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Geometric Properties of the $\widetilde{W}^{(\alpha_i)}$ -Curvature Tensor Family on Kenmotsu Manifolds

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Abstract

In this paper, we investigate the family of curvature tensors $\widetilde{W}^{(\alpha_i)}$, $i = 1, 2, \dots, 12$ under the Schouten–van Kampen connection $\widetilde{\nabla}$ on Kenmotsu manifolds. Our focus is directed toward evaluating diverse curvature conditions, such as $\widetilde{W}^{(\alpha_j)}$ -flatness, ξ - $\widetilde{W}^{(\alpha_j)}$ -flatness, quasi $\widetilde{W}^{(\alpha_j)}$ -flatness, and ϕ - $\widetilde{W}^{(\alpha_j)}$ -flatness for $j = 5, 8$. We show that these curvature restrictions imply that the corresponding Kenmotsu manifolds are Einstein or η -Einstein manifolds. Furthermore, we examine the derivation condition $\widetilde{W}^{(\alpha_j)}(Z_1, Z_2) \widetilde{W}^{(\alpha_i)} = 0$, $j = 5, 8$, $i = 1, 2, \dots, 12$ and prove that any Kenmotsu manifold satisfying this condition is an Einstein manifold.

Keywords: Kenmotsu manifolds; Einstein manifolds; η -Einstein manifolds; $\widetilde{W}^{(\alpha_i)}$ -curvature tensor family; Schouten–van Kampen connection

MSC: 53D10; 53C25; 53C05; 53C35

1. Introduction

In 1972, Katsuei Kenmotsu pioneered a remarkably important class of almost contact metric spaces, now known as Kenmotsu manifolds [1]. Such a manifold is equipped with an almost contact metric structure (ϕ, ξ, η, g) , in which the characteristic Reeb vector field ξ is non-Killing. The underlying geometry is governed by the fundamental structural equations $\nabla_{Z_1} \xi = Z_1 - \eta(Z_1)\xi$ and $(\nabla_{Z_1} \phi)Z_2 = -g(Z_1, \phi Z_2)\xi - \eta(Z_2)\phi Z_1$. Due to their intrinsic connection to hyperbolic geometry and warped product spaces, these manifolds have been extensively studied across various disciplines of modern differential geometry [2,3].

Exploring the intricate geometric and topological characteristics of a manifold heavily relies on choosing a suitable linear connection. From this perspective, the Schouten–van Kampen (SvK) connection dates back to 1930, when it was formulated to explore the geometry of non-holonomic manifolds in a systematic manner [4]. The defining feature of this connection is its ability to keep two mutually complementary distributions invariant under parallel transport on a differentiable manifold [5,6]. In 2013, Olszak extended the application of the SvK connection to the domain of almost contact metric structures [7], revealing its essential curvature behavior. Building upon these developments, this connection has become a powerful tool for analyzing various contact metric frameworks, including Sasakian and f -Kenmotsu manifolds [8,9].

Curvature tensors are indispensable geometric objects for characterizing local geometry and gravitational configurations on a manifold. While classical curvature tensors—such as the Weyl conformal, concircular, and conharmonic curvature tensors—provide distinct geometric perspectives, Tripathi and Gupta [10] introduced the generalized T -curvature



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tensor, which unifies a wide range of well-known curvature tensors, including the Weyl conformal curvature tensor and the family of W_i -curvature tensors ($i = 0, \dots, 9$). More recently, comparative studies have revealed significant relationships between the Weyl conformal curvature tensors associated with the Levi-Civita and the SvK connections on Kenmotsu manifolds [11].

A prominent direction in contemporary literature involves examining the flatness and derivation conditions of these curvature tensors to classify underlying manifold structures. Imposing vanishing criteria, such as ξ -concurvature flatness, ϕ -concurvature flatness, or quasi- T -flatness, allow researchers to reduce complex structures into simpler Einstein or η -Einstein spaces [12,13]. For instance, if the curvature tensor with respect to the SvK connection vanishes identically, the Kenmotsu manifold becomes locally isometric to the hyperbolic space $H^{2n+1}(-1)$ [14]. Furthermore, recent investigations involving W_1 , W_2 , and Q -curvature tensors have highlighted how specific flatness constraints enforce rigid conditions on the ambient geometric structure [15–17]. These ongoing explorations of generalized curvature conditions continue to provide profound insights into the structure of Kenmotsu manifolds [11,18].

However, despite these developments, a systematic investigation of the $\tilde{W}^{(\alpha_i)}$ -curvature tensor family, $i = 1, 2, \dots, 12$, associated with the generalized T -curvature tensor under the SvK connection on Kenmotsu manifolds has not yet been carried out. In [11], W_0 , W_0^* , W_2 , and W_4 curvature tensors are examined and compared under the SvK and the Levi-Civita connections. Moreover, previous studies have mainly focused on individual members of the generalized curvature tensor family or on their counterparts with respect to the Levi-Civita connection. Motivated by these results, the present paper extends this line of research by focusing on the tensors $\tilde{W}^{(\alpha_5)}$ and $\tilde{W}^{(\alpha_8)}$, two members of the $\tilde{W}^{(\alpha_i)}$ -curvature tensor family, $i = 1, 2, \dots, 12$, associated with the generalized T -curvature tensor, and investigating their curvature restrictions as well as their derivation conditions with respect to the entire $\tilde{W}^{(\alpha_i)}$ -curvature tensor family, $i = 1, 2, \dots, 12$, under the SvK connection on Kenmotsu manifolds.

The paper is organized as follows. Section 3 focuses on the family of curvature tensors $\tilde{W}^{(\alpha_i)}$, $i = 1, 2, \dots, 12$, with respect to the SvK connection $\tilde{\nabla}$. Various curvature conditions, including $\tilde{W}^{(\alpha_j)}$ -flatness, ξ - $\tilde{W}^{(\alpha_j)}$ -flatness, quasi- $\tilde{W}^{(\alpha_j)}$ -flatness, and ϕ - $\tilde{W}^{(\alpha_j)}$ -flatness, are investigated, leading to Einstein and η -Einstein characterizations of Kenmotsu manifolds. Section 4 deals with the derivation condition $\tilde{W}^{(\alpha_j)}(Z_1, Z_2) \tilde{W}^{(\alpha_i)} = 0$, $j = 5, 8$, $i = 1, 2, \dots, 12$ under which Einstein characterizations are obtained. The obtained Einstein and η -Einstein characterizations demonstrate that the considered curvature restrictions impose strong restrictions on the Ricci tensor of Kenmotsu manifolds. In particular, the Einstein condition indicates that the Ricci tensor is completely determined by the metric, whereas the η -Einstein condition allows the Ricci tensor to be decomposed into a metric component and an additional component in the direction of the structure vector field ξ . Therefore, the investigated curvature conditions reveal significant geometric restrictions arising from the Schouten–van Kampen connection.

2. Preliminaries

Let M denote a smooth manifold of dimension $(2n + 1)$. An almost contact metric structure on M is specified by a quadruplet (ϕ, ξ, η, g) , where ϕ represents a $(1, 1)$ -type tensor field, ξ stands for the characteristic vector field, η is a 1-form, and g signifies a Riemannian metric on M satisfying the conditions

$$\phi^2(Z_1) = -Z_1 + \eta(Z_1)\xi, \quad \eta \circ \phi = 0, \quad (1)$$

for any vector field $Z_1 \in \chi(M)$, along with the requirements

$$\eta(\xi) = 1, \quad \phi\xi = 0. \quad (2)$$

Furthermore, the Riemannian metric g satisfies

$$g(\phi Z_1, \phi Z_2) = g(Z_1, Z_2) - \eta(Z_1)\eta(Z_2). \quad (3)$$

As an immediate consequence of these structural definitions, the following relations hold

$$g(Z_1, \phi Z_2) = -g(\phi Z_1, Z_2), \quad g(Z_1, \xi) = \eta(Z_1) \quad (4)$$

for all $Z_1, Z_2 \in \chi(M)$. Under these circumstances, g is a compatible Riemannian metric with respect to the almost contact structure (ϕ, ξ, η) on $\chi(M)$ [3,19].

An almost contact metric manifold (M, ϕ, ξ, η, g) is classified as an almost Kenmotsu manifold provided that the relations

$$d\eta = 0 \quad \text{and} \quad d\Phi = 2\eta \wedge \Phi \quad (5)$$

are satisfied, where $\Phi(Z_1, Z_2) = g(Z_1, \phi Z_2)$ represents the fundamental 2-form.

More precisely, following the geometric criteria given in [3], an almost Kenmotsu manifold reduces to a Kenmotsu manifold if the condition

$$(\nabla_{Z_1}\phi)Z_2 = -\eta(Z_2)\phi Z_1 - g(Z_1, \phi Z_2)\xi \quad (6)$$

holds for any $Z_1, Z_2 \in \chi(M)$, where ∇ denotes the Levi-Civita connection associated with g .

On a Kenmotsu manifold M , the identities regarding the Levi-Civita connection ∇ , the Riemannian curvature tensor R , and the Ricci operator S satisfy the following standard identities [1,20,21]

$$\nabla_{Z_1}\xi = Z_1 - \eta(Z_1)\xi, \quad (7)$$

$$(\nabla_{Z_1}\eta)Z_2 = g(\nabla_{Z_1}\xi, Z_2), \quad (8)$$

$$R(Z_1, Z_2)\xi = \eta(Z_1)Z_2 - \eta(Z_2)Z_1, \quad (9)$$

$$R(\xi, Z_1)Z_2 = \eta(Z_2)Z_1 - g(Z_1, Z_2)\xi, \quad (10)$$

$$S(Z_1, \xi) = -2n\eta(Z_1), \quad (11)$$

$$Q\xi = -2n\xi. \quad (12)$$

Here Q is the $(1,1)$ type Ricci operator which is intrinsically related to the Ricci tensor through the metric via $S(Z_1, Z_2) = g(QZ_1, Z_2)$. It is worth noting that the Reeb vector field ξ plays a distinguished role in the curvature structure of Kenmotsu manifolds. Indeed, relation (12) shows that ξ is an eigenvector of the Ricci operator corresponding to the eigenvalue $-2n$. This property further emphasizes the geometric significance of the Reeb vector field in Kenmotsu manifolds.

