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Analytic Umbral Transmutations and Bessel Moments

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Abstract

We apply the recently proposed analytic extension of formal indicial umbral calculus to the evaluation and structural interpretation of Bessel moments, replacing formal symbolic constructions with Mellin–Barnes analytic transmutations. The classical umbral representation of J_0 converts products of Bessel functions into Gaussian integrals involving sums of independent symbolic operators. This formal mechanism may reproduce correct identities in suitable convergence chambers, but it may also lead to non-admissible hypergeometric expansions at the physically relevant parameter values. The cubic moment already exhibits this difficulty: the formal Appell F_4 expansion associated with the equilateral case lies outside its domain of convergence. We address this obstruction by replacing the formal expansion with Mellin–Barnes representations of the corresponding umbral pairings. In this formulation, Ramanujan’s Master Theorem selects the analytic ground state associated with a Bessel product. The factorisation $J_0^3 = J_0 J_0^2$ fuses the elementary Bessel state with the square state and gives the cubic moment as a one-dimensional Meijer–Barnes function. The same mechanism yields a scaled cubic formula and a fourth-moment Meijer–Barnes representation whose residues give a convergent harmonic-number expansion. The fifth moment marks the first higher-rank case: the natural grouping $J_0^5 = J_0^2 J_0^2 J_0$ leads to a bivariate Barnes transmutation rather than to an ordinary Meijer G -function. Finally, real powers J_0^α , $\alpha > 2$, are interpreted through Mellin-selected ground states, which need not reduce to finite Gamma products. Thus, Bessel moments provide a concrete hierarchy of analytic umbral representations, from rank-one Meijer–Barnes functions to higher-rank Barnes structures, and distinguish the global analytic meaning of an umbral construction from the local convergence of its residue expansions.

Keywords: indicial umbral calculus; analytic umbral pairing; Mellin transform; Barnes integrals; Bessel moments



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1. Introduction

The interest in umbral calculus, in its distinctive indicial formulation, was initially motivated by the evaluation of infinite integrals [1], within a formalism naturally encoding Ramanujan’s Master Theorem [2].

This program was carried out, for instance in [3,4], using integrals involving Bessel functions as benchmark examples. The guiding paradigm is the use of the Bessel umbral image

$$J_0(x) = \exp\left(-u \frac{x^2}{4}\right) \varphi,$$

where u is the formal indicial umbral operator [4,5], which, after expanding the exponential in Taylor series, acts by evaluation on the corresponding vacuum φ according to the prescription

$$u^r \varphi = \frac{1}{\Gamma(r+1)}, \quad r \in \mathbb{N}.$$

One can then apply ordinary Gaussian integration methods.

In the following, we apply this method to integrals involving powers of Bessel functions, analyse the limits of the method, and propose a possible way out based on the point of view recently developed in [6].

The starting example is the evaluation of integrals containing the product $J_0(ax)J_0(bx)$. Using independent umbral vacua, products of Bessel functions may be reduced to Gaussian integrals involving fractional powers of sums of umbral operators:

$$J_0(ax) = \exp\left(-u_1 \frac{a^2 x^2}{4}\right) \varphi_1, \quad J_0(bx) = \exp\left(-u_2 \frac{b^2 x^2}{4}\right) \varphi_2.$$

Thus, for constant a, b , one formally finds

$$\begin{aligned} I_2(a, b) &= \int_{-\infty}^{\infty} J_0(ax) J_0(bx) dx \\ &= 2\sqrt{\pi} \left(u_1 a^2 + u_2 b^2\right)^{-1/2} \varphi_1 \varphi_2. \end{aligned} \quad (1)$$

Assuming $|a| > |b|$, this gives

$$I_2(a, b) = \frac{2}{|a|} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \left(\frac{1}{2}\right)_r}{(r!)^2} \left(\frac{b^2}{a^2}\right)^r.$$

The assumption $|a| > |b|$ is made solely to justify the expansion. The final expression is analytically continued to all admissible values of a and b , yielding the symmetric result in terms of the complete elliptic integral (see below).

Equivalently,

$$I_2(a, b) = \frac{2}{|a|} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{b^2}{a^2}\right).$$

Using

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right) = \frac{2}{\pi} K(k),$$

where

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

is the complete elliptic integral of the first kind with modulus k , the previous integral can be written as

$$I_2(a, b) = \frac{4}{\pi A} K\left(\frac{B}{A}\right), \quad A = \max(|a|, |b|), \quad B = \min(|a|, |b|).$$

The correctness of the result is readily verified. As a sanity check, we note that the series diverges for $|a| = |b|$, coherently with the fact that the integral $I_2(1, 1)$ is not finite.

We now consider

$$I_3(\alpha, \beta, \gamma) = \int_{-\infty}^{\infty} J_0(\alpha x) J_0(\beta x) J_0(\gamma x) dx. \quad (2)$$

Using three independent vacua, we can formally write the integral as

$$I_3(\alpha, \beta, \gamma) = 2\sqrt{\pi} \left(u_1 \alpha^2 + u_2 \beta^2 + u_3 \gamma^2 \right)^{-1/2} \prod_{j=1}^3 \varphi_j.$$

Assuming

$$|\alpha| > |\beta| + |\gamma|, \quad (3)$$

one obtains

$$I_3(\alpha, \beta, \gamma) = \frac{2}{|\alpha|} F_4 \left(\frac{1}{2}, \frac{1}{2}; 1, 1; \frac{\beta^2}{\alpha^2}, \frac{\gamma^2}{\alpha^2} \right).$$

Here, F_4 denotes the Appell hypergeometric function

$$F_4(a, b; c, c'; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{(c)_m (c')_n} \frac{x^m y^n}{m! n!},$$

which converges for

$$\sqrt{|x|} + \sqrt{|y|} < 1,$$

consistently with the assumption above. The Appell function F_4 belongs to the broader family of generalised hypergeometric functions, which continue to find applications well beyond classical special-function theory, including recent developments in geometric function theory [7,8].

