

Article

Structural Properties and Zero Distributions of q -Laguerre–Hahn–Tricomi–Appell Polynomials in the Framework of Quantum q -Calculus

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Abstract

We introduce a bivariate Hahn-factorial q -Laguerre–Tricomi–Appell polynomial class obtained by multiplying a nonsingular Appell factor by a Hahn-factorial deformation of the two-variable q -Laguerre–Tricomi generating kernel. The construction is interpreted formally, coefficientwise, and is not presented as an orthogonality or Laguerre–Hahn functional characterization. Its defining product yields a finite q -binomial convolution, which is the principal mechanism for transferring algebraic and operational properties from the base family to the Appell deformation. We establish direct and inverse connection formulas, a Hessenberg determinant representation, recurrence and higher Hahn-difference formulas, Hahn-shift identities, operational transfer formulas, and quasi-monomiality relations through a basis-defined raising operator. We also derive reductions to the underlying Hahn-factorial q -Laguerre–Tricomi family, one-variable Hahn–Appell and q -Appell families, translated q -Laguerre specializations, admissible Bernoulli–Euler-type subclasses, a singular Genocchi-type convolution, and the joint classical limit $q \rightarrow 1^-$, $w \rightarrow 0$. A finite coefficient scheme is then used to study representative zero distributions and graphical behavior. The results show that the proposed class is a coherent Appell-type deformation of a Hahn-factorial q -Laguerre–Tricomi kernel and that its structural identities follow from explicitly invertible q -binomial transforms whenever the Appell factor has a nonzero constant term.

Keywords: Hahn-factorial q -Laguerre–Tricomi polynomials; q -Appell polynomials; q -binomial convolution; monomiality principle; differential equations; formal Hahn difference relations; zero distribution



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1. Introduction

Appell-type polynomial sequences, Gould–Hopper polynomials, and their q -extensions play an important role in the theory of special functions, generating functions, and operational methods. Their significance lies in the fact that they connect algebraic structures

with analytic tools such as recurrence relations, operational identities, and differential or q -difference equations. For this reason, they arise naturally in combinatorics, approximation theory, mathematical physics, and operator analysis; see, for example [1–7].

In recent years, considerable attention has been devoted to hybrid polynomial families obtained by combining an Appell kernel with an additional generating factor. Such constructions often preserve the Appell character while enriching the underlying algebraic and operational structure. This viewpoint has led to a number of new polynomial classes together with explicit formulas, determinant representations, recurrence relations, and differential or q -difference equations. Recent examples include q -Laguerre-based, Laguerre-based Appell, truncated-exponential-based, and convolution-type Appell constructions [8–14]; related limit phenomena for q -Laguerre polynomials also form part of the classical background [15]. Applications of q -exponential kernels to related trigonometric, Hermite–Appell, and Genocchi-type constructions may be found in [7,16,17].

Although the present construction uses the phrase Laguerre–Hahn, it is used here for a Hahn-factorial deformation of a Laguerre–Tricomi generating kernel, not as a full orthogonality or moment-functional characterization. The genuine H_q -Laguerre–Hahn and H_q -semiclassical orthogonal-polynomial background is well developed in [18–21]. General q -classical difference-calculus treatments and Hahn/Jackson calculus references may be found in [3,5,22,23]. This positioning is used below to keep the Appell deformation separate from the stronger orthogonality theory.

Throughout this paper, we assume that $0 < q < 1$ and use the standard terminology and notation of [1,15]. The q -shifted factorial $(\alpha; q)_\theta$ is defined by

$$(\alpha; q)_\theta = \prod_{s=0}^{\theta-1} (1 - \alpha q^s), \quad \theta \in \mathbb{N}_0, \quad \alpha \in \mathbb{C}, \quad 0 < q < 1, \quad (1)$$

with the empty product convention $(\alpha; q)_0 = 1$.

The q -number is defined by

$$[d]_q = \frac{1 - q^d}{1 - q}, \quad d \in \mathbb{C}, \quad 0 < q < 1. \quad (2)$$

The corresponding q -factorial is

$$[\theta]_q! = \begin{cases} \prod_{v=1}^{\theta} [v]_q, & \theta \in \mathbb{N}, \\ 1, & \theta = 0, \end{cases} \quad 0 < q < 1. \quad (3)$$

For complex d , we use $q^d := \exp(d \log q)$, where $\log q < 0$ is the real logarithm. For integers $0 \leq k \leq n$, the Gaussian q -binomial coefficient is

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

For the convention used in this paper, the Gauss q -binomial power is defined by

$$(\mu \pm a)_q^\theta := \prod_{j=0}^{\theta-1} (\mu \pm q^j a).$$

Its expansion may be written in either of the equivalent forms

$$\begin{aligned} (\mu \pm a)_q^\theta &= \sum_{\nu=0}^{\theta} \binom{\theta}{\nu}_q q^{\binom{\theta-\nu}{2}} \mu^\nu (\pm a)^{\theta-\nu} \\ &= \sum_{k=0}^{\theta} \binom{\theta}{k}_q q^{k(k-1)/2} \mu^{\theta-k} (\pm a)^k. \end{aligned} \quad (4)$$

Thus the exponent in the first line is $\binom{\theta-\nu}{2}$; after the reindexing $k = \theta - \nu$, it is the standard exponent $k(k-1)/2$.

The two standard q -exponential functions used below are [1,3,5]:

$$e_q(\mu) = \sum_{\theta=0}^{\infty} \frac{\mu^\theta}{[\theta]_q!}, \quad 0 < q < 1, \quad |\mu| < \frac{1}{1-q}, \quad (5)$$

and

$$E_q(\eta) = \sum_{\theta=0}^{\infty} \frac{q^{\binom{\theta}{2}} \eta^\theta}{[\theta]_q!}, \quad 0 < q < 1, \quad \eta \in \mathbb{C}. \quad (6)$$

To distinguish the relevant operation from ordinary addition, define the q -addition associated with the product $e_q E_q$ by

$$(\mu \oplus_q \eta)^\theta := \sum_{\nu=0}^{\theta} \binom{\theta}{\nu}_q q^{\binom{\theta-\nu}{2}} \mu^\nu \eta^{\theta-\nu}.$$

The q -Cauchy product of (5) and (6) then gives

$$e_q(\mu) E_q(\eta) = \sum_{\theta=0}^{\infty} \frac{(\mu \oplus_q \eta)^\theta}{[\theta]_q!}. \quad (7)$$

The factor $q^{\binom{\theta-\nu}{2}}$ is essential here because the second factor is E_q ; the exponent-free q -binomial convolution corresponds instead to a product of two e_q -series. Consequently,

$$e_q(\gamma) E_q(-\gamma) = e_q(-\gamma) E_q(\gamma) = 1, \quad |\gamma| < \frac{1}{1-q}. \quad (8)$$

The standard q -derivative operator $\widehat{D}_{q,\mu}$ acting on a function $f(\mu)$ is defined as follows; see [1,15,24]:

$$\widehat{D}_{q,\mu} f(\mu) = \begin{cases} \frac{f(\mu) - f(q\mu)}{(1-q)\mu}, & \mu \neq 0, \\ f'(0), & \mu = 0, \end{cases} \quad 0 < q < 1, \quad (9)$$

provided that f is differentiable at the origin. The second line is the continuous extension of the first, since

$$\lim_{\mu \rightarrow 0} \frac{f(\mu) - f(q\mu)}{(1-q)\mu} = f'(0).$$

For a formal power series $f(\mu) = \sum_{n \geq 0} a_n \mu^n$, the same convention is interpreted coefficient-wise; in particular, $(\widehat{D}_{q,\mu} f)(0) = a_1$.

The following standard q -derivative identities will be used:

$$\widehat{D}_{q,\mu} \mu^\theta = [\theta]_q \mu^{\theta-1}, \quad (10)$$

$$\widehat{D}_{q,\mu} e_q(\alpha \mu) = \alpha e_q(\alpha \mu), \quad \alpha \in \mathbb{C}, \quad (11)$$

and

$$\widehat{D}_{q,\mu}^k e_q(\alpha\mu) = \alpha^k e_q(\alpha\mu), \quad k \in \mathbb{N}, \quad \alpha \in \mathbb{C}. \quad (12)$$

For two functions $f(\mu)$ and $g(\mu)$, the q -Leibniz rule used in [11,25] is

$$\widehat{D}_{q,\mu}(f(\mu)g(\mu)) = f(\mu)\widehat{D}_{q,\mu}g(\mu) + g(q\mu)\widehat{D}_{q,\mu}f(\mu). \quad (13)$$

Cao et al. [25] introduced the two-variable q -Laguerre polynomials $L_{n,q}(\mu, \eta)$ through the generating function:

$$C_{0,q}(\mu\delta)e_q(\eta\delta) = \sum_{\theta=0}^{\infty} L_{\theta,q}(\mu, \eta) \frac{\delta^\theta}{[\theta]_q!}, \quad (14)$$

where $C_{0,q}$ is the zeroth-order q -Bessel–Tricomi function [24], defined by

$$C_{0,q}(\mu\delta) = \sum_{k=0}^{\infty} \frac{(-1)^k \mu^k \delta^k}{([k]_q!)^2}. \quad (15)$$

The operational identity for the q -Laguerre polynomials $L_{\theta,q}(\mu, \eta)$ is given by [25]:

$$e_q(-\delta\widehat{D}_{q,\mu}^{-1})e_q(\eta\delta)\{1\} = \sum_{\theta=0}^{\infty} L_{\theta,q}(\mu, \eta) \frac{\delta^\theta}{[\theta]_q!}. \quad (16)$$

For $m \in \mathbb{N}$, if the normalized right inverse satisfies $\widehat{D}_{q,\mu}^{-1}\{1\} = \mu$, then

$$(\widehat{D}_{q,\mu}^{-1})^m\{1\} = \frac{\mu^m}{[m]_q!}. \quad (17)$$

The zeroth-order Tricomi kernel satisfies the following q -differential identities, obtained by differentiating the series term by term:

$$\widehat{D}_{q,\mu}(\mu\widehat{D}_{q,\mu}C_{0,q}(\mu\delta)) = -\delta C_{0,q}(\mu\delta), \quad (18)$$

and, more generally,

$$\widehat{D}_{q,\mu}(\mu\widehat{D}_{q,\mu}C_{0,q}(\alpha\mu\delta)) = -\alpha\delta C_{0,q}(\alpha\mu\delta). \quad (19)$$

The q -dilatation operator T_μ acts on a function $f(\mu)$ as follows; see [25]:

$$T_\mu^k f(\mu) = f(q^k \mu), \quad k \in \mathbb{R}, \quad 0 < q < 1, \quad (20)$$

where

$$T_\mu^{-1}T_\mu^1 f(\mu) = f(\mu). \quad (21)$$

The q -derivative of $e_q(\eta\delta^m)$ is given by [25]:

$$\widehat{D}_{q,\delta}e_q(\eta\delta^m) = \eta\delta^{m-1}T_{(\eta;m)}e_q(\eta\delta^m). \quad (22)$$

Here

$$T_{(\eta;m)} = \frac{1 - q^m T_\eta^m}{1 - q T_\eta} = 1 + q T_\eta + \cdots + q^{m-1} T_\eta^{m-1}.$$

Al-Salam introduced the q -Appell polynomials $\mathcal{A}_{n,q}(x)$ through the generating relation [26,27]:

$$\mathcal{A}_q(\delta)e_q(\mu\delta) = \sum_{\theta=0}^{\infty} \mathcal{A}_{\theta,q}(\mu) \frac{\delta^\theta}{[\theta]_q!}. \quad (23)$$

Here

$$\mathcal{A}_q(\delta) = \sum_{\theta=0}^{\infty} \mathcal{A}_{\theta,q} \frac{\delta^\theta}{[\theta]_q!}, \quad \mathcal{A}_{0,q} \neq 0. \quad (24)$$

Equation (23) is equivalent to the defining q -Appell lowering relation. Indeed, applying $\widehat{D}_{q,\mu}$ coefficientwise to both sides of (23), and using (11) with $\alpha = \delta$, gives

$$\widehat{D}_{q,\mu}[\mathcal{A}_q(\delta)e_q(\mu\delta)] = \delta \mathcal{A}_q(\delta)e_q(\mu\delta).$$

Comparison of coefficients of $\delta^\theta / [\theta]_q!$ yields the explicit q -difference relation

$$\widehat{D}_{q,\mu}\mathcal{A}_{\theta,q}(\mu) = [\theta]_q\mathcal{A}_{\theta-1,q}(\mu), \quad \theta \geq 1,$$

with $\widehat{D}_{q,\mu}\mathcal{A}_{0,q}(\mu) = 0$. Thus the generating relation (23) and this lowering property describe the same q -Appell sequence. The normalization $\mathcal{A}_{0,q} = 1$ is imposed only when explicitly stated; the inverse and transfer results require only the nonsingularity condition $\mathcal{A}_{0,q} \neq 0$.

For a function of a variable z , we distinguish the standard q -derivative from the Hahn difference operator

$$\widehat{D}_{q,w;z}f(z) := \frac{f(z) - f(qz + w)}{(1-q)z - w}, \quad z \neq \frac{w}{1-q}, \quad (25)$$

with the value at the fixed point understood by the corresponding limiting derivative whenever an analytic interpretation is used. Algebraically, (25) is used through its action on the Hahn factorial bases below. The two Hahn-type q -exponential functions are used in the formal sense of Hahn-Appell calculus; see the Hahn-difference framework in [22,28]. We write

$$e_q^{wt}(zt) := \sum_{n=0}^{\infty} (z)_{q,w}^n \frac{t^n}{[n]_q!}, \quad (26)$$

and

$$E_q^{wt}(zt) := \sum_{n=0}^{\infty} [z]_{q,w}^n \frac{t^n}{[n]_q!}. \quad (27)$$

The closed product forms of these kernels depend on the precise normalization of the Hahn factorials; the present paper uses only the coefficientwise series forms and the formal lowering and Hahn-addition rules stated in (52)–(53). With this convention the inverse relation is written as

$$E_q^{-w}(t)e_q^w(t) = 1, \quad (28)$$

and the limiting cases are

$$e_q^0(t) = E_q^0(t) = e_q(t), \quad e_1^0(t) = E_1^0(t) = e^t,$$

$$e_1^w(t) = (1+w)^{\frac{t}{w}}, \quad E_1^w(t) = (1-w)^{-\frac{t}{w}}.$$

The polynomial bases of Hahn calculus are $\{(t)_{q,w}^n\}_{n \geq 0}$ and $\{[t]_{q,w}^n\}_{n \geq 0}$, given by

$$(t)_{q,w}^n = t(t-w)(t-[2]_qw)(t-[3]_qw) \cdots (t-[n-1]_qw), \quad (29)$$

and

$$[t]_{q,w}^n = t(qt+w)(q^2t+[2]_qw) \cdots (q^{n-1}t+[n-1]_qw). \quad (30)$$

Applying the Hahn difference operator to (29) and (30), we obtain

$$\widehat{D}_{q,w;t}(t)_{q,w}^n = [n]_q(t)_{q,w}^{n-1},$$

$$\widehat{\mathcal{D}}_{q,w;t}[t]_{q,w}^n = [n]_q [qt + w]_{q,w}^{n-1},$$

where $w \in \mathbb{C}$, $q \in (0, 1)$, and $n \geq 1$.

For fixed $0 < q < 1$ and $w \in \mathbb{C}$, the normalized inverse Hahn operator used below is the linear map $\widehat{\mathcal{J}}_{q,w;x} : \mathbb{C}[x] \rightarrow \mathbb{C}[x]$ defined on the falling Hahn basis by

$$\widehat{\mathcal{J}}_{q,w;x}(x)_{q,w}^n := \frac{(x)_{q,w}^{n+1}}{[n+1]_q}, \quad n \geq 0. \quad (31)$$

Hence $\widehat{\mathcal{D}}_{q,w;x}\widehat{\mathcal{J}}_{q,w;x} = I$ on $\mathbb{C}[x]$, and $\widehat{\mathcal{J}}_{q,w;x}p$ is the unique polynomial g satisfying $\widehat{\mathcal{D}}_{q,w;x}g = p$ and $g(0) = 0$. Existence follows from (31); uniqueness follows because the kernel of $\widehat{\mathcal{D}}_{q,w;x}$ on polynomials consists of constants. The definition extends coefficientwise to formal power series. In particular,

$$(\widehat{\mathcal{J}}_{q,w;x})^m \{1\} = \frac{(x)_{q,w}^m}{[m]_q!}, \quad m \geq 0, \quad (32)$$

and therefore

$$C_{0,q}^{wt}(xt) = e_q(-t\widehat{\mathcal{J}}_{q,w;x})\{1\}. \quad (33)$$

Any unnormalized right inverse differs from $\widehat{\mathcal{J}}_{q,w;x}$ by an arbitrary constant; the condition $g(0) = 0$ removes precisely this ambiguity.

The associated one-variable Hahn–Appell kernel, in the normalization of [28], is defined by

$$\mathcal{A}_q(t)e_q^{wt}(xt) = \sum_{n=0}^{\infty} \mathcal{P}_n(x;w,q) \frac{t^n}{[n]_q!}. \quad (34)$$

The monomiality principle is a useful technique for studying special polynomial sequences through their lowering and raising operators. It provides a convenient way to construct new q -special polynomial families and to identify the quasi-monomial structure of existing q -extensions. For further information, one may consult [11,29–31]. In the q -setting, this principle gives operator forms of q -difference and q -partial difference equations whenever the relevant raising and lowering operators have concrete realizations. Cao and colleagues [25] recently used this viewpoint in quantum calculus. Let $\{s_{\theta,q}(\mu)\}_{\theta \geq 0}$ be a polynomial basis with $s_{0,q}(\mu) = 1$. We use $\widehat{M}_q$ and $\widehat{P}_q$ as the algebraic raising and lowering operators of the quasi-monomiality framework [25], defined on this basis by

$$\widehat{M}_q \{s_{\theta,q}(\mu)\} = s_{\theta+1,q}(\mu), \quad (35)$$

and

$$\widehat{P}_q \{s_{\theta,q}(\mu)\} = [\theta]_q s_{\theta-1,q}(\mu), \quad \theta \geq 1, \quad (36)$$

with $\widehat{P}_q \{s_{0,q}(\mu)\} = 0$. In particular, $\widehat{M}_q$ is a basis-shift (raising) operator; it is not a multiplicative derivative in the sense of multiplicative calculus. Accordingly, the construction neither invokes a q -logarithm nor relies on an inverse relation between e_q and a q -logarithm. Any concrete differential or q -difference realization of $\widehat{M}_q$ or $\widehat{P}_q$ will be stated separately.

The definitions (35)–(36) imply the q -oscillator identity

$$(\widehat{P}_q \widehat{M}_q - q \widehat{M}_q \widehat{P}_q) \{s_{\theta,q}(\mu)\} = s_{\theta,q}(\mu), \quad (37)$$

since $[\theta+1]_q - q[\theta]_q = 1$. They also give the number-operator identities

$$\widehat{M}_q \widehat{P}_q \{s_{\theta,q}(\mu)\} = [\theta]_q s_{\theta,q}(\mu), \quad (38)$$

and

$$\widehat{P}_q \widehat{M}_q \{s_{\theta,q}(\mu)\} = [\theta + 1]_q s_{\theta,q}(\mu). \quad (39)$$

Consequently, Equation (40) is the ordinary commutator of the lowering and raising operators, not a multiplicative derivative:

$$(\widehat{P}_q \widehat{M}_q - \widehat{M}_q \widehat{P}_q) \{s_{\theta,q}(\mu)\} = ([\theta + 1]_q - [\theta]_q) s_{\theta,q}(\mu) = q^\theta s_{\theta,q}(\mu). \quad (40)$$

Iterating (35) gives

$$\widehat{M}_q^r \{s_{\theta,q}\} = s_{\theta+r,q}(\mu). \quad (41)$$

In particular,

$$s_{\theta,q}(\mu) = \widehat{M}_q^\theta \{s_{0,q}\} = \widehat{M}_q^\theta \{1\}, \quad (42)$$

where $s_{0,q}(\mu) = 1$. The resulting polynomial sequence therefore has the generating representation

$$e_q(\widehat{M}_q \delta) \{1\} = \sum_{\theta=0}^{\infty} s_{\theta,q}(\mu) \frac{\delta^\theta}{[\theta]_q!}. \quad (43)$$

In this paper, we define a class of Hahn-factorial q -Laguerre–Tricomi–Appell polynomials by multiplying an Appell series by the generating function of an underlying Hahn-factorial q -Laguerre–Tricomi sequence. The resulting family is connected to the base sequence by a q -binomial convolution, which transfers structural properties from the original family to its Appell deformation. Specifically, we derive direct and inverse connection formulas, summation identities, operational representations, a Rodrigues-type formula, induced lowering and raising relations, and a quasi-monomial framework. We then examine classical reductions, a finite coefficient scheme, zero-location results, and the joint limit $q \rightarrow 1^-$, $w \rightarrow 0$.

2. Hahn-Factorial q -Laguerre–Tricomi–Appell Polynomials

In this section, we introduce the Hahn-factorial q -Laguerre–Tricomi base polynomials $\mathcal{L}_n(x, y; w, q)$ and the associated two-variable Hahn-factorial q -Laguerre–Tricomi–Appell polynomials $\mathcal{L}\mathcal{A}_n(x, y; w, q)$. The terminology emphasizes that the construction is a Hahn-factorial Appell deformation of a Laguerre–Tricomi kernel; it is not a Laguerre–Hahn orthogonality or moment-functional characterization.

Let $x, y, w \in \mathbb{C}$ and $0 < q < 1$. We define the two-variable Hahn-factorial q -Laguerre–Tricomi polynomials $\mathcal{L}_n(x, y; w, q)$ by

$$e_q^{wt}(yt) C_{0,q}^{wt}(xt) = e_q^{wt}(yt) e_q(-t \widehat{\mathcal{J}}_{q,w;x}) \{1\} = \sum_{n=0}^{\infty} \mathcal{L}_n(x, y; w, q) \frac{t^n}{[n]_q!}, \quad (44)$$

where $C_{0,q}^{wt}(xt)$ denotes the zeroth-order Hahn–Tricomi function

$$C_{0,q}^{wt}(xt) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)_{q,w}^n t^n}{([n]_q!)^2}. \quad (45)$$

We note that

$$\mathcal{L}_n(x, y; w, q) = [n]_q! \sum_{k=0}^n \frac{(-1)^{n-k} (x)_{q,w}^{n-k} (y)_{q,w}^k}{[k]_q! ([n-k]_q!)^2}. \quad (46)$$

The two-variable Hahn-factorial q -Laguerre–Tricomi polynomials $\mathcal{L}_n(x, y; w, q)$ are quasi-monomial at the algebraic level. Indeed, (46) shows that

$$\mathcal{L}_n(x, y; w, q) = (y)_{q,w}^n + \text{terms of degree at most } n-1 \text{ in } y.$$

Since $(y)_{q,w}^n$ is monic of degree n , the sequence $\{\mathcal{L}_n\}_{n \geq 0}$ is a basis of $\mathbb{C}(x, w, q)[y]$. Its lowering operator is

$$\hat{P}_{qLHP} = \hat{\mathcal{D}}_{q,w;y}, \quad \hat{P}_{qLHP} \mathcal{L}_n(x, y; w, q) = [n]_q \mathcal{L}_{n-1}(x, y; w, q), \quad (47)$$

and there exists a unique linear raising operator on this basis, defined by

$$\hat{M}_{qLHP} \mathcal{L}_n(x, y; w, q) = \mathcal{L}_{n+1}(x, y; w, q), \quad n \geq 0. \quad (48)$$

Consequently,

$$\left(\hat{M}_{qLHP} \hat{\mathcal{D}}_{q,w;y} - [n]_q \right) \mathcal{L}_n(x, y; w, q) = 0. \quad (49)$$

Thus existence and uniqueness of the abstract raising operator are proved by triangularity. What is not asserted universally is a closed realization of $\hat{M}_{qLHP}$ in terms of x, y, T_x, T_y , and powers of $\hat{\mathcal{D}}_{q,w;y}$; such a realization requires additional commutation data for a concrete operator model.

Motivated by Equations (34) and (44), we define the two-variable Hahn-factorial q -Laguerre–Tricomi–Appell polynomials, denoted by ${}_{\mathcal{L}}\mathcal{A}_n(x, y; w, q)$. We interpret all generating functions coefficientwise in the formal power-series ring $\mathbb{C}[[t]]$. Whenever an analytic interpretation is required, we additionally assume that the relevant kernels converge in a common neighborhood of the origin; reciprocal-series arguments are then restricted to the nonsingular case $\mathcal{A}_{0,q} \neq 0$. The defining generating function is

$$\mathcal{A}_q(t) e_q^{wt}(yt) C_{0,q}^{wt}(xt) = \sum_{n=0}^{\infty} {}_{\mathcal{L}}\mathcal{A}_n(x, y; w, q) \frac{t^n}{[n]_q!}, \quad (50)$$

or equivalently, by (33),

$$\mathcal{A}_q(t) e_q^{wt}(yt) e_q(-t \hat{\mathcal{J}}_{q,w;x}) \{1\} = \sum_{n=0}^{\infty} {}_{\mathcal{L}}\mathcal{A}_n(x, y; w, q) \frac{t^n}{[n]_q!}. \quad (51)$$

Here $\hat{\mathcal{J}}_{q,w;x}$ is the normalized right inverse defined in (31); no two-sided inverse of $\hat{\mathcal{D}}_{q,w;x}$ on all polynomials is claimed.

In the operational formulas below, we work at the level of formal power series and use the Hahn identities

$$\hat{\mathcal{D}}_{q,w;\xi} e_q^{wt}(\xi t) = t e_q^{wt}(\xi t), \quad \hat{\mathcal{D}}_{q,w;\xi}^r e_q^{wt}(\xi t) = t^r e_q^{wt}(\xi t), \quad r \in \mathbb{N}_0, \quad (52)$$

and the multiplicative law with respect to the formal Hahn sum $\oplus_{q,w}$

$$e_q^{wt}((\xi \oplus_{q,w} \eta)t) = e_q^{wt}(\xi t) e_q^{wt}(\eta t). \quad (53)$$

We also write $\ominus_{q,w} \delta$ for the formal Hahn inverse of a Hahn increment δ , characterized by

$$e_q^{wt}((\ominus_{q,w} \delta)t) e_q^{wt}(\delta t) = 1.$$

These relations are used as defining formal rules for the Hahn-type translation kernel throughout the operational part of the paper. Consequently, every identity involving (50), (51), and (53) is interpreted coefficientwise in the ring of formal power series. The normalized integration operator $\hat{\mathcal{J}}_{q,w;x}$ is distinct from the formal inverse increment $\ominus_{q,w} \delta$: the former is a rigorously defined right inverse of $\hat{\mathcal{D}}_{q,w;x}$ with the normalization $g(0) = 0$, whereas the latter belongs only to the formal Hahn-addition law.

Equivalently, applying the normalized Hahn integration exponential to the one-variable Appell kernel gives

$$e_q(-t\widehat{\mathcal{J}}_{q,w;x})\left\{\mathcal{A}_q(t)e_q^{wt}(yt)\right\}=\sum_{n=0}^{\infty}\mathcal{L}\mathcal{A}_n(x,y;w,q)\frac{t^n}{[n]_q!}. \quad (54)$$

Remark 1. As $w \rightarrow 0^+$, Equation (50) reduces formally to the two-variable q -Laguerre–Appell family [11,25]

$$\mathcal{A}_q(t)e_q(yt)C_{0,q}(xt)=\sum_{n=0}^{\infty}\mathcal{L}\mathcal{A}_{n,q}(x,y)\frac{t^n}{[n]_q!}. \quad (55)$$

Remark 2. If, in addition, $q \rightarrow 1^-$, then (50) tends formally to the classical two-variable Laguerre–Appell family $\mathcal{L}\mathcal{A}_n(x,y)$ (see [12,31]).

Remark 3 (Relation with genuine H_q -Laguerre–Hahn orthogonal polynomials). The terminology in the present paper does not identify the sequence (50) with the genuine H_q -Laguerre–Hahn orthogonal polynomials studied in [18–21]. In that theory, the defining data are a regular orthogonality functional and an H_q -Laguerre–Hahn or semiclassical functional equation, from which orthogonality and recurrence properties follow. Here, the defining data are instead the product of a Hahn-factorial Laguerre–Tricomi kernel and an arbitrary Appell series. We do not establish a regular moment functional, a Favard three-term recurrence, positivity, or an H_q -Laguerre–Hahn class equation. Therefore the relation is structural and terminological only. An intersection with the genuine orthogonal theory could occur for a special choice of $\mathcal{A}_q$, x , w , and parameters, but only after those additional orthogonality and functional-equation conditions are proved separately.

Theorem 1. For every $n \geq 0$, the two-variable Hahn-factorial q -Laguerre–Tricomi–Appell polynomials are connected to the base Hahn-factorial q -Laguerre–Tricomi family by the finite q -binomial transform

$$\mathcal{L}\mathcal{A}_n(x,y;w,q)=\sum_{r=0}^n\binom{n}{r}_q\mathcal{A}_{r,q}\mathcal{L}_{n-r}(x,y;w,q). \quad (56)$$

Proof. Write the two factors in exponential q -factorial form:

$$\mathcal{A}_q(t)=\sum_{r=0}^{\infty}\mathcal{A}_{r,q}\frac{t^r}{[r]_q!}, \quad e_q^{wt}(yt)C_{0,q}^{wt}(xt)=\sum_{s=0}^{\infty}\mathcal{L}_s(x,y;w,q)\frac{t^s}{[s]_q!}.$$

Their product is

$$\begin{aligned} \mathcal{A}_q(t)e_q^{wt}(yt)C_{0,q}^{wt}(xt) &= \left(\sum_{r=0}^{\infty}\mathcal{A}_{r,q}\frac{t^r}{[r]_q!}\right)\left(\sum_{s=0}^{\infty}\mathcal{L}_s(x,y;w,q)\frac{t^s}{[s]_q!}\right) \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^n\frac{\mathcal{A}_{r,q}\mathcal{L}_{n-r}(x,y;w,q)}{[r]_q![n-r]_q!}t^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{r=0}^n\frac{[n]_q!}{[r]_q![n-r]_q!}\mathcal{A}_{r,q}\mathcal{L}_{n-r}(x,y;w,q)\right)\frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{r=0}^n\binom{n}{r}_q\mathcal{A}_{r,q}\mathcal{L}_{n-r}(x,y;w,q)\right)\frac{t^n}{[n]_q!}. \end{aligned}$$

The second equality groups all pairs (r,s) with $r+s=n$, and the third uses $\binom{n}{r}_q=[n]_q!/([r]_q![n-r]_q!)$. Comparison with the coefficient of $t^n/[n]_q!$ in the defining generating function (50) yields (56). $\square$

Assume now that $\mathcal{A}_{0,q} \neq 0$. For the inverse formula we introduce the reciprocal Appell factor

$$\mathcal{B}_q(t) := \frac{1}{\mathcal{A}_q(t)} = \sum_{r=0}^{\infty} \beta_{r,q} \frac{t^r}{[r]_q!}. \quad (57)$$

The coefficients $\beta_{r,q}$ are determined recursively by

$$\sum_{r=0}^n \binom{n}{r}_q \mathcal{A}_{r,q} \beta_{n-r,q} = \delta_{n,0}, \quad (58)$$

where $\delta_{n,0}$ is the Kronecker delta. In particular,

$$\beta_{0,q} = \mathcal{A}_{0,q}^{-1}, \quad \beta_{n,q} = -\mathcal{A}_{0,q}^{-1} \sum_{r=1}^n \binom{n}{r}_q \mathcal{A}_{r,q} \beta_{n-r,q}, \quad n \geq 1. \quad (59)$$

Theorem 2. Assume $\mathcal{A}_{0,q} \neq 0$. For every $n \geq 0$,

$$\mathcal{L}_n(x, y; w, q) = \sum_{r=0}^n \binom{n}{r}_q \beta_{r,q} \mathcal{L}_{n-r}(x, y; w, q). \quad (60)$$

Consequently,

$$\mathcal{L} \mathcal{A}_0(x, y; w, q) = \beta_{0,q}^{-1} \mathcal{L}_0(x, y; w, q), \quad (61)$$

and for $n \geq 1$,

$$\mathcal{L} \mathcal{A}_n(x, y; w, q) = \beta_{0,q}^{-1} \left[\mathcal{L}_n(x, y; w, q) - \sum_{r=1}^n \binom{n}{r}_q \beta_{r,q} \mathcal{L}_{n-r}(x, y; w, q) \right]. \quad (62)$$

Proof. Multiplying (50) by $\mathcal{B}_q(t)$ gives

$$\mathcal{B}_q(t) \sum_{n \geq 0} \mathcal{L} \mathcal{A}_n(x, y; w, q) \frac{t^n}{[n]_q!} = e_q^{wt}(yt) C_{0,q}^{wt}(xt) = \sum_{n \geq 0} \mathcal{L}_n(x, y; w, q) \frac{t^n}{[n]_q!}.$$

Comparing coefficients gives (60). Isolating the term $r = 0$ in (60) gives (61) and (62). The recursion (59) follows in the same way from $\mathcal{A}_q(t) \mathcal{B}_q(t) = 1$. $\square$

The next formula is a q -Hessenberg determinant version of the inverse triangular system (60). Determinant representations of this type are common in Appell and Sheffer theory [31–33]; here the ordinary binomial coefficients are replaced by q -binomial coefficients.

Theorem 3. Assume $\beta_{0,q} \neq 0$. For $n \geq 1$,

$$\mathcal{L} \mathcal{A}_n(x, y; w, q) = \frac{1}{\beta_{0,q}^{n+1}} \begin{vmatrix} \beta_{0,q} & 0 & 0 & \cdots & 0 & \mathcal{L}_0(x, y; w, q) \\ \binom{1}{1}_q \beta_{1,q} & \beta_{0,q} & 0 & \cdots & 0 & \mathcal{L}_1(x, y; w, q) \\ \binom{2}{2}_q \beta_{2,q} & \binom{2}{1}_q \beta_{1,q} & \beta_{0,q} & \cdots & 0 & \mathcal{L}_2(x, y; w, q) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{n-1}{n-1}_q \beta_{n-1,q} & \binom{n-1}{n-2}_q \beta_{n-2,q} & \binom{n-1}{n-3}_q \beta_{n-3,q} & \cdots & \beta_{0,q} & \mathcal{L}_{n-1}(x, y; w, q) \\ \binom{n}{n}_q \beta_{n,q} & \binom{n}{n-1}_q \beta_{n-1,q} & \binom{n}{n-2}_q \beta_{n-2,q} & \cdots & \binom{n}{1}_q \beta_{1,q} & \mathcal{L}_n(x, y; w, q) \end{vmatrix}. \quad (63)$$

For $n = 0$, the same convention gives $\mathcal{L} \mathcal{A}_0 = \beta_{0,q}^{-1} \mathcal{L}_0$.

Proof. For $0 \leq i \leq n$, formula (60) may be written as the triangular linear system

$$\mathcal{L}_i(x, y; w, q) = \sum_{j=0}^i \binom{i}{i-j}_q \beta_{i-j,q} \mathcal{L}_j(x, y; w, q). \quad (64)$$

The coefficient matrix of the unknown vector $(\mathcal{L}\mathcal{A}_0, \mathcal{L}\mathcal{A}_1, \dots, \mathcal{L}\mathcal{A}_n)^T$ is lower triangular and has diagonal entries $\beta_{0,q}$. Its determinant is therefore $\beta_{0,q}^{n+1}$. Applying Cramer's rule to the last unknown $\mathcal{L}\mathcal{A}_n$ replaces the last column of this lower triangular coefficient matrix by the right-hand side vector $(\mathcal{L}_0, \dots, \mathcal{L}_n)^T$. The resulting determinant is exactly the determinant displayed in (63). Dividing by $\beta_{0,q}^{n+1}$ proves the assertion. $\square$

Theorem 4. Assume $\mathcal{A}_{0,q} \neq 0$. Then the sequence $\mathcal{L}\mathcal{A}_n(x, y; w, q)$ satisfies the triangular recurrence (62). Independently of this nonsingularity assumption, its Hahn-difference lowering relation with respect to the Appell variable y is

$$\widehat{\mathcal{D}}_{q,w;y} \mathcal{L}\mathcal{A}_n(x, y; w, q) = [n]_q \mathcal{L}\mathcal{A}_{n-1}(x, y; w, q), \quad n \geq 1, \quad (65)$$

and $\widehat{\mathcal{D}}_{q,w;y} \mathcal{L}\mathcal{A}_0(x, y; w, q) = 0$. If the raising operator is defined on this basis by

$$\widehat{M}_{q,LHA} \mathcal{L}\mathcal{A}_n(x, y; w, q) = \mathcal{L}\mathcal{A}_{n+1}(x, y; w, q), \quad (66)$$

then the formal monomiality equation is

$$(\widehat{M}_{q,LHA} \widehat{\mathcal{D}}_{q,w;y} - [n]_q) \mathcal{L}\mathcal{A}_n(x, y; w, q) = 0. \quad (67)$$

Proof. The triangular recurrence is the isolated form (62) of the inverse connection formula. For (65), apply $\widehat{\mathcal{D}}_{q,w;y}$ to both sides of (50). Since $\mathcal{A}_q(t)$ and $C_{0,q}^{wt}(xt)$ do not depend on y , and since (52) gives $\widehat{\mathcal{D}}_{q,w;y} e_q^{wt}(yt) = t e_q^{wt}(yt)$, we obtain

$$\sum_{n \geq 0} \widehat{\mathcal{D}}_{q,w;y} \mathcal{L}\mathcal{A}_n(x, y; w, q) \frac{t^n}{[n]_q!} = t \sum_{n \geq 0} \mathcal{L}\mathcal{A}_n(x, y; w, q) \frac{t^n}{[n]_q!}.$$

The right-hand side equals

$$\sum_{n \geq 1} [n]_q \mathcal{L}\mathcal{A}_{n-1}(x, y; w, q) \frac{t^n}{[n]_q!}.$$

Coefficient comparison proves (65). Finally, (67) follows immediately from (65) and the defining raising relation (66), as in the monomiality principle for q -Appell and Sheffer-type polynomial sets [27,32,34]. $\square$

Corollary 1. For $r, n \in \mathbb{N}_0$,

$$\widehat{\mathcal{D}}_{q,w;y}^r \mathcal{L}\mathcal{A}_n(x, y; w, q) = \begin{cases} \frac{[n]_q!}{[n-r]_q!} \mathcal{L}\mathcal{A}_{n-r}(x, y; w, q), & 0 \leq r \leq n, \\ 0, & r > n. \end{cases} \quad (68)$$

Proof. Apply (65) repeatedly. Equivalently, apply $\widehat{\mathcal{D}}_{q,w;y}^r$ directly to the generating function (50); by (52), this multiplies the generating function by t^r . Thus

$$\sum_{n \geq 0} \widehat{\mathcal{D}}_{q,w;y}^r \mathcal{L}\mathcal{A}_n(x, y; w, q) \frac{t^n}{[n]_q!} = t^r \sum_{n \geq 0} \mathcal{L}\mathcal{A}_n(x, y; w, q) \frac{t^n}{[n]_q!},$$

and comparison of the coefficient of t^n yields (68). $\square$

The simple Hahn translation acts on the Hahn–Appell coordinate. In the present bivariate notation this coordinate is y ; if one uses the conventional one-variable Appell notation x in (34), the same identity is the Hahn x -shift formula.

Theorem 5. For every $n \geq 0$ and every formal Hahn increment ξ ,

$$\mathcal{L}\mathcal{A}_n(x, y \oplus_{q,w} \xi; w, q) = \sum_{k=0}^n \binom{n}{k}_q \mathcal{L}\mathcal{A}_k(x, y; w, q) (\xi)_{q,w}^{n-k}. \quad (69)$$

In particular, the associated one-variable Hahn–Appell polynomials (34) satisfy the Hahn x -shift identity

$$\mathcal{P}_n(x \oplus_{q,w} \xi; w, q) = \sum_{k=0}^n \binom{n}{k}_q \mathcal{P}_k(x; w, q) (\xi)_{q,w}^{n-k}. \quad (70)$$

Proof. By (50) and the Hahn addition rule (53),

$$\begin{aligned} \sum_{n \geq 0} \mathcal{L}\mathcal{A}_n(x, y \oplus_{q,w} \xi; w, q) \frac{t^n}{[n]_q!} &= \mathcal{A}_q(t) C_{0,q}^{wt}(xt) e_q^{wt}((y \oplus_{q,w} \xi)t) \\ &= \left(\sum_{k \geq 0} \mathcal{L}\mathcal{A}_k(x, y; w, q) \frac{t^k}{[k]_q!} \right) \left(\sum_{j \geq 0} (\xi)_{q,w}^j \frac{t^j}{[j]_q!} \right). \end{aligned}$$

The Cauchy product gives (69). Formula (70) is proved in the same way from (34); this is the standard Hahn–Appell translation argument of [28]. $\square$

Corollary 2. For every $n \geq 0$,

$$\sum_{k=0}^n \binom{n}{k}_q (\xi)_{q,w}^{n-k} \mathcal{L}\mathcal{A}_k(x, y; w, q) = \mathcal{L}\mathcal{A}_n(x, y \oplus_{q,w} \xi; w, q). \quad (71)$$

Moreover, combining the shift formula with the direct connection formula gives

$$\mathcal{L}\mathcal{A}_n(x, y \oplus_{q,w} \xi; w, q) = \sum_{r=0}^n \binom{n}{r}_q \mathcal{A}_{r,q} \sum_{j=0}^{n-r} \binom{n-r}{j}_q \mathcal{L}_j(x, y; w, q) (\xi)_{q,w}^{n-r-j}. \quad (72)$$

Finally, using the inverse coefficients $\beta_{r,q}$,

$$\sum_{r=0}^n \binom{n}{r}_q \beta_{r,q} \mathcal{L}\mathcal{A}_{n-r}(x, y \oplus_{q,w} \xi; w, q) = \mathcal{L}_n(x, y \oplus_{q,w} \xi; w, q). \quad (73)$$

Proof. Equation (71) is just (69) rewritten as a finite summation formula. To prove (72), substitute the direct connection formula (56) into the left-hand side of (71) and use the elementary q -binomial identity

$$\binom{n}{k}_q \binom{k}{r}_q = \binom{n}{r}_q \binom{n-r}{k-r}_q, \quad 0 \leq r \leq k \leq n.$$

After changing the index $j = k - r$, the result is (72). Equation (73) follows by applying the inverse connection formula (60) to the shifted coordinate $y \oplus_{q,w} \xi$. $\square$

3. Operational Identities and Quasi-Monomiality Principle

This section records the operator form of the construction introduced in Section 2. The guiding point is that the Appell factor in (50) acts as an invertible transfer operator on

the underlying Hahn-factorial q -Laguerre–Tricomi base sequence. This is the q -analogue of the operational treatment of Appell and Sheffer sequences developed in the umbral and monomiality literature [31–33,35], with the q -Appell part following Al-Salam [27] and Ihrig–Ismail [34]. The Hahn translation conventions are those of Varma, Yaşar and Özarslan [28]; the Laguerre-type quasi-monomial viewpoint is consistent with the recent q -Laguerre framework of Cao, Raza and Fadel [25].

Let

$$\mathcal{P} := \widehat{\mathcal{D}}_{q,w;y} \quad (74)$$

act on the polynomial variable y . Since $\mathcal{P}$ lowers the degree in y , every formal power series in $\mathcal{P}$ acts locally as a finite operator on each polynomial. We therefore define the Appell transfer operator and its inverse by

$$\mathfrak{A} := \mathcal{A}_q(\mathcal{P}) = \sum_{r=0}^{\infty} \mathcal{A}_{r,q} \frac{\mathcal{P}^r}{[r]_q!}, \quad \mathfrak{B} := \mathcal{B}_q(\mathcal{P}) = \sum_{r=0}^{\infty} \beta_{r,q} \frac{\mathcal{P}^r}{[r]_q!}, \quad (75)$$

where $\mathcal{B}_q(t) = 1/\mathcal{A}_q(t)$ is the reciprocal Appell factor from (57); this requires $\mathcal{A}_{0,q} \neq 0$. Thus

$$\mathfrak{A}\mathfrak{B} = \mathfrak{B}\mathfrak{A} = I \quad (76)$$

on the polynomial space under consideration.

Theorem 6. Assume $\mathcal{A}_{0,q} \neq 0$. For every $n \geq 0$, the Hahn-factorial q -Laguerre–Tricomi–Appell polynomials satisfy

$$\mathcal{L}\mathcal{A}_n(x, y; w, q) = \mathfrak{A}\mathcal{L}_n(x, y; w, q), \quad (77)$$

and the inverse identity

$$\mathcal{L}_n(x, y; w, q) = \mathfrak{B}\mathcal{L}\mathcal{A}_n(x, y; w, q). \quad (78)$$

Equivalently,

$$\mathcal{L}\mathcal{A}_n(x, y; w, q) = \sum_{r=0}^n \binom{n}{r}_q \mathcal{A}_{r,q} \mathcal{L}_{n-r}(x, y; w, q), \quad (79)$$

and

$$\mathcal{L}_n(x, y; w, q) = \sum_{r=0}^n \binom{n}{r}_q \beta_{r,q} \mathcal{L}\mathcal{A}_{n-r}(x, y; w, q). \quad (80)$$

Proof. By (52),

$$\mathcal{P}^r \left\{ e_q^{wt}(yt) C_{0,q}^{wt}(xt) \right\} = t^r e_q^{wt}(yt) C_{0,q}^{wt}(xt), \quad r \geq 0.$$

Applying $\mathfrak{A}$ to the generating function (44) therefore gives

$$\mathfrak{A} \left\{ \sum_{n=0}^{\infty} \mathcal{L}_n(x, y; w, q) \frac{t^n}{[n]_q!} \right\} = \left(\sum_{r=0}^{\infty} \mathcal{A}_{r,q} \frac{t^r}{[r]_q!} \right) e_q^{wt}(yt) C_{0,q}^{wt}(xt).$$

The right-hand side is exactly the defining generating function (50). Comparing the coefficients of $t^n/[n]_q!$ proves (77). Expanding the operator $\mathfrak{A}$ in (77) and using

$$\mathcal{P}^r \mathcal{L}_n(x, y; w, q) = \begin{cases} \frac{[n]_q!}{[n-r]_q!} \mathcal{L}_{n-r}(x, y; w, q), & 0 \leq r \leq n, \\ 0, & r > n, \end{cases}$$

gives (79). The inverse statements follow by replacing $\mathcal{A}_q(t)$ with $\mathcal{B}_q(t) = 1/\mathcal{A}_q(t)$ and using (76); coefficient comparison gives (78) and (80). These are precisely the direct and inverse q -binomial transforms obtained in Theorems 1 and 2. $\square$

Corollary 3 (Operational differentiation rule). For $m, n \in \mathbb{N}_0$,

$$\mathcal{P}^m \mathcal{L} \mathcal{A}_n(x, y; w, q) = \begin{cases} \frac{[n]_q!}{[n-m]_q!} \mathcal{L} \mathcal{A}_{n-m}(x, y; w, q), & 0 \leq m \leq n, \\ 0, & m > n. \end{cases} \quad (81)$$

Proof. The operator $\mathfrak{A} = \mathcal{A}_q(\mathcal{P})$ is a formal series in $\mathcal{P}$, so it commutes with $\mathcal{P}^m$. Hence

$$\mathcal{P}^m \mathcal{L} \mathcal{A}_n = \mathcal{P}^m \mathfrak{A} \mathcal{L}_n = \mathfrak{A} \mathcal{P}^m \mathcal{L}_n.$$

Using the corresponding lowering rule for the base sequence and then applying (77) gives (81). Equivalently, one may apply $\mathcal{P}^m$ to the generating function (50), which multiplies it by t^m , and then compare coefficients. This recovers Corollary 1 in operator form. $\square$

The next identity expresses Hahn translation as an exponential of the lowering operator. For a formal Hahn increment ξ , set

$$\mathfrak{E}_\xi := e_q^{wt}(\xi \mathcal{P}) := \sum_{r=0}^{\infty} (\xi)_{q,w}^r \frac{\mathcal{P}^r}{[r]_q!}. \quad (82)$$

Theorem 7. For every $n \geq 0$,

$$\mathfrak{E}_\xi \mathcal{L} \mathcal{A}_n(x, y; w, q) = \mathcal{L} \mathcal{A}_n(x, y \oplus_{q,w} \xi; w, q). \quad (83)$$

Consequently,

$$\mathcal{L} \mathcal{A}_n(x, y \oplus_{q,w} \xi; w, q) = \sum_{r=0}^n \binom{n}{r}_q (\xi)_{q,w}^r \mathcal{L} \mathcal{A}_{n-r}(x, y; w, q). \quad (84)$$

Proof. Apply $\mathfrak{E}_\xi$ coefficientwise to (50). Since $\mathcal{P}^r$ multiplies the generating function by t^r , we obtain

$$\mathfrak{E}_\xi \left\{ \sum_{n=0}^{\infty} \mathcal{L} \mathcal{A}_n(x, y; w, q) \frac{t^n}{[n]_q!} \right\} = e_q^{wt}(\xi t) \mathcal{A}_q(t) e_q^{wt}(yt) C_{0,q}^{wt}(xt).$$

Using the Hahn addition rule (53), the last expression is

$$\mathcal{A}_q(t) e_q^{wt}((y \oplus_{q,w} \xi)t) C_{0,q}^{wt}(xt) = \sum_{n=0}^{\infty} \mathcal{L} \mathcal{A}_n(x, y \oplus_{q,w} \xi; w, q) \frac{t^n}{[n]_q!}.$$

Coefficient comparison proves (83). Expanding $\mathfrak{E}_\xi$ and using (81) gives the finite identity (84). Reindexing $k = n - r$ yields the equivalent form (69). $\square$

We now state the quasi-monomiality principle for the present family. Let

$$\mathcal{V}_{\mathcal{L}} := \text{span}\{\mathcal{L}_n(x, y; w, q) : n \geq 0\}, \quad \mathcal{V}_{\mathcal{L}\mathcal{A}} := \text{span}\{\mathcal{L} \mathcal{A}_n(x, y; w, q) : n \geq 0\}.$$

By the triangularity displayed after (46), $\{\mathcal{L}_n\}_{n \geq 0}$ is a basis, so the linear map

$$\mathcal{P} \mathcal{L}_n = [n]_q \mathcal{L}_{n-1}, \quad \mathcal{M} \mathcal{L}_n = \mathcal{L}_{n+1}, \quad (85)$$

with $\mathcal{L}_{-1} = 0$, exists and is unique on $\mathcal{V}_{\mathcal{L}}$. Thus the existence of the abstract raising operator is proved, not assumed. This basis-defined operator automatically satisfies $\mathcal{P}\mathcal{M}_{\mathcal{L}} - q\mathcal{M}_{\mathcal{L}}\mathcal{P} = I$ on $\mathcal{V}_{\mathcal{L}}$, because $[n+1]_q - q[n]_q = 1$. Only a concrete realization of $\mathcal{M}_{\mathcal{L}}$ as a finite or convergent combination of multiplication, shifts, and Hahn differences remains an additional assumption when such an explicit formula is desired.

Theorem 8. Assume $\mathcal{A}_{0,q} \neq 0$, so that $\mathfrak{B} = \mathfrak{A}^{-1}$ is defined on the polynomial space. Define

$$\mathcal{P}_{\mathcal{L}A} := \mathcal{P}, \quad \mathcal{M}_{\mathcal{L}A} := \mathfrak{A}\mathcal{M}_{\mathcal{L}}\mathfrak{B}. \quad (86)$$

Then the family $\{\mathcal{L}A_n(x, y; w, q)\}_{n \geq 0}$ is quasi-monomial and satisfies

$$\mathcal{P}_{\mathcal{L}A} \mathcal{L}A_n(x, y; w, q) = [n]_q \mathcal{L}A_{n-1}(x, y; w, q), \quad (87)$$

$$\mathcal{M}_{\mathcal{L}A} \mathcal{L}A_n(x, y; w, q) = \mathcal{L}A_{n+1}(x, y; w, q). \quad (88)$$

Consequently,

$$(\mathcal{M}_{\mathcal{L}A} \mathcal{P}_{\mathcal{L}A} - [n]_q I) \mathcal{L}A_n(x, y; w, q) = 0, \quad (89)$$

and

$$(\mathcal{P}_{\mathcal{L}A} \mathcal{M}_{\mathcal{L}A} - [n+1]_q I) \mathcal{L}A_n(x, y; w, q) = 0. \quad (90)$$

Proof. The nonsingularity condition $\mathcal{A}_{0,q} \neq 0$ makes $\mathcal{A}_q(z)$ a unit in the formal power-series ring. Since $\mathcal{P}$ is locally nilpotent on each polynomial, both $\mathfrak{A} = \mathcal{A}_q(\mathcal{P})$ and $\mathfrak{B} = \mathcal{B}_q(\mathcal{P})$ act by finite sums on every basis element, and $\mathfrak{A}\mathfrak{B} = \mathfrak{B}\mathfrak{A} = I$. Consequently, the conjugation $\mathfrak{A}\mathcal{M}_{\mathcal{L}}\mathfrak{B}$ is a well-defined linear operator on $\mathcal{V}_{\mathcal{L}A}$.

Because $\mathfrak{A}$ is a formal series in $\mathcal{P}$, it commutes with $\mathcal{P}$. Using (77) and (85),

$$\mathcal{P}_{\mathcal{L}A} \mathcal{L}A_n = \mathcal{P}\mathfrak{A}\mathcal{L}_n = \mathfrak{A}\mathcal{P}\mathcal{L}_n = [n]_q \mathfrak{A}\mathcal{L}_{n-1} = [n]_q \mathcal{L}A_{n-1},$$

which proves (87). Similarly, the inverse transfer identity gives

$$\mathcal{M}_{\mathcal{L}A} \mathcal{L}A_n = \mathfrak{A}\mathcal{M}_{\mathcal{L}}\mathfrak{B}\mathcal{L}A_n = \mathfrak{A}\mathcal{M}_{\mathcal{L}}\mathcal{L}_n = \mathfrak{A}\mathcal{L}_{n+1} = \mathcal{L}A_{n+1},$$

which is (88). The two number-operator equations follow by applying the lowering and raising relations in the indicated order. This establishes both the existence of the conjugated raising operator and the claimed quasi-monomiality transfer. $\square$

Corollary 4. On $\mathcal{V}_{\mathcal{L}A}$,

$$\mathcal{P}_{\mathcal{L}A} \mathcal{M}_{\mathcal{L}A} - q\mathcal{M}_{\mathcal{L}A} \mathcal{P}_{\mathcal{L}A} = I. \quad (91)$$

Equivalently, the ordinary commutator acts diagonally as

$$(\mathcal{P}_{\mathcal{L}A} \mathcal{M}_{\mathcal{L}A} - \mathcal{M}_{\mathcal{L}A} \mathcal{P}_{\mathcal{L}A}) \mathcal{L}A_n = q^n \mathcal{L}A_n. \quad (92)$$

Proof. Using (87) and (88),

$$\mathcal{P}_{\mathcal{L}A} \mathcal{M}_{\mathcal{L}A} \mathcal{L}A_n = [n+1]_q \mathcal{L}A_n, \quad \mathcal{M}_{\mathcal{L}A} \mathcal{P}_{\mathcal{L}A} \mathcal{L}A_n = [n]_q \mathcal{L}A_n.$$

Therefore

$$(\mathcal{P}_{\mathcal{L}A} \mathcal{M}_{\mathcal{L}A} - q\mathcal{M}_{\mathcal{L}A} \mathcal{P}_{\mathcal{L}A}) \mathcal{L}A_n = ([n+1]_q - q[n]_q) \mathcal{L}A_n = \mathcal{L}A_n,$$

because $[n+1]_q - q[n]_q = 1$. This proves (91). The ordinary commutator formula follows similarly from $[n+1]_q - [n]_q = q^n$. $\square$

The conjugated raising operator can be written explicitly in terms of the base raising operator and the Appell factor. Let z be an auxiliary variable and write

$$R_A(z) := \frac{\mathcal{A}_q(qz)}{\mathcal{A}_q(z)}, \quad S_A(z) := \frac{\widehat{D}_{q,z}\mathcal{A}_q(z)}{\mathcal{A}_q(z)}, \quad (93)$$

where $\widehat{D}_{q,z}\mathcal{A}_q(z)$ is computed formally from $\widehat{D}_{q,z}z^r = [r]_q z^{r-1}$. Since $\mathcal{A}_{0,q} \neq 0$, both quotient series are well defined near $z = 0$ as formal power series.

Proposition 1. Assume that the base operators satisfy the q -oscillator relation

$$\mathcal{P}\mathcal{M}_{\mathcal{L}} - q\mathcal{M}_{\mathcal{L}}\mathcal{P} = I \quad (94)$$

on $\mathcal{V}_{\mathcal{L}}$. Then

$$\mathcal{M}_{\mathcal{L}A} = \mathcal{M}_{\mathcal{L}}R_A(\mathcal{P}) + S_A(\mathcal{P}). \quad (95)$$

Proof. From (94), an induction gives

$$\mathcal{P}^r \mathcal{M}_{\mathcal{L}} = q^r \mathcal{M}_{\mathcal{L}} \mathcal{P}^r + [r]_q \mathcal{P}^{r-1}, \quad r \geq 1, \quad (96)$$

with the obvious interpretation for $r = 0$. Indeed, the case $r = 1$ is (94); multiplying the induction hypothesis by $\mathcal{P}$ on the left and using (94) gives the step $r \mapsto r + 1$ together with $[r + 1]_q = 1 + q[r]_q$.

Let $f(z) = \sum_{r \geq 0} f_r z^r / [r]_q!$ be any formal series. Multiplying (96) by $f_r / [r]_q!$ and summing over r yields

$$f(\mathcal{P})\mathcal{M}_{\mathcal{L}} = \mathcal{M}_{\mathcal{L}}f(q\mathcal{P}) + (\widehat{D}_{q,z}f)(\mathcal{P}). \quad (97)$$

Taking $f = \mathcal{A}_q$ and multiplying on the right by $\mathfrak{B} = \mathcal{B}_q(\mathcal{P}) = 1/\mathcal{A}_q(\mathcal{P})$, we obtain

$$\mathfrak{A}\mathcal{M}_{\mathcal{L}}\mathfrak{B} = \mathcal{M}_{\mathcal{L}}\mathcal{A}_q(q\mathcal{P})\mathcal{B}_q(\mathcal{P}) + (\widehat{D}_{q,z}\mathcal{A}_q)(\mathcal{P})\mathcal{B}_q(\mathcal{P}).$$

Using (93) gives (95). This proves the stated explicit realization. $\square$

Theorem 9. For $n \geq 0$,

$${}_{\mathcal{L}}\mathcal{A}_n(x, y; w, q) = \mathcal{M}_{\mathcal{L}A}^n {}_{\mathcal{L}}\mathcal{A}_0(x, y; w, q). \quad (98)$$

If the standard normalization $\mathcal{A}_{0,q} = 1$ is imposed, then ${}_{\mathcal{L}}\mathcal{A}_0(x, y; w, q) = 1$ and

$${}_{\mathcal{L}}\mathcal{A}_n(x, y; w, q) = \mathcal{M}_{\mathcal{L}A}^n \{1\}. \quad (99)$$

Moreover,

$$e_q(t\mathcal{M}_{\mathcal{L}A}) {}_{\mathcal{L}}\mathcal{A}_0(x, y; w, q) = \sum_{n=0}^{\infty} {}_{\mathcal{L}}\mathcal{A}_n(x, y; w, q) \frac{t^n}{[n]_q!}. \quad (100)$$

Proof. Repeated application of the raising relation (88) gives

$$\mathcal{M}_{\mathcal{L}A}^n {}_{\mathcal{L}}\mathcal{A}_0 = {}_{\mathcal{L}}\mathcal{A}_n,$$

which proves (98). The equality ${}_{\mathcal{L}}\mathcal{A}_0 = \mathcal{A}_{0,q} {}_{\mathcal{L}}\mathcal{A}_0$ follows from (79); since $\mathcal{L}_0 = 1$, the normalization $\mathcal{A}_{0,q} = 1$ gives (99). Finally,

$$e_q(t\mathcal{M}_{\mathcal{L}A})\mathcal{L}\mathcal{A}_0 = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \mathcal{M}_{\mathcal{L}A}^n \mathcal{L}\mathcal{A}_0 = \sum_{n=0}^{\infty} \mathcal{L}\mathcal{A}_n \frac{t^n}{[n]_q!},$$

which is (100). Thus the sequence is generated from its initial polynomial exactly as required by the quasi-monomiality principle. $\square$

Remark 4. When a concrete q -difference realization of $\mathcal{M}_{\mathcal{L}}$ is available for a specific Hahn-factorial q -Laguerre–Tricomi family, Proposition 1 turns (89) into an explicit q -difference equation. Thus the operator identities above provide a transfer mechanism: any known raising operator for the base Hahn-factorial family produces, by conjugation with $\mathfrak{A}$, the raising operator, q -oscillator relation, and Rodrigues-type formula for the corresponding Appell deformation.

4. Reductions to Classical Families

This section records the main reductions of the family introduced in (50). These reductions are useful for two reasons. First, they show that the construction is compatible with the Hahn-factorial base family, the q -Laguerre–Appell, Hahn–Appell, and ordinary q -Appell cases. Second, they identify the joint non- q /non-Hahn limit as the classical Laguerre–Appell convolution. All limits in this section are meant coefficientwise in the corresponding formal power series; under the usual convergence assumptions, the same statements hold analytically in a neighborhood of the origin.

For clarity we write

$$\mathcal{G}_{q,w}^{\mathcal{A}}(x, y; t) := \mathcal{A}_q(t) e_q^{wt}(yt) C_{0,q}^{wt}(xt) = \sum_{n=0}^{\infty} \mathcal{L}\mathcal{A}_n(x, y; w, q) \frac{t^n}{[n]_q!}. \quad (101)$$

The base sequence is generated by

$$\mathcal{L}_{q,w}(x, y; t) := e_q^{wt}(yt) C_{0,q}^{wt}(xt) = \sum_{n=0}^{\infty} \mathcal{L}_n(x, y; w, q) \frac{t^n}{[n]_q!}. \quad (102)$$

Theorem 10. For every $n \geq 0$, the following reductions hold.

1. If $\mathcal{A}_q(t) \equiv 1$, then

$$\mathcal{L}\mathcal{A}_n(x, y; w, q) = \mathcal{L}_n(x, y; w, q). \quad (103)$$

2. If $x = 0$, then the Tricomi factor disappears and

$$\mathcal{A}_q(t) e_q^{wt}(yt) = \sum_{n=0}^{\infty} \mathcal{P}_n(y; w, q) \frac{t^n}{[n]_q!}, \quad (104)$$

where

$$\mathcal{L}\mathcal{A}_n(0, y; w, q) = \mathcal{P}_n(y; w, q) = \sum_{r=0}^n \binom{n}{r}_q \mathcal{A}_{r,q}(y)_{q,w}^{n-r}. \quad (105)$$

Thus the bivariate family reduces to a one-variable Hahn–Appell family.

3. In the coefficientwise limit $w \rightarrow 0^+$, one obtains the two-variable q -Laguerre–Appell family

$$\mathcal{A}_q(t) e_q(yt) C_{0,q}(xt) = \sum_{n=0}^{\infty} \mathcal{L}_q \mathcal{A}_n(x, y; q) \frac{t^n}{[n]_q!}, \quad (106)$$

where

$$\mathcal{L}_q \mathcal{A}_n(x, y; q) = [n]_q! \sum_{r+j+k=n} \frac{\mathcal{A}_{r,q}(-1)^j x^j y^k}{[r]_q! [k]_q! ([j]_q!)^2}. \quad (107)$$

4. Combining $x = 0$ with $w \rightarrow 0^+$ gives the ordinary one-variable q -Appell family

$$\mathcal{A}_q(t)e_q(yt) = \sum_{n=0}^{\infty} \mathcal{A}_n(y; q) \frac{t^n}{[n]_q!}, \quad (108)$$

with

$$\mathcal{A}_n(y; q) = \sum_{r=0}^n \binom{n}{r}_q \mathcal{A}_{r,q} y^{n-r}. \quad (109)$$

In particular, if also $\mathcal{A}_q(t) \equiv 1$, then $\mathcal{A}_n(y; q) = y^n$.

Proof. For (1), substituting $\mathcal{A}_q(t) = 1$ in (101) gives exactly the base generating function (102). Since formal power series have unique coefficients, the coefficient of $t^n / [n]_q!$ on both sides is the same, which proves (103).

For (2), put $x = 0$ in the Hahn–Tricomi expansion (45). All terms with $j \geq 1$ contain the factor $(0)_{q,w}^j$ and vanish, while the term $j = 0$ equals one. Hence $C_{0,q}^{wt}(0) = 1$, and (101) becomes (104). Multiplying

$$\mathcal{A}_q(t) = \sum_{r \geq 0} \mathcal{A}_{r,q} \frac{t^r}{[r]_q!} \quad \text{and} \quad e_q^{wt}(yt) = \sum_{k \geq 0} (y)_{q,w}^k \frac{t^k}{[k]_q!}$$

and comparing the coefficient of $t^n / [n]_q!$ gives the finite q -binomial convolution (105).

For (3), use the defining coefficientwise reduction of the Hahn factors as $w \rightarrow 0^+$:

$$e_q^{wt}(yt) \longrightarrow e_q(yt), \quad C_{0,q}^{wt}(xt) \longrightarrow C_{0,q}(xt) = \sum_{j=0}^{\infty} \frac{(-1)^j x^j t^j}{([j]_q!)^2}.$$

Therefore (101) converges coefficientwise to (106). Expanding the three factors

$$\mathcal{A}_q(t), \quad e_q(yt) = \sum_{k \geq 0} y^k \frac{t^k}{[k]_q!}, \quad C_{0,q}(xt) = \sum_{j \geq 0} \frac{(-1)^j x^j t^j}{([j]_q!)^2},$$

the coefficient of t^n is the sum over all triples $r + j + k = n$. Multiplication by $[n]_q!$ gives (107).

Finally, (4) follows from (3) by setting $x = 0$, so that $C_{0,q}(0) = 1$. The resulting Cauchy product of $\mathcal{A}_q(t)$ and $e_q(yt)$ gives (109). If $\mathcal{A}_q(t) = 1$, then only the term $r = 0$ remains and $\mathcal{A}_n(y; q) = y^n$. $\square$

Theorem 11. Let a be a formal Hahn increment and choose

$$\mathcal{A}_q(t) = e_q^{wt}(at) = \sum_{r=0}^{\infty} (a)_{q,w}^r \frac{t^r}{[r]_q!}. \quad (110)$$

Then

$$\mathcal{L} \mathcal{A}_n(x, y; w, q) = \mathcal{L}_n(x, y \oplus_{q,w} a; w, q), \quad n \geq 0. \quad (111)$$

Consequently, in the limit $w \rightarrow 0^+$,

$$\mathcal{L}_q \mathcal{A}_n(x, y; q) = \sum_{r=0}^n \binom{n}{r}_q a^r \mathcal{L}_{n-r}(x, y; q), \quad (112)$$

which is the q -binomial translation of the two-variable q -Laguerre family.

Proof. Substitution of (110) into the defining generating function (101) gives

$$\sum_{n \geq 0} \mathcal{L} \mathcal{A}_n(x, y; w, q) \frac{t^n}{[n]_q!} = e_q^{wt}(at) e_q^{wt}(yt) C_{0,q}^{wt}(xt).$$

By the Hahn addition rule (53),

$$e_q^{wt}(at) e_q^{wt}(yt) = e_q^{wt}((y \oplus_{q,w} a)t).$$

Hence

$$\sum_{n \geq 0} \mathcal{L} \mathcal{A}_n(x, y; w, q) \frac{t^n}{[n]_q!} = e_q^{wt}((y \oplus_{q,w} a)t) C_{0,q}^{wt}(xt).$$

The last expression is precisely the generating function of $\mathcal{L}_n(x, y \oplus_{q,w} a; w, q)$, and coefficient comparison proves (111).

When $w \rightarrow 0^+$, the kernel (110) becomes $e_q(at)$, so its coefficients are $\mathcal{A}_{r,q} = a^r$. Substituting these coefficients into the direct connection formula (56) gives (112). Equivalently, it is obtained by taking the coefficientwise limit of (111). $\square$

The preceding result shows that one important choice of Appell factor does not create a new base family; it translates the Hahn–Appell coordinate. Other choices recover the standard Appell subfamilies attached to Bernoulli, Euler, and Genocchi kernels.

Proposition 2. Use distinct symbols for the three kernels:

$$\mathcal{A}_q^{(B)}(t) := \frac{t}{e_q(t) - 1} = \sum_{r=0}^{\infty} \mathfrak{B}_{r,q} \frac{t^r}{[r]_q!}, \quad \mathcal{A}_q^{(E)}(t) := \frac{2}{e_q(t) + 1} = \sum_{r=0}^{\infty} \mathfrak{E}_{r,q} \frac{t^r}{[r]_q!}, \quad (113)$$

$$\mathcal{A}_q^{(G)}(t) := \frac{2t}{e_q(t) + 1} = \sum_{r=0}^{\infty} \mathfrak{G}_{r,q} \frac{t^r}{[r]_q!}. \quad (114)$$

These kernels use the same e_q normalization as (5). The Bernoulli normalization is consistent with Al-Salam’s q -Bernoulli convention [26]. The Bernoulli and Euler kernels are units because $\mathcal{A}_q^{(B)}(0) = \mathcal{A}_q^{(E)}(0) = 1$, whereas $\mathcal{A}_q^{(G)}(0) = 0$.

The corresponding direct convolution formulas are

$$\mathcal{L} \mathfrak{B}_n(x, y; w, q) = \sum_{r=0}^n \binom{n}{r}_q \mathfrak{B}_{r,q} \mathcal{L}_{n-r}(x, y; w, q), \quad (115)$$

$$\mathcal{L} \mathfrak{E}_n(x, y; w, q) = \sum_{r=0}^n \binom{n}{r}_q \mathfrak{E}_{r,q} \mathcal{L}_{n-r}(x, y; w, q), \quad (116)$$

$$\mathcal{L} \mathfrak{G}_n(x, y; w, q) = \sum_{r=0}^n \binom{n}{r}_q \mathfrak{G}_{r,q} \mathcal{L}_{n-r}(x, y; w, q). \quad (117)$$

The first two are nonsingular Appell deformations. The third is a singular direct convolution and is excluded from statements requiring a reciprocal kernel.

Proof. For $X \in \{B, E, G\}$, multiply the expansion of $\mathcal{A}_q^{(X)}(t)$ by the base generating function (102). The same q -Cauchy product as in Theorem 1 gives the corresponding formulas (115), (116), or (117). No reciprocal series is needed for these direct identities. $\square$

Remark 5 (What the Genocchi singularity permits). Because $\mathcal{A}_q^{(G)}(0) = 0$, it is not invertible in $\mathbb{C}[[t]]$. Equivalently, the triangular Genocchi connection matrix has zero diagonal, so the inverse

connection and the Hessenberg determinant derived from a nonzero diagonal cannot hold in their original form. This obstruction is intrinsic to the unshifted Genocchi family. Nevertheless,

$$\mathcal{A}_q^{(G)}(t) = t\mathcal{A}_q^{(E)}(t), \quad (118)$$

which gives the exact shifted regularization

$$\mathcal{L}\mathfrak{G}_0(x, y; w, q) = 0, \quad \mathcal{L}\mathfrak{G}_n(x, y; w, q) = [n]_q \mathcal{L}\mathfrak{G}_{n-1}(x, y; w, q), \quad n \geq 1. \quad (119)$$

Thus inverse and determinant formulas may be applied to the regularized sequence $\mathcal{L}\mathfrak{G}_{n+1}/[n+1]_q = \mathcal{L}\mathfrak{G}_n$, but not directly to the original Genocchi indexing with its zero constant term.

Theorem 12. Assume that $q \rightarrow 1^-$ and $w = w(q) \rightarrow 0$ in such a way that, coefficientwise,

$$\mathcal{A}_q(t) \longrightarrow \mathcal{A}(t) = \sum_{r=0}^{\infty} A_r \frac{t^r}{r!}, \quad e_q^{wt}(yt) \longrightarrow e^{yt}, \quad (120)$$

$$C_{0,q}^{wt}(xt) \longrightarrow C_0(xt) := \sum_{j=0}^{\infty} \frac{(-1)^j x^j t^j}{(j!)^2}. \quad (121)$$

Then

$$\mathcal{A}(t)e^{yt}C_0(xt) = \sum_{n=0}^{\infty} \mathcal{L}\mathcal{A}_n^{\text{cl}}(x, y) \frac{t^n}{n!}, \quad (122)$$

and, for every fixed n ,

$$\lim_{\substack{q \rightarrow 1^- \\ w \rightarrow 0}} \mathcal{L}\mathcal{A}_n(x, y; w, q) = \mathcal{L}\mathcal{A}_n^{\text{cl}}(x, y). \quad (123)$$

Moreover,

$$\mathcal{L}\mathcal{A}_n^{\text{cl}}(x, y) = \sum_{r=0}^n \binom{n}{r} A_r \mathcal{L}_{n-r}^{\text{cl}}(x, y), \quad (124)$$

where

$$e^{yt}C_0(xt) = \sum_{n=0}^{\infty} \mathcal{L}_n^{\text{cl}}(x, y) \frac{t^n}{n!}. \quad (125)$$

Equivalently,

$$\mathcal{L}\mathcal{A}_n^{\text{cl}}(x, y) = n! \sum_{r+j+k=n} \frac{A_r (-1)^j x^j y^k}{r! k! (j!)^2}. \quad (126)$$

In particular, $x = 0$ gives the ordinary Appell family generated by $\mathcal{A}(t)e^{yt}$, and if also $\mathcal{A}(t) = 1$, then the limit is the monomial sequence y^n .

Proof. By the hypotheses (120) and (121), the generating function (101) converges coefficientwise to the product (122). Since coefficientwise limits commute with extraction of any fixed coefficient, the coefficient of $t^n/n!$ in the limiting product is the limit of the coefficient of $t^n/[n]_q!$ in (101); this proves (123).

To prove the convolution formula, multiply

$$\mathcal{A}(t) = \sum_{r \geq 0} A_r \frac{t^r}{r!} \quad \text{and} \quad e^{yt}C_0(xt) = \sum_{s \geq 0} \mathcal{L}_s^{\text{cl}}(x, y) \frac{t^s}{s!}.$$

The coefficient of t^n in the Cauchy product is

$$\sum_{r=0}^n \frac{A_r \mathcal{L}_{n-r}^{\text{cl}}(x, y)}{r!(n-r)!}.$$

Multiplying by $n!$ gives (124). Finally, expanding also

$$e^{yt} = \sum_{k \geq 0} y^k \frac{t^k}{k!}, \quad C_0(xt) = \sum_{j \geq 0} \frac{(-1)^j x^j t^j}{(j!)^2},$$

shows that the coefficient of t^n in $\mathcal{A}(t)e^{yt}C_0(xt)$ is the triple sum over $r + j + k = n$. Multiplication by $n!$ gives (126). Setting $x = 0$ leaves $C_0(0) = 1$, and setting $\mathcal{A}(t) = 1$ after that leaves e^{yt} , whose coefficients are y^n . $\square$

5. Distribution of Zeros and Graphical Representations

This section complements the structural results obtained above with a finite coefficient scheme, zero-location information, and reproducible graphical representations. For an arbitrary Appell factor $\mathcal{A}_q(t)$, a universal zero theorem is not expected: the zeros change with the coefficients $\mathcal{A}_{r,q}$ and with the chosen Hahn parameters. Therefore we first give a general plotting formula that follows from the convolution formula, and then we state rigorous zero information for a canonical computable specialization. This is consistent with the general philosophy used for zeros of special and orthogonal polynomials: the generating function gives the coefficient array, while the geometric location of zeros is studied after a concrete parameter specialization; see, for instance, the standard treatments in [36,37] and the polynomial-zero background in [38].

The explicit expansion obtained from (50) may be reorganized as a polynomial in the Hahn factorial basis $(x)_{q,w}^j$. This form is useful because every graph in this section is obtained by evaluating finite sums; no truncation of an infinite generating function is involved.

Proposition 3. *For each $n \geq 0$, the Hahn-factorial q -Laguerre–Tricomi–Appell polynomial may be written as*

$$\mathcal{L}\mathcal{A}_n(x, y; w, q) = \sum_{j=0}^n (-1)^j \Gamma_{n,j}^{(\mathcal{A})}(y; w, q) (x)_{q,w}^j, \quad (127)$$

where

$$\Gamma_{n,j}^{(\mathcal{A})}(y; w, q) = \frac{[n]_q!}{([j]_q!)^2} \sum_{r=0}^{n-j} \frac{\mathcal{A}_{r,q}(y)_{q,w}^{n-j-r}}{[r]_q! [n-j-r]_q!}, \quad 0 \leq j \leq n. \quad (128)$$

In particular, when $w = 0$, this becomes an ordinary polynomial in x ,

$$\mathcal{L}_q \mathcal{A}_n(x, y; q) = \sum_{j=0}^n (-1)^j \Gamma_{n,j}^{(\mathcal{A})}(y; 0, q) x^j. \quad (129)$$

Proof. Starting from (50), fix the exponent j of the Hahn factorial term $(x)_{q,w}^j$. The remaining two indices, say r and k , satisfy $r + k = n - j$. Thus

$$\mathcal{L}\mathcal{A}_n(x, y; w, q) = [n]_q! \sum_{j=0}^n \frac{(-1)^j (x)_{q,w}^j}{([j]_q!)^2} \sum_{r+k=n-j} \frac{\mathcal{A}_{r,q}(y)_{q,w}^k}{[r]_q! [k]_q!}.$$

Writing $k = n - j - r$ gives (127) and (128). If $w = 0$, the Hahn basis satisfies $(x)_{q,0}^j = x^j$ and $(y)_{q,0}^m = y^m$, so (129) follows. $\square$

Remark 6. Formula (127) is the practical starting point for numerical work with a general Appell factor. For fixed n , it gives all coefficients by finite arithmetic from $\mathcal{A}_{0,q}, \dots, \mathcal{A}_{n,q}$. The graphical panels below were produced from this formula and from its translated specialization (132); hence the computations are independent of any numerical approximation to the generating series.

For explicit zero statements and reproducible plots, we use the translated slice obtained by taking

$$w = 0, \quad \mathcal{A}_q(t) = e_q(at), \quad (130)$$

so that the Appell factor acts as a q -translation of the Appell coordinate, as in Theorem 11. Let Y denote the resulting effective translated coordinate. In other words, the generating function to be plotted is

$$e_q(Yt)C_{0,q}(xt) = \sum_{n=0}^{\infty} P_n(x; Y, q) \frac{t^n}{[n]_q!}, \quad 0 < q < 1. \quad (131)$$

Here Y may be read as the real value of the translated coordinate after the Hahn translation has been fixed; it should not be confused with the ordinary sum $y + a$ before passing to the classical limit. The next proposition gives the finite expression used in the figures.

Proposition 4. For $n \geq 0$,

$$P_n(x; Y, q) = [n]_q! \sum_{j=0}^n \frac{(-1)^j x^j Y^{n-j}}{[n-j]_q! ([j]_q!)^2}. \quad (132)$$

Equivalently,

$$P_n(x; Y, q) = \sum_{j=0}^n (-1)^j c_{n,j}(q) Y^{n-j} x^j, \quad c_{n,j}(q) := \frac{[n]_q!}{[n-j]_q! ([j]_q!)^2}. \quad (133)$$

The first five polynomials are

$$\begin{aligned} P_0(x; Y, q) &= 1, \\ P_1(x; Y, q) &= Y - x, \\ P_2(x; Y, q) &= Y^2 - [2]_q Yx + \frac{x^2}{[2]_q}, \\ P_3(x; Y, q) &= Y^3 - [3]_q Y^2 x + \frac{[3]_q}{[2]_q} Yx^2 - \frac{x^3}{[2]_q [3]_q}, \\ P_4(x; Y, q) &= Y^4 - [4]_q Y^3 x + \frac{[3]_q [4]_q}{[2]_q^2} Y^2 x^2 - \frac{[4]_q}{[2]_q [3]_q} Yx^3 + \frac{x^4}{[2]_q [3]_q [4]_q}. \end{aligned} \quad (134)$$

Proof. For (132), put $w = 0$ and use the translated generating function (131). Since

$$e_q(Yt) = \sum_{k=0}^{\infty} Y^k \frac{t^k}{[k]_q!}, \quad C_{0,q}(xt) = \sum_{j=0}^{\infty} \frac{(-1)^j x^j t^j}{([j]_q!)^2},$$

the coefficient of t^n in the product is

$$\sum_{j=0}^n \frac{(-1)^j x^j Y^{n-j}}{[n-j]_q! ([j]_q!)^2}.$$

Multiplication by $[n]_q!$ gives (132). Formula (133) is a direct rewriting of the same expression. Finally, substituting $n = 0, 1, 2, 3, 4$ and using $[m]_q! = [1]_q [2]_q \cdots [m]_q$ gives the displayed examples in (134). $\square$

The zeros of $P_n(x; Y, q)$ are the zeros of a finite polynomial whose coefficients are explicit functions of q and Y . This permits two useful statements: an annular bound and continuity of the zero set under changes of q .

Theorem 13. Let $n \geq 1$, $0 < q < 1$, and $Y > 0$. The polynomial $P_n(x; Y, q)$ has degree n , has no negative real zero, and its non-real zeros occur in conjugate pairs. Moreover, every zero ζ of $P_n(\cdot; Y, q)$ satisfies

$$Y \min_{0 \leq j \leq n-1} \frac{[j+1]_q^2}{[n-j]_q} \leq |\zeta| \leq Y \max_{0 \leq j \leq n-1} \frac{[j+1]_q^2}{[n-j]_q}. \quad (135)$$

Proof. The leading coefficient in (132) is $(-1)^n/[n]_q!$, which is nonzero; therefore the degree is exactly n . Since the coefficients are real when $q, Y \in \mathbb{R}$, non-real zeros occur in conjugate pairs.

Set $x = -Yz$. Then (133) gives

$$Y^{-n} P_n(-Yz; Y, q) = Q_n(z) := \sum_{j=0}^n c_{n,j}(q) z^j, \quad (136)$$

where every coefficient $c_{n,j}(q)$ is positive. Hence $Q_n(z) > 0$ for $z > 0$, and therefore $P_n(x; Y, q)$ has no zero on the negative real axis.

The Eneström–Kakeya theorem applied to the positive-coefficient polynomial Q_n gives

$$\min_{0 \leq j \leq n-1} \frac{c_{n,j}(q)}{c_{n,j+1}(q)} \leq |z| \leq \max_{0 \leq j \leq n-1} \frac{c_{n,j}(q)}{c_{n,j+1}(q)}.$$

Using

$$\frac{c_{n,j}(q)}{c_{n,j+1}(q)} = \frac{[j+1]_q^2}{[n-j]_q}, \quad 0 \leq j \leq n-1,$$

and the relation $|x| = Y|z|$ gives (135). $\square$

Proposition 5. Fix $n \geq 1$ and $Y > 0$. On every compact interval $q \in [q_0, q_1] \subset (0, 1)$, the multiset of zeros of $P_n(\cdot; Y, q)$ depends continuously on q .

Proof. After division by its leading coefficient $(-1)^n/[n]_q!$, the polynomial $P_n(x; Y, q)$ becomes monic. Its remaining coefficients are finite expressions in $[1]_q, \dots, [n]_q$, and hence are continuous functions of q on $[q_0, q_1]$. The zeros of a monic polynomial depend continuously, as an unordered multiset, on the coefficients. Equivalently, the zeros are the eigenvalues of the companion matrix, whose entries are continuous functions of the coefficients. This proves the assertion. $\square$

Example 1. Verified numerical computation. For the parameter values used in the zero plot, namely $q = 0.6$ and $Y = 1.2$, formula (132) gives

$$P_5(x; 1.2, 0.6) = \sum_{j=0}^5 (-1)^j c_{5,j}(0.6) 1.2^{5-j} x^j.$$

The coefficients and roots were independently recomputed from this finite formula using 50-digit arithmetic. The unrounded roots have polynomial residuals below 10^{-50} ; Table 1 displays values rounded to nine decimal places.

Table 1. Verified zeros of $P_5(x; 1.2, 0.6)$, rounded to nine decimal places.

Zero Type	Verified Approximate Zero
Real zero	0.996059170
Complex-conjugate pair	$0.545573923 \pm 2.582541030 i$
Complex-conjugate pair	$2.145871309 \pm 1.018132702 i$

The conjugate-pair structure reported in Table 1 agrees with Theorem 13. The annular estimate is deliberately conservative, but it provides a parameter-explicit region that contains all zeros.

The following figures illustrate the translated slice (131). Figure 1 compares the normalized real profiles for degrees 0 through 5, whereas Figure 2 shows how the degree-five profile changes as q varies. Figure 3 displays the complex zeros for degrees 2 through 8 and superimposes the Eneström–Kakeya bounds for $n = 8$; the degree-five points are consistent with the numerical values in Table 1. Figure 4 provides the corresponding two-parameter surface for P_4 . These panels should not be read as zero plots for an arbitrary Appell factor; rather, they are representative of the graphical information obtainable from (127) after a concrete specialization. For a different Appell kernel, one replaces the coefficients $\mathcal{A}_{r,q}$ in (128); the same finite coefficient computation then gives the corresponding curves and zeros.

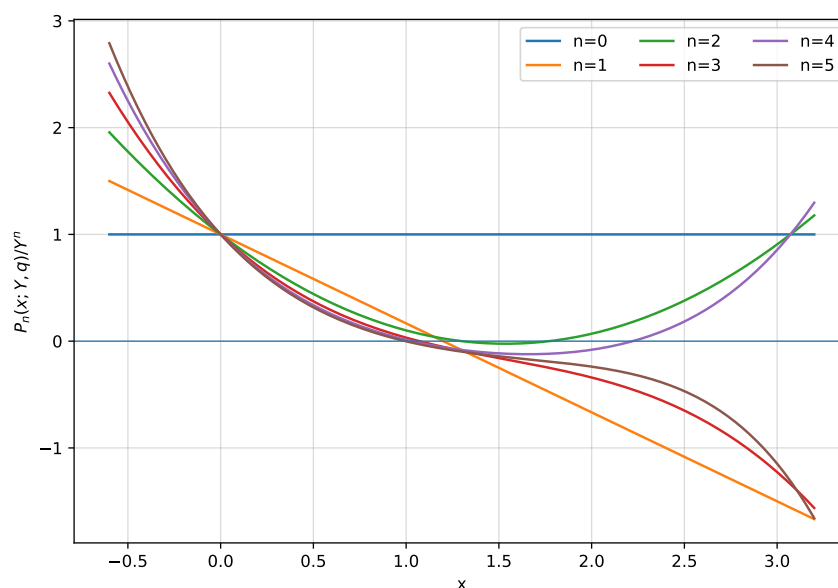


Figure 1. Normalized profiles of $P_n(x; 1.2, 0.6)/1.2^n$ for $0 \leq n \leq 5$.

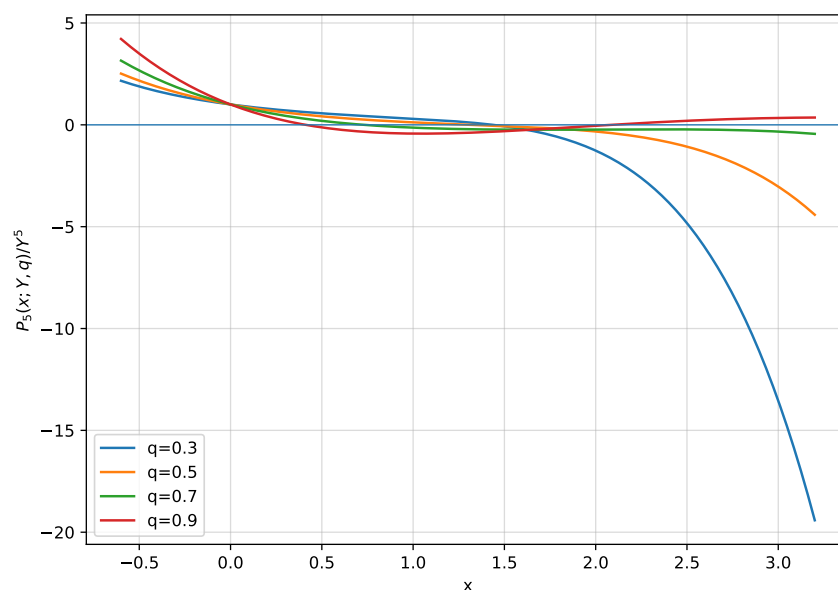


Figure 2. Variation of $P_5(x; 1.2, q)/1.2^5$ for $q \in \{0.3, 0.5, 0.7, 0.9\}$.

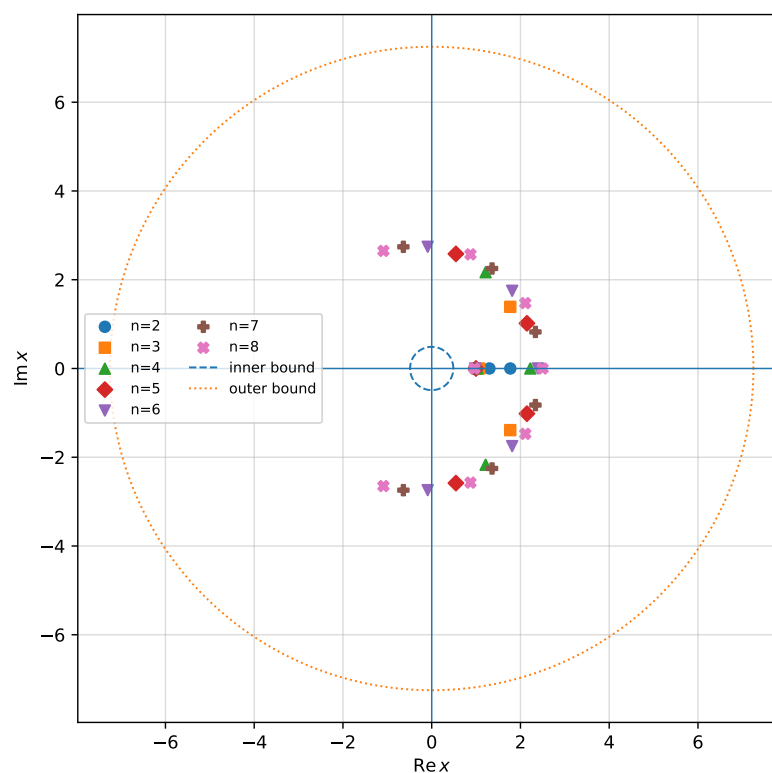


Figure 3. Zeros of $P_n(x; 1.2, 0.6)$ for $2 \leq n \leq 8$, together with the Eneström–Kakeya annulus from Theorem 13 for $n = 8$.

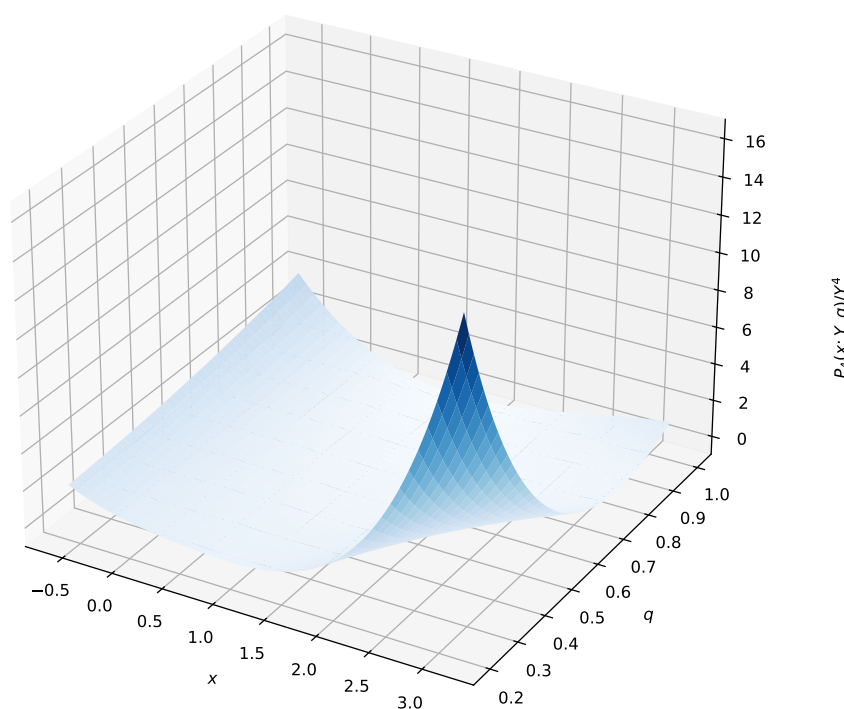


Figure 4. Surface of $P_4(x; 1.2, q)/1.2^4$ over the (x, q) -plane; the endpoint $q = 1$ is interpreted by continuous extension.

6. Conclusions

We have constructed a two-variable Hahn-factorial q -Laguerre–Tricomi–Appell family by adjoining an arbitrary nonsingular Appell factor to a Hahn-factorial q -Laguerre–Tricomi generating function. The central structural feature is the finite q -binomial connection

between the deformed sequence and the base sequence. Because the Appell factor is invertible as a formal power series, this connection has an explicit inverse, and both transforms are valid coefficientwise without imposing additional analytic assumptions beyond those needed for convergence interpretations.

The main consequences established in Sections 2 and 3 are direct and inverse connection formulas, a Hessenberg determinant representation, triangular recurrences, higher Hahn-difference formulas, Hahn-shift identities, and summation formulas. In the operator form, the Appell factor acts as the transfer operator $\mathfrak{A} = \mathcal{A}_q(\mathcal{P})$, where $\mathcal{P} = \widehat{D}_{q,w;y}$ is the Hahn lowering operator in the Appell coordinate. This formulation gives an inverse operator, a translation operator, conjugated raising operators, number-operator equations, a q -oscillator relation, and a Rodrigues-type representation. The abstract raising operator follows from triangularity, whereas a concrete difference-operator realization requires additional commutation data for the chosen base model.

The reduction results in Section 4 show that the construction is compatible with several established subclasses. Special choices of the Appell factor recover the underlying Hahn-factorial base sequence, one-variable Hahn–Appell polynomials, ordinary q -Appell polynomials, translated q -Laguerre families, admissible Bernoulli–Euler variants, and a singular Genocchi-type direct convolution. The coefficientwise joint limit $q \rightarrow 1^-$, $w \rightarrow 0$ gives the corresponding classical Laguerre–Appell convolution. Finally, Section 5 shows that the finite coefficient array provides a reproducible basis for plotting and for locating zeros in concrete parameter regimes; the annular zero bound and continuity statement illustrate how qualitative information can be extracted once the Appell data and parameters are fixed.

Future work may focus on orthogonality or quasi-orthogonality for distinguished Appell factors, explicit recurrence and linearization coefficients for concrete subclasses, and sharper asymptotic or zero-location results. These questions would further clarify when Appell deformations of Hahn-factorial q -Laguerre kernels interact with the broader theory of genuine q -Laguerre–Hahn and semiclassical orthogonal polynomial systems.

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