Furthermore, by exploiting the compatibility of the Kenmotsu structure with the tensor fields, the Ricci tensor under the action of the structure tensor ϕ satisfies the subsequent property

$$S(\phi Z_1, \phi Z_2) = S(Z_1, Z_2) + 2n\eta(Z_1)\eta(Z_2). \quad (13)$$

An η -Einstein structure on a Kenmotsu manifold M is characterized by the Ricci tensor expansion

$$S(Z_1, Z_2) = ag(Z_1, Z_2) + b\eta(Z_1)\eta(Z_2) \quad (14)$$

where a, b are smooth functions on M . Specifying $b = 0$ instantly gives an Einstein structure [1,22].

In our subsequent computations, we utilize the following fundamental trace identities derived on a $(2n + 1)$ -dimensional Kenmotsu manifold with respect to the local orthonormal frame [3]

$$\sum_{i=1}^{2n} g(e_i, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n, \quad (15)$$

$$\sum_{i=1}^{2n} g(\phi e_i, Z_3)S(Z_2, \phi e_i) = S(Z_2, Z_3) + 2n\eta(Z_2)\eta(Z_3), \quad (16)$$

$$\sum_{i=1}^{2n} g(\phi e_i, \phi Z_3)S(Z_2, \phi e_i) = S(Z_2, \phi Z_3), \quad (17)$$

$$\sum_{i=1}^{2n} g(R(\phi e_i, \phi Z_2)\phi Z_3, \phi e_i) = S(\phi Z_2, \phi Z_3) + g(\phi Z_2, \phi Z_3), \quad (18)$$

$$\sum_{i=1}^{2n} S(\phi e_i, \phi e_i) = r + 2n, \quad (19)$$

$$\sum_{i=1}^{2n} S(\phi Z_2, \phi e_i)g(\phi e_i, \phi Z_3) = S(\phi Z_2, \phi Z_3), \quad (20)$$

$$\sum_{i=1}^{2n} g(\phi Z_2, \phi e_i)g(\phi e_i, \phi Z_3) = g(\phi Z_2, \phi Z_3). \quad (21)$$

Throughout the paper, geometric objects associated with the Schouten–van Kampen connection are distinguished by a tilde, such as $\tilde{\nabla}$, $\tilde{R}$, $\tilde{S}$, $\tilde{Q}$, and $\tilde{r}$, whereas the classical notations without a tilde refer to the conventional Levi-Civita connection. Following [7], the connection $\tilde{\nabla}$ is related to the connection ∇ via

$$\tilde{\nabla}_{Z_1} Z_2 = \nabla_{Z_1} Z_2 - \eta(Z_2)\nabla_{Z_1} \xi + (\nabla_{Z_1} \eta)(Z_2)\xi. \quad (22)$$

By substituting (7) and (8) into (22), we immediately have

$$\tilde{\nabla}_{Z_1} Z_2 = \nabla_{Z_1} Z_2 + g(Z_1, Z_2)\xi - \eta(Z_2)Z_1. \quad (23)$$

Setting $Z_2 = \xi$ in (23) and taking into account (7), one easily finds

$$\tilde{\nabla}_{Z_1} \xi = 0. \quad (24)$$

The curvature operators $\tilde{R}$ and R are conventionally given by

$$\tilde{R}(Z_1, Z_2) = [\tilde{\nabla}_{Z_1}, \tilde{\nabla}_{Z_2}] - \tilde{\nabla}_{[Z_1, Z_2]}, \quad (25)$$

$$R(Z_1, Z_2) = [\nabla_{Z_1}, \nabla_{Z_2}] - \nabla_{[Z_1, Z_2]}. \quad (26)$$

Then, a direct calculation using (23) leads to the relation

$$\tilde{R}(Z_1, Z_2)Z_3 = R(Z_1, Z_2)Z_3 + g(Z_2, Z_3)Z_1 - g(Z_1, Z_3)Z_2. \quad (27)$$

By contracting (27) over a local orthonormal frame, the Ricci tensor $\tilde{S}$ is given by

$$\tilde{S}(Z_1, Z_2) = S(Z_1, Z_2) + 2ng(Z_1, Z_2). \quad (28)$$

Equivalently, the corresponding Ricci operator $\tilde{Q}$ satisfies the equation

$$\tilde{Q}Z_1 = QZ_1 + 2nZ_1. \quad (29)$$

Finally, by taking the trace of the Ricci operator $\tilde{Q}$, the scalar curvature $\tilde{r}$ is obtained as

$$\tilde{r} = r + 2n(2n + 1). \quad (30)$$

Apart from the well-known Riemannian curvature tensor, several alternative types of curvature tensors exist, such as the Weyl conformal, conharmonic, and concircular curvature tensors, each providing distinct geometric characteristics. Let R , S , Q and r denote the Riemannian curvature tensor, the Ricci tensor, the Ricci operator, and the scalar curvature, respectively. The generalized $\mathcal{T}$ -curvature tensor is defined by

$$\begin{aligned} \mathcal{T}(Z_1, Z_2)Z_3 = & a_0R(Z_1, Z_2)Z_3 + a_1S(Z_2, Z_3)Z_1 + a_2S(Z_1, Z_3)Z_2 \\ & + a_3S(Z_1, Z_2)Z_3 + a_4g(Z_2, Z_3)QZ_1 + a_5g(Z_1, Z_3)QZ_2 \\ & + a_6g(Z_1, Z_2)QZ_3 + a_7r(g(Z_2, Z_3)Z_1 - g(Z_1, Z_3)Z_2) \end{aligned} \quad (31)$$

where $a_0, \dots, a_7$ are smooth functions on M [10].

3. $\tilde{W}^{(\alpha_i)}$ -Curvature Tensor Family on Kenmotsu Manifolds with Flatness Properties of $\tilde{W}^{(\alpha_5)}$ and $\tilde{W}^{(\alpha_8)}$

In this section, we introduce the $W^{(\alpha_i)}$ -curvature tensor family on Kenmotsu manifolds under the Schouten-van Kampen connection, where each tensor is denoted by $\tilde{W}^{(\alpha_i)}$ for $i = 1, \dots, 12$. We then investigate their flatness properties, with a particular emphasis on the cases $j = 5$ and $j = 8$. Specifically, we examine the geometric consequences associated with several notions of flatness, namely $\tilde{W}^{(\alpha_j)}$ -flat, ξ - $\tilde{W}^{(\alpha_j)}$ -flat, quasi $\tilde{W}^{(\alpha_j)}$ -flat, and ϕ - $\tilde{W}^{(\alpha_j)}$ -flat structures for $j = 5, 8$.

Among the special cases of the generalized $\mathcal{T}$ -curvature tensor mentioned in (31), we consider the family of $W^{(\alpha_i)}$ -curvature tensors ($i = 1, \dots, 12$), obtained by setting $a_0 = 1$, choosing exactly two nonzero coefficients from the set $\{a_1, \dots, a_6\}$, each equal to $\pm \frac{1}{2n}$, and setting all remaining coefficients, including a_7 , equal to zero.

In this case, we write

$$W^{(\alpha_i)} = \mathcal{T}_{\alpha_i} = \mathcal{T}_{(a_1, a_2, a_3, a_4, a_5, a_6)}, \quad i = 1, \dots, 12,$$

to denote the family of curvature tensors corresponding to the ordered coefficient tuples

$$\alpha_i = (a_1, a_2, a_3, a_4, a_5, a_6),$$

where the $W^{(\alpha_i)}$ correspond respectively to the classical curvature tensors W_0 , W_0^* , W_1 , W_1^* , and W_j ($j = 2, \dots, 9$), while the corresponding ordered coefficient tuples α_i ($i = 1, \dots, 12$) are explicitly listed in Table 1. Here, the configurations corresponding to W_0 , W_0^* , W_1 , W_1^* , W_3 , and W_4 are adopted from [23], whereas those corresponding to W_2 and W_5 – W_9 are obtained from [24] and [25], respectively.

Table 1. The ordered 6-tuples α_i defining the $W^{(\alpha_i)}$ -curvature tensors.

α_i	Corresponding Ordered 6-Tuples
α_1	$\left(\frac{-1}{2n}, 0, 0, 0, \frac{1}{2n}, 0\right)$
α_2	$\left(\frac{1}{2n}, 0, 0, 0, \frac{-1}{2n}, 0\right)$
α_3	$\left(\frac{1}{2n}, \frac{-1}{2n}, 0, 0, 0, 0\right)$
α_4	$\left(\frac{-1}{2n}, \frac{1}{2n}, 0, 0, 0, 0\right)$
α_5	$\left(0, 0, 0, \frac{-1}{2n}, \frac{1}{2n}, 0\right)$
α_6	$\left(0, \frac{-1}{2n}, 0, \frac{1}{2n}, 0, 0\right)$
α_7	$\left(0, 0, 0, 0, \frac{1}{2n}, \frac{-1}{2n}\right)$
α_8	$\left(0, \frac{-1}{2n}, 0, 0, \frac{1}{2n}, 0\right)$
α_9	$\left(\frac{-1}{2n}, 0, 0, 0, 0, \frac{1}{2n}\right)$
α_{10}	$\left(\frac{-1}{2n}, 0, 0, \frac{1}{2n}, 0, 0\right)$
α_{11}	$\left(\frac{-1}{2n}, 0, \frac{1}{2n}, 0, 0, 0\right)$
α_{12}	$\left(0, 0, \frac{1}{2n}, \frac{-1}{2n}, 0, 0\right)$

Definition 1. The $\tilde{W}^{(\alpha_i)}$ -curvature tensor family ($i = 1, \dots, 12$) is defined by

$$\begin{aligned}\tilde{W}^{(\alpha_i)}(Z_1, Z_2)Z_3 &= \tilde{T}_{\alpha_i}(Z_1, Z_2)Z_3 \\ &= \tilde{R}(Z_1, Z_2)Z_3 + a_1 \tilde{S}(Z_2, Z_3)Z_1 + a_2 \tilde{S}(Z_1, Z_3)Z_2 + a_3 \tilde{S}(Z_1, Z_2)Z_3 \\ &\quad + a_4 g(Z_2, Z_3)\tilde{Q}Z_1 + a_5 g(Z_1, Z_3)\tilde{Q}Z_2 + a_6 g(Z_1, Z_2)\tilde{Q}Z_3.\end{aligned}\quad (32)$$

for all $Z_1, Z_2, Z_3 \in \chi(M)$, where $\tilde{R}$, $\tilde{S}$ and $\tilde{Q}$ denote the curvature tensor, the Ricci tensor, and the Ricci operator of $\tilde{\nabla}$, respectively. Moreover, $\alpha_i = (a_1, a_2, a_3, a_4, a_5, a_6)$ are the ordered 6-tuples given in Table 1 for $i = 1, \dots, 12$.