The conclusion of the formal umbral evaluation would therefore seem to be that the integral obtained by setting $\alpha = \beta = \gamma = 1$ in Equation (2), namely

$$I_3(1, 1, 1) = \int_{-\infty}^{\infty} J_0(x)^3 dx$$

does not provide a finite result. This, however, is not true: the integral sums to

$$\int_{-\infty}^{\infty} J_0(x)^3 dx = \frac{\Gamma(1/6)^2}{2^{2/3} \sqrt{3} \pi^{3/2} \Gamma(5/6)} \simeq 1.79288 \dots$$

(see e.g., [9–11]), as can also be verified with an actual numerical computation.

This is precisely where the naive expansion, deriving from the solution encoded in the formal expression above, reveals its weakness, although it does produce correct results under suitable conditions on the triple (α, β, γ) .

The remainder of this paper develops the analytic umbral framework introduced in [6], showing how it resolves this difficulty through Mellin–Barnes analytic continuation.

The paper is organised as follows. Section 2 reviews the analytic umbral framework and its Mellin–Barnes interpretation. Section 3 introduces the elementary umbral representations of the Bessel function and its square. Sections 4 and 5 develop the cubic moment and its scaled generalisation, while Section 6 derives the fourth moment together with its residue expansion. Section 7 discusses the fifth moment, showing the transition from one-variable Meijer–Barnes representations to a bivariate Barnes transmutation. Section 8 extends the construction to real fractional powers of the Bessel function through Ramanujan-selected ground states, and Section 9 summarises the main conclusions.

2. The Analytic Umbral Framework

In the analytic umbral framework, the umbral operator u is reinterpreted as a functional in the Borel additive complex variable u , namely

$$u \mapsto e^u.$$

For a suitable f , the formal operator function $f(\zeta u)$ thus corresponds to the composed umbral Borel functional $f(\zeta e^u)$. After introducing the multiplicative Borel variable $z = e^u$, one defines the spectral jump function associated with $f(\zeta e^u)$ as

$$J_f(t; \zeta) := \mathcal{M}_z[f(\zeta z)](-t) = \int_0^\infty f(\zeta z) z^{-t-1} dz,$$

where $\mathcal{M}$ denotes the Mellin transform operator. The umbral pairing with an admissible ground state function $\varphi(t)$ is then defined by the Mellin–Barnes integral (see e.g., [12])

$$F(\zeta) = \langle f(\zeta e^u), \varphi \rangle := \frac{1}{2\pi i} \int_{C_t} J_f(t; \zeta) \varphi(t) dt. \quad (4)$$

Here, C_t denotes a vertical Mellin–Barnes contour

$$t = c + i\tau, \quad \tau \in \mathbb{R}$$

chosen so as to separate the two families of poles of the integrand. The precise value of c depends on the particular Mellin–Barnes representation under consideration and will be specified whenever necessary. An admissible $\varphi(t)$ is typically a meromorphic function with at most exponential growth on vertical lines compatible with the Barnes kernel in Equation (4).

In particular, when $f(\zeta e^u) = \exp(\pm \zeta e^u)$, we have that

$$J_{\exp_\pm}(t; \zeta) = (\mp \zeta)^t \Gamma(-t).$$

With the adopted contour orientation conventions, the formal umbral pairing $f(\zeta u) \varphi$ corresponds to the series obtained by closing the contour to the right and evaluating the integral in Equation (4) by residues. The analytic and formal evaluations coincide when $F(\zeta)$ is whole; otherwise, the formal evaluation only yields the (possibly divergent) expansion of $F(\zeta)$ at the origin.

In this sense, the analytic umbral framework provides a rigorous justification and realises a genuine extension of formal umbral calculus. We refer the reader to [6] for further details.

In typical applications, given $F(\zeta)$ and $\varphi(t)$, one aims at identifying the umbral Borel functional that makes Equation (4) satisfied. By reverting the point of view and fixing the jump function entering Equation (4) to $J_{\exp_-}(t; \zeta)$, we can interpret the analytic pairing as a distinguished transform applied to the ground state—typically called *spectral clock* in this context—namely the *exponential umbral transmutation* of $\varphi(t)$:

$$F(\zeta) = \mathfrak{U}_{\exp_-}[\varphi](\zeta) := \frac{1}{2\pi i} \int_{C_t} \zeta^t \Gamma(-t) \varphi(t) dt.$$

The aim is now to identify the spectral clock that satisfies the previous identity. This problem can often be solved by relying on Ramanujan’s Master Theorem.

This is the viewpoint adopted in the following sections.

Remark 1 (Admissible exchanges and analytic continuation). *In the computations below, interchanges between spatial integration and Mellin–Barnes pairing are understood in the following standard sense. One first places the Barnes contours in strips where the Gamma factors provide sufficient decay on vertical lines and where the Gaussian integral produced by the exponential kernel is absolutely convergent. In this regularised domain, the exchange follows from absolute convergence. The resulting Barnes integral then provides the analytic continuation of the expression to the parameter values of interest, provided no pole-crossing is left unaccounted for.*

Thus, when we write identities such as

$$\int_{-\infty}^{\infty} \langle K(x, \exp u), \varphi(t) \rangle dx = \left\langle \int_{-\infty}^{\infty} K(x, \exp u) dx, \varphi(t) \right\rangle,$$

the equality is to be read as an equality first in a common domain of absolute convergence and then by Mellin–Barnes analytic continuation. This convention is the analytic counterpart of the formal umbral manipulation.

3. Analytic Umbral Transmutation and Ramanujan Selection

We use exponential umbral transmutations of the form

$$F(y) = \mathfrak{L}_{\exp_-}[\Phi](y) = \langle \exp(-y e^u), \Phi(t) \rangle.$$

Formally, this means that

$$F(y) = \sum_{m=0}^{\infty} \frac{(-y)^m}{m!} \Phi(m).$$

Ramanujan’s Master Theorem provides the inverse analytic selection rule for the ground state (for a modern account of Ramanujan’s Master Theorem and its Mellin-transform interpretation, see [13]). Namely, if the Mellin transform of F exists in the relevant strip, then

$$\mathcal{M}[F](s) = \Gamma(s) \Phi(-s),$$

where

$$\mathcal{M}[F](s) = \int_0^{\infty} y^{s-1} F(y) dy.$$

Thus, the Mellin-admissible continuation of the coefficient clock $\Phi(m)$ is

$$\Phi(t) = \frac{\mathcal{M}[F](-t)}{\Gamma(-t)}.$$

This is the sense in which Ramanujan’s theorem acts as the inverse form of the analytic umbral transmutation.

Remark 2 (Selection rather than formal summation). *In the present use, Ramanujan’s Master Theorem is not invoked merely as a formal summation rule. It is used as an analytic selection principle: the Mellin transform of F , whenever it exists in a non-empty strip and admits the required meromorphic continuation, determines the admissible continuation of the coefficient clock. Thus, the object*

$$\Phi(t) = \frac{\mathcal{M}[F](-t)}{\Gamma(-t)}$$

is the Mellin-selected ground state associated with F . For integer Bessel powers, this state may reduce to a finite Gamma quotient. For fractional powers, it generally does not, and the selected state should be regarded as an analytic continuation datum rather than as a globally convergent Taylor coefficient function.

For Bessel moments, it is convenient to use

$$y = x^2.$$

If

$$F_{\alpha}(y) := J_0(\sqrt{y})^{\alpha},$$

then, since J_0 is even,

$$\int_{-\infty}^{\infty} J_0(x)^\alpha dx = \mathcal{M}[F_\alpha]\left(\frac{1}{2}\right),$$

whenever the integral is convergent with the chosen branch convention. Therefore, the Ramanujan-selected ground state satisfies

$$\int_{-\infty}^{\infty} J_0(x)^\alpha dx = \sqrt{\pi} \Phi_\alpha\left(-\frac{1}{2}\right),$$

with

$$\Phi_\alpha(t) = \frac{\mathcal{M}[J_0(\sqrt{y})^\alpha](-t)}{\Gamma(-t)}.$$

For integer powers, this state may reduce to a finite Barnes or Meijer-type object. For generic fractional powers, it should instead be regarded as a genuinely Ramanujan-selected spectral state.

4. The Elementary and Square Bessel Clocks

Let

$$\mathcal{C}_0(z) = J_0(2\sqrt{z}) = \sum_{m=0}^{\infty} \frac{(-z)^m}{\Gamma(1+m)^2}.$$

Thus,

$$J_0(x) = \mathcal{C}_0\left(\frac{x^2}{4}\right).$$

We use standard notation and normalisations for Bessel functions as in [9,14,15].

The elementary Bessel–Clifford clock is

$$\varphi_1(t) = \frac{1}{\Gamma(1+t)},$$

and therefore,

$$J_0(x) = \left\langle \exp\left(-\frac{x^2}{4} \exp u\right), \varphi_1(t) \right\rangle.$$

The square has a different clock. Since

$$\mathcal{C}_0(z)^2 = \sum_{m=0}^{\infty} (-z)^m \sum_{p+q=m} \frac{1}{\Gamma(1+p)^2 \Gamma(1+q)^2}$$

and

$$\sum_{p=0}^m \binom{m}{p}^2 = \binom{2m}{m},$$

one obtains

$$\mathcal{C}_0(z)^2 = \sum_{m=0}^{\infty} (-z)^m \frac{\Gamma(2m+1)}{\Gamma(m+1)^4}.$$

Writing this as

$$\mathcal{C}_0(z)^2 = \langle \exp(-z \exp v), \varphi_2(s) \rangle$$

forces

$$\frac{\varphi_2(m)}{\Gamma(1+m)} = \frac{\Gamma(2m+1)}{\Gamma(m+1)^4}.$$

Therefore,

$$\varphi_2(s) = \frac{\Gamma(2s+1)}{\Gamma(s+1)^3},$$

and

$$J_0(x)^2 = \left\langle \exp\left(-\frac{x^2}{4} \exp v\right), \varphi_2(s) \right\rangle.$$

5. The Cubic Moment as a Two-Clock Contraction

We now consider

$$I_3 := I_3(1, 1, 1) = \int_{-\infty}^{\infty} J_0(x)^3 dx.$$

The useful factorisation is

$$J_0(x)^3 = J_0(x)J_0(x)^2.$$

Using the clocks φ_1 and φ_2 , this gives

$$J_0(x)^3 = \left\langle \left\langle \exp\left(-\frac{x^2}{4}(\exp u + \exp v)\right), \varphi_1(t) \varphi_2(s) \right\rangle \right\rangle.$$

After the usual admissible exchange between the spatial integral and the analytic pairing, the Gaussian integral gives

$$\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{4}(\exp u + \exp v)\right) dx = 2\sqrt{\pi} (\exp u + \exp v)^{-1/2}.$$

Hence,

$$I_3 = 2\sqrt{\pi} \left\langle \left\langle (\exp u + \exp v)^{-1/2}, \varphi_1(t) \varphi_2(s) \right\rangle \right\rangle.$$

This formula is analogous to the formal pairing of Equation (1) for $a = b = 1$, but its analytic character enables the limitations of the formal approach to be bypassed—at this stage, the Mellin–Barnes representation provides the analytic continuation of the formal expansion, so the restriction $|a| > |b|$ is no longer required.

