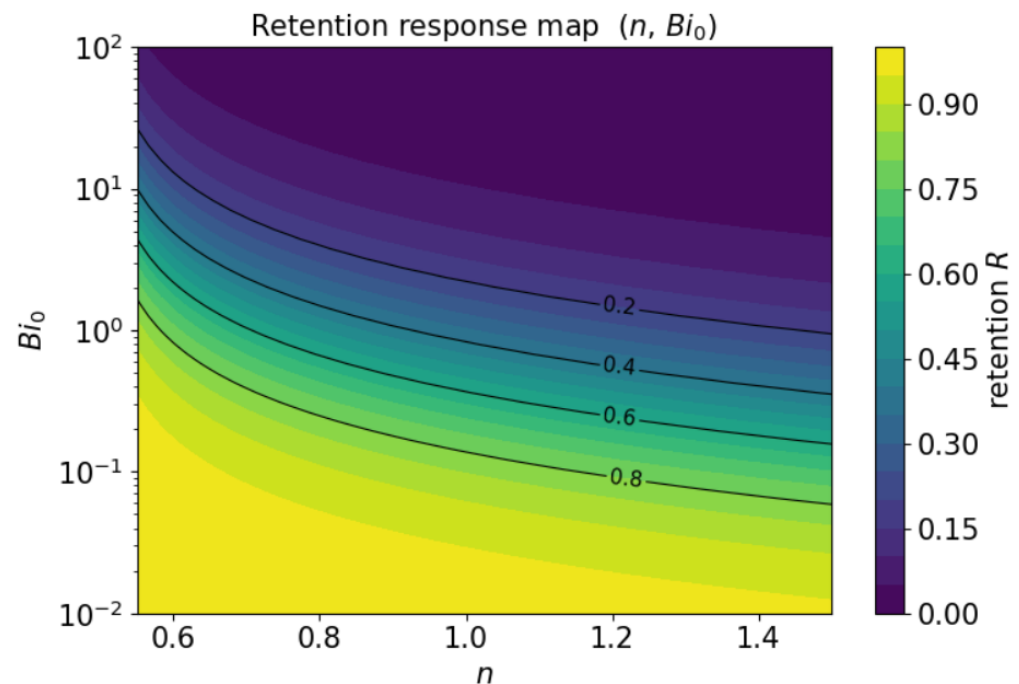


Figure 7. Thermal-retention parametric response map $R = G(k)/G(0)$ over the (n, Bi_0) plane.

To determine how much heat is retained across the dynamically active jet, we calculate the mean temperature across the jet using a velocity-weighted mean temperature since temperature alone does not tell us how much heat is actually being transported. The velocity-weighted mean temperature, $\bar{G}_u(\xi)$, is defined by

$$\bar{G}_u = \frac{\int_0^k F'(\xi)G(\xi)d\xi}{\int_0^k F'(\xi)d\xi} \quad (92)$$

where $F(\xi)$ is given in Table 1 and $G(\xi)$ in (80).

The velocity-weighted mean temperature is plotted against the shearing index n for fixed Biot numbers and also against the Biot number for a fixed n in Figure 8.

Since shear-thinning fluids spread heat more effectively towards cooled regions, the fast-moving part of the jet participates more actively in heat loss accounting for the lower velocity-weighted mean temperature, while for shear-thickening fluids thermal distribution is restricted. As a result, more heat remains trapped in the core of the jet, resulting in a larger velocity-weighted mean temperature.

For shear-thinning fluids for low Biot numbers, the shear-thinning profile remains above the shear-thickening and Newtonian profiles. However, at higher Biot numbers, the shear-thickening profile sits above the shear-thinning profile. This behaviour can be explained since for low Biot numbers, convective cooling is weaker and so the fast-moving region of the jet remains relatively warm while at higher Biot numbers, heat is transported to the heat sink more effectively thereby reducing overall temperature.