Substituting Equations (27)–(29) into Equation (32) yields the $\tilde{W}^{(\alpha_i)}$ -curvature tensor family ($i = 1, \dots, 12$) in terms of ∇ on M , obtained as

$$\begin{aligned}\tilde{W}^{(\alpha_i)}(Z_1, Z_2)Z_3 &= R(Z_1, Z_2)Z_3 + a_1 S(Z_2, Z_3)Z_1 + a_2 S(Z_1, Z_3)Z_2 + a_3 S(Z_1, Z_2)Z_3 \\ &\quad + a_4 g(Z_2, Z_3)QZ_1 + a_5 g(Z_1, Z_3)QZ_2 + a_6 g(Z_1, Z_2)QZ_3 \\ &\quad + 2n\left(\frac{1}{2n} + a_1 + a_4\right)g(Z_2, Z_3)Z_1 \\ &\quad + 2n\left(-\frac{1}{2n} + a_2 + a_5\right)g(Z_1, Z_3)Z_2 \\ &\quad + 2n(a_3 + a_6)g(Z_1, Z_2)Z_3.\end{aligned}\quad (33)$$

The explicit forms of the $\tilde{W}^{(\alpha_i)}$ -curvature tensors ($i = 1, \dots, 12$) in Equation (33) are listed below. Although these expressions are obtained directly by substituting the corresponding ordered 6-tuples given in Table 1 into Equation (33), they are presented explicitly for the convenience of the subsequent computations.

$$\tilde{W}^{(\alpha_1)} = R(Z_1, Z_2)Z_3 - \frac{1}{2n}S(Z_2, Z_3)Z_1 + \frac{1}{2n}g(Z_1, Z_3)QZ_2 \quad (34)$$

$$\tilde{W}^{(\alpha_2)} = R(Z_1, Z_2)Z_3 + 2g(Z_2, Z_3)Z_1 - 2g(Z_1, Z_3)Z_2 + \frac{1}{2n}[S(Z_2, Z_3)Z_1 - g(Z_1, Z_3)QZ_2] \quad (35)$$

$$\tilde{W}^{(\alpha_3)} = R(Z_1, Z_2)Z_3 + 2g(Z_2, Z_3)Z_1 - 2g(Z_1, Z_3)Z_2 + \frac{1}{2n}S(Z_2, Z_3)Z_1 - \frac{1}{2n}S(Z_1, Z_3)Z_2 \quad (36)$$

$$\tilde{W}^{(\alpha_4)} = R(Z_1, Z_2)Z_3 - \frac{1}{2n}S(Z_2, Z_3)Z_1 + \frac{1}{2n}S(Z_1, Z_3)Z_2 \quad (37)$$

$$\tilde{W}^{(\alpha_5)} = R(Z_1, Z_2)Z_3 - \frac{1}{2n}g(Z_2, Z_3)QZ_1 + \frac{1}{2n}g(Z_1, Z_3)QZ_2 \quad (38)$$

$$\tilde{W}^{(\alpha_6)} = R(Z_1, Z_2)Z_3 + 2g(Z_2, Z_3)Z_1 - 2g(Z_1, Z_3)Z_2 - \frac{1}{2n}S(Z_1, Z_3)Z_2 + \frac{1}{2n}g(Z_2, Z_3)QZ_1 \quad (39)$$

$$\tilde{W}^{(\alpha_7)} = R(Z_1, Z_2)Z_3 + g(Z_2, Z_3)Z_1 - g(Z_1, Z_2)Z_3 + \frac{1}{2n}g(Z_1, Z_3)QZ_2 - \frac{1}{2n}g(Z_1, Z_2)QZ_3 \quad (40)$$

$$\tilde{W}^{(\alpha_8)} = R(Z_1, Z_2)Z_3 + g(Z_2, Z_3)Z_1 - g(Z_1, Z_3)Z_2 - \frac{1}{2n}S(Z_1, Z_3)Z_2 + \frac{1}{2n}g(Z_1, Z_3)QZ_2 \quad (41)$$

$$\tilde{W}^{(\alpha_9)} = R(Z_1, Z_2)Z_3 + g(Z_1, Z_2)Z_3 - g(Z_1, Z_3)Z_2 - \frac{1}{2n}S(Z_2, Z_3)Z_1 + \frac{1}{2n}g(Z_1, Z_2)QZ_3 \quad (42)$$

$$\tilde{W}^{(\alpha_{10})} = R(Z_1, Z_2)Z_3 + g(Z_2, Z_3)Z_1 - g(Z_1, Z_3)Z_2 - \frac{1}{2n}S(Z_2, Z_3)Z_1 + \frac{1}{2n}g(Z_2, Z_3)QZ_1 \quad (43)$$

$$\tilde{W}^{(\alpha_{11})} = R(Z_1, Z_2)Z_3 + g(Z_1, Z_2)Z_3 - g(Z_1, Z_3)Z_2 - \frac{1}{2n}S(Z_2, Z_3)Z_1 + \frac{1}{2n}S(Z_1, Z_2)Z_3 \quad (44)$$

$$\tilde{W}^{(\alpha_{12})} = R(Z_1, Z_2)Z_3 + g(Z_1, Z_2)Z_3 - g(Z_1, Z_3)Z_2 + \frac{1}{2n}S(Z_1, Z_2)Z_3 - \frac{1}{2n}g(Z_2, Z_3)QZ_1 \quad (45)$$

Using Equations (4) and (9)–(12), together with straightforward computations, we obtain the action of the $\tilde{W}^{(\alpha_i)}$ -curvature tensors when one or more arguments are ξ . As a representative example,

$$\begin{aligned} \tilde{W}^{(\alpha_i)}(Z_1, Z_2)\xi = & R(Z_1, Z_2)\xi + a_1S(Z_2, \xi)Z_1 + a_2S(Z_1, \xi)Z_2 + a_3S(Z_1, Z_2)\xi \\ & + a_4g(Z_2, \xi)QZ_1 + a_5g(Z_1, \xi)QZ_2 + a_6g(Z_1, Z_2)Q\xi \\ & + 2n\left(\frac{1}{2n} + a_1 + a_4\right)g(Z_2, \xi)Z_1 \\ & + 2n\left(-\frac{1}{2n} + a_2 + a_5\right)g(Z_1, \xi)Z_2 \\ & + 2n(a_3 + a_6)g(Z_1, Z_2)\xi. \end{aligned}$$

Then, using Equations (4), (9)–(12), we obtain the simplified form

$$\tilde{W}^{(\alpha_i)}(Z_1, Z_2)\xi = a_3(S(Z_1, Z_2) + 2ng(Z_1, Z_2))\xi + a_4(2nZ_1 + QZ_1)\eta(Z_2) + a_5(2nZ_2 + QZ_2)\eta(Z_1).$$

The remaining identities follow from analogous computations and are stated in the following lemma.

Lemma 1. *Let ξ be the characteristic vector field. Then, the following identities hold:*

$$\begin{aligned} \tilde{W}^{(\alpha_i)}(Z_1, Z_2)\xi = & a_3(S(Z_1, Z_2) + 2ng(Z_1, Z_2))\xi + a_4(2nZ_1 + QZ_1)\eta(Z_2) \\ & + a_5(2nZ_2 + QZ_2)\eta(Z_1), \end{aligned} \quad (46)$$

$$\begin{aligned} \tilde{W}^{(\alpha_i)}(Z_1, \xi)Z_3 = & a_2(S(Z_1, Z_3) + 2ng(Z_1, Z_3))\xi + a_4(2nZ_1 + QZ_1)\eta(Z_3) \\ & + a_6(2nZ_3 + QZ_3)\eta(Z_1), \end{aligned} \quad (47)$$

$$\begin{aligned} \tilde{W}^{(\alpha_i)}(\xi, Z_2)Z_3 = & a_1(S(Z_2, Z_3) + 2ng(Z_2, Z_3))\xi + a_5(2nZ_2 + QZ_2)\eta(Z_3) \\ & + a_6(2nZ_3 + QZ_3)\eta(Z_2), \end{aligned} \quad (48)$$

$$\tilde{W}^{(\alpha_i)}(\xi, Z_2)\xi = a_5(2nZ_2 + QZ_2), \quad (49)$$

$$\tilde{W}^{(\alpha_i)}(Z_1, \xi)\xi = a_4(2nZ_1 + QZ_1). \quad (50)$$

Definition 2. A Kenmotsu manifold M is said to be:

1. $\tilde{W}^{(\alpha_j)}$ -flat for $j = 5, 8$ if

$$\tilde{W}^{(\alpha_j)}(Z_1, Z_2)Z_3 = 0,$$
2. $\xi\text{-}\tilde{W}^{(\alpha_j)}$ -flat for $j = 5, 8$ if

$$\tilde{W}^{(\alpha_j)}(Z_1, Z_2)\xi = 0,$$
3. quasi $\tilde{W}^{(\alpha_j)}$ -flat for $j = 5, 8$ if

$$g(\tilde{W}^{(\alpha_j)}(\phi Z_1, Z_2)Z_3, \phi Z_4) = 0,$$
4. $\phi\text{-}\tilde{W}^{(\alpha_j)}$ -flat for $j = 5, 8$ if

$$g(\tilde{W}^{(\alpha_j)}(\phi Z_1, \phi Z_2)\phi Z_3, \phi Z_4) = 0$$

for all $Z_1, Z_2, Z_3, Z_4 \in \chi(M)$.