Specifically, we use the Mellin–Barnes decomposition

$$(a + b)^{-1/2} = \frac{1}{\sqrt{\pi}} \frac{1}{2\pi i} \int_{C_r} \Gamma(-r) \Gamma\left(r + \frac{1}{2}\right) a^r b^{-r-1/2} dr,$$

where the contour separates the poles of $\Gamma(-r)$ from those of $\Gamma(r + 1/2)$. With $a = \exp u$ and $b = \exp v$, the two clocks evaluate at complementary spectral values:

$$\langle \exp(ru), \varphi_1(t) \rangle = \varphi_1(r),$$

and

$$\left\langle \exp\left(\left(-r - \frac{1}{2}\right)v\right), \varphi_2(s) \right\rangle = \varphi_2\left(-r - \frac{1}{2}\right).$$

Thus,

$$I_3 = \frac{2}{2\pi i} \int_{C_r} \Gamma(-r) \Gamma\left(r + \frac{1}{2}\right) \varphi_1(r) \varphi_2\left(-r - \frac{1}{2}\right) dr.$$

Since,

$$\varphi_1(r) = \frac{1}{\Gamma(1+r)}$$

and

$$\varphi_2\left(-r - \frac{1}{2}\right) = \frac{\Gamma(-2r)}{\Gamma\left(\frac{1}{2} - r\right)^3},$$

we obtain

$$I_3 = \frac{2}{2\pi i} \int_{C_r} \Gamma(-r) \Gamma\left(r + \frac{1}{2}\right) \frac{1}{\Gamma(1+r)} \frac{\Gamma(-2r)}{\Gamma\left(\frac{1}{2}-r\right)^3} dr.$$

Using

$$\Gamma(-2r) = 2^{-2r-1} \pi^{-1/2} \Gamma(-r) \Gamma\left(\frac{1}{2}-r\right),$$

this becomes

$$I_3 = \frac{1}{\sqrt{\pi}} \frac{1}{2\pi i} \int_{C_r} 4^{-r} \frac{\Gamma(-r)^2 \Gamma\left(r + \frac{1}{2}\right)}{\Gamma(1+r) \Gamma\left(\frac{1}{2}-r\right)^2} dr.$$

Equivalently, after setting $r = -t$,

$$I_3 = \frac{1}{\sqrt{\pi}} \frac{1}{2\pi i} \int_{C_t} 4^t \frac{\Gamma(t)^2 \Gamma\left(\frac{1}{2}-t\right)}{\Gamma(1-t) \Gamma\left(\frac{1}{2}+t\right)^2} dt.$$

This Barnes integral can also be directly obtained from the Mellin–Parseval formula. Indeed,

$$\mathcal{M}[J_0](s) = 2^{s-1} \frac{\Gamma(s/2)}{\Gamma(1-s/2)},$$

and

$$\mathcal{M}[J_0^2](s) = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(s/2) \Gamma(1/2-s/2)}{\Gamma(1-s/2)^2}.$$

Therefore,

$$I_3 = 2 \frac{1}{2\pi i} \int_{C_s} \mathcal{M}[J_0](s) \mathcal{M}[J_0^2](1-s) ds,$$

and the change of variable $s = 2t$ gives precisely the preceding Barnes integral.

This observation should not be read as reducing the present construction to a disguised use of Mellin–Parseval. Mellin–Parseval gives the same Barnes integral once the Mellin transforms of the factors are already known. The analytic-umbral construction has a different role: it explains how these Mellin data arise from elementary spectral clocks and how product integrals are generated by fusing such clocks through Barnes decompositions of the algebraic kernels produced by the spatial integration.

In this sense Mellin–Parseval is the rank-one analytic shadow of the umbral mechanism. For the cubic moment, the binary fusion of the elementary clock φ_1 and the square clock φ_2 produce precisely the Mellin–Barnes object, which a direct Mellin–Parseval computation would recover a posteriori. The advantage of the umbral formulation is therefore structural rather than merely evaluative: it provides an algebraic blueprint for constructing the Barnes integrand from simpler ground states.

In Meijer–Barnes notation,

$$I_3 = \frac{1}{\sqrt{\pi}} G_{3,3}^{2,1} \left(\frac{1}{4} \middle| \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right).$$

Our conventions for Meijer–Barnes integrals and Meijer G-functions are those of [15,16]; for a broader Mellin–Barnes framework; see also [17].

The numeric value of I_3 , as verified with the aid of Mathematica™, is

$$\frac{1}{\sqrt{\pi}} G_{3,3}^{2,1} \left(\frac{1}{4} \middle| \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \simeq 1.79288 \dots$$

This numerically agrees with classical Watson's evaluation reported in Section 1:

$$I_3 = \frac{\Gamma(1/6)^2}{2^{2/3}\sqrt{3}\pi^{3/2}\Gamma(5/6)}.$$

Remark 3. The Mellin–Barnes value of the cubic Bessel moment can be written in the hypergeometric form

$$I_3 = \sqrt{3} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; \frac{1}{4}\right).$$

By Clausen's identity [18,19],

$${}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; z\right) = \left[{}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; z\right)\right]^2.$$

Therefore,

$$I_3 = \sqrt{3} \left[{}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; \frac{1}{4}\right)\right]^2.$$

The singular-modulus evaluation [20]

$${}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; \frac{1}{4}\right) = \frac{\Gamma\left(\frac{1}{6}\right)}{2^{1/3}\sqrt{3}\pi^{3/4}\Gamma\left(\frac{5}{6}\right)^{1/2}}$$

then gives

$$I_3 = \frac{\Gamma\left(\frac{1}{6}\right)^2}{2^{2/3}\sqrt{3}\pi^{3/2}\Gamma\left(\frac{5}{6}\right)}.$$

This identifies the Mellin–Barnes continuation of the formal cubic umbral expression with the classical closed value of the three-Bessel moment.