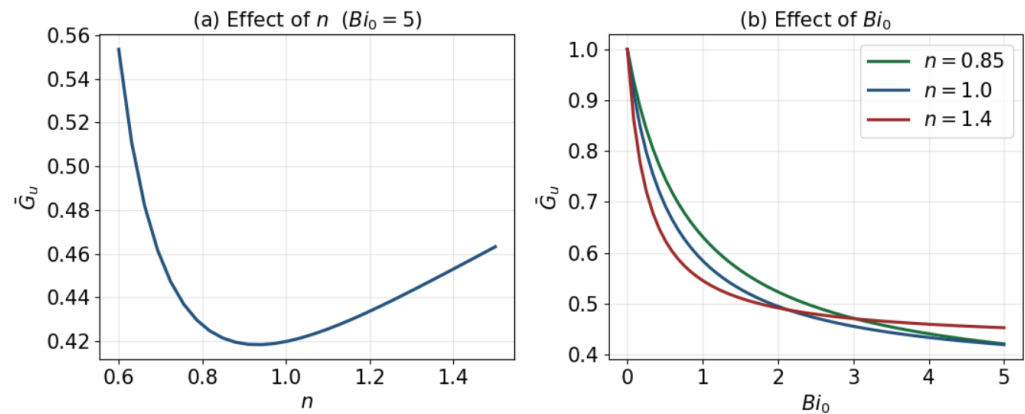


Figure 8. Velocity-weighted (bulk) mean temperature $\bar{G}_u = \int F' G d\xi / \int F' d\xi$ of the fluid transported downstream. (a) Variation with n at fixed $Bi_0 = 5$. (b) Variation with Bi_0 for three rheologies.

7. Verification of the Analytical Solution

7.1. Numerical Verification

To verify the analytical temperature solution given by (89), reduced energy Equation (65) was solved independently as a two-point boundary-value problem using the `solve_bvp` routine available in SciPy [31]. The numerical solver used only the governing differential equation together with the prescribed boundary conditions, and the analytical solution was not supplied during the computation. The resulting numerical temperature profiles were then compared directly with the analytical solution.

Excellent agreement was obtained between the two solutions over the entire computational domain. For the representative case $n = 0.7$ and $Bi_0 = 2$, the maximum absolute difference between the analytical and numerical solutions was 3.2×10^{-8} , confirming the accuracy of the closed-form expression. Furthermore, the solution correctly recovered the expected limiting behaviour. This is clearly shown in Figures 9 and 10. As $Bi_0 \rightarrow 0$, the temperature approached the uniform adiabatic state, while for the Newtonian limit ($n = 1$), the expected temperature distribution was recovered. These comparisons provide strong independent verification of the analytical solution. The results can be found in Table 2.

Table 2. Verification metrics for the representative case $n = 0.7$, $Pr_e = 1$, $Bi_0 = 2$.

Quantity	Value
Max. deviation $ G_{\text{num}} - G_{\text{analytic}} $	3.2×10^{-8}
RMS deviation	1.4×10^{-8}
Constancy of flux $(F')^{n-1} G'$ (spread)	3.9×10^{-13}
Surface condition residual $ q(k) + Bi_0 G(k) $	machine precision
Adiabatic limit $\max G - 1 $ as $Bi_0 \rightarrow 0$	6.4×10^{-9}

The results can be seen graphically below.