In the following theorems, we investigate the $\tilde{W}^{(\alpha_j)}$ -flatness conditions introduced above for $j = 5, 8$.

Theorem 1. The $\tilde{W}^{(\alpha_5)}$ -flatness condition on a Kenmotsu manifold M implies the Einstein character of M [11].

Theorem 2. The $\tilde{W}^{(\alpha_8)}$ -flatness condition on a Kenmotsu manifold M implies the Einstein character of M .

Proof. We consider a $\tilde{W}^{(\alpha_8)}$ -flat Kenmotsu manifold M , that is $\tilde{W}^{(\alpha_8)}(Z_1, Z_2)Z_3 = 0$, by List 1 in Definition 2. Consequently, it follows from (41) that the curvature tensor satisfies

$$R(Z_1, Z_2)Z_3 = \frac{1}{2n}[S(Z_1, Z_3)Z_2 - g(Z_1, Z_3)QZ_2] + g(Z_1, Z_3)Z_2 - g(Z_2, Z_3)Z_1. \quad (51)$$

Setting $Z_2 = \xi$ in (51), and using (10), we obtain

$$-\eta(Z_3)Z_1 + g(Z_1, Z_3)\xi = \frac{1}{2n}[S(Z_1, Z_3)\xi - g(Z_1, Z_3)Q\xi] + g(Z_1, Z_3)\xi - g(\xi, Z_3)Z_1. \quad (52)$$

Using (4) and (12), and applying $g(\cdot, \xi)$ to both sides of (52), we get

$$S(Z_1, Z_3) = -2ng(Z_1, Z_3).$$

Therefore, M has an Einstein characteristic. $\square$

Theorem 3. The $\xi - \tilde{W}^{(\alpha_5)}$ -flatness condition on a Kenmotsu manifold M implies the Einstein character of M .

Proof. Let M be $\xi - \tilde{W}^{(\alpha_5)}$ -flat. Then, $\tilde{W}^{(\alpha_5)}(Z_1, Z_2)\xi = 0$ for all $Z_1, Z_2 \in \chi(M)$, which follows from List 2 in Definition 2. Substituting $\alpha_i = \alpha_5 = (0, 0, 0, \frac{-1}{2n}, \frac{1}{2n}, 0)$ into Equation (46), we obtain

$$\tilde{W}^{(\alpha_5)}(Z_1, Z_2)\xi = \eta(Z_1)Z_2 - \eta(Z_2)Z_1 - \frac{1}{2n}\eta(Z_2)QZ_1 + \frac{1}{2n}\eta(Z_1)QZ_2 \quad (53)$$

Taking the g inner product of both sides of (53) with an arbitrary vector field Z_5 , we get

$$g\left(\tilde{W}^{(\alpha_5)}(Z_1, Z_2)\xi, Z_5\right) = \eta(Z_1)g(Z_2, Z_5) - \eta(Z_2)g(Z_1, Z_5) - \frac{1}{2n}\eta(Z_2)g(QZ_1, Z_5) + \frac{1}{2n}\eta(Z_1)g(QZ_2, Z_5). \quad (54)$$

Setting $Z_2 = \xi$ in the resulting equation, and using $g\left(\tilde{W}^{(\alpha_5)}(Z_1, Z_2)\xi, Z_5\right) = 0$, together with (4) and (12), we obtain

$$S(Z_1, Z_5) = 2ng(Z_1, Z_5).$$

Thus, M is an Einstein manifold, as required. $\square$

The $\xi - \tilde{W}^{(\alpha_j)}$ -flatness condition is only considered for α_5 , since $\tilde{W}^{(\alpha_8)}(Z_1, Z_2)\xi$ vanishes identically in the case of α_8 . Consequently, the $\xi - \tilde{W}^{(\alpha_8)}$ -flatness condition is automatically satisfied and does not lead to any additional Einstein or η -Einstein characterization.

The proofs of the following curvature characterizations are based on a common computational strategy. We evaluate the corresponding curvature conditions with respect to suitable ϕ -adapted orthonormal frames and take traces over the contact distribution. By using the fundamental identities of Kenmotsu manifolds, together with the relations involving the Ricci operator and the Reeb vector field ξ , the obtained tensorial equations are reduced to algebraic conditions on the Ricci tensor. In this way, the coefficients in the decomposition

$$S = ag + b\eta \otimes \eta$$

are determined, which leads to the Einstein and η -Einstein characterizations established below.

Theorem 4. *The quasi $\tilde{W}^{(\alpha_5)}$ -flatness condition on a Kenmotsu manifold M implies the η -Einstein character of M .*

Proof. Let M be quasi $\tilde{W}^{(\alpha_5)}$ -flat. Then, by (38), we have

$$g\left(\tilde{W}^{(\alpha_5)}(\phi Z_1, Z_2)Z_3, \phi W\right) = g(R(\phi Z_1, Z_2)Z_3, \phi W) - \frac{1}{2n}g(Z_2, Z_3)g(\phi QZ_1, \phi W) + \frac{1}{2n}g(\phi Z_1, Z_3)g(QZ_2, \phi W).$$

By List 3 in Definition 2, it follows that

$$g(R(\phi Z_1, Z_2)Z_3, \phi W) = \frac{1}{2n}g(Z_2, Z_3)g(\phi QZ_1, \phi W) - \frac{1}{2n}g(\phi Z_1, Z_3)g(QZ_2, \phi W)$$

Taking the trace with respect to the local orthonormal frame, we obtain

$$\sum_{i=1}^{2n} g(R(\phi e_i, Z_2)Z_3, \phi e_i) = \frac{1}{2n} \sum_{i=1}^{2n} g(Z_2, Z_3)g(\phi Qe_i, \phi e_i) - \frac{1}{2n} \sum_{i=1}^{2n} g(\phi e_i, Z_3)g(QZ_2, \phi e_i).$$

Using (3), (16), (18), and (19), we derive

$$S(Z_2, Z_3) - \eta(Z_3)\eta(Z_2) + g(Z_2, Z_3) = \frac{r+2n}{2n}g(Z_2, Z_3) - \frac{1}{2n}S(Z_2, Z_3) - 2n\eta(Z_2)\eta(Z_3).$$

Rearranging the terms yields

$$S(Z_2, Z_3) = \frac{r}{2n+1}g(Z_2, Z_3) + \frac{-4n^2+2n}{2n+1}\eta(Z_2)\eta(Z_3).$$

Comparing the above equation with (14), we obtain

$$a = \frac{r}{2n+1}, \quad b = \frac{-4n^2 + 2n}{2n+1}.$$

Consequently, M admits an η -Einstein metric. $\square$

Theorem 5. The quasi $\tilde{W}^{(\alpha_8)}$ -flatness condition on a Kenmotsu manifold M implies the η -Einstein character of M .

Proof. Let M be quasi $\tilde{W}^{(\alpha_8)}$ -flat. Combining (41) with List 3 in Definition 2, we have

$$\begin{aligned} g(R(\phi Z_1, Z_2)Z_3, \phi W) = & -g(Z_2, Z_3)g(\phi Z_1, \phi W) + g(\phi Z_1, Z_3)g(Z_2, \phi W) \\ & + \frac{1}{2n}S(\phi Z_1, Z_3)g(Z_2, \phi W) - \frac{1}{2n}g(\phi Z_1, Z_3)g(QZ_2, \phi W) \end{aligned}$$

Thus taking the trace with respect to the local orthonormal frame, we obtain

$$\begin{aligned} \sum_{i=1}^{2n} g(R(\phi e_i, Z_2)Z_3, \phi e_i) = & -\sum_{i=1}^{2n} g(Z_2, Z_3)g(\phi e_i, \phi e_i) + \sum_{i=1}^{2n} g(\phi e_i, Z_3)g(Z_2, \phi e_i) \\ & + \frac{1}{2n}\sum_{i=1}^{2n} S(\phi e_i, Z_3)g(Z_2, \phi e_i) - \frac{1}{2n}\sum_{i=1}^{2n} g(\phi e_i, Z_3)g(QZ_2, \phi e_i). \end{aligned}$$

By virtue of the relations (15), (16), and (21), together with the definition of Ricci operator Q , given by $S(Z_1, Z_2) = g(QZ_1, Z_2)$, we evaluate,

$$\begin{aligned} S(Z_2, Z_3) - \eta(Z_3)\eta(Z_2) + g(Z_2, Z_3) = & -2ng(Z_2, Z_3) + g(Z_2, Z_3) \\ & + \frac{1}{2n}(S(Z_2, Z_3) + 2n\eta(Z_3)\eta(Z_2)) - \frac{1}{2n}g(QZ_2, Z_3), \end{aligned}$$

which means

$$S(Z_2, Z_3) = -2ng(Z_2, Z_3) + 2\eta(Z_2)\eta(Z_3).$$

Comparing the above equation with (14), we obtain

$$a = -2n, \quad b = 2.$$

Hence, an η -Einstein metric is induced on M . $\square$

Theorem 6. The ϕ - $\tilde{W}^{(\alpha_5)}$ -flatness condition on a Kenmotsu manifold M implies the η -Einstein character of M .

