

Article

Theoretical Analysis of Barrow Holographic Dark Energy in Fractal Cosmology

Hanshu Zhao and Weiqiang Yang * 

Department of Physics, Liaoning Normal University, Dalian 116029, China

* Correspondence: d11102004@163.com

Abstract

We investigate interacting Barrow holographic dark energy (BHDE) with the Hubble horizon as the infrared cutoff in a fractal cosmological background. The fractal measure modifies the Friedmann sector and leads to the nonstandard closure relation $\Omega_{\text{dm}} + \Omega_{\text{de}} = 1 + \gamma$. Using Planck-inspired present-day normalization $\Omega_{\text{de}0} = 0.6847$ and the benchmark parameters $(\Delta, \omega, \beta) = (0.8, 0.263, 0.123)$, we study four linear and nonlinear dark-sector interactions over $-0.99 \leq z \leq 3$. The benchmark solutions remain within the physical background domain throughout this interval: the density fractions are non-negative, the relevant denominators remain positive, and the nonlinear Q_3 and Q_4 terms remain real. All four prescriptions exhibit a transition from decelerated to accelerated expansion. The non-interacting limit gives $z_t \simeq 0.86$, whereas for $\zeta = 0.12$ the transition redshifts are approximately 2.02, 1.97, 1.26, and 1.87 for Q_1 , Q_2 , Q_3 , and Q_4 , respectively; in particular, the Q_4 solution recovers a finite transition within the plotted redshift range. A stronger positive coupling generally shifts acceleration onset to a higher redshift, with the strongest response for Q_1 and the weakest for Q_3 . The statefinder quantities $S_3^{(1)}$, $S_3^{(2)}$, and the s – r trajectories distinguish the interaction structures through their finite-redshift evolution and present-day values. A fixed-background SN Ia comparison using 1046 selected Pantheon supernovae, diagonal FITRES uncertainties, and an analytically profiled additive nuisance parameter gives the lowest information criteria for flat Λ CDM. Among the BHDE benchmarks that interact, Q_4 is the closest case, with $\Delta\text{AIC} = \Delta\text{BIC} \simeq 1.21$. This comparison is not intended as a global posterior constraint on the model parameters.

Keywords: Barrow holographic dark energy; fractal cosmology; interacting dark energy; SN Ia; statefinder diagnostic



Academic Editor: Stefano Profumo

Received: 29 June 2026

Revised: 26 July 2026

Accepted: 4 August 2026

Published: 5 August 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

1. Introduction

The late-time accelerated expansion of the universe is supported by the original Type Ia supernova observations and subsequent precision cosmological measurements [1–3]. This discovery placed dark energy and possible modifications of gravity at the center of modern cosmology, although the physical mechanism responsible for the acceleration remains unresolved.

The conceptual foundations of modern cosmology were established through Einstein's relativistic cosmological model [4], the expanding-universe solutions of Friedmann and Lemaître [5,6], and Hubble's distance–velocity relation [7]. The hot Big Bang picture was subsequently strengthened by primordial-nucleosynthesis calculations [8] and the discovery of the cosmic microwave background [9]. Inflation later provided a mechanism for

addressing the horizon and flatness problems [10], although its detailed microphysical origin remains under investigation.

Broadly speaking, two classes of explanations for the present acceleration have been pursued: the cosmic energy budget may contain a component with sufficiently negative pressure, or gravity may deviate from general relativity on cosmological scales. The Λ CDM model [11,12] successfully describes a wide range of observations, but the fine-tuning and coincidence problems [11], together with tensions involving H_0 and the growth of structure [13], continue to motivate extended cosmological scenarios.

A broad class of dynamical dark-energy models has therefore been developed, including quintessence [14–17], Chaplygin-gas models [18,19], and holographic dark energy (HDE) [20,21]. In Barrow holographic dark energy (BHDE), quantum-gravitational effects deform the black-hole entropy–area relation through the parameter Δ [22,23]. Fractal cosmology provides a complementary modification by replacing the standard spacetime integration measure with an effective fractal measure [24,25]. In the combined framework, the fractal measure changes the Friedmann sector and the closure relation, while the Barrow deformation modifies the scaling of the dark-energy density with the infrared cutoff.

Dark-energy models previously studied in fractal cosmology include Tsallis agegraphic and holographic scenarios [26,27], reconstructed BHDE models [28], and two-component fractal cosmologies [29]. These studies show that entropy deformations and modified cosmological measures can jointly produce background dynamics that differ from those of standard FRW cosmology.

Recent investigations of holographic dark energy have also emphasized thermodynamic consistency, stability, viscosity, and observational diagnostics. Barrow HDE has been studied through generalized thermodynamics, viscous effects, equation-of-state behavior, and statefinder diagnostics [30]. Related work includes viscous holographic $f(Q)$ cosmology [31], thermodynamic and stability analyses of Tsallis HDE in $F(R, T)$ gravity [32], generalized HDE models based on corrected entropies [33], and interacting Barrow–Ricci–Gauss–Bonnet HDE in symmetric teleparallel gravity [34]. Interacting holographic scenarios have also been examined through the equation of state, the deceleration parameter, statefinder and $O_m(z)$ diagnostics, and chi-square comparisons [35]. Statistical tests using observational Hubble data and joint late-time datasets have been applied to related dark-energy and modified-gravity models [36,37].

Viscous-fluid cosmologies provide a complementary phenomenological setting and have been compared with BAO, Pantheon, and $H(z)$ observations [38]. Holographic energy densities have also been used as effective sources in recent relativistic wormhole constructions [39–42]. Although these static configurations address a different physical sector, they illustrate the broader use of holographic energy densities in relativistic gravity.

The present work combines the Barrow entropy deformation, a fractal cosmological measure, and four linear and nonlinear dark-sector interactions within a common background framework. This construction allows the interaction prescriptions to be compared using the modified closure relation, the evolution of Ω_{de} , w_{de} , and q , the acceleration-transition redshift, the physical-domain conditions of the nonlinear Q_3 and Q_4 couplings, and the statefinder hierarchy. The same benchmark backgrounds are also examined through a profiled SN Ia comparison and one-parameter distance-modulus sensitivity tests. The novelty of the analysis lies in applying these dynamical, geometric, and observational diagnostics consistently to four interaction structures within the same fractal BHDE system.

The paper is organized as follows. Section 2 introduces the fractal cosmological framework. Section 3 derives the interacting BHDE equations and presents the background and SN Ia analyses. Section 4 discusses the statefinder diagnostics. Section 5 summarizes the main conclusions, while Appendix A compares the model with selected alternative

holographic dark-energy scenarios and Appendix B provides details of the metric variation leading to the modified Friedmann equations.