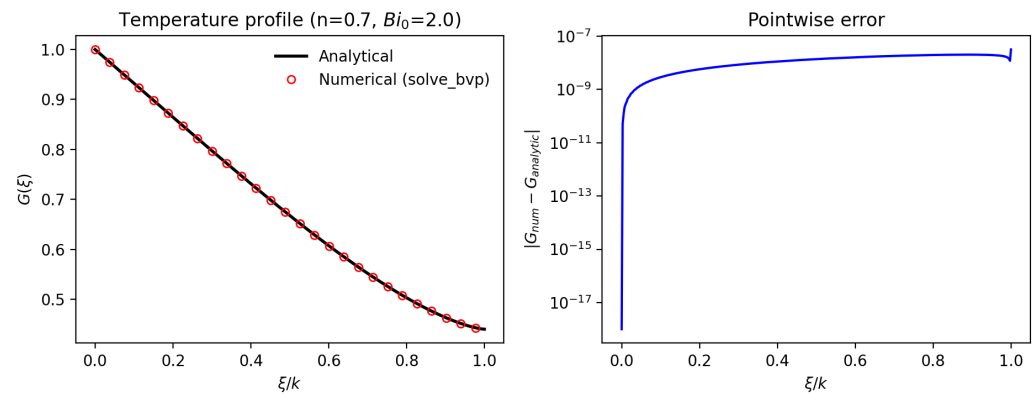


Figure 9. Verification of the closed-form temperature solution for $n = 0.7$, $Pr_e = 1$, $Bi_0 = 2$. **(left)** Similarity in temperature $G(\xi)$ against the scaled transverse coordinate ξ/k : the analytical solution (89) (solid line) and the independent numerical solution of reduced energy Equation (65) obtained with `solve_bvp` (markers). **(right)** Pointwise absolute deviation $|G_{\text{num}} - G_{\text{analytic}}|$, which remains below 3.2×10^{-8} across the domain and is governed by the interpolation of the velocity field rather than the solver tolerance.

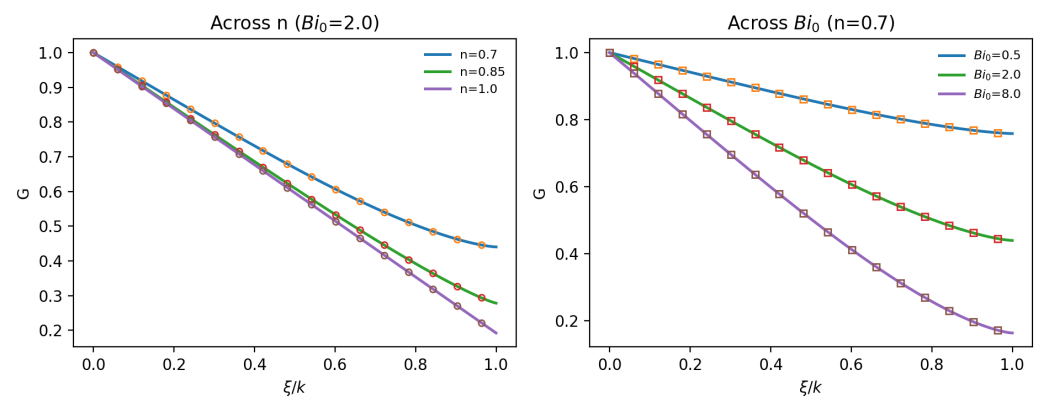


Figure 10. Agreement between the analytical (lines) and numerical (markers) temperature profiles $G(\xi)$ over a range of parameters. **(left)** Effect of the power-law index for $n = 0.7, 0.85, 1.0$ at fixed $Bi_0 = 2$; the Newtonian case $n = 1$ gives a linear distribution, while shear-thinning indices ($n < 1$) flatten the profile near the free surface. **(right)** Effect of the surface heat loss for $Bi_0 = 0.5, 2, 8$ at fixed $n = 0.7$; increasing Bi_0 lowers the free-surface temperature, approaching the isothermal limit. In all cases the numerical solution reproduces the closed form to within the discretisation error.

7.2. Limiting Cases and Verification Against Existing Literature

The solution admits several physically meaningful limits, which serve both to validate the analysis and to locate its contribution. For $n = 1$ the constitutive law becomes Newtonian and the shear-dependent diffusivity reduces to a constant, so that the temperature field recovers the classical thermal liquid jet problem studied for Newtonian fluids in [32,33]. Suppressing the shear dependence of the diffusivity for general n returns the constant-property energy balance. In the limit $Bi_0 \rightarrow 0$, the convective surface flux vanishes and the cross-stream temperature becomes uniform, recovering the adiabatic free-surface jet, whereas $Bi_0 \rightarrow \infty$ pins the free-surface temperature to the ambient value. Each limit reduces the present result to a previously known or elementary case. To the author's knowledge, however, no prior study has investigated the temperature field of a free-surface power-law liquid jet that simultaneously incorporates convective free-surface heat loss and a shear-rate-dependent thermal diffusivity. The present closed-form solution therefore addresses a configuration not previously available in the literature. A summary of the work carried out on thermal liquid jets is given below in Table 3 below.

Table 3. Position of the present work relative to existing studies on jet, film, and power-law fluid heat transfer.

Study	Liquid Jet	Free Surface	Power-Law Fluid	Convective Surface Loss	Shear-Dep. Diffusivity
Chaudhury [32]	Yes	Yes	No	No	No
Shu & Wilks [33,34]	Yes	Yes	No	No	No
Lienhard [35]	Yes	Yes	No	Wall [†]	No
Shu & Wilks [36]	Film	Yes	No	No	No
Zheng et al. [37,38]	No	No	Yes	No	Yes
Present work	Yes	Yes	Yes	Yes	Yes

[†] Heat transfer is mainly to the impinging wall rather than a convective free surface.

8. Conclusions

In the present analysis, the planar thermal liquid jet for a power-law fluid is investigated. It is shown that the behaviour of such fluids can be modelled using a shear-rate-dependent thermal diffusivity which contributes a novel description of the thermal evolution of power-law fluids downstream. Parametric solutions to the equations in the representative range $\frac{1}{2} < n < \infty$ are found for the thermal field for both adiabatic and convective heat loss boundary conditions. The parametric study showed that the thermal behaviour is governed by the coupled influence of the power-law index and the Biot number. Beyond sufficiently large Biot numbers, both the free-surface heat loss and conductive heat flux approached asymptotic limits, indicating that the cooling process becomes limited by the rate of thermal transport within the fluid rather than by the external convective conditions.

The rheology was found to strongly influence the redistribution of thermal energy within the jet. Shear-thinning fluids promoted more effective thermal redistribution, resulting in broader temperature variations throughout the jet and greater sensitivity to convective cooling under strong heat loss conditions. In contrast, shear-thickening fluids restricted internal thermal transport, retaining more thermal energy within the advecting core while concentrating thermal gradients near the free surface. These competing mechanisms produced distinct crossover behaviour in the temperature profiles and velocity-weighted temperatures, demonstrating a transition from redistribution-dominated heat transfer at weak cooling to surface-loss-dominated heat transfer at strong cooling.

Overall, the results highlight the significant role of non-Newtonian rheology in controlling thermal transport within liquid jets and provide analytical insight into the coupled effects of shear-dependent thermal diffusion and convective cooling. The present solutions furnish useful benchmark results for future studies of thermally coupled non-Newtonian free-surface flows and offer guidance for engineering applications where controlling thermal retention or heat removal is of practical importance.

The present study is subject to several modelling assumptions that provide opportunities for future investigation. First, the analysis is restricted to a two-dimensional, steady laminar liquid jet, whereas many practical free-surface flows exhibit three-dimensional effects, flow unsteadiness and, at sufficiently large Reynolds numbers, transition to turbulence. Secondly, the fluid rheology is represented by the power-law constitutive model, which accurately captures shear-thinning and shear-thickening behaviour over an intermediate shear-rate range but does not account for yield stress or finite shear-rate viscosity limits that arise in more complex non-Newtonian fluids. The thermal analysis also neglects viscous dissipation, buoyancy and surface tension, all of which may become important in applications involving large temperature gradients or highly viscous fluids. Furthermore,

the convective boundary condition employs an effective Biot number to preserve similarity invariance, providing an analytically tractable representation of spatially varying heat transfer but not capturing more general external thermal environments. Finally, although the analytical solution was independently verified against numerical solutions of the reduced boundary-value problem, experimental validation and comparison with direct numerical simulations would provide further confidence in the model and help assess its applicability to practical thermal liquid jet systems.

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