Proof. Let M be ϕ - $\tilde{W}^{(\alpha_5)}$ -flat. Then by (38) and List 4 in Definition 2, we have

$$g(R(\phi Z_1, \phi Z_2)\phi Z_3, \phi W) = \frac{1}{2n}g(\phi Z_2, \phi Z_3)g(\phi QZ_1, \phi W) - \frac{1}{2n}g(\phi Z_1, \phi Z_3)g(\phi QZ_2, \phi W).$$

Thus, for the local orthonormal frame of M , setting $Z_1 = W = e_i$, we get

$$\sum_{i=1}^{2n} g(R(\phi e_i, \phi Z_2)\phi Z_3, \phi e_i) = \frac{1}{2n}\sum_{i=1}^{2n} g(\phi Z_2, \phi Z_3)g(\phi Qe_i, \phi e_i) - \frac{1}{2n}\sum_{i=1}^{2n} g(\phi e_i, \phi Z_3)g(\phi QZ_2, \phi e_i).$$

Using (19), (20) and the relation $S(Z_1, Z_2) = g(QZ_1, Z_2)$, we derive

$$S(\phi Z_2, \phi Z_3) + g(\phi Z_2, \phi Z_3) = \left(\frac{2n+r}{2n}\right)g(\phi Z_2, \phi Z_3) - \frac{1}{2n}S(\phi Z_2, \phi Z_3).$$

Now using (3) and (13), we obtain

$$\left(\frac{2n+1}{2n}\right)S(Z_2, Z_3) + (2n+1)\eta(Z_2)\eta(Z_3) = \frac{r}{2n}g(Z_2, Z_3) - \frac{r}{2n}\eta(Z_2)\eta(Z_3),$$

or equivalently

$$S(Z_2, Z_3) = \left(\frac{r}{2n+1}\right)g(Z_2, Z_3) - \left(\frac{4n^2+2n+r}{2n+1}\right)\eta(Z_2)\eta(Z_3).$$

Comparing the above equation with (14), we obtain

$$a = \frac{r}{2n+1}, \quad b = \frac{4n^2+2n+r}{2n+1}.$$

This yields an η -Einstein metric on M , concluding the proof. $\square$

Theorem 7. The $\phi\text{-}\tilde{W}^{(\alpha_8)}$ -flatness condition on a Kenmotsu manifold M implies the Einstein character of M .

Proof. Assume that M is a $\phi\text{-}\tilde{W}^{(\alpha_8)}$ -flat Kenmotsu manifold. Then, from (41) and List 4 in Definition 2, we get

$$\begin{aligned} g(R(\phi Z_1, \phi Z_2)\phi Z_3, \phi W) = & -g(\phi Z_2, \phi Z_3)g(\phi Z_1, \phi W) + g(\phi Z_1, \phi Z_3)g(\phi Z_2, \phi W) \\ & + \frac{1}{2n}S(\phi Z_1, \phi Z_3)g(\phi Z_2, \phi W) - \frac{1}{2n}g(\phi Z_1, \phi Z_3)g(Q\phi Z_2, \phi W) \end{aligned}$$

Now, for a local orthonormal frame of M , setting $Z_1 = W = e_i$, we have

$$\begin{aligned} \sum_{i=1}^{2n} g(R(\phi e_i, \phi Z_2)\phi Z_3, \phi e_i) = & -\sum_{i=1}^{2n} g(\phi Z_2, \phi Z_3)g(\phi e_i, \phi e_i) + \sum_{i=1}^{2n} g(\phi e_i, \phi Z_3)g(\phi Z_2, \phi e_i) \\ & + \frac{1}{2n}\sum_{i=1}^{2n} S(\phi e_i, \phi Z_3)g(\phi Z_2, \phi e_i) - \frac{1}{2n}\sum_{i=1}^{2n} g(\phi e_i, \phi Z_3)g(Q\phi Z_2, \phi e_i). \end{aligned}$$

Using (15), (18), (20), (21), and the relation $S(Z_1, Z_2) = g(QZ_1, Z_2)$, we obtain

$$S(\phi Z_2, \phi Z_3) + g(\phi Z_2, \phi Z_3) = (-2n+1)g(\phi Z_2, \phi Z_3)$$

Applying (3) and (13), it follows that

$$S(Z_2, Z_3) = -2ng(Z_2, Z_3).$$

Hence, an Einstein metric is induced on M , as required. $\square$

4. Induced Einstein Structures via $\tilde{W}^{(\alpha_5)}$ and $\tilde{W}^{(\alpha_8)}$ Curvature Derivations

The main objective of this section is to analyze how the conditions $\tilde{W}^{(\alpha_5)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_i)} = 0$ and $\tilde{W}^{(\alpha_8)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_i)} = 0$, where $i = 1, \dots, 12$ characterize the underlying geometry of Kenmotsu manifolds. The subsequent theorems provide the necessary and sufficient conditions for $\tilde{W}^{(\alpha_j)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_i)} = 0$ for $i = 1, \dots, 12, j = 5, 8$ to hold on M . To maintain conciseness, the curvature tensors are investigated in distinct groups based on the similarity of their proof methods. The grouping is determined by the common reduction steps appearing in the corresponding computations. The proofs within each group follow the same general procedure involving suitable choices of vector fields associated with ζ , taking a g -inner product when necessary, and using the fundamental identities of Kenmotsu manifolds.

Following the standard notion of tensor derivations introduced in [26–29], the action of a $(1, 3)$ -type tensor $\mathcal{T}_1$ on another $(1, 3)$ -type tensor $\mathcal{T}_2$ on M is defined by

$$(\mathcal{T}_1(Z_1, Z_2)\mathcal{T}_2)(Z_3, Z_4)Z_5 = \mathcal{T}_1(Z_1, Z_2)\mathcal{T}_2(Z_3, Z_4)Z_5 - \mathcal{T}_2(\mathcal{T}_1(Z_1, Z_2)Z_3, Z_4)Z_5 - \mathcal{T}_2(Z_3, \mathcal{T}_1(Z_1, Z_2)Z_4)Z_5 - \mathcal{T}_2(Z_3, Z_4)\mathcal{T}_1(Z_1, Z_2)Z_5 \quad (55)$$

Theorem 8. For $\tilde{W}^{(\alpha_5)}$ -curvature tensor on a Kenmotsu manifold M , satisfying $\tilde{W}^{(\alpha_5)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_i)} = 0$ for $i = 1, 2, 5, 8, 10$ is equivalent to M being an Einstein manifold.

Proof. Suppose that $\tilde{W}^{(\alpha_5)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_1)} = 0$. Setting $\mathcal{T}_1 = \tilde{W}^{(\alpha_5)}$, $\mathcal{T}_2 = \tilde{W}^{(\alpha_1)}$, and choosing $Z_1 = Z_5 = \xi$ in (55) we get

$$\begin{aligned} (\tilde{W}^{(\alpha_5)}(\xi, Z_2)\tilde{W}^{(\alpha_1)})(Z_3, Z_4)\xi &= \tilde{W}^{(\alpha_5)}(\xi, Z_2)\tilde{W}^{(\alpha_1)}(Z_3, Z_4)\xi - \tilde{W}^{(\alpha_1)}(\tilde{W}^{(\alpha_5)}(\xi, Z_2)Z_3, Z_4)\xi \\ &\quad - \tilde{W}^{(\alpha_1)}(Z_3, \tilde{W}^{(\alpha_5)}(\xi, Z_2)Z_4)\xi - \tilde{W}^{(\alpha_1)}(Z_3, Z_4)\tilde{W}^{(\alpha_5)}(\xi, Z_2)\xi \\ &= 0 \end{aligned} \quad (56)$$

for any $Z_1, Z_2, Z_3, Z_4, Z_5 \in \chi(M)$. Substituting $\alpha_1 = \left(\frac{-1}{2n}, 0, 0, 0, \frac{1}{2n}, 0\right)$ into (46), and $\alpha_5 = \left(0, 0, 0, \frac{-1}{2n}, \frac{1}{2n}, 0\right)$ into (48) and (49), and combining these equations with (56), we obtain

$$\begin{aligned} (\tilde{W}^{(\alpha_5)}(\xi, Z_2)\tilde{W}^{(\alpha_1)})(Z_3, Z_4)\xi &= \eta(Z_3)\tilde{W}^{(\alpha_5)}(\xi, Z_2)Z_4 + \frac{1}{2n}\eta(Z_3)\tilde{W}^{(\alpha_5)}(\xi, Z_2)QZ_4 \\ &\quad - \eta(Z_3)\tilde{W}^{(\alpha_1)}(Z_2, Z_4)\xi - \frac{1}{2n}\eta(Z_3)\tilde{W}^{(\alpha_1)}(QZ_2, Z_4)\xi \\ &\quad - \eta(Z_4)\tilde{W}^{(\alpha_1)}(Z_3, Z_2)\xi - \frac{1}{2n}\eta(Z_4)\tilde{W}^{(\alpha_1)}(Z_3, QZ_2)\xi \\ &\quad - \tilde{W}^{(\alpha_1)}(Z_3, Z_4)Z_2 - \frac{1}{2n}\tilde{W}^{(\alpha_1)}(Z_3, Z_4)QZ_2 = 0. \end{aligned} \quad (57)$$

Again, using (48) with α_5 , (46) and (47) with α_1 , together with (4) and (12), and setting $Z_4 = \xi$ in (57), we derive

$$\eta(Z_3)Z_2 + \frac{1}{n}\eta(Z_3)QZ_2 + \frac{1}{4n^2}\eta(Z_3)Q^2Z_2 = 0.$$

Next, setting $Z_3 = \xi$, we arrive at the following quadratic relation for the Ricci operator Q :

$$Q^2 + 4nQ + 4n^2I = 0.$$

Since the Ricci operator Q is self-adjoint with respect to the Riemannian metric g , the operator $Q + 2nI$ is also self-adjoint. Moreover, since a self-adjoint nilpotent operator on a real inner-product space must vanish, the relation $(Q + 2nI)^2 = 0$ implies $Q + 2nI = 0$. Hence, we obtain $Q = -2nI$, which gives $S(Z_2, Z_3) = -2ng(Z_2, Z_3)$. Thus, M is an Einstein manifold, completing the proof.

In the converse direction, assume that M is Einstein with $S(Z_2, Z_3) = -2ng(Z_2, Z_3)$. Then $QZ_2 = -2nZ_2$. Consequently (34) and (38) yield

$$\tilde{W}^{(\alpha_5)}(Z_1, Z_2)Z_3 = \tilde{W}^{(\alpha_1)}(Z_1, Z_2)Z_3 = R(Z_1, Z_2)Z_3 + g(Z_2, Z_3)Z_1 - g(Z_1, Z_3)Z_2.$$

Substituting the above relation into (55) and applying (46) with $Z_5 = \xi$ in the cases $\alpha_i = \alpha_1$ and $\alpha_i = \alpha_5$, we deduce that $\tilde{W}^{(\alpha_5)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_1)} = 0$.

The other cases can be obtained through the same computational sequence by taking $\tilde{W}^{(\alpha_i)}$ for $i = 2, 5, 8, 10$ instead of $\tilde{W}^{(\alpha_1)}$. Thus, the proof is complete. $\square$

Theorem 9. For $\tilde{W}^{(\alpha_5)}$ -curvature tensor on a Kenmotsu manifold M , satisfying $\tilde{W}^{(\alpha_5)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_i)} = 0$ for $i = 4, 6, 11, 12$ is equivalent to M being an Einstein manifold.

Proof. Suppose that $\tilde{W}^{(\alpha_5)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_6)} = 0$. Setting $\mathcal{T}_1 = \tilde{W}^{(\alpha_5)}$, $\mathcal{T}_2 = \tilde{W}^{(\alpha_6)}$, and choosing $Z_1 = Z_5 = \xi$ in (55), we get

$$\begin{aligned} \left(\tilde{W}^{(\alpha_5)}(\xi, Z_2) \tilde{W}^{(\alpha_6)} \right) (Z_3, Z_4) \xi &= \tilde{W}^{(\alpha_5)}(\xi, Z_2) \tilde{W}^{(\alpha_6)}(Z_3, Z_4) \xi - \tilde{W}^{(\alpha_6)} \left(\tilde{W}^{(\alpha_5)}(\xi, Z_2) Z_3, Z_4 \right) \xi \\ &\quad - \tilde{W}^{(\alpha_6)} \left(Z_3, \tilde{W}^{(\alpha_5)}(\xi, Z_2) Z_4 \right) \xi - \tilde{W}^{(\alpha_6)}(Z_3, Z_4) \tilde{W}^{(\alpha_5)}(\xi, Z_2) \xi \\ &= 0, \end{aligned} \quad (58)$$

for any $Z_1, Z_2, Z_3, Z_4, Z_5 \in \chi(M)$. Substituting $\alpha_6 = \left(0, \frac{-1}{2n}, 0, \frac{1}{2n}, 0, 0\right)$ into (46), and $\alpha_5 = \left(0, 0, 0, \frac{-1}{2n}, \frac{1}{2n}, 0\right)$ into (48) and (49), and combining these equations with (58), we obtain

$$\begin{aligned} \left(\tilde{W}^{(\alpha_5)}(\xi, Z_2) \tilde{W}^{(\alpha_6)} \right) (Z_3, Z_4) \xi &= \eta(Z_4) \tilde{W}^{(\alpha_5)}(\xi, Z_2) Z_3 + \frac{1}{2n} \eta(Z_4) \tilde{W}^{(\alpha_5)}(\xi, Z_2) Q Z_3 \\ &\quad - \eta(Z_3) \tilde{W}^{(\alpha_6)}(Z_2, Z_4) \xi - \frac{1}{2n} \eta(Z_3) \tilde{W}^{(\alpha_6)}(Q Z_2, Z_4) \xi \\ &\quad - \eta(Z_4) \tilde{W}^{(\alpha_6)}(Z_3, Z_2) \xi - \frac{1}{2n} \eta(Z_4) \tilde{W}^{(\alpha_6)}(Z_3, Q Z_2) \xi \\ &\quad - \tilde{W}^{(\alpha_6)}(Z_3, Z_4) Z_2 - \frac{1}{2n} \tilde{W}^{(\alpha_6)}(Z_3, Z_4) Q Z_2 = 0. \end{aligned} \quad (59)$$

Again, using (48) with α_5 , (46), (47) and (50) with α_6 , together with (4), (12), and setting $Z_4 = \xi$ in (59), we derive

$$\frac{1}{4n^2} S(Q Z_2, Z_3) \xi + \frac{1}{n} S(Z_2, Z_3) \xi + g(Z_2, Z_3) \xi - \eta(Z_3) Z_2 - \frac{1}{n} \eta(Z_3) Q Z_2 - \frac{1}{4n^2} \eta(Z_3) Q^2 Z_2 = 0 \quad (60)$$

Next, taking the g inner product of both sides of (60) with ξ , we get

$$\frac{1}{4n^2} S(Q Z_2, Z_3) + \frac{1}{n} S(Z_2, Z_3) + g(Z_2, Z_3) = 0$$

or equivalently

$$\frac{1}{4n^2} g(Q^2 Z_2, Z_3) + \frac{1}{n} g(Q Z_2, Z_3) + g(Z_2, Z_3) = 0 \quad (61)$$

Thus, the Equation (61) yields the quadratic relation for the Ricci operator Q :

$$Q^2 + 4nQ + 4n^2I = 0.$$

Solving the above equation yields $Q = -2nI$. Consequently, $S(Z_2, Z_3) = -2ng(Z_2, Z_3)$. Hence, M is indeed an Einstein manifold.

For the converse case, assume that M is an Einstein Kenmotsu manifold satisfying $S(Z_2, Z_3) = -2ng(Z_2, Z_3)$, so that $Q Z_2 = -2nZ_2$. Substituting these expressions for S and Q into (38) and (39), combining the resulting identities with (55), and applying (46) with $Z_5 = \xi$ in the cases $\alpha_i = \alpha_5$ and $\alpha_i = \alpha_6$, we obtain $\tilde{W}^{(\alpha_5)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_6)} = 0$.

The other cases can be obtained through the same computational sequence by taking $\tilde{W}^{(\alpha_i)}$ for $i = 4, 11, 12$ instead of $\tilde{W}^{(\alpha_6)}$. Thus, the proof is complete. $\square$

Theorem 10. For $\tilde{W}^{(\alpha_5)}$ -curvature tensor on a Kenmotsu manifold M , satisfying $\tilde{W}^{(\alpha_5)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_i)} = 0$ for $i = 3, 9$ is equivalent to M being an Einstein manifold.

Proof. Suppose that $\tilde{W}^{(\alpha_5)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_9)} = 0$. Setting $\mathcal{T}_1 = \tilde{W}^{(\alpha_5)}$, $\mathcal{T}_2 = \tilde{W}^{(\alpha_9)}$, and choosing $Z_1 = Z_3 = \xi$ in (55) we get

$$\begin{aligned} \left(\tilde{W}^{(\alpha_5)}(\xi, Z_2) \tilde{W}^{(\alpha_9)} \right) (\xi, Z_4) Z_5 &= \tilde{W}^{(\alpha_5)}(\xi, Z_2) \tilde{W}^{(\alpha_9)}(\xi, Z_4) Z_5 - \tilde{W}^{(\alpha_9)} \left(\tilde{W}^{(\alpha_5)}(\xi, Z_2) \xi, Z_4 \right) Z_5 \\ &\quad - \tilde{W}^{(\alpha_9)} \left(\xi, \tilde{W}^{(\alpha_5)}(\xi, Z_2) Z_4 \right) Z_5 - \tilde{W}^{(\alpha_9)}(\xi, Z_4) \tilde{W}^{(\alpha_5)}(\xi, Z_2) Z_5 \\ &= 0, \end{aligned} \quad (62)$$

for any $Z_1, Z_2, Z_3, Z_4, Z_5 \in \chi(M)$. Substituting $\alpha_9 = \left(\frac{-1}{2n}, 0, 0, 0, 0, \frac{1}{2n}\right)$ into (48), and $\alpha_5 = \left(0, 0, 0, \frac{-1}{2n}, \frac{1}{2n}, 0\right)$ into (48) and (49), and combining these equations with (62), we obtain

$$\begin{aligned} \left(\tilde{W}^{(\alpha_5)}(\xi, Z_2)\tilde{W}^{(\alpha_9)}\right)(Z_3, Z_4)\xi = & -g(Z_4, Z_5)\tilde{W}^{(\alpha_5)}(\xi, Z_2)\xi + \eta(Z_4)\tilde{W}^{(\alpha_5)}(\xi, Z_2)Z_5 \\ & - \frac{1}{2n}S(Z_4, Z_5)\tilde{W}^{(\alpha_5)}(\xi, Z_2)\xi + \frac{1}{2n}\eta(Z_4)\tilde{W}^{(\alpha_5)}(\xi, Z_2)QZ_5 \\ & - \tilde{W}^{(\alpha_9)}(Z_2, Z_4)Z_5 - \frac{1}{2n}\tilde{W}^{(\alpha_9)}(QZ_2, Z_4)Z_5 \\ & - \eta(Z_4)\tilde{W}^{(\alpha_9)}(\xi, Z_2)Z_5 - \frac{1}{2n}\eta(Z_4)\tilde{W}^{(\alpha_9)}(\xi, QZ_2)Z_5 \\ & - \eta(Z_5)\tilde{W}^{(\alpha_9)}(\xi, Z_4)Z_2 - \frac{1}{2n}\eta(Z_5)\tilde{W}^{(\alpha_9)}(\xi, Z_4)QZ_2 = 0. \end{aligned} \quad (63)$$

Again, using (48) and (49) with α_5 , (47) and (48) with α_9 , together with (4), (12), and setting $Z_4 = \xi$ in (63), we derive

$$\frac{1}{4n^2}S(QZ_2, Z_5)\xi + \frac{1}{n}S(Z_2, Z_5)\xi + g(Z_2, Z_5)\xi - \eta(Z_5)Z_2 - \frac{1}{n}\eta(Z_5)QZ_2 - \frac{1}{4n^2}\eta(Z_5)Q^2Z_2 = 0 \quad (64)$$

Next, taking the g inner product of both sides of (64) with ξ , we get

$$\frac{1}{4n^2}S(QZ_2, Z_5) + \frac{1}{n}S(Z_2, Z_5) + g(Z_2, Z_5) = 0$$

or equivalently

$$\frac{1}{4n^2}g(Q^2Z_2, Z_5) + \frac{1}{n}g(QZ_2, Z_5) + g(Z_2, Z_5) = 0 \quad (65)$$

Thus, the Equation (65) yields the quadratic relation for the Ricci operator Q

$$Q^2 + 4nQ + 4n^2I = 0.$$

Solving the above equation yields $Q = -2nI$, which implies $S(Z_2, Z_5) = -2ng(Z_2, Z_5)$. Thus, M is an Einstein manifold, as required.

For the converse statement, suppose that M is an Einstein Kenmotsu manifold satisfying $S(Z_2, Z_5) = -2ng(Z_2, Z_5)$, so that $QZ_2 = -2nZ_2$. Substituting these relations into the definitions of (38) and (42) and subsequently into (55), and applying (46) for $\alpha_i = \alpha_5$ and $\alpha_i = \alpha_9$ with $Z_5 = \xi$, we conclude that $\tilde{W}^{(\alpha_5)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_9)} = 0$.

The other case can be obtained through the same computational sequence by taking $\tilde{W}^{(\alpha_3)}$ instead of $\tilde{W}^{(\alpha_9)}$. Thus, the proof is complete. $\square$

Theorem 11. For $\tilde{W}^{(\alpha_8)}$ -curvature tensor on a Kenmotsu manifold M , satisfying $\tilde{W}^{(\alpha_8)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_i)} = 0$ for $i = 1, 3, 4, 6, 8, 9, 11, 12$ is equivalent to M being an Einstein manifold.

Proof. Suppose that $\tilde{W}^{(\alpha_8)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_{11})} = 0$. Setting $\mathcal{T}_1 = \tilde{W}^{(\alpha_8)}$, $\mathcal{T}_2 = \tilde{W}^{(\alpha_{11})}$, and choosing $Z_1 = Z_5 = \xi$ in (55) we get

$$\begin{aligned} \left(\tilde{W}^{(\alpha_8)}(\xi, Z_2)\tilde{W}^{(\alpha_{11})}\right)(Z_3, Z_4)\xi = & \tilde{W}^{(\alpha_8)}(\xi, Z_2)\tilde{W}^{(\alpha_{11})}(Z_3, Z_4)\xi - \tilde{W}^{(\alpha_{11})}\left(\tilde{W}^{(\alpha_8)}(\xi, Z_2)Z_3, Z_4\right)\xi \\ & - \tilde{W}^{(\alpha_{11})}\left(Z_3, \tilde{W}^{(\alpha_8)}(\xi, Z_2)Z_4\right)\xi - \tilde{W}^{(\alpha_{11})}(Z_3, Z_4)\tilde{W}^{(\alpha_8)}(\xi, Z_2)\xi \\ = & 0, \end{aligned} \quad (66)$$

for any $Z_1, Z_2, Z_3, Z_4, Z_5 \in \chi(M)$. Substituting $\alpha_8 = \left(0, \frac{-1}{2n}, 0, 0, 0, \frac{1}{2n}\right)$ into (49), and $\alpha_{11} = \left(\frac{-1}{2n}, 0, \frac{1}{2n}, 0, 0, 0\right)$ into (46), and combining these equations with (66), we obtain

$$\begin{aligned}
\left(\tilde{W}^{(\alpha_8)}(\xi, Z_2)\tilde{W}^{(\alpha_{11})}\right)(Z_3, Z_4)\xi = & g(Z_3, Z_4)\tilde{W}^{(\alpha_8)}(\xi, Z_2)\xi + \frac{1}{2n}S(Z_3, Z_4)\tilde{W}^{(\alpha_8)}(\xi, Z_2)\xi \\
& - \eta(Z_3)\tilde{W}^{(\alpha_{11})}(Z_2, Z_4)\xi - \frac{1}{2n}\eta(Z_3)\tilde{W}^{(\alpha_{11})}(QZ_2, Z_4)\xi \\
& - \eta(Z_4)\tilde{W}^{(\alpha_{11})}(Z_3, Z_2)\xi - \frac{1}{2n}\eta(Z_4)\tilde{W}^{(\alpha_{11})}(Z_3, QZ_2)\xi \\
& - \tilde{W}^{(\alpha_{11})}(Z_3, Z_4)Z_2 - \frac{1}{2n}\tilde{W}^{(\alpha_{11})}(Z_3, Z_4)QZ_2 = 0.
\end{aligned} \tag{67}$$

Again, using (49) with α_8 , (46), (47), and (50) with α_{11} , together with (4), (12), and setting $Z_4 = \xi$ in (67), we derive

$$\frac{1}{4n^2}S(QZ_2, Z_3)\xi + \frac{1}{n}S(Z_2, Z_3)\xi + g(Z_2, Z_3)\xi = 0 \tag{68}$$

Next, taking the g inner product of both sides of (68) with ξ , we get

$$\frac{1}{4n^2}S(QZ_2, Z_3) + \frac{1}{n}S(Z_2, Z_3) + g(Z_2, Z_3) = 0$$

or equivalently

$$\frac{1}{4n^2}g(Q^2Z_2, Z_3) + \frac{1}{n}g(QZ_2, Z_3) + g(Z_2, Z_3) = 0 \tag{69}$$

Thus, the Equation (69) yields the quadratic relation for the Ricci operator Q :

$$Q^2 + 4nQ + 4n^2I = 0.$$

From the above equation, we obtain $Q = -2nI$, which in turn implies $S(Z_2, Z_3) = -2ng(Z_2, Z_3)$. Therefore, M is an Einstein manifold, as required.

For the converse case, let M be an Einstein Kenmotsu manifold satisfying $S(Z_2, Z_3) = -2ng(Z_2, Z_3)$, so that $QZ_2 = -2nZ_2$. Inserting these relations into the definitions of (41) and (44) and then employing the resulting expressions in (55), we apply (46) with $Z_5 = \xi$ for $\alpha_i = \alpha_8$ and $\alpha_i = \alpha_{11}$. Consequently, $\tilde{W}^{(\alpha_8)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_{11})} = 0$.

The other cases can be obtained through the same computational sequence by taking $\tilde{W}^{(\alpha_i)}$ for $i = 1, 3, 4, 6, 8, 9, 12$ instead of $\tilde{W}^{(\alpha_{11})}$. Thus, the proof is complete. $\square$

Theorem 12. For $\tilde{W}^{(\alpha_8)}$ -curvature tensor on a Kenmotsu manifold M , satisfying $\tilde{W}^{(\alpha_8)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_i)} = 0$ for $i = 2, 10$ is equivalent to M being an Einstein manifold.

Proof. Suppose that $\tilde{W}^{(\alpha_8)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_{10})} = 0$. Setting $\mathcal{T}_1 = \tilde{W}^{(\alpha_8)}$, $\mathcal{T}_2 = \tilde{W}^{(\alpha_{10})}$, and choosing $Z_1 = Z_5 = \xi$ in (55) we get

$$\begin{aligned}
\left(\tilde{W}^{(\alpha_8)}(\xi, Z_2)\tilde{W}^{(\alpha_{10})}\right)(Z_3, Z_4)\xi = & \tilde{W}^{(\alpha_8)}(\xi, Z_2)\tilde{W}^{(\alpha_{10})}(Z_3, Z_4)\xi - \tilde{W}^{(\alpha_{10})}\left(\tilde{W}^{(\alpha_8)}(\xi, Z_2)Z_3, Z_4\right)\xi \\
& - \tilde{W}^{(\alpha_{10})}\left(Z_3, \tilde{W}^{(\alpha_8)}(\xi, Z_2)Z_4\right)\xi - \tilde{W}^{(\alpha_{10})}(Z_3, Z_4)\tilde{W}^{(\alpha_8)}(\xi, Z_2)\xi \\
= & 0,
\end{aligned} \tag{70}$$

for any $Z_1, Z_2, Z_3, Z_4, Z_5 \in \chi(M)$. Substituting $\alpha_8 = \left(0, \frac{-1}{2n}, 0, 0, \frac{1}{2n}, 0\right)$ into (49), and $\alpha_{10} = \left(\frac{-1}{2n}, 0, 0, \frac{1}{2n}, 0, 0\right)$ into (46), and combining these equations with (70), we obtain

$$\begin{aligned}
\left(\tilde{W}^{(\alpha_8)}(\xi, Z_2)\tilde{W}^{(\alpha_{10})}\right)(Z_3, Z_4)\xi = & -\frac{1}{2n}S(Z_3, Z_4)\tilde{W}^{(\alpha_8)}(\xi, Z_2)\xi - g(Z_3, Z_4)\tilde{W}^{(\alpha_8)}(\xi, Z_2)\xi \\
& - \eta(Z_3)\tilde{W}^{(\alpha_{10})}(Z_2, Z_4)\xi - \frac{1}{2n}\eta(Z_3)\tilde{W}^{(\alpha_{10})}(QZ_2, Z_4)\xi \\
& - \eta(Z_4)\tilde{W}^{(\alpha_{10})}(Z_3, Z_2)\xi - \frac{1}{2n}\eta(Z_4)\tilde{W}^{(\alpha_{10})}(Z_3, QZ_2)\xi \\
& - \tilde{W}^{(\alpha_{10})}(Z_3, Z_4)Z_2 - \frac{1}{2n}\tilde{W}^{(\alpha_{10})}(Z_3, Z_4)QZ_2 = 0.
\end{aligned} \tag{71}$$

Again, using (49) with α_8 , (49), (47) and (50) with α_{10} , together with (4), (12), and setting $Z_4 = \xi$ in (71), we derive

$$\eta(Z_3)Z_2 + \frac{1}{n}\eta(Z_3)QZ_2 + \frac{1}{4n^2}\eta(Z_3)Q^2Z_2 = 0.$$

Next, setting $Z_3 = \xi$, we arrive at the following quadratic relation for the Ricci operator Q :

$$Q^2 + 4nQ + 4n^2I = 0.$$

The above equation leads to $Q = -2nI$, which implies $S(Z_2, Z_3) = -2ng(Z_2, Z_3)$. Consequently, M reduces to an Einstein manifold.

In the converse direction, assume that M is an Einstein Kenmotsu manifold with $S(Z_2, Z_3) = -2ng(Z_2, Z_3)$, and hence $QZ_2 = -2nZ_2$. Substituting these relations for S and Q into (41) and (43), and employing the resulting expressions in (55), we apply (46) with $Z_5 = \xi$ for $\alpha_i = \alpha_8$ and $\alpha_i = \alpha_{10}$. Consequently, we arrive at $\tilde{W}^{(\alpha_8)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_{10})} = 0$.

The other case can be obtained through the same computational sequence by taking $\tilde{W}^{(\alpha_2)}$ instead of $\tilde{W}^{(\alpha_{10})}$. Thus, the proof is complete. $\square$

Remark 1. Following the same computational scheme as in the preceding theorems for $\tilde{W}^{(\alpha_5)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_7)} = 0$, $\tilde{W}^{(\alpha_8)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_5)} = 0$, and $\tilde{W}^{(\alpha_8)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_7)} = 0$, together with (2), (4), (9)–(12), and (55), the resulting equations reduce to identities, independently of the position of the structure vector field ξ among Z_1, Z_2, Z_3, Z_4, Z_5 . Hence, no nontrivial Ricci operator relation or curvature restriction is obtained. Consequently, these cases do not impose any additional restriction on the geometry of the manifold, and therefore no analogous geometric characterization can be derived.

Example 1. The following three-dimensional Kenmotsu manifold was also considered in [11]. Let

$$M = \{(u, v, w) \in \mathbb{R}^3 : w \neq 0\},$$

be the three-dimensional manifold with the standard coordinates (u, v, w) in $\mathbb{R}^3$. The linearly independent vector fields E_1, E_2, E_3 on M are defined by

$$E_1 = e^{-w} \frac{\partial}{\partial u}, \quad E_2 = e^{-w} \frac{\partial}{\partial v}, \quad E_3 = \frac{\partial}{\partial w}.$$

Define a Riemannian metric g on M by

$$g = e^{2w}(du \otimes du + dv \otimes dv) + dw \otimes dw.$$

Since

$$g(E_i, E_j) = \delta_{ij}, \quad i, j = 1, 2, 3,$$

the set $\{E_1, E_2, E_3\}$ forms an orthonormal frame.

Let η and ϕ be the 1-form and $(1, 1)$ -tensor field, respectively, given by

$$\eta(Z_1) = g(Z_1, E_3), \quad \phi(E_1) = E_2, \quad \phi(E_2) = -E_1, \quad \phi(E_3) = 0,$$

for any vector field Z_1 on M . Then we have

$$\phi^2 Z_1 = -Z_1 + \eta(Z_1)E_3, \quad g(\phi Z_1, \phi Z_2) = g(Z_1, Z_2) - \eta(Z_1)\eta(Z_2),$$

showing that $M(\phi, \xi, \eta, g)$ is an almost-contact metric manifold, where $E_3 = \xi$. The fundamental form Φ is defined by

$$\Phi(Z_1, Z_2) = g(Z_1, \phi Z_2).$$

Computing for the basis vectors

$$\Phi\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = g\left(\phi \frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = g(E_2, e^{2w} E_2) = e^{2w}$$

is obtained. Hence, $\Phi = e^{2w} du \wedge dv$. Differentiating gives

$$d\Phi = d(e^{2w} du \wedge dv) = 2e^{2w} dw \wedge du \wedge dv = 2\eta \wedge \Phi,$$

so M is a Kenmotsu manifold, since $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$.

The Lie brackets of the vector fields E_1, E_2, E_3 are

$$[E_1, E_2] = 0, \quad [E_1, E_3] = E_1, \quad [E_2, E_3] = E_2.$$

The Levi-Civita connection ∇ satisfies

$$\nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_2} E_2 = -E_3, \quad \nabla_{E_1} E_3 = E_1, \quad \nabla_{E_2} E_3 = E_2.$$

The non-zero components of the Riemann tensor in the orthonormal frame $\{E_1, E_2, E_3\}$ are

$$\begin{aligned} R(E_1, E_2)E_1 &= E_2, & R(E_1, E_2)E_2 &= -E_1, & R(E_1, E_3)E_1 &= E_3, & R(E_2, E_3)E_2 &= E_3, \\ R(E_2, E_1)E_1 &= -E_2, & R(E_2, E_1)E_2 &= E_1, & R(E_1, E_3)E_3 &= -E_1, & R(E_2, E_3)E_3 &= -E_2. \end{aligned}$$

All other components vanish. Consequently, the components of the Ricci tensor S are

$$S(E_i, E_j) = -2\delta_{ij}, \quad i, j = 1, 2, 3.$$

This implies

$$S(Z_1, Z_2) = -2g(Z_1, Z_2).$$

From

$$S(Z_1, Z_2) = g(QZ_1, Z_2),$$

we conclude

$$Q = -2I.$$

Specifically,

$$QE_1 = -2E_1, \quad QE_2 = -2E_2, \quad QE_3 = -2E_3.$$

Since $\dim M = 3$, we have $n = 1$. Thus the $\tilde{W}^{(\alpha_5)}$ -curvature tensor is given by

$$\tilde{W}^{(\alpha_5)}(Z_1, Z_2)Z_3 = R(Z_1, Z_2)Z_3 - \frac{1}{2}g(Z_2, Z_3)QZ_1 + \frac{1}{2}g(Z_1, Z_3)QZ_2.$$

As $\xi = E_3$, evaluating the components $\tilde{W}^{(\alpha_5)}(E_i, E_j)\xi$ with respect to the orthonormal frame $\{E_1, E_2, E_3\}$, using $Q = -2I$ together with the curvature components of the above Kenmotsu manifold, we obtain

$$\tilde{W}^{(\alpha_5)}(E_i, E_j)\xi = 0, \quad i, j = 1, 2, 3.$$

Therefore, the manifold is $\xi - \tilde{W}^{(\alpha_5)}$ -flat.

Moreover, since

$$S = -2g,$$

the manifold is an Einstein manifold.

Example 2. Using the same Kenmotsu manifold introduced in Example 1, we now examine the $\phi - \tilde{W}^{(\alpha_8)}$ -flatness condition.

Since $\dim M = 3$, we have $n = 1$. Moreover, we know that

$$S = -2g, \quad Q = -2I.$$

Hence, the $\tilde{W}^{(\alpha_8)}$ -curvature tensor is given by

$$\tilde{W}^{(\alpha_8)}(Z_1, Z_2)Z_3 = R(Z_1, Z_2)Z_3 + g(Z_2, Z_3)Z_1 - g(Z_1, Z_3)Z_2.$$

Using

$$\phi(E_1) = E_2, \quad \phi(E_2) = -E_1, \quad \phi(E_3) = 0,$$

and evaluating $g(\tilde{W}^{(\alpha_8)}(\phi E_i, \phi E_j)\phi E_k, \phi E_l)$ for all possible choices of $E_i, E_j, E_k, E_l \in \{E_1, E_2, E_3\}$, we obtain

$$g(\tilde{W}^{(\alpha_8)}(\phi E_i, \phi E_j)\phi E_k, \phi E_l) = 0, \quad i, j, k, l = 1, 2, 3.$$

Therefore, the manifold is $\phi - \tilde{W}^{(\alpha_8)}$ -flat.

5. Conclusions

In this paper, we have detailed the geometric structure and classification of Kenmotsu manifolds by explicitly focusing on the curvature tensor family $\tilde{W}^{(\alpha_i)}$ ($i = 1, 2, \dots, 12$) under the connection $\tilde{\nabla}$. Based on our comprehensive analytical derivations, we have established how various flatness and derivation-type algebraic constraints force the underlying manifold to admit specific geometric profiles.

The principal findings of this family-based investigation can be summarized through the distinct behaviors of the $\tilde{W}^{(\alpha_5)}$ and $\tilde{W}^{(\alpha_8)}$ curvature operators:

- The total flatness condition for both tensors forces the underlying structure to be an Einstein manifold.
- The partial ξ -flatness condition is meaningful only for $\tilde{W}^{(\alpha_5)}$ and ensures an Einstein metric, whereas for $\tilde{W}^{(\alpha_8)}$, it does not yield a definite geometric reduction.
- Regarding weaker constraints, the quasi-flatness condition for both tensors leads to an Einstein manifold, while the ϕ -flatness condition induces an η -Einstein character for $\tilde{W}^{(\alpha_5)}$ and an Einstein structure for $\tilde{W}^{(\alpha_8)}$.
- For $j \in \{5, 8\}$ and $i \in \{1, \dots, 12\}$, the vanishing of the derivation condition $\tilde{W}^{(\alpha_j)}(Z_1, Z_2) \cdot \tilde{W}^{(\alpha_i)} = 0$ forces the manifold to be an Einstein space in all scenarios, with the sole exceptions of the pairs $(j, i) \in \{(5, 7), (8, 5), (8, 7)\}$ which do not provide a geometric reduction.

These classifications reveal the algebraic behavior of generalized curvature tensors under a non-symmetric connection. For future research, this systematic methodology can be extended to study different curvature tensor families on nearly cosymplectic or trans-Sasakian manifolds.

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