6. Scaled Cubic Products and an Effective One-Clock Function

The same computation applies to the scaled product integral

$$I(a, b) := \int_{-\infty}^{\infty} J_0(ax) J_0(bx)^2 dx.$$

The two-clock representation is

$$J_0(ax) J_0(bx)^2 = \left\langle \left\langle \exp\left(-\frac{x^2}{4}(a^2 \exp u + b^2 \exp v)\right), \varphi_1(t) \varphi_2(s) \right\rangle \right\rangle.$$

The Gaussian integral and the same Barnes decomposition yield

$$I(a, b) = \frac{1}{\sqrt{\pi}} \frac{1}{b} \frac{1}{2\pi i} \int_{C_r} \left(\frac{a^2}{4b^2}\right)^r \frac{\Gamma(-r)^2 \Gamma\left(\frac{1}{2} + r\right)}{\Gamma(1+r) \Gamma\left(\frac{1}{2} - r\right)^2} dr.$$

Define

$$\mathcal{H}(\zeta) := \frac{1}{\sqrt{\pi}} \frac{1}{2\pi i} \int_{C_r} \zeta^r \frac{\Gamma(-r)^2 \Gamma\left(\frac{1}{2} + r\right)}{\Gamma(1+r) \Gamma\left(\frac{1}{2} - r\right)^2} dr.$$

Here, C_r is a vertical Mellin–Barnes contour of the form $r = c + i\tau$, with $\tau \in \mathbb{R}$ and

$$-\frac{1}{2} < c < 0$$

separating the double poles of $\Gamma(-r)^2$ at $r = 0, 1, 2, \dots$ from the poles of $\Gamma(\frac{1}{2} + r)$ at $r = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots$

Then,

$$I(a, b) = \frac{1}{b} \mathcal{H}\left(\frac{a^2}{4b^2}\right).$$

Equivalently,

$$\mathcal{H}(\zeta) = \frac{1}{\sqrt{\pi}} G_{3,3}^{2,1}\left(\zeta \left| \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 0, 0, 0 \end{matrix} \right.\right),$$

so that

$$I(a, b) = \frac{1}{\sqrt{\pi} b} G_{3,3}^{2,1}\left(\frac{a^2}{4b^2} \left| \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 0, 0, 0 \end{matrix} \right.\right),$$

with the usual Barnes-contour prescription and analytic continuation in the ratio a/b where necessary.

The same function can be interpreted as an effective one-clock transmutation:

$$\mathcal{H}(\zeta) = \langle \exp(-\zeta \exp u), \varphi_{\text{eff}}(t) \rangle,$$

where

$$\varphi_{\text{eff}}(t) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(-t)\Gamma(\frac{1}{2} + t)}{\Gamma(1+t)\Gamma(\frac{1}{2} - t)^2}.$$

This effective clock contains the spectral convolution of the elementary clock φ_1 and the square clock φ_2 .

7. The Fourth Moment

We now consider

$$I_4 := \int_{-\infty}^{\infty} J_0(x)^4 dx.$$

The natural grouping is symmetric:

$$J_0(x)^4 = J_0(x)^2 J_0(x)^2.$$

Using the square clock φ_2 , we obtain

$$J_0(x)^4 = \left\langle \left\langle \exp\left(-\frac{x^2}{4}(\exp u + \exp v)\right), \varphi_2(t) \varphi_2(s) \right\rangle \right\rangle.$$

The same Gaussian integration and Barnes decomposition give

$$I_4 = 2 \frac{1}{2\pi i} \int_{C_r} \Gamma(-r) \Gamma\left(r + \frac{1}{2}\right) \varphi_2(r) \varphi_2\left(-r - \frac{1}{2}\right) dr.$$

Substitution of

$$\varphi_2(r) = \frac{\Gamma(2r+1)}{\Gamma(r+1)^3}$$

and

$$\varphi_2\left(-r - \frac{1}{2}\right) = \frac{\Gamma(-2r)}{\Gamma\left(\frac{1}{2} - r\right)^3}$$

yields

$$I_4 = \frac{2}{2\pi i} \int_{C_r} \Gamma(-r) \Gamma\left(r + \frac{1}{2}\right) \frac{\Gamma(2r+1)}{\Gamma(r+1)^3} \frac{\Gamma(-2r)}{\Gamma\left(\frac{1}{2}-r\right)^3} dr.$$

Using

$$\Gamma(2r+1) = 2^{2r} \pi^{-1/2} \Gamma\left(r + \frac{1}{2}\right) \Gamma(r+1)$$

and

$$\Gamma(-2r) = 2^{-2r-1} \pi^{-1/2} \Gamma(-r) \Gamma\left(\frac{1}{2}-r\right),$$

the powers of 2 cancel the external factor 2, and one obtains

$$I_4 = \frac{1}{\pi} \frac{1}{2\pi i} \int_{C_r} \frac{\Gamma(-r)^2 \Gamma\left(r + \frac{1}{2}\right)^2}{\Gamma(r+1)^2 \Gamma\left(\frac{1}{2}-r\right)^2} dr.$$

Define the effective fourth-moment function

$$\mathcal{H}_4(\zeta) := \frac{1}{\pi} \frac{1}{2\pi i} \int_{C_r} \zeta^r \frac{\Gamma(-r)^2 \Gamma\left(r + \frac{1}{2}\right)^2}{\Gamma(r+1)^2 \Gamma\left(\frac{1}{2}-r\right)^2} dr.$$

Then,

$$I_4 = \mathcal{H}_4(1).$$

As a one-clock transmutation,

$$\mathcal{H}_4(\zeta) = \langle \exp(-\zeta \exp u), \varphi_{\text{eff},4}(t) \rangle,$$

where

$$\varphi_{\text{eff},4}(t) = \frac{1}{\pi} \frac{\Gamma(-t) \Gamma\left(t + \frac{1}{2}\right)^2}{\Gamma(t+1)^2 \Gamma\left(\frac{1}{2}-t\right)^2}.$$

In Meijer–Barnes notation,

$$\mathcal{H}_4(\zeta) = \frac{1}{\pi} G_{4,4}^{2,2} \left(\zeta \left| \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 0, 0, 0, 0 \end{matrix} \right. \right).$$

Therefore,

$$\int_{-\infty}^{\infty} J_0(x)^4 dx = \frac{1}{\pi} G_{4,4}^{2,2} \left(1 \left| \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 0, 0, 0, 0 \end{matrix} \right. \right),$$

and numerically

$$\int_{-\infty}^{\infty} J_0(x)^4 dx = 1.80545 \dots$$

Residue expansion. . Keep ζ as an auxiliary parameter and assume first that $|\zeta| < 1$. Closing the Barnes contour to the right, the enclosed poles are the double poles at $r = n$, $n = 0, 1, 2, \dots$, coming from $\Gamma(-r)^2$. With the usual right-closure orientation, one obtains

$$\mathcal{H}_4(\zeta) = -\frac{1}{\pi} \sum_{n=0}^{\infty} \text{Res}_{r=n} \left[\zeta^r \frac{\Gamma(-r)^2 \Gamma\left(r + \frac{1}{2}\right)^2}{\Gamma(r+1)^2 \Gamma\left(\frac{1}{2}-r\right)^2} \right].$$

The residue calculation is a standard double-pole Barnes computation; compare the Meijer G-function residue conventions in [15,16]. We record the elementary ingredients. Near $r = n$, one has

$$\Gamma(-r)^2 = \frac{1}{(n!)^2(r-n)^2} + \frac{2(H_n - \gamma)}{(n!)^2(r-n)} + O(1).$$

Moreover,

$$\frac{\Gamma\left(n + \frac{1}{2}\right)^2}{\Gamma(n+1)^2\Gamma\left(\frac{1}{2} - n\right)^2} = \frac{\left(\frac{2n}{n}\right)^4}{256^n},$$

and the logarithmic derivative of the remaining analytic factor gives

$$\frac{d}{dr} \log \left[\zeta^r \frac{\Gamma\left(r + \frac{1}{2}\right)^2}{\Gamma(r+1)^2\Gamma\left(\frac{1}{2} - r\right)^2} \right]_{r=n} = \log \zeta + 8H_{2n} - 6H_n - 8\log 2 + 2\gamma.$$

Combining this derivative with the simple-pole coefficient of $\Gamma(-r)^2$, the Euler constants cancel and one obtains

$$\operatorname{Res}_{r=n} \left[\zeta^r \frac{\Gamma(-r)^2\Gamma\left(r + \frac{1}{2}\right)^2}{\Gamma(r+1)^2\Gamma\left(\frac{1}{2} - r\right)^2} \right] = \frac{\left(\frac{2n}{n}\right)^4}{256^n} \zeta^n [\log \zeta + 8(H_{2n} - H_n - \log 2)].$$

Therefore,

$$\mathcal{H}_4(\zeta) = -\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\left(\frac{2n}{n}\right)^4}{256^n} \zeta^n [\log \zeta + 8(H_{2n} - H_n - \log 2)].$$

Taking the limit $\zeta \rightarrow 1^-$, the logarithmic term disappears and

$$I_4 = \mathcal{H}_4(1) = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{\left(\frac{2n}{n}\right)^4}{256^n} (\log 2 + H_n - H_{2n}).$$

The convergence follows from

$$\frac{\left(\frac{2n}{n}\right)^4}{256^n} \sim \frac{1}{\pi^2 n^2}$$

and

$$\log 2 + H_n - H_{2n} \sim \frac{1}{4n},$$

so the summand is $O(n^{-3})$. Numerically, this residue expansion gives

$$\frac{8}{\pi} \sum_{n=0}^{\infty} \frac{\left(\frac{2n}{n}\right)^4}{256^n} (\log 2 + H_n - H_{2n}) = 1.80545 \dots$$

Remark 4 (Right closure of the Barnes contour). For $0 < \zeta < 1$, the right closure of the Barnes contour in the representation of $\mathcal{H}_4(\zeta)$ is justified by the standard Stirling estimates for Gamma ratios on vertical lines. More precisely, take the closing contours through vertical lines $\operatorname{Re} r = N + 1/2$, avoiding the poles at the non-negative integers. On such lines, the factor ζ^r gives exponential decay in N , while the Gamma quotient has at most polynomial growth in N and polynomial decay on horizontal sides. Hence, the contribution of the closing sides tends to zero as $N \rightarrow \infty$. The enclosed poles are therefore exactly the double poles of $\Gamma(-r)^2$ at $r = n$, $n \in \mathbb{N}$.

At the endpoint $\zeta = 1$, the series is understood as the Abel limit $\zeta \rightarrow 1^-$. The resulting harmonic-number series is absolutely convergent, since its n -th summand is $O(n^{-3})$.

8. Beyond Meijer G: The Fifth Bessel Moment as a Bivariate Barnes Transmutation

The preceding examples show that the cubic and quartic Bessel moments admit particularly efficient one-dimensional Barnes representations. This is not because the analytic umbral method is intrinsically restricted to third and fourth powers but because in those cases, the relevant product can be reduced to a binary fusion of two single-state analytic transmutations. The binomial Barnes representation then introduces only one auxiliary Barnes variable, and the result is naturally expressible as a Meijer G-function.

The fifth moment is the first case where the same mechanism naturally produces a genuinely multivariate Barnes object. We group

$$J_0(x)^5 = J_0(x)^2 J_0(x)^2 J_0(x)$$

and use two copies of the square clock

$$\varphi_2(t) = \frac{\Gamma(2t+1)}{\Gamma(t+1)^3}$$

together with the elementary clock

$$\varphi_1(t) = \frac{1}{\Gamma(1+t)}.$$

In umbral form, this means that

$$J_0(x)^2 = \left\langle \exp\left(-\frac{x^2}{4} \exp u\right), \varphi_2(t) \right\rangle$$

and

$$J_0(x) = \left\langle \exp\left(-\frac{x^2}{4} \exp u\right), \varphi_1(t) \right\rangle.$$

Therefore, the fifth power is represented by a three-clock pairing. After the Gaussian integration in the spatial variable, the kernel becomes

$$(\exp u + \exp v + \exp w)^{-1/2}.$$

This is the point at which the fifth moment departs from the Meijer G-type behaviour of the cubic and quartic cases.

The appropriate Barnes representation is now the multinomial one:

$$(a+b+c)^{-1/2} = \frac{1}{\Gamma(1/2)} \frac{1}{(2\pi i)^2} \int_{C_r} \int_{C_s} \Gamma(-r)\Gamma(-s)\Gamma\left(\frac{1}{2}+r+s\right) a^r b^s c^{-1/2-r-s} dr ds.$$

Choosing

$$a = \exp u, \quad b = \exp v, \quad c = \exp w$$

shows that the three clocks are evaluated respectively at

$$r, \quad s, \quad -\frac{1}{2} - r - s.$$

Consequently, the full fifth moment can be written formally as

$$I_5 = 2 \frac{1}{(2\pi i)^2} \int_{C_r} \int_{C_s} \Gamma(-r)\Gamma(-s)\Gamma\left(\frac{1}{2}+r+s\right) \varphi_2(r)\varphi_2(s)\varphi_1\left(-\frac{1}{2}-r-s\right) dr ds.$$

Substituting the two clocks gives the cleaner bivariate Barnes representation

$$I_5 = 2 \frac{1}{(2\pi i)^2} \int_{C_r} \int_{C_s} \Gamma(-r) \Gamma(-s) \frac{\Gamma\left(\frac{1}{2} + r + s\right)}{\Gamma\left(\frac{1}{2} - r - s\right)} \frac{\Gamma(2r+1)}{\Gamma(r+1)^3} \frac{\Gamma(2s+1)}{\Gamma(s+1)^3} dr ds.$$

This formula should be read with the same Barnes-contour conventions used in the preceding one-dimensional cases but now with two independent contour variables.

It is useful to introduce two auxiliary parameters and regard the last expression as the value at $(1, 1)$ of a bivariate Barnes function. Define

$$\mathcal{H}_5(\zeta, \eta) = \frac{1}{(2\pi i)^2} \int_{C_r} \int_{C_s} \zeta^r \eta^s \Gamma(-r) \Gamma(-s) \Phi_5^{\text{eff}}(r, s) dr ds,$$

where the bivariate effective ground state is

$$\Phi_5^{\text{eff}}(r, s) = 2 \frac{\Gamma\left(\frac{1}{2} + r + s\right)}{\Gamma\left(\frac{1}{2} - r - s\right)} \frac{\Gamma(2r+1)}{\Gamma(r+1)^3} \frac{\Gamma(2s+1)}{\Gamma(s+1)^3}.$$

This is the bivariate analogue of the one-clock Mellin–Barnes transmutations occurring in the cubic and quartic cases. The variables ζ and η keep track of the two independent Barnes directions produced by the ternary kernel.

Then,

$$I_5 = \mathcal{H}_5(1, 1).$$

In this sense, the fifth moment is still governed by an effective umbral state, but the state is now bivariate. The ordinary Meijer G -function is replaced by a two-variable Mellin–Barnes function.

Closing both Barnes contours to the right gives local Horn-type expansions for $\mathcal{H}_5(\zeta, \eta)$, rather than a directly convergent formula at the physical point $(1, 1)$. Indeed, the right poles are

$$r = m, \quad s = n, \quad m, n \in \mathbb{N}$$

and the resulting residue expansion is a local expansion in the variables ζ and η . Unlike the fourth-moment case, the endpoint $(\zeta, \eta) = (1, 1)$ does not lie in the domain where this naive right-residue expansion gives an efficient numerical summation. Thus, the formula

$$I_5 = \mathcal{H}_5(1, 1)$$

should be interpreted as an analytic-continuation statement for a bivariate Barnes transmutation, not as a directly useful residue series.

This gives the natural hierarchy behind the preceding computations:

$n = 3, 4$: binary fusion $\longrightarrow$ one effective clock $\longrightarrow$ one-variable Meijer G ,

$n = 5$: ternary fusion $\longrightarrow$ bivariate effective clock $\longrightarrow$ two-variable Barnes function.

Thus, the obstruction at the fifth moment is not a failure of the analytic umbral method. Rather, it is the first appearance of the higher-rank Barnes structure, which is suppressed in the cubic and quartic cases by the availability of a binary reduction.

The role of the fifth moment in the present note is therefore classificatory. It shows that the Meijer G -phenomenon observed for the cubic and quartic moments is not a universal feature of Bessel moments but the consequence of binary fusion. The moment I_5 is the first

case in which the natural factorisation produces a ternary kernel and hence a rank-two Barnes transmutation.

From this point of view, the failure of the naive right-residue expansion at $(\zeta, \eta) = (1, 1)$ is not a breakdown of the analytic-umbral construction. It indicates that the physical point lies outside the local Horn chamber obtained by closing both contours to the right. The bi-variate Barnes representation still identifies the global analytic object whose boundary value is I_5 . What changes is the nature of the special function: the ordinary one-variable Meijer G -function is replaced by a higher-rank Barnes function, and its local residue expansions no longer provide a direct endpoint summation.

Thus, the fifth moment marks the boundary between Meijer-reducible Bessel moments and genuinely multivariate Barnes structures. This distinction is invisible if one looks only for numerical quadratures, but it is naturally detected by the analytic-umbral factorisation of clocks.

9. Fractional Powers

We now return to

$$I_\alpha := \int_{-\infty}^{\infty} J_0(x)^\alpha dx,$$

with the principal branch convention

$$J_0(x)^\alpha := \exp(\alpha \operatorname{Log} J_0(x)).$$

For non-integer α , the zeros of J_0 become branch points of the integrand. If $j_0 = 0 < j_1 < j_2 < \dots$ are the non-negative zeros of J_0 , then for real α , the Mellin transform decomposes as

$$\mathcal{M}[F_\alpha](s) = 2 \sum_{k=0}^{\infty} e^{i\pi\alpha k} \int_{j_k}^{j_{k+1}} x^{2s-1} |J_0(x)|^\alpha dx,$$

because the sign of J_0 alternates from interval to interval.

The convergence strip follows from the endpoint behaviour. At the origin,

$$J_0(x) = 1 + O(x^2), \quad x \rightarrow 0,$$

so one needs

$$\operatorname{Re} s > 0.$$

At infinity,

$$J_0(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\pi}{4}\right).$$

For real non-integer α , there is no systematic cancellation sufficient to improve the absolute estimate, and the tail is absolutely integrable when

$$2 \operatorname{Re} s - 1 - \frac{\alpha}{2} < -1,$$

that is,

$$\operatorname{Re} s < \frac{\alpha}{4}.$$

Thus, a natural absolute-convergence strip is

$$0 < \operatorname{Re} s < \frac{\alpha}{4}.$$

In particular, the value $s = 1/2$ belongs to the strip exactly when

$$\alpha > 2.$$

The zeros of J_0 impose no stronger condition in this range: the zeros are simple, and locally, $|J_0(x)|^\alpha$ is integrable whenever $\alpha > -1$.

Therefore, for real $\alpha > 2$, the principal-branch integral is absolutely convergent and

$$I_\alpha = \sqrt{\pi} \Phi_\alpha \left(-\frac{1}{2} \right),$$

where

$$\Phi_\alpha(t) = \frac{\mathcal{M}[J_0(\sqrt{y})^\alpha](-t)}{\Gamma(-t)}. \quad (5)$$

This formula is exact, but it is a structural reduction rather than an automatic closed-form evaluation. The integer cases are special because the branch obstruction disappears and the effective state may collapse to a finite Barnes kernel.

Remark 5. For $\alpha = 1$, the full-line integral is conditionally convergent and equals 2. For $\alpha = 2$, the integral diverges logarithmically, consistently with the boundary $\alpha > 2$. The cases $\alpha = 3$ and $\alpha = 4$ are the first absolutely convergent integer cases and are treated explicitly in this article.

For non-integer α , the selected state Φ_α should not be expected to collapse to a finite Gamma product. The zeros of $J_0(\sqrt{y})$ on the positive real axis introduce branch transitions in the principal-branch function $J_0(\sqrt{y})^\alpha$. Consequently, the Mellin-selected state contains global information about the oscillatory zero structure of the Bessel factor. Equation (5) is therefore not used as a closed evaluation but as the definition of the admissible spectral ground state associated with the fractional Bessel product. The point is that this state exists and can be analysed as a Mellin object even when no finite umbral clock factorisation is available.

10. Conclusions

We have shown how a consistent analytic reformulation of the umbral framework makes it possible to bypass the limitations of the formal umbral approach in the determination of the Bessel moments by reproducing and reinterpreting classical results.

The main conclusion we can draw from the carried out analysis is structural. Products of Bessel functions can be reorganised into exponential umbral transmutations with distinct spectral clocks. Spatial integration then produces elementary algebraic kernels whose Barnes decompositions fuse the clocks into effective umbral states.

For J_0^3 , the relevant clocks are φ_1 and φ_2 , and the binomial Barnes decomposition produces the one-variable Meijer–Barnes function $\mathcal{H}$. For the scaled product $J_0(ax)J_0(bx)^2$, the entire dependence on the ratio of scales enters through the single argument $\frac{a^2}{4b^2}$. Thus, the scaled moment is controlled by the same effective function $\mathcal{H}$.

For J_0^4 , the symmetric pair of square clocks φ_2, φ_2 again leads to a binary fusion. The resulting effective function $\mathcal{H}_4$ is a one-variable Meijer–Barnes function, and its value at 1 gives the fourth Bessel moment. In this case, closing the Barnes contour to the right gives a useful Abel endpoint expansion: the double poles produce a rapidly convergent harmonic-number series for I_4 .

The fifth moment clarifies the scope of this Meijer G-phenomenon. Grouping

$$J_0(x)^5 = J_0(x)^2 J_0(x)^2 J_0(x)$$

still gives a perfectly natural analytic umbral representation, but the spatial integral now produces a ternary kernel. Its multinomial Barnes decomposition introduces two independent Barnes variables. Consequently, the effective object is no longer a one-clock state and no longer an ordinary Meijer G -function. Instead, I_5 is realised as the value

$$I_5 = \mathcal{H}_5(1, 1)$$

of a bivariate Barnes transmutation. Closing the two Barnes contours gives local Horn-type expansions rather than a directly convergent residue series at the physical point. Thus, the fifth moment separates the structural analytic umbral representation from the most efficient numerical representation, which is better obtained from the angular product formula for J_0^2 .

The fractional-power discussion explains why the same language persists beyond integer moments. For real $\alpha > 2$, the principal-branch moment is given by the Mellin value

$$\mathcal{M}[J_0(\sqrt{y})^\alpha] \left(\frac{1}{2} \right)$$

or, equivalently, by

$$\sqrt{\pi} \Phi_\alpha \left(-\frac{1}{2} \right),$$

where Φ_α is selected by Ramanujan's Master Theorem. In general, this selected state is not a finite Gamma product. Nevertheless, the analytic umbral interpretation remains the same: Bessel moments are Mellin values of Ramanujan-selected ground states, and the passage from $n = 3, 4$ to $n = 5$ displays the transition from rank-one Meijer–Barnes transmutations to higher-rank Barnes structures.

As a final remark, we observe that although the naive Appell series associated with the formal evaluation of $I_3(1, 1, 1)$, namely

$$F_4 \left(\frac{1}{2}, \frac{1}{2}; 1, 1, 1, 1 \right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} \right)_{m+n}^2}{(m!)^2 (n!)^2},$$

does not converge, the underlying umbral contraction still identifies the correct analytic object. The failure is not in the umbral factorisation itself, but in the particular Appell series used to realise it locally. That series represents only one convergence chamber of the analytic continuation and ceases to be admissible at the equilateral point.

Equivalently, one may say that the umbral representation gives the global Mellin–Barnes object, while the Appell realisation is only a local expansion of that object. In other words, the umbral identity remains meaningful, whereas the particular Appell series chosen to realise it is only a local representation and ceases to be admissible at the equilateral point.

Equivalently, one may say that the umbral representation exists globally, while its Appell realisation is restricted to a local convergence chamber.

The same Mellin-selection mechanism suggests a broader class of fractional Franel-type functionals [21,22], in which integer combinatorial coefficients are replaced by analytically selected ground states. We leave this direction for a separate study, since its natural questions concern the singular structure and admissibility of the selected Mellin states rather than the Bessel-moment hierarchy developed here.

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