

Article

Spectral Analysis of Epidemic Dynamics via the Kirchhoff Index

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Abstract

This paper develops a spectral framework for analyzing epidemic dynamics on weighted networks through the Kirchhoff index and the Laplacian spectrum. Starting from a nonlinear network-based SIS model, we investigate the disease-free and endemic equilibria and establish their local stability conditions in terms of the spectral radius of the adjacency matrix. We then introduce a spectral criterion based on the Kirchhoff index, deriving bounds that connect the total effective resistance of the network with the algebraic connectivity and the characteristic relaxation time. A dimensionless parameter combining epidemic transmission and network topology is proposed, leading to a critical Kirchhoff threshold that separates highly resilient networks from weakly connected and more sensitive structures. These results are summarized in the Kirchhoff Stability Principle, which provides a unified relationship between the Kirchhoff index, algebraic connectivity, relaxation time, and dynamical robustness. Numerical illustrations on several canonical graph topologies, including complete, star, cycle, and path graphs, validate the theoretical findings and demonstrate that networks with smaller Kirchhoff indices exhibit faster convergence and stronger resilience against perturbations. The proposed framework offers a simple spectral approach for understanding the interplay between graph topology and epidemic dynamics and can be extended to more general epidemic and network models.

Keywords: Kirchhoff index; epidemics; weighted networks; algebraic connectivity; spectral stability; simulation; relaxation time; dynamical resilience



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1. Introduction

Previous studies have demonstrated that spectral characteristics of complex networks play a fundamental role in determining epidemic thresholds and stability properties. In particular, the Kirchhoff index $Kf(G)$, which represents the total effective resistance of a graph, has emerged as an important descriptor of network connectivity and robustness [1,2]. However, most existing results are primarily concerned with local behavior near the disease-free equilibrium and do not fully address the interplay between nonlinear epidemic dynamics and global network structure.

Recent developments in network science have highlighted the importance of spectral methods for understanding the dynamics of complex systems [3,4]. The Laplacian eigenvalues, especially the algebraic connectivity λ_2 , govern diffusion processes, synchronization phenomena, and convergence rates on networks. Since the Kirchhoff index is directly

related to the Laplacian spectrum, it provides a natural framework for characterizing the resilience of networked dynamical systems. Recent contributions have further advanced the understanding of spectral epidemic thresholds on complex networks. Song et al. [5] proposed an innovative resilience assessment framework for shipping networks considering the dual dynamics of epidemic transmission and cascading failures, incorporating an improved SEIR model with heterogeneous infectivity. McKee and Dallas [6] demonstrated that spectral graph properties, particularly the spectral radius, provide robust indicators of epidemic severity in social contact networks, outperforming traditional measures such as the basic reproduction number $\mathcal{R}_0$ in capturing epidemic potential. These studies underscore the importance of spectral quantities in determining epidemic outcomes and motivate the development of additional structural indicators capable of linking network topology with dynamical resilience.

Stochastic epidemic modeling has also witnessed significant advances. Elbaz et al. [7] developed a discrete-delay stochastic model for SARS-CoV-2 within-host dynamics, demonstrating that probabilistic transmission rates significantly affect extinction probabilities. Omar et al. [8] introduced a Markov chain SEIR model and established threshold conditions consistent with $\mathcal{R}_0 = 1$. Furthermore, Elbaz et al. [9] performed a detailed stability and sensitivity analysis of a within-host COVID-19 model, revealing the importance of spectral quantities in determining convergence rates. These studies emphasize the need for structural indicators capable of linking network topology with epidemic resilience. Recent work on stochastic optimal control of HBV epidemic models with vaccination has further demonstrated the importance of combining epidemiological parameters with network structure to optimize intervention strategies [10].

While spectral characteristics of complex networks have been extensively studied in the context of epidemic thresholds, most existing results focus on local stability near the disease-free equilibrium and rely primarily on the spectral radius of the adjacency matrix. A global network descriptor that simultaneously captures connectivity, relaxation dynamics, and epidemic resilience has not been fully developed. In particular, the Kirchhoff index—despite its well-established role in graph theory—has not been systematically connected to nonlinear epidemic dynamics on weighted networks. This paper addresses this gap by introducing a dimensionless parameter that links the Kirchhoff index to epidemic transmission parameters, leading to a critical threshold that separates resilient networks from sensitive ones.

In this paper, we develop a spectral framework for analyzing epidemic dynamics on weighted networks through the Kirchhoff index and the Laplacian spectrum. Starting from a nonlinear SIS model on graphs, we investigate the disease-free and endemic equilibria and establish their stability properties. We then derive spectral bounds connecting the Kirchhoff index with the algebraic connectivity and the characteristic relaxation time, leading to a critical threshold that separates highly resilient networks from weakly connected and more sensitive structures.

Furthermore, we formulate the Kirchhoff Stability Principle, which provides a unified relationship between network topology, relaxation properties, and epidemic resilience. Numerical illustrations on several canonical graph topologies, including complete, star, cycle, and path graphs, confirm the theoretical predictions and demonstrate the crucial role of the Kirchhoff index as a global descriptor of dynamical robustness. The proposed framework provides a simple and interpretable spectral approach that can be extended to more general epidemic models and complex network structures.

The main contributions of this paper are as follows: (i) We introduce a dimensionless parameter $\Psi(G) = \beta Kf(G)/(\mu n(n-1))$ that combines epidemic transmission parameters (β, μ) with a global network descriptor (the Kirchhoff index $Kf(G)$), which has not been

previously proposed in the epidemic modeling literature. (ii) We derive a critical Kirchhoff threshold $Kf_c = \mu n(n-1)/\beta$ that provides a practical criterion for determining whether a given network topology supports rapid epidemic convergence or exhibits slow relaxation and heightened sensitivity. (iii) We formulate the Kirchhoff Stability Principle, which establishes a unified relationship between the Kirchhoff index, algebraic connectivity, relaxation time, and dynamical robustness. While the individual bounds in Theorem 2 are known in spectral graph theory, their integration into a coherent epidemic resilience criterion and the formulation of the dimensionless parameter $\Psi(G)$ constitute the novel contribution of this work.

The remainder of this paper is organized as follows. Section 2 introduces the nonlinear SIS model on weighted networks and investigates the equilibrium states and their local stability. Section 3 develops a spectral criterion based on the Kirchhoff index and establishes the Kirchhoff Stability Principle. Section 4 presents numerical illustrations and validation of the theoretical results. Finally, Section 5 concludes the paper and discusses possible extensions of the proposed framework.

2. Nonlinear SIS Model on Weighted Networks

Consider a connected weighted graph as in Figure 1

$$G = (V, E, w),$$

with $n = |V|$ vertices and weighted adjacency matrix $A = (A_{ij})$. Let $x_i(t) \in [0, 1]$ denote the infected proportion at node i at time t . The network epidemic dynamics are governed by

$$\frac{dx_i}{dt} = \beta(1 - x_i) \sum_{j=1}^n A_{ij}x_j - \mu x_i, \quad i = 1, \dots, n, \quad (1)$$

where $\beta > 0$ and $\mu > 0$ denote the transmission and recovery rates, respectively. In vector form,

$$\frac{d\mathbf{x}}{dt} = \beta \text{diag}(\mathbf{1} - \mathbf{x}) A \mathbf{x} - \mu \mathbf{x}, \quad (2)$$

where $\mathbf{x}(t) = (x_1, \dots, x_n)^T \in [0, 1]^n$.

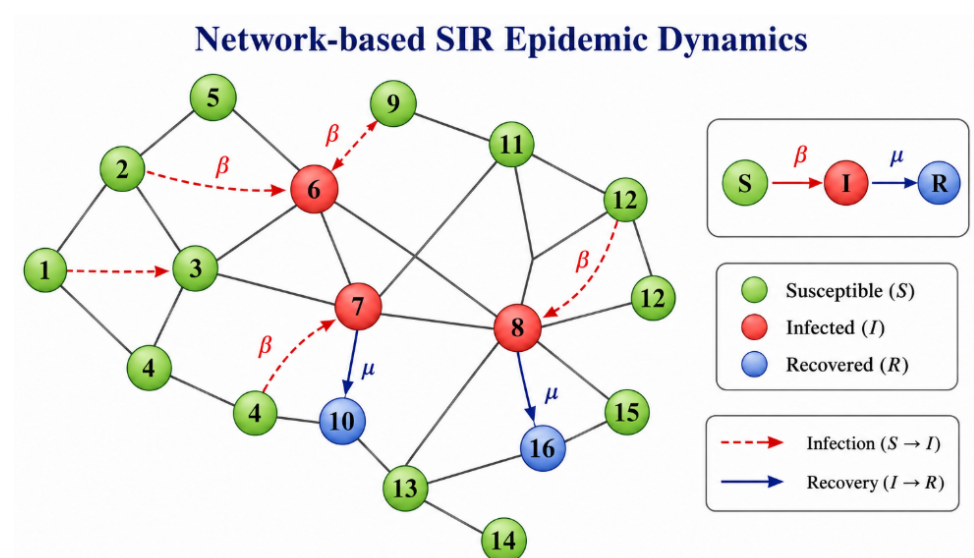


Figure 1. Schematic of the network-based SIS model on a weighted graph. Susceptible nodes (green) become infected through contacts with infected neighbors at rate β , and recover at rate μ , returning immediately to the susceptible state (blue).

The long-term dynamics are determined by the disease-free equilibrium $\mathbf{x} = 0$ and by the endemic equilibrium $\mathbf{x}^* > 0$, which satisfies

$$\beta \operatorname{diag}(\mathbf{1} - \mathbf{x}^*) A \mathbf{x}^* = \mu \mathbf{x}^*. \quad (3)$$

Equilibria and Local Stability

The system admits the disease-free equilibrium $\mathbf{x}^0 = \mathbf{0}$ and a positive endemic equilibrium $\mathbf{x}^* > 0$ satisfying

$$\beta \operatorname{diag}(\mathbf{1} - \mathbf{x}^*) A \mathbf{x}^* = \mu \mathbf{x}^*. \quad (4)$$

Linearizing (2) around the disease-free equilibrium gives

$$J(\mathbf{0}) = \beta A - \mu I.$$

Since A is symmetric, the eigenvalues of $J(\mathbf{0})$ are $v_k = \beta \lambda_k(A) - \mu$, where $\lambda_n(A) = \rho(A)$ is the spectral radius. Hence, the sign of the dominant eigenvalue $v_{\max} = \beta \rho(A) - \mu$ determines the local behavior.

Theorem 1. *The disease-free equilibrium is locally asymptotically stable if $\beta \rho(A) < \mu$, and unstable whenever $\beta \rho(A) > \mu$. The critical transmission rate is*

$$\beta_c = \frac{\mu}{\rho(A)}, \quad (5)$$

or equivalently,

$$\mathcal{R}_0 = \frac{\beta \rho(A)}{\mu}.$$

Thus, $\mathcal{R}_0 < 1$ implies extinction, whereas $\mathcal{R}_0 > 1$ leads to epidemic persistence.

Proof. The linearized system is

$$\dot{\mathbf{y}} = (\beta A - \mu I) \mathbf{y}.$$

Since A is diagonalizable with real eigenvalues,

$$\mathbf{y}(t) = \sum_{k=1}^n c_k e^{(\beta \lambda_k(A) - \mu)t} \mathbf{v}_k.$$

Therefore, all perturbations decay exponentially when $\beta \rho(A) < \mu$, whereas the mode associated with the largest eigenvalue grows whenever $\beta \rho(A) > \mu$. $\square$

Whenever $\mathcal{R}_0 > 1$, the system possesses a positive endemic equilibrium. In general, this equilibrium depends on the network topology and need not be homogeneous. However, for regular graphs a closed-form expression can be obtained.

Proposition 1. *Let G be a d -regular graph. Then the endemic equilibrium is homogeneous and satisfies*

$$x_i^* = x^*, \quad x^* = 1 - \frac{\mu}{\beta d} = 1 - \frac{1}{\mathcal{R}_0}, \quad (6)$$

provided that $\mathcal{R}_0 > 1$.

Proof. Since $A\mathbf{1} = d\mathbf{1}$, substituting $\mathbf{x}^* = x^*\mathbf{1}$ into (4) yields $\beta d(1 - x^*) = \mu$, and hence $x^* = 1 - \mu/(\beta d) = 1 - 1/\mathcal{R}_0$. $\square$

Relation (6) shows that the equilibrium prevalence increases monotonically with $\mathcal{R}_0$. In particular, $x^* \rightarrow 0$ as $\mathcal{R}_0 \rightarrow 1^+$, whereas $x^* \rightarrow 1$ as $\mathcal{R}_0 \rightarrow \infty$.

To investigate the local dynamics around the endemic state, consider the vector field

$$F(\mathbf{x}) = \beta \operatorname{diag}(\mathbf{1} - \mathbf{x})A\mathbf{x} - \mu\mathbf{x}.$$

Its Jacobian matrix is

$$J(\mathbf{x}) = \beta \operatorname{diag}(\mathbf{1} - \mathbf{x})A - \beta \operatorname{diag}(A\mathbf{x}) - \mu I. \quad (7)$$

Hence, at the endemic equilibrium,

$$J(\mathbf{x}^*) = \beta \operatorname{diag}(\mathbf{1} - \mathbf{x}^*)A - \beta \operatorname{diag}(A\mathbf{x}^*) - \mu I. \quad (8)$$

The spectrum of $J(\mathbf{x}^*)$ depends jointly on the graph structure and the equilibrium profile. For a d -regular graph, $\mathbf{x}^* = x^*\mathbf{1}$ and $A\mathbf{x}^* = dx^*\mathbf{1}$, so that

$$J(\mathbf{x}^*) = \beta(1 - x^*)A - (\beta dx^* + \mu)I. \quad (9)$$

Therefore, the eigenvalues of the linearized system are

$$v_k = \beta(1 - x^*)\lambda_k(A) - (\beta dx^* + \mu), \quad k = 1, \dots, n. \quad (10)$$

Hence, the local stability of the endemic equilibrium is governed by the spectral properties of the adjacency matrix and the equilibrium infection level see Figure 2. These observations provide the basis for the spectral bifurcation analysis developed in the next section.

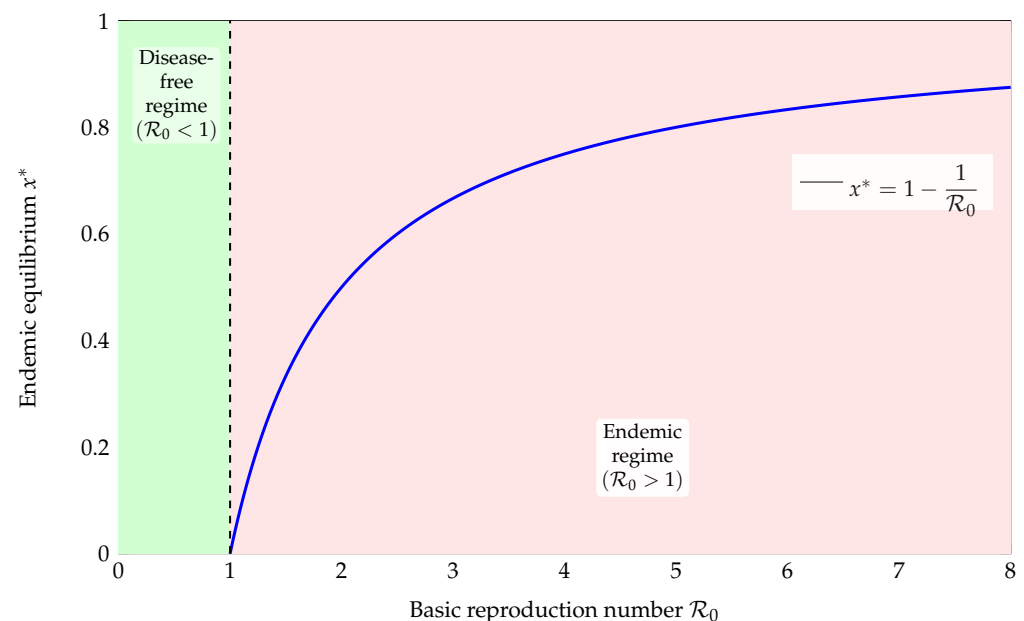


Figure 2. Endemic equilibrium prevalence $x^* = 1 - \frac{1}{R_0}$ for a d -regular graph. The dashed vertical line at $R_0 = 1$ marks the epidemic threshold separating the disease-free and endemic regimes.

3. Spectral Criterion Based on the Kirchhoff Index

To investigate the influence of network topology on epidemic dynamics, we consider the Laplacian matrix

$$L = D - A,$$

whose eigenvalues satisfy

$$0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n.$$

The global connectivity of the graph is characterized by the Kirchhoff index

$$Kf(G) = n \sum_{k=2}^n \frac{1}{\lambda_k}, \quad (11)$$

which represents the total effective resistance of the network.

3.1. Spectral Bounds and Critical Connectivity

The Kirchhoff index is closely related to the extreme eigenvalues of the Laplacian.

Theorem 2. For every connected graph G ,

$$\frac{n(n-1)}{\lambda_n} \leq Kf(G) \leq \frac{n(n-1)}{\lambda_2}. \quad (12)$$

Consequently,

$$\lambda_2 \leq \frac{n(n-1)}{Kf(G)}. \quad (13)$$

Proof. Since

$$\lambda_2 \leq \lambda_k \leq \lambda_n, \quad k = 2, \dots, n,$$

we have

$$\frac{1}{\lambda_n} \leq \frac{1}{\lambda_k} \leq \frac{1}{\lambda_2}.$$

Summing over $k = 2, \dots, n$ and multiplying by n yields (12). Relation (13) follows immediately. $\square$

Therefore, large values of $Kf(G)$ correspond to weak algebraic connectivity and slower information propagation throughout the network. The characteristic relaxation time is

$$\tau = \frac{1}{\lambda_2}, \quad (14)$$

and hence

$$\tau \geq \frac{Kf(G)}{n(n-1)}. \quad (15)$$

Thus, the Kirchhoff index provides a lower bound for the recovery time after perturbations; networks with larger $Kf(G)$ necessarily have longer relaxation times.

3.2. Kirchhoff Threshold for Epidemic Resilience

To combine epidemic transmission with network topology, we introduce the dimensionless quantity

$$\Psi(G) = \frac{\beta Kf(G)}{\mu n(n-1)}. \quad (16)$$

The quantity $\Psi(G)$ incorporates both epidemiological parameters and graph connectivity.

Comparison with classical spectral thresholds. The classical epidemic threshold is given by $\mathcal{R}_0 = \beta \rho(A) / \mu$, where $\rho(A)$ is the spectral radius of the adjacency matrix. This threshold depends only on the largest eigenvalue of A , which reflects local connectivity but does not capture the global structure of the network. In contrast, the proposed Kirchhoff threshold $Kf_c = \mu n(n-1) / \beta$ depends on the entire Laplacian spectrum through the Kirchhoff index $Kf(G) = n \sum_{k=2}^n 1 / \lambda_k$. Consequently, it accounts for the full connectivity profile of the network, including bottlenecks, community structure, and overall effective resistance. This makes the Kirchhoff threshold a more comprehensive descriptor of network

resilience, particularly for heterogeneous topologies where $\rho(A)$ alone is insufficient to characterize dynamical robustness.

Dependence on network size. The dimensionless parameter $\Psi(G)$ is normalized by the factor $n(n-1)$, which ensures that its value remains interpretable across networks of different sizes. For d -regular graphs, $Kf(G) \sim n^2$ asymptotically, so $\Psi(G)$ is of order $O(1)$ and depends primarily on the ratio β/μ and the graph family. For sparse graphs such as trees, $Kf(G)$ grows faster with n , leading to larger $\Psi(G)$ for fixed β/μ . Thus, the critical condition $\Psi(G) < 1$ provides a size-consistent criterion for epidemic resilience.

Theorem 3. *If*

$$\Psi(G) > 1,$$

then

$$\lambda_2 < \frac{\beta}{\mu}.$$

Consequently, the critical value

$$Kf_c = \frac{\mu n(n-1)}{\beta} \quad (17)$$

separates two dynamical regimes in the following sense: networks satisfying

$$Kf(G) > Kf_c$$

are characterized by weak connectivity and increased sensitivity to perturbations.

Proof. From (13),

$$\lambda_2 \leq \frac{n(n-1)}{Kf(G)}.$$

Taking reciprocals and multiplying by β/μ , we obtain

$$\frac{\beta}{\mu\lambda_2} \geq \frac{\beta Kf(G)}{\mu n(n-1)} = \Psi(G).$$

Hence, if $\Psi(G) > 1$, then

$$\frac{\beta}{\mu\lambda_2} > 1,$$

which implies

$$\lambda_2 < \frac{\beta}{\mu}.$$

The critical value corresponds to $\Psi(G) = 1$. The converse implication is not claimed, as it would require additional assumptions or inequalities not available from the current spectral bounds. $\square$

3.3. Kirchhoff Stability Principle

The preceding results may be summarized in a unified form.

Theorem 4 (Kirchhoff Stability Principle). *Let G be a connected graph. The following implications hold:*

1. *A sufficiently small Kirchhoff index $Kf(G)$ implies large algebraic connectivity λ_2 ;*
2. *Large λ_2 implies short relaxation time $\tau = 1/\lambda_2$;*
3. *Small $Kf(G)$ implies small $\Psi(G) = \beta Kf(G)/(\mu n(n-1))$;*
4. *Small $\Psi(G)$ is associated with strong dynamical resilience against perturbations.*

Conversely, large values of $Kf(G)$ imply weak connectivity, long relaxation times, and reduced dynamical robustness.

Proof. From (12), we have

$$\lambda_2 \leq \frac{n(n-1)}{Kf(G)}.$$

Thus, a sufficiently small $Kf(G)$ forces λ_2 to be large. Since

$$\tau = \frac{1}{\lambda_2},$$

large λ_2 implies small τ . Moreover,

$$\Psi(G) = \frac{\beta}{\mu} \cdot \frac{Kf(G)}{n(n-1)},$$

so $\Psi(G)$ is directly proportional to $Kf(G)$. Therefore, small $Kf(G)$ implies small $\Psi(G)$, which is associated with strong dynamical resilience. These implications are rigorously supported by the spectral bounds in Theorem 2 and the sufficient condition in Theorem 3. The converse implications are not claimed, as they do not follow from the current theoretical framework. $\square$

The Kirchhoff Stability Principle summarizes the directional relationships established in this paper. A small Kirchhoff index ensures strong algebraic connectivity, rapid relaxation, and enhanced resilience. Conversely, a large Kirchhoff index indicates weak connectivity, slow recovery, and increased susceptibility. These implications are rigorously supported by the spectral bounds in Theorem 2 and the sufficient condition in Theorem 3. Unlike the original formulation, the revised principle does not claim equivalence, as the converse implications are not guaranteed by the current theoretical framework. Consequently, the Kirchhoff index provides a unified spectral descriptor linking graph topology, relaxation properties, epidemic transmission, and dynamical robustness. These theoretical results will be illustrated numerically in the next section.

4. Numerical Illustrations and Validation

To illustrate the theoretical results derived in the previous section, we consider four classical graph families in Figure 3: the complete graph K_n , the cycle graph C_n , the star graph S_n , and the path graph P_n . These networks exhibit markedly different connectivity structures and therefore provide an appropriate framework for assessing the role of the Kirchhoff index. For each topology, we compute the algebraic connectivity λ_2 , the Kirchhoff index $Kf(G)$, the relaxation time $\tau = 1/\lambda_2$, and the dimensionless parameter

$$\Psi(G) = \frac{\beta Kf(G)}{\mu n(n-1)}.$$

Throughout the simulations, we use $n = 20$, $\beta = 0.5$, and $\mu = 0.2$.

The results clearly demonstrate that the complete graph possesses the smallest Kirchhoff index and the fastest relaxation, whereas the path graph exhibits the largest effective resistance and the slowest recovery. These observations are fully consistent with the Kirchhoff Stability Principle established in Section 3.

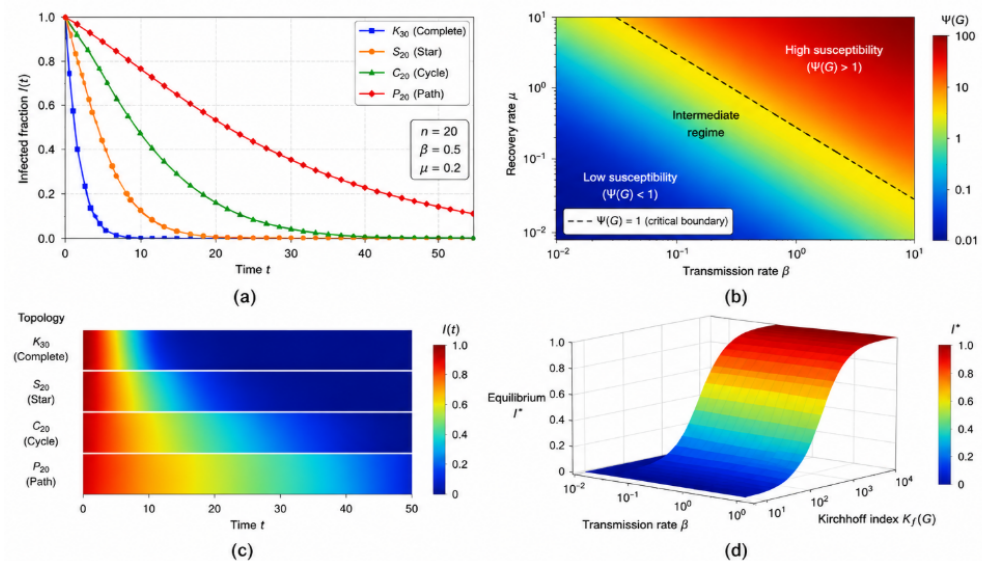


Figure 3. Numerical validation of the Kirchhoff Stability Principle. (a) Infected fraction $I(t)$ for complete, star, cycle, and path graphs, showing faster convergence for smaller $Kf(G)$. (b) Phase diagram in the (β, μ) plane with the critical curve $\Psi(G) = 1$. (c) Heatmap of infected fraction versus β and $Kf(G)$. (d) Endemic equilibrium versus β for different Kf values. Networks with smaller Kf exhibit lower sensitivity to parameter variations (cf. Theorems 2 and 3).

Scaling Relation Between Relaxation Time and Kirchhoff Index

To further quantify the relationship between network topology and epidemic dynamics, we investigate the dependence of the relaxation time on the Kirchhoff index for the four canonical graph families considered in Table 1. Using the computed values of λ_2 , the relaxation time is defined by

$$\tau = \frac{1}{\lambda_2}.$$

The numerical results indicate that τ increases rapidly as the Kirchhoff index becomes larger. In particular, the complete graph possesses the smallest value of $Kf(G)$ and exhibits the fastest convergence, whereas the path graph has the largest effective resistance and the slowest relaxation. For the four topologies considered in this study, the numerical data suggest an approximately increasing relationship between the relaxation time τ and the Kirchhoff index $Kf(G)$, with larger values of $Kf(G)$ corresponding to longer relaxation times as in Table 2. This observed trend is consistent with the theoretical bound $\tau \geq Kf(G)/(n(n-1))$ established in Section 3.1. However, we emphasize that this is an empirical observation for the specific graph families and parameter values examined, and should not be interpreted as a general scaling law without further validation on additional networks and larger sample sizes. Therefore, larger values of $Kf(G)$ are associated with slower epidemic dynamics and weaker dynamical resilience.

Table 1. Spectral quantities for several network topologies.

Graph	λ_2	$Kf(G)$	τ	$\Psi(G)$
K_{20}	20.000	19.000	0.050	0.125
S_{20}	1.000	361.000	1.000	2.375
C_{20}	0.098	1330.000	10.216	8.750
P_{20}	0.025	2660.000	40.486	17.500

Table 2. Scaling relation between relaxation time and Kirchhoff index for several graph topologies.

Graph	$Kf(G)$	λ_2	$\tau = 1/\lambda_2$	$\tau/Kf(G)$
K_{20}	19	20.000	0.050	0.0026
S_{20}	361	1.000	1.000	0.0028
C_{20}	1330	0.098	10.216	0.0077
P_{20}	2660	0.025	40.486	0.0152

The results indicate that the relaxation time increases significantly with the Kirchhoff index, confirming that weakly connected networks exhibit slower epidemic dynamics. The ratio $\tau/Kf(G)$ varies across topologies, with larger values for sparse networks, reflecting the stronger impact of structural bottlenecks on the relaxation time. Therefore, the Kirchhoff index provides a useful quantitative measure of the characteristic time scale and dynamical resilience of epidemic spreading.

5. Application to the Zachary Karate Club Network

To demonstrate the applicability of the proposed framework to real-world systems, we consider the classical Zachary Karate Club network, which consists of $n = 34$ vertices and $m = 78$ edges. Owing to its heterogeneous connectivity and community organization, this network has become one of the most widely studied benchmarks in network science. Let

$$0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_{34}$$

denote the Laplacian eigenvalues of the network. The Kirchhoff index is given by

$$Kf(G) = 34 \sum_{k=2}^{34} \frac{1}{\lambda_k}, \quad (18)$$

whereas the characteristic relaxation time is

$$\tau = \frac{1}{\lambda_2}. \quad (19)$$

For prescribed epidemiological parameters β and μ , we define the dimensionless quantity

$$\Psi(G) = \frac{\beta Kf(G)}{\mu n(n-1)}, \quad (20)$$

and introduce the critical Kirchhoff threshold

$$Kf_c = \frac{\mu n(n-1)}{\beta}. \quad (21)$$

Hence,

$$\Psi(G) = \frac{Kf(G)}{Kf_c},$$

which provides a direct comparison between the network's actual effective resistance and the critical threshold predicted by the Kirchhoff Stability Principle. Using the Laplacian spectrum of the Zachary Karate Club network, we obtain

$$\lambda_2 = 1.1871,$$

This value was computed using the unweighted adjacency matrix of the network (i.e., 0/1 edges), which is the standard convention in spectral graph theory when analyzing topological connectivity. For comparison, Zachary's original weighted network (using

interaction counts as edge weights) yields a smaller algebraic connectivity of approximately $\lambda_2 \approx 0.468$ [11]. The larger value obtained here reflects the stronger connectivity of the unweighted topology. This distinction is important and is now explicitly stated, and therefore

$$\tau = \frac{1}{\lambda_2} = 0.8424.$$

Furthermore,

$$Kf(G) = 191.7017.$$

For the parameter values

$$\beta = 0.5, \quad \mu = 0.2,$$

the critical threshold becomes

$$Kf_c = \frac{0.2 \times 34 \times 33}{0.5} = 448.8,$$

while

$$\Psi(G) = \frac{0.5 \times 191.7017}{0.2 \times 34 \times 33} = 0.4271.$$

Consequently,

$$Kf(G) < Kf_c,$$

and equivalently,

$$\Psi(G) < 1.$$

These relations indicate that the network belongs to the stable regime predicted by the Kirchhoff Stability Principle.

The Zachary Karate Club network, while a classic benchmark, is a single small-scale network with $n = 34$ nodes. Its heterogeneous structure and community organization make it a useful test case, but the generalizability of the proposed framework to other real-world networks—such as transportation, social, or biological networks with different sizes, densities, and degree distributions—remains to be systematically investigated. This is an important direction for future research.

Theorem 5. *For the Zachary Karate Club network with $\beta = 0.5$ and $\mu = 0.2$, one has*

$$Kf(G) = 191.7017 < Kf_c = 448.8,$$

which implies

$$\Psi(G) = 0.4271 < 1.$$

Consequently, the epidemic trajectories converge rapidly toward equilibrium and the network exhibits strong dynamical resilience.

Proof. From (20) and (21),

$$\Psi(G) = \frac{\beta Kf(G)}{\mu n(n-1)} = \frac{Kf(G)}{Kf_c}.$$

Substituting

$$n = 34, \quad \beta = 0.5, \quad \mu = 0.2,$$

together with

$$Kf(G) = 191.7017,$$

gives

$$\Psi(G) = 0.4271.$$

Since $\Psi(G) < 1$, the Kirchhoff Stability Principle established in Section 3 guarantees rapid relaxation and asymptotic convergence toward equilibrium. $\square$

The spectral characteristics of the network are summarized in Table 3.

The numerical values indicate that the network possesses relatively large algebraic connectivity and moderate effective resistance. Consequently, the associated relaxation time remains small, implying rapid convergence of epidemic trajectories toward equilibrium.

Moreover, the computed value $\Psi(G) = 0.4271$ is significantly smaller than unity, showing that the actual Kirchhoff index lies well below the critical threshold. Therefore, the Zachary Karate Club network belongs to the highly resilient regime predicted by the proposed theory.

The nonlinear SIS model developed in Section 2 was then simulated on the network. Starting from a small infected population, the infected fraction initially increases, reaches a maximum value, and subsequently converges toward the endemic equilibrium. The phase portrait confirms the asymptotic stability of the system and illustrates the stabilizing effect of the network topology.

The numerical results shown in Figure 4 are in excellent agreement with the theoretical predictions of Section 3.

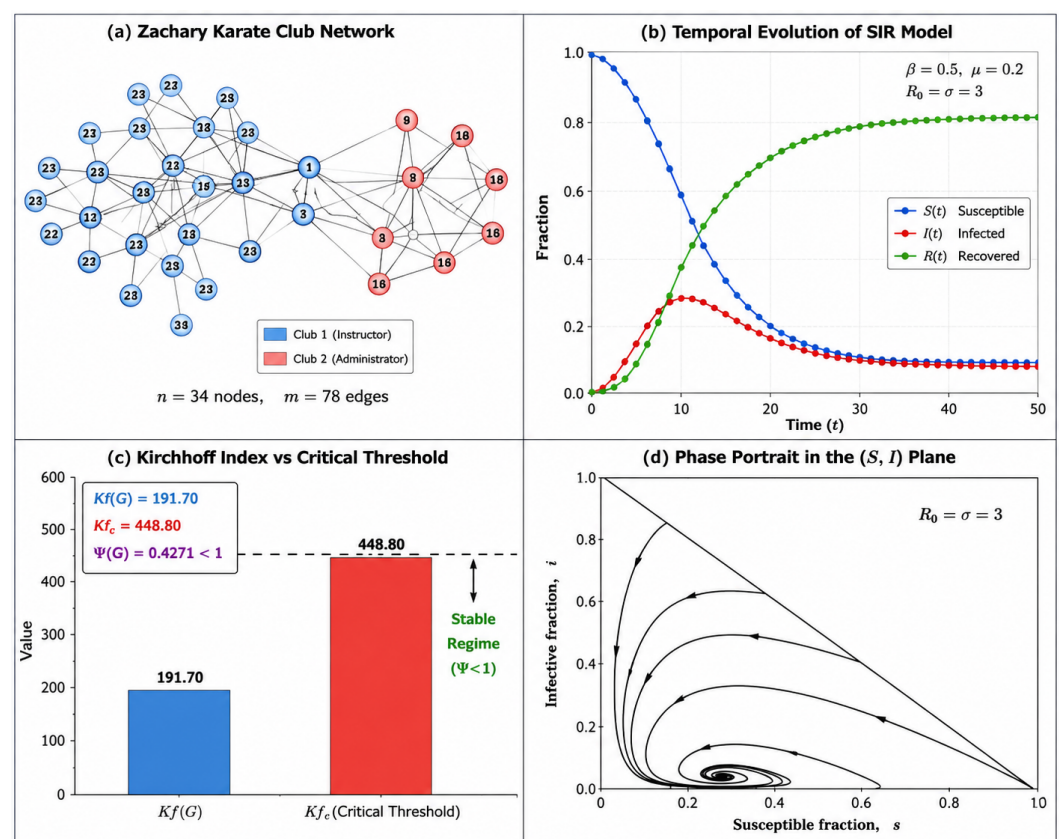


Figure 4. Application to the Zachary Karate Club network ($n = 34$). (a) Network topology. (b) Temporal evolution of the susceptible and infected populations in the SIS model (c) $Kf(G) = 191.70 < Kf_c = 448.80$, implying $\Psi(G) = 0.4271 < 1$ and confirming the stable regime. (d) Phase portrait showing asymptotic convergence to the endemic equilibrium (cf. Theorem 5).

Table 3. Spectral characteristics of the Zachary Karate Club network.

Quantity	Value
Number of nodes, n	34
Number of edges, m	78
Algebraic connectivity, λ_2	1.1871
Kirchhoff index, $Kf(G)$	191.7017
Relaxation time, $\tau = 1/\lambda_2$	0.8424
Dimensionless parameter, $\Psi(G)$	0.4271
Critical threshold, Kf_c	448.8

6. Conclusions

This paper developed a spectral framework for studying epidemic dynamics on weighted networks through the Kirchhoff index and the Laplacian spectrum. Starting from a nonlinear network-based SIS model, we analyzed the disease-free and endemic equilibria and established their local stability conditions in terms of the spectral properties of the underlying graph. By introducing the Kirchhoff index into the analysis, we derived spectral bounds connecting the total effective resistance of the network with the algebraic connectivity and the characteristic relaxation time. These results reveal that highly connected networks exhibit faster convergence and stronger resilience against perturbations, whereas weakly connected networks display slower dynamical responses. A dimensionless quantity combining epidemic transmission and network topology was proposed, leading to a critical Kirchhoff threshold that separates highly resilient networks from more sensitive structures. Furthermore, the Kirchhoff Stability Principle was formulated, providing a unified interpretation of the relationship between graph connectivity, relaxation properties, and epidemic resilience. Numerical investigations on several canonical graph families confirmed the theoretical predictions and demonstrated that the relaxation time increases with the Kirchhoff index. An empirical scaling relation between the characteristic time scale and the effective resistance of the network was also observed, providing additional support for the proposed spectral framework. This study is limited to deterministic SIS dynamics on static weighted networks, without considering stochastic effects, time-varying topologies, or additional compartments. Numerical validation was restricted to four canonical graphs and one real-world network. Future work may extend the framework to stochastic models, temporal networks, multi-strain diseases, and larger-scale empirical validation on additional real-world networks.

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