2. Fractal Cosmology Theoretical Background

Fractal cosmology extends conventional spacetime geometry by generalizing the standard Riemannian integration measure. The fundamental scaling relation for isotropic coordinates is defined as

$$[x^\mu] = -1, \quad \mu = 0, 1, \dots, D-1, \quad (1)$$

where D represents the topological dimension. This framework is implemented through a modified integration measure:

$$d^D x \rightarrow d\varrho(x) = d^D x v(x), \quad [\varrho] = -D\alpha \neq -D, \quad (2)$$

with α ($0 < \alpha < 1$) quantifying the deviation from standard integer dimensionality induced by fractal effects.

Throughout this work, the fractal measure weight is denoted consistently by v . The function $v(x)$ encodes the fractal contribution to the geometry, and its time-dependent cosmological form is written as $v(t)$. To avoid unphysical behavior such as divergences as $t \rightarrow 0$, we adopt the commonly used phenomenological choice

$$v = a^{-\beta}, \quad (3)$$

where a is the scale factor and $\beta \equiv D(1 - \alpha)$ characterizes the fractal properties.

The gravitational dynamics follow from the total action

$$S = S_g + S_m, \quad (4)$$

which comprises the gravitational sector

$$S_g = \frac{M_p^2}{2} \int d\varrho(x) \sqrt{-g} (R - \omega \partial_\mu v \partial^\mu v), \quad (5)$$

and the matter sector

$$S_m = \int d\varrho(x) \sqrt{-g} \mathcal{L}_m. \quad (6)$$

Here, g is the determinant of $g_{\mu\nu}$, $M_p^{-2} = 8\pi G$ is the reduced Planck mass, ω is the fractal parameter, R is the Ricci scalar, and $\mathcal{L}_m$ is the matter Lagrangian density.

For a spatially flat Friedmann–Robertson–Walker (FRW) universe with metric

$$ds^2 = -dt^2 + a^2(t) (dr^2 + r^2 d\Omega^2), \quad (7)$$

variation of the action yields the modified Friedmann equations

$$H^2 + H \frac{\dot{v}}{v} - \frac{\omega}{6} \dot{v}^2 = \frac{1}{3M_p^2} \rho, \quad (8)$$

$$\dot{H} + H^2 - H \frac{\dot{v}}{v} + \frac{\omega}{3} \dot{v}^2 - \frac{\square v}{2v} = -\frac{1}{6M_p^2} (\rho + 3p), \quad (9)$$

where $H = \dot{a}/a$, ρ and p are the total energy density and pressure, and

$$\square v = -(\ddot{v} + 3H\dot{v}). \quad (10)$$

In the limit $v = 1$, Equations (8)–(10) reduce to the standard FRW equations.

For a universe containing dark energy and dark matter ($\rho = \rho_{de} + \rho_{dm}$, $p = p_{de}$), the conservation equation becomes

$$\dot{\rho} = -\left(3H + \frac{\dot{v}}{v}\right)(\rho + p). \quad (11)$$

Using the specific form (3), Equation (8) can be rewritten as

$$H^2 \left(1 - \beta - \frac{\beta^2 \omega a^{-2\beta}}{6}\right) = \frac{1}{3M_p^2}(\rho_{de} + \rho_{dm}), \quad (12)$$

while the separated conservation equations read

$$\dot{\rho}_{dm} = -(3 - \beta)H\rho_{dm}, \quad (13)$$

$$\dot{\rho}_{de} = -(1 + w_{de})(3 - \beta)H\rho_{de}, \quad (14)$$

where $w_{de} = p_{de}/\rho_{de}$ is the dark-energy equation-of-state parameter. This setup allows one to embed BHDE into a fractal cosmological background.

3. Dynamical Evolution Analysis

We now consider Barrow holographic dark energy (BHDE), in which quantum-gravitational corrections are encoded through a deformation of black-hole entropy. Following Barrow's proposal, quantum effects can induce a "rough" or fractal-like horizon structure, leading to a modified entropy–area relation

$$S_B = \left(\frac{A}{A_0}\right)^{1+\Delta/2}, \quad (15)$$

where A is the horizon area, A_0 is the Planck area, and Δ ($0 \leq \Delta \leq 1$) quantifies the deformation. The standard Bekenstein–Hawking entropy is recovered at $\Delta = 0$, while $\Delta = 1$ corresponds to the maximal deformation limit.

In holographic dark energy, one imposes the relation $\rho L^4 \leq S$, with L a characteristic infrared length scale. Using the Barrow entropy (15) yields the BHDE density

$$\rho_{de} = CL^{\Delta-2}, \quad (16)$$

where C is a constant. For a flat FRW universe, choosing the Hubble horizon as the infrared cutoff ($L = H^{-1}$), one obtains

$$\rho_{de} = CH^{2-\Delta}. \quad (17)$$

Differentiating gives

$$\dot{\rho}_{de} = (2 - \Delta)\rho_{de} \frac{\dot{H}}{H}. \quad (18)$$

In standard non-interacting holographic dark energy with the Hubble radius as the infrared cutoff, the scaling $\rho_{de} \propto H^2$ makes the dark-energy density track the pressureless matter density and generally does not produce accelerated expansion [20,43]. The standard argument is modified in the present framework because the Barrow deformation changes the scaling to $\rho_{de} \propto H^{2-\Delta}$, the fractal measure modifies the Friedmann and conservation equations, and the interaction term permits energy exchange between the dark components. The combined system can therefore support accelerating benchmark solutions even when the Hubble horizon is used as the infrared cutoff.

The Hubble cutoff is adopted here because it gives a local autonomous background system and allows the effects of the Barrow deformation, the fractal measure, and the interaction prescription to be isolated. Alternative choices, such as the future event horizon, Ricci scale, or Granda–Oliveros cutoff, would change the dark-energy density and its evolution equation. They could consequently alter the transition redshifts and the detailed behavior of the trajectories. The conclusions of the present analysis are therefore specific to the Hubble-horizon cutoff and should not be interpreted as cutoff-independent predictions.

3.1. Interacting Models Dynamical Evolution

To investigate the background dynamics, we numerically integrate the autonomous evolution equation for Ω_{de} derived below for each interaction prescription.

Throughout this subsection, we adopt the benchmark parameter set $\Delta = 0.8$, $\omega = 0.263$, and $\beta = 0.123$. The Barrow exponent is restricted theoretically to $0 \leq \Delta \leq 1$, and $\Delta = 0.8$ is chosen to display a clearly non-negligible entropy deformation. The values of ω and β are representative choices used in related fractal dark-energy studies [24,27,28]. They are employed here to examine the dynamical response of the model and should not be interpreted as posterior constraints from a joint cosmological likelihood.

The interaction strength is varied over $\zeta \in [0, 0.12]$, including the non-interacting limit. This interval defines a benchmark sequence from weak to comparatively strong positive coupling and is used to compare the sensitivity of the four interaction prescriptions. The strong-coupling endpoint is illustrative and is not assumed a priori to be observationally preferred.

For the numerical normalization, we take $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$ and $\Omega_{de0} = 0.6847$. The latter is a Planck-inspired choice based on the base- Λ CDM estimate $\Omega_{\Lambda 0} = 0.6847 \pm 0.0073$ [3]; it is used only as the present-day normalization of the interacting BHDE solutions. The value of H_0 sets the distance scale, while its degeneracy with the absolute SN Ia magnitude is absorbed into the additive nuisance parameter $\mathcal{M}$.

For analytical simplicity, the background is treated as a reduced two-component system consisting of interacting BHDE and an effective pressureless matter sector. Baryons and radiation are not evolved as separately conserved components. The resulting model is therefore intended as a late-time benchmark rather than a complete reconstruction of the cosmic energy budget.

We allow for an energy exchange term Q between the two dark-sector components [44–57]. The conservation equations are then

$$\dot{\rho}_{dm} + (3 - \beta)H\rho_{dm} = Q, \quad (19)$$

$$\dot{\rho}_{de} + (1 + w_{de})(3 - \beta)H\rho_{de} = -Q. \quad (20)$$

We examine four commonly used phenomenological forms: $Q_1 = 3\zeta H\rho_{de}$, $Q_2 = 3\zeta H\rho_{dm}$, $Q_3 = 3\zeta H \frac{\rho_{de}\rho_{dm}}{\rho_{de} + \rho_{dm}}$, and $Q_4 = 3\zeta H \sqrt{\rho_{de}\rho_{dm}}$, where ζ is the coupling parameter and $\zeta = 0$ corresponds to the non-interacting limit.

With the sign convention adopted in Equations (19) and (20), $Q > 0$ corresponds to energy transfer from dark energy to dark matter, whereas $Q < 0$ describes transfer in the opposite direction. For an expanding background with positive dark-sector densities, all four interaction functions have the same sign as ζ . Hence, the range $\zeta \geq 0$ considered here represents energy transfer from BHDE to the effective pressureless matter component. A negative value of ζ would instead describe energy transfer from matter to BHDE.

Using $\rho_{cr} = \frac{3H^2}{8\pi G}$, the density parameters are

$$\Omega_{dm} = \frac{\rho_{dm}}{\rho_{cr}}, \quad \Omega_{de} = \frac{\rho_{de}}{\rho_{cr}}. \quad (21)$$

Combining Equation (21) with the modified Friedmann equation yields the effective closure relation

$$\Omega_{dm} + \Omega_{de} = 1 + \gamma, \quad (22)$$

$$\gamma = -\beta - \frac{\beta^2 \omega (1+z)^{2\beta}}{6}, \quad (23)$$

where the redshift is defined by $1+z \equiv a^{-1}$ (with $a_0 = 1$). The parameter γ encodes the deviation from the standard closure relation induced by the fractal measure.

For the adopted values $\beta = 0.123$ and $\omega = 0.263$, the present fractal correction is

$$\gamma_0 = -\beta - \frac{\beta^2 \omega}{6} = -0.123663. \quad (24)$$

Together with the Planck-inspired normalization $\Omega_{de0} = 0.6847$, the modified closure relation gives

$$\Omega_{dm0} = 1 + \gamma_0 - \Omega_{de0} = 0.191637. \quad (25)$$

This effective pressureless matter fraction is below the base- Λ CDM estimate $\Omega_{m0} = 0.315 \pm 0.007$ [3]. The difference follows directly from the negative fractal contribution γ_0 and shows that the adopted normalization does not reproduce the standard present-day matter budget. Accordingly, $\Omega_{dm0} = 0.191637$ is interpreted as the effective matter fraction of the selected fractal benchmark, not as an independent measurement of the observed matter abundance.

If baryons and radiation were introduced explicitly, the closure relation would become

$$\Omega_{dm} + \Omega_b + \Omega_r + \Omega_{de} = 1 + \gamma. \quad (26)$$

For the same γ_0 and Ω_{de0} , however, the total non-dark-energy fraction would still satisfy

$$\Omega_{dm0} + \Omega_{b0} + \Omega_{r0} = 1 + \gamma_0 - \Omega_{de0} = 0.191637. \quad (27)$$

Separating baryons and radiation would therefore redistribute the available non-dark-energy density but would not remove the reduced matter budget associated with the chosen fractal closure relation. This limitation is taken into account when interpreting the background and SN Ia comparisons below.

Differentiating the modified Friedmann equation with respect to time and using Equations (20) and (22), one obtains

$$\frac{\dot{H}}{H^2} = \frac{\frac{Q}{3M_p^2 H^3} + (\beta - 3)(1 + \gamma - \Omega_{de}) + 2\beta(\gamma + \beta)}{2 + 2\gamma - \Omega_{de}(2 - \Delta)}. \quad (28)$$

From $\Omega_{de} = \rho_{de}/\rho_{cr} \propto H^{-\Delta}$, one finds

$$\dot{\Omega}_{de} = -\Delta \Omega_{de} \frac{\dot{H}}{H}. \quad (29)$$

Hereafter, a prime denotes differentiation with respect to $\ln a$, namely $\Omega'_{de} = d\Omega_{de}/d \ln a$.

$$\Omega'_{de} = -\Delta \Omega_{de} \frac{\frac{Q}{3M_p^2 H^3} + (\beta - 3)(1 + \gamma - \Omega_{de}) + 2\beta(\gamma + \beta)}{2 + 2\gamma - \Omega_{de}(2 - \Delta)}. \quad (30)$$

Combining Equations (20), (22) and (28) yields the BHDE equation of state:

$$w_{de} = \frac{Q}{3(\beta-3)\Omega_{de}H^3M_p^2} + \frac{(2-\Delta)\frac{Q}{3M_p^2H^3} + (\beta-3)(1+\gamma-\Omega_{de}) + 2\beta(\gamma+\beta)}{(\beta-3)(2+2\gamma-\Omega_{de}(2-\Delta))} - 1. \quad (31)$$

From the deceleration parameter definition $q \equiv -\frac{\ddot{a}}{aH^2}$, we obtain

$$q = -1 - \frac{\dot{H}}{H^2} = -1 - \frac{\frac{Q}{3M_p^2H^3} + (\beta-3)(1+\gamma-\Omega_{de}) + 2\beta(\gamma+\beta)}{2+2\gamma-\Omega_{de}(2-\Delta)}. \quad (32)$$

Inserting Q_1 – Q_4 into Equations (30)–(32) yields

$$(1) \quad Q_1 = 3H\zeta\rho_{de}$$

$$\Omega'_{de} = -\Delta\Omega_{de}\frac{3\zeta\Omega_{de} + (\beta-3)(1+\gamma-\Omega_{de}) + 2\beta(\gamma+\beta)}{2+2\gamma-\Omega_{de}(2-\Delta)}, \quad (33)$$

$$w_{de} = \frac{3\zeta}{(\beta-3)} + \frac{(2-\Delta)\frac{3\zeta\Omega_{de} + (\beta-3)(1+\gamma-\Omega_{de}) + 2\beta(\gamma+\beta)}{2+2\gamma-\Omega_{de}(2-\Delta)}}{(\beta-3)} - 1, \quad (34)$$

$$q = -1 - \frac{3\zeta\Omega_{de} + (\beta-3)(1+\gamma-\Omega_{de}) + 2\beta(\gamma+\beta)}{2+2\gamma-\Omega_{de}(2-\Delta)}. \quad (35)$$

$$(2) \quad Q_2 = 3H\zeta\rho_{dm}$$

$$\Omega'_{de} = -\Delta\Omega_{de}\frac{(3\zeta+\beta-3)(1+\gamma-\Omega_{de}) + 2\beta(\gamma+\beta)}{2+2\gamma-\Omega_{de}(2-\Delta)}, \quad (36)$$

$$w_{de} = \frac{3\zeta(1+\gamma-\Omega_{de})}{(\beta-3)\Omega_{de}} + \frac{(2-\Delta)\frac{(3\zeta+\beta-3)(1+\gamma-\Omega_{de}) + 2\beta(\gamma+\beta)}{2+2\gamma-\Omega_{de}(2-\Delta)}}{(\beta-3)} - 1, \quad (37)$$

$$q = -1 - \frac{(3\zeta+\beta-3)(1+\gamma-\Omega_{de}) + 2\beta(\gamma+\beta)}{2+2\gamma-\Omega_{de}(2-\Delta)}. \quad (38)$$

$$(3) \quad Q_3 = 3\zeta H \frac{\rho_{de}\rho_{dm}}{\rho_{de}+\rho_{dm}}$$

$$\Omega'_{de} = -\Delta\Omega_{de}\frac{3\zeta\Omega_{de}(1+\gamma-\Omega_{de}) + (\beta-3)(1+\gamma-\Omega_{de})(1+\gamma) + 2\beta(\gamma+\beta)(1+\gamma)}{[2+2\gamma-\Omega_{de}(2-\Delta)](1+\gamma)}, \quad (39)$$

$$w_{de} = \frac{3\zeta(1+\gamma-\Omega_{de})}{(\beta-3)(1+\gamma)} + \frac{(2-\Delta)\frac{3\zeta\Omega_{de}(1+\gamma-\Omega_{de}) + (\beta-3)(1+\gamma-\Omega_{de})(1+\gamma) + 2\beta(\gamma+\beta)(1+\gamma)}{[2+2\gamma-\Omega_{de}(2-\Delta)](1+\gamma)}}{(\beta-3)} - 1, \quad (40)$$

$$q = -1 - \frac{3\zeta\Omega_{de}(1+\gamma-\Omega_{de}) + (\beta-3)(1+\gamma-\Omega_{de})(1+\gamma) + 2\beta(\gamma+\beta)(1+\gamma)}{[2+2\gamma-\Omega_{de}(2-\Delta)](1+\gamma)}. \quad (41)$$

$$(4) \quad Q_4 = 3\zeta H \sqrt{\rho_{de}\rho_{dm}}$$

$$\Omega'_{de} = -\Delta\Omega_{de}\frac{3\zeta[\Omega_{de}(1+\gamma-\Omega_{de})]^{1/2} + (\beta-3)(1+\gamma-\Omega_{de}) + 2\beta(\gamma+\beta)}{2+2\gamma-\Omega_{de}(2-\Delta)}, \quad (42)$$

$$w_{de} = \frac{3\zeta[\Omega_{de}(1+\gamma-\Omega_{de})]^{1/2}}{(\beta-3)\Omega_{de}} + \frac{(2-\Delta)\frac{3\zeta[\Omega_{de}(1+\gamma-\Omega_{de})]^{1/2} + (\beta-3)(1+\gamma-\Omega_{de}) + 2\beta(\gamma+\beta)}{2+2\gamma-\Omega_{de}(2-\Delta)}}{(\beta-3)} - 1, \quad (43)$$

$$q = -1 - \frac{3\zeta[\Omega_{de}(1+\gamma-\Omega_{de})]^{1/2} + (\beta-3)(1+\gamma-\Omega_{de}) + 2\beta(\gamma+\beta)}{2+2\gamma-\Omega_{de}(2-\Delta)}. \quad (44)$$

The autonomous equations are integrated using $\ln a$ as the independent variable over $\ln(0.01) \leq \ln a \leq \ln(100)$, with the present-day initial condition $\Omega_{\text{de}}(\ln a = 0) = \Omega_{\text{de}}(a = 1) = 0.6847$. We use an adaptive stiffness-switching algorithm with working precision 30 and accuracy and precision goals equal to 14. Because the step size is selected adaptively by the solver, no fixed integration step is imposed. The displayed interval is $-0.99 \leq z \leq 3$, where $1 + z = a^{-1}$. The transition redshift is obtained by locating a sign change of $q(z)$ in $0 \leq z \leq 3$ and refining the root of $q(z_t) = 0$ within the corresponding bracket.

The physical background domain was checked over the same numerical interval for every interaction prescription and coupling strength. The solutions satisfy $\Omega_{\text{de}} > 0$, $\Omega_{\text{dm}} = 1 + \gamma - \Omega_{\text{de}} \geq 0$, and $2 + 2\gamma - \Omega_{\text{de}}(2 - \Delta) > 0$. The quantities w_{de} and q remain real and finite. For Q_3 , the additional factor $1 + \gamma$ remains positive, while the radicand $\Omega_{\text{de}}\Omega_{\text{dm}}$ entering Q_4 is non-negative. Thus, the nonlinear interaction terms remain real and the benchmark solutions remain within the required physical background domain over $-0.99 \leq z \leq 3$. These checks concern the homogeneous background only and do not constitute a perturbative stability test.

The physical-domain conditions were evaluated quantitatively over all four interaction prescriptions, $\zeta = 0, 0.04, 0.08$, and 0.12 , and over $-0.99 \leq z \leq 3$. The global minimum of the effective matter fraction is

$$\min \Omega_{\text{dm}} = 6.265589 \times 10^{-7} > 0, \quad (45)$$

with the minimum attained by the common non-interacting solution at the future numerical boundary $z = -0.99$. The interacting solutions remain above this limiting value. The principal evolution denominator satisfies

$$\min[2 + 2\gamma - (2 - \Delta)\Omega_{\text{de}}] = 0.701393714 > 0, \quad (46)$$

where the global minimum occurs for the same non-interacting solution at $z \simeq -0.96881$. For the nonlinear interactions, the additional Q_3 factor and the Q_4 radicand satisfy

$$\min(1 + \gamma) = 0.876067344 > 0, \quad \min(\Omega_{\text{de}}\Omega_{\text{dm}}) = 5.493579 \times 10^{-7} > 0. \quad (47)$$

The minimum Q_4 radicand is likewise attained in the non-interacting limit at $z = -0.99$. These numerical minima explicitly confirm that the benchmark solutions remain real and within the physical background domain throughout the investigated redshift interval.

The nonlinear prescriptions may nevertheless develop additional instabilities outside the verified background domain. The Q_3 interaction becomes singular if the total dark-sector density $\rho_{\text{de}} + \rho_{\text{dm}}$ approaches zero, whereas Q_4 requires both densities to remain non-negative and is non-analytic when either density approaches zero. At the perturbative level, variations of these nonlinear functions may contain inverse-density or square-root factors and can amplify fluctuations near such boundaries. Sufficiently large couplings may also drive a component toward a negative density or produce non-adiabatic large-scale instabilities. The numerical bounds above exclude these singular behaviors at the homogeneous-background level over the investigated interval, but they do not test ghost, gradient, sound-speed, or cosmological-perturbation stability. A complete perturbative stability analysis is left for future work.

Figure 1 shows the redshift evolution of Ω_{de} for the benchmark parameters and the common normalization $\Omega_{\text{de}0} = 0.6847$. The curves therefore meet at $z = 0$. In all four interaction models, the dark-energy fraction decreases toward the past and increases toward the future boundary of the numerical interval. At $z > 0$, a larger positive coupling generally increases Ω_{de} relative to the non-interacting solution, whereas at $z < 0$ it suppresses the

dark-energy fraction. The coupling dependence is strongest for Q_1 , weakest for Q_3 , and intermediate for Q_2 and Q_4 .

The transition redshift provides a useful kinematic characterization of the change from decelerated to accelerated expansion [58].

Figure 2 displays the deceleration parameter. Each interaction prescription evolves from $q > 0$ at sufficiently high redshift within the plotted interval to $q < 0$ at the present epoch. The common non-interacting solution gives $z_t = 0.85845$. Table 1 displays the transition redshift for the different coupling parameter, such as $\xi = 0.12$, the transition redshifts are 2.02175, 1.97160, 1.25782, and 1.87260 for Q_1 , Q_2 , Q_3 , and Q_4 , respectively. For Q_4 , a finite transition is obtained for every coupling considered, and the solution has returned to the decelerating branch by $z = 3$. The values near $z_t \simeq 2$ at the strong-coupling endpoint are illustrative and are not presented as observationally preferred transition epochs.

Figure 3 presents the evolution of w_{de} . The equation of state remains real and finite over the numerical interval. Its stronger high-redshift variation, particularly for interaction forms involving the matter density, must be interpreted within the reduced two-component late-time model and does not establish compatibility with early-universe constraints. Some strongly coupled solutions approach or cross $w_{de} = -1$ near the future boundary of the numerical interval; this background behavior alone neither demonstrates a perturbative instability nor proves the absence or presence of a finite-time big-rip singularity. A separate perturbative and asymptotic analysis would be required.

Because the calculation is restricted to $z \leq 3$ and does not evolve baryons and radiation separately, its domain of applicability is the late-time homogeneous background. It does not describe recombination, primordial nucleosynthesis, or the radiation-dominated epoch.

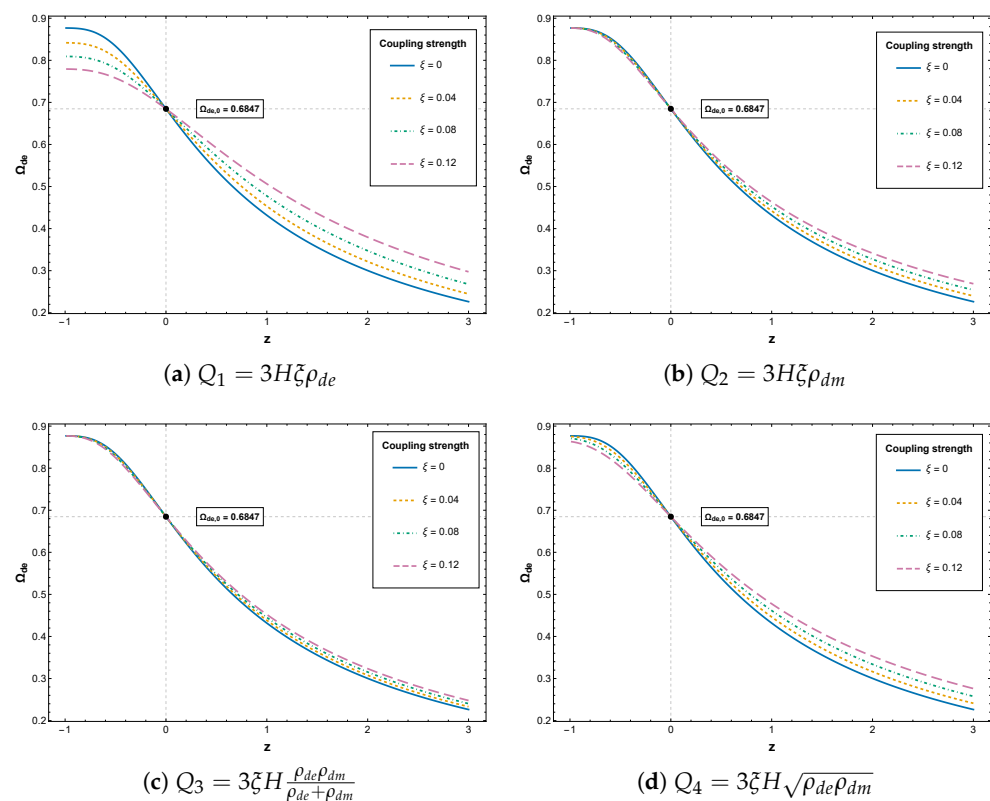


Figure 1. Evolution of $\Omega_{de}(z)$ for the four interaction forms and $\xi = 0, 0.04, 0.08$, and 0.12 , evaluated at $(\Delta, \omega, \beta) = (0.8, 0.263, 0.123)$ and $\Omega_{de0} = 0.6847$.

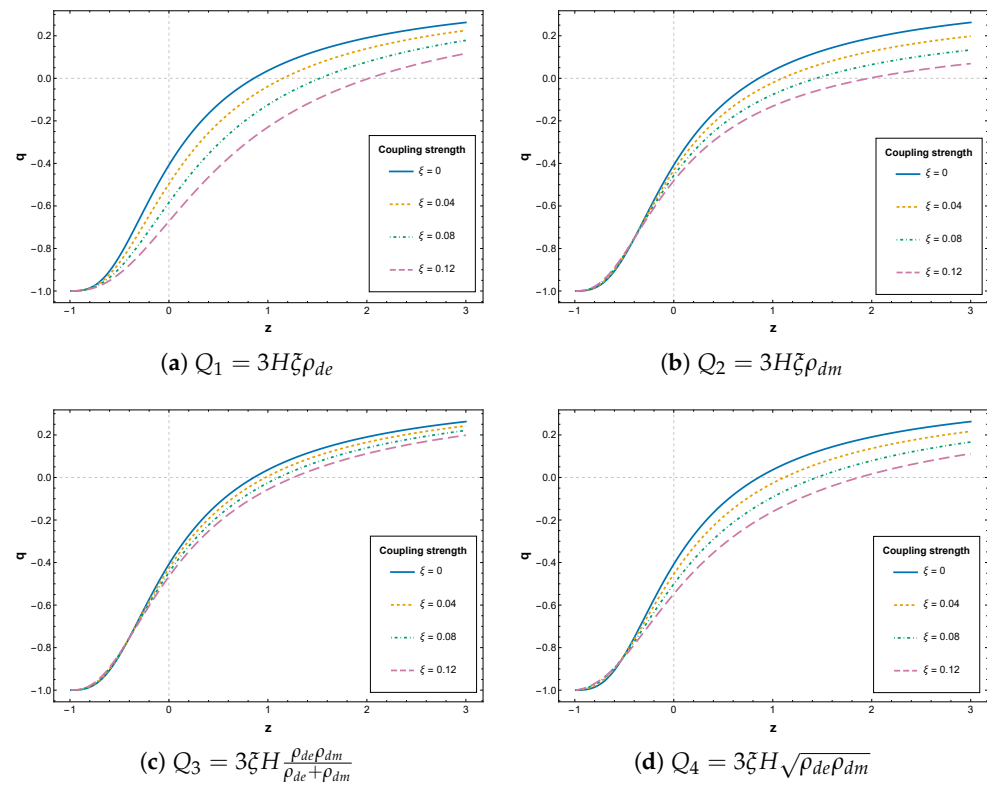


Figure 2. Evolution of the deceleration parameter $q(z)$ for the four interaction forms and coupling strengths, evaluated at $(\Delta, \omega, \beta) = (0.8, 0.263, 0.123)$.

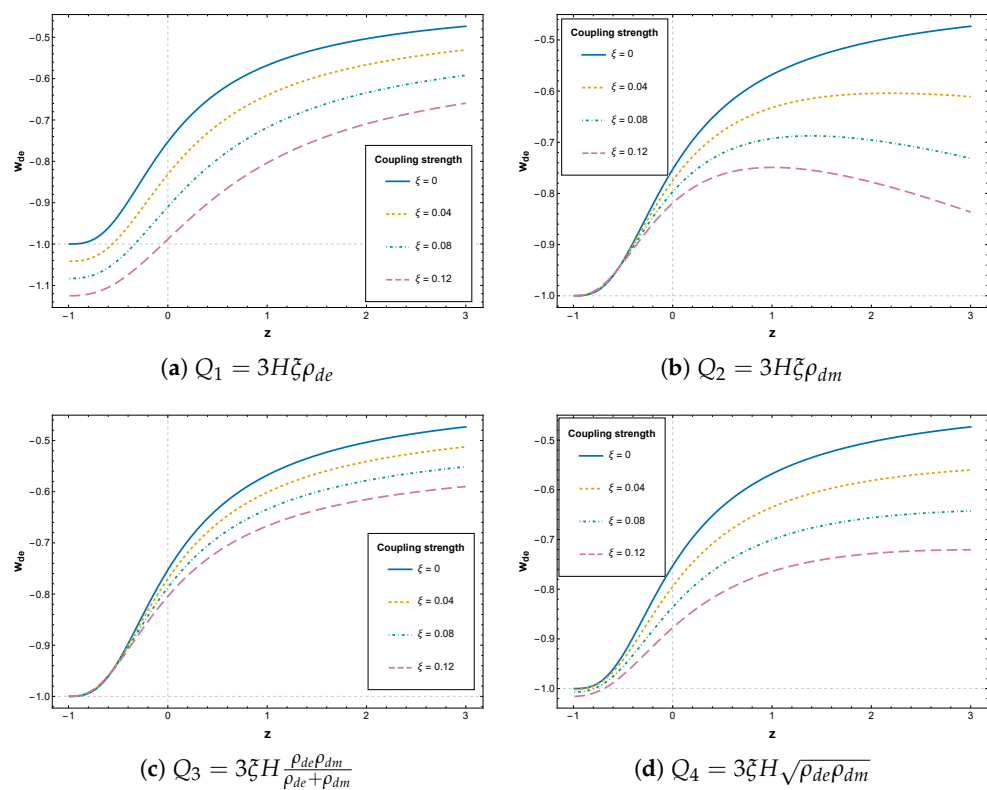


Figure 3. Evolution of the dark-energy equation-of-state parameter $w_{de}(z)$ for the four interaction forms and coupling strengths, evaluated at $(\Delta, \omega, \beta) = (0.8, 0.263, 0.123)$.

Table 1. Acceleration-transition redshift z_t , defined by $q(z_t) = 0$, for the four interaction forms and coupling strengths.

Model	$\xi = 0$	$\xi = 0.04$	$\xi = 0.08$	$\xi = 0.12$
Q_1	0.85845	1.15252	1.52466	2.02175
Q_2	0.85845	1.09535	1.43838	1.97160
Q_3	0.85845	0.97610	1.10864	1.25782
Q_4	0.85845	1.11495	1.44445	1.87260

3.2. SN Ia Distance Modulus and Parameter Sensitivity

To examine the late-time background consistency of the interacting fractal BHDE scenarios, we compare their luminosity-distance predictions with the Pantheon SN Ia sample [59]. After parsing the FITRES table and retaining valid entries in $0.01 \leq z \leq 2.5$, the numerical analysis contains $N = 1046$ supernovae. The observable distance modulus is

$$\mu(z) = 5 \log_{10} \left[\frac{d_L(z)}{\text{Mpc}} \right] + 25, \quad (48)$$

where

$$d_L(z) = (1+z)c \int_0^z \frac{dz'}{H(z')}. \quad (49)$$

For the interacting BHDE models, $H(z)$ is reconstructed directly from the background quantity $B(z) \equiv \dot{H}/H^2$ through

$$\frac{d \ln H}{dz} = -\frac{B(z)}{1+z}, \quad E(z) \equiv \frac{H(z)}{H_0} = \exp \left[\int_0^z -\frac{B(z')}{1+z'} dz' \right]. \quad (50)$$

This construction uses the same interacting background equations as the dynamical analysis and does not impose a separately conserved $\Omega_{\text{dm}0}(1+z)^3$ term.

The absolute SN magnitude and the Hubble-scale normalization enter the distance modulus through an additive nuisance parameter $\mathcal{M}$. For the diagonal FITRES uncertainties used here, the profile chi-square is

$$\chi^2(\mathcal{M}) = \sum_{i=1}^N \frac{[\mu_{\text{obs},i} - \mu_{\text{th},i} - \mathcal{M}]^2}{\sigma_i^2}. \quad (51)$$

Writing $d_i = \mu_{\text{obs},i} - \mu_{\text{th},i}$ and $w_i = \sigma_i^{-2}$, the nuisance parameter is analytically profiled as

$$\mathcal{M}_{\text{best}} = \frac{\sum_i w_i d_i}{\sum_i w_i}. \quad (52)$$

For the Q_1 – Q_4 entries in Table 2, we fix $\xi = 0.04$, $(\Delta, \omega, \beta) = (0.8, 0.263, 0.123)$, and $\Omega_{\text{de}0} = 0.6847$. The cosmological background is therefore not fitted; only the additive nuisance parameter $\mathcal{M}$ is profiled. Hence $k = 1$ and

$$\text{dof} = N - k = 1045. \quad (53)$$

The information criteria are evaluated from

$$\text{AIC} = \chi_{\text{prof}}^2 + 2k, \quad \text{BIC} = \chi_{\text{prof}}^2 + k \ln N. \quad (54)$$

Table 2 gives a fixed-background comparison under the adopted diagonal-error likelihood. All five cases have reduced chi-square values close to unity. The flat Λ CDM reference gives the smallest χ^2 , AIC, and BIC values. Among the interacting BHDE benchmarks, Q_4 is closest to Λ CDM, followed by Q_2 , Q_3 , and Q_1 . The corresponding information-criterion differences are $\Delta\text{AIC} = \Delta\text{BIC} = 1.20913, 2.22817, 3.10085$, and 3.33059 , respectively. These

values compare only the selected background curves under the simplified likelihood; they do not constitute a global fit of the BHDE parameter space or a complete test of observational viability.

Table 2. Fixed-parameter comparison with the selected Pantheon SN Ia sample. The interacting Q_1 – Q_4 entries use $\xi = 0.04$, $(\Delta, \omega, \beta) = (0.8, 0.263, 0.123)$, and $\Omega_{\text{de}0} = 0.6847$. The additive nuisance parameter $\mathcal{M}$ is analytically profiled out, and diagonal FITRES uncertainties are used.

Model	$\mathcal{M}_{\text{best}}$	χ^2_{prof}	$\chi^2_{\text{prof}}/\text{dof}$	AIC	BIC	ΔAIC
ΛCDM	0.010279	1054.76507	1.00934	1056.76507	1061.71780	0.00000
Q_1	0.007745	1058.09566	1.01253	1060.09566	1065.04839	3.33059
Q_2	0.022976	1056.99324	1.01148	1058.99324	1063.94597	2.22817
Q_3	0.025185	1057.86592	1.01231	1059.86592	1064.81865	3.10085
Q_4	0.017461	1055.97420	1.01050	1057.97420	1062.92693	1.20913

Figure 4 shows one-parameter sensitivity tests for the Q_1 interaction with $\xi = 0.04$. The upper panels display the SN Ia Hubble diagram, and the lower panels show $\Delta\mu = \mu_{\text{model}} - \mu_{\Lambda\text{CDM}}$. The nuisance parameter is profiled independently for each curve, so the residuals primarily compare the redshift-dependent shape of the distance relation. The models are nearly degenerate at low redshift and separate gradually toward the high-redshift end. Within the displayed parameter ranges, variations of Δ and β produce clearer changes than variations of ω . These plots characterize local sensitivity around the benchmark point rather than posterior parameter constraints.

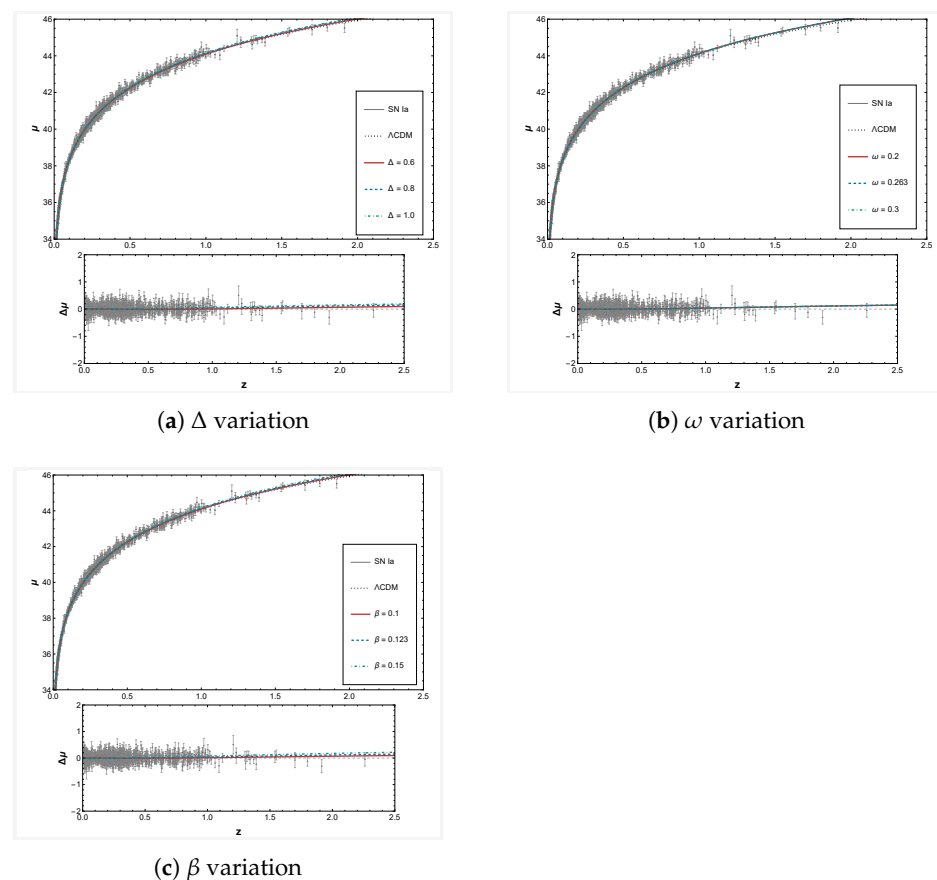


Figure 4. SN Ia distance-modulus and residual comparisons for one-parameter variations around the fiducial interacting configuration $Q_1 = 3\xi H\rho_{\text{de}}$, $\xi = 0.04$, and $(\Delta, \omega, \beta) = (0.8, 0.263, 0.123)$. Gray points and thin error bars denote the selected SN Ia data, the black dotted line is the flat ΛCDM reference with $(\Omega_{m0}, \Omega_{\Lambda0}) = (0.3153, 0.6847)$, and the lower panels show $\Delta\mu = \mu_{\text{model}} - \mu_{\Lambda\text{CDM}}$.

4. Statefinder Diagnostic Examination

Geometric diagnostics provide a complementary way to compare dark-energy models at the level of background kinematics [60,61]. Defining $H = \dot{a}/a$ and $q = -\ddot{a}/(aH^2)$, the statefinder pair is

$$r \equiv \frac{\ddot{a}}{aH^3} = 2q^2 + q - \frac{\dot{q}}{H} = 2q^2 + q - a \frac{dq}{da}, \quad (55)$$

$$s \equiv \frac{r-1}{3\left(q-\frac{1}{2}\right)}. \quad (56)$$

The statefinder hierarchy has been widely applied to distinguish dynamical and holographic dark-energy models [62,63]. The quantities used below are

$$S_3^{(1)} \equiv r, \quad S_3^{(2)} \equiv \frac{S_3^{(1)} - 1}{3\left(q - \frac{1}{2}\right)} = s. \quad (57)$$

For a spatially flat Λ CDM background, $S_3^{(1)} = 1$, $S_3^{(2)} = 0$, and the corresponding fixed point in the s - r plane is $(0, 1)$. For the numerical evaluation, $q(\ln a)$ is constructed from the same high-precision background solution used for $\Omega_{\text{de}}(\ln a)$. Since $a dq/da = dq/d \ln a$, the derivative entering Equation (55) is obtained directly from the derivative of the interpolating solution rather than by finite differencing the plotted data. The statefinder calculation uses the same adaptive stiffness-switching integration, working precision 30, and accuracy and precision goals of 14 as the background analysis. Since higher derivatives are especially sensitive near the boundary of an interpolation interval, the physical discussion is based on the smooth finite-redshift trajectories and the explicitly evaluated present-day values; no conclusion is drawn from small-scale features extremely close to $z = -1$.

As a numerical convergence test, the statefinder trajectories were recalculated using working precision 40 and accuracy and precision goals equal to 18, and were compared with the results obtained using working precision 30 and accuracy and precision goals equal to 14. Over the smooth finite-redshift interval $-0.9 \leq z \leq 3$, including all four interaction prescriptions and all coupling strengths considered, the maximum absolute differences were

$$\max |\delta r| = 4.50384 \times 10^{-9}, \quad \max |\delta s| = 1.67530 \times 10^{-9}. \quad (58)$$

The statefinder trajectories and the present-day values reported in Table 3 are therefore numerically stable to well beyond the quoted precision.

Substituting interaction forms Q_1 – Q_4 into the background equation gives

$$\left. \frac{\dot{H}}{H^2} \right|_{Q_1} = \frac{3\zeta\Omega_{\text{de}} + 2\beta(\gamma + \beta) + (\beta - 3)(1 + \gamma - \Omega_{\text{de}})}{2 + 2\gamma - \Omega_{\text{de}}(2 - \Delta)}, \quad (59)$$

$$\left. \frac{\dot{H}}{H^2} \right|_{Q_2} = \frac{2\beta(\gamma + \beta) + (3\zeta + \beta - 3)(1 + \gamma - \Omega_{\text{de}})}{2 + 2\gamma - \Omega_{\text{de}}(2 - \Delta)}, \quad (60)$$

$$\left. \frac{\dot{H}}{H^2} \right|_{Q_3} = \frac{3\zeta\Omega_{\text{de}}(1 + \gamma - \Omega_{\text{de}}) + 2\beta(\gamma + \beta)(1 + \gamma) + (\beta - 3)(1 + \gamma - \Omega_{\text{de}})(1 + \gamma)}{[2 + 2\gamma - \Omega_{\text{de}}(2 - \Delta)](1 + \gamma)}, \quad (61)$$

$$\left. \frac{\dot{H}}{H^2} \right|_{Q_4} = \frac{3\zeta[\Omega_{\text{de}}(1 + \gamma - \Omega_{\text{de}})]^{1/2} + 2\beta(\gamma + \beta) + (\beta - 3)(1 + \gamma - \Omega_{\text{de}})}{2 + 2\gamma - \Omega_{\text{de}}(2 - \Delta)}. \quad (62)$$

Table 3. Present-day dynamical and statefinder quantities for the four interaction forms. The last row gives the corresponding flat Λ CDM benchmark.

Model	ξ	$w_{\text{de}0}$	q_0	z_t	r_0	s_0
Q_1	0.00	−0.752928	−0.407645	0.85845	0.679369	0.117752
	0.04	−0.831448	−0.495895	1.15252	0.705092	0.098708
	0.08	−0.909967	−0.584146	1.52466	0.738510	0.080398
	0.12	−0.988486	−0.672396	2.02175	0.779622	0.062658
Q_2	0.00	−0.752928	−0.407645	0.85845	0.679369	0.117752
	0.04	−0.774904	−0.432345	1.09535	0.634521	0.130667
	0.08	−0.796881	−0.457045	1.43838	0.594739	0.141150
	0.12	−0.818857	−0.481745	1.97160	0.560024	0.149386
Q_3	0.00	−0.752928	−0.407645	0.85845	0.679369	0.117752
	0.04	−0.770099	−0.426944	0.97610	0.652739	0.124877
	0.08	−0.787269	−0.446242	1.10864	0.628606	0.130831
	0.12	−0.804440	−0.465541	1.25782	0.606970	0.135686
Q_4	0.00	−0.752928	−0.407645	0.85845	0.679369	0.117752
	0.04	−0.794468	−0.454333	1.11495	0.645439	0.123842
	0.08	−0.836008	−0.501021	1.44445	0.621637	0.125992
	0.12	−0.877548	−0.547709	1.87260	0.607963	0.124728
Λ CDM	–	−1.000000	−0.527050	0.63156	1.000000	0.000000

Figure 5 shows the evolution of $S_3^{(1)} = r$. Over the displayed interval, the trajectories exhibit a pronounced finite-redshift minimum and move toward the Λ CDM reference value as z approaches -0.99 . The effect of the coupling depends on the interaction form. For Q_1 , increasing ξ raises the present value of r from 0.679369 to 0.779622, thereby moving the current state closer to unity, although the ordering changes along the full trajectory. For Q_2 , Q_3 , and Q_4 , a stronger coupling lowers the present value of r . The trajectory separation is largest for Q_2 , weakest for Q_3 , and intermediate for Q_4 .

Figure 6 displays positive values of $S_3^{(2)} = s$ throughout the finite-redshift domain and a decrease toward zero as z approaches -0.99 . In the Q_1 case, increasing ξ lowers $S_3^{(2)}$ over the main redshift interval and shifts the present state closer to the Λ CDM value. The Q_2 , Q_3 , and Q_4 cases show redshift-dependent curve crossings, so their coupling dependence cannot be represented by a single ordering over the entire evolution. The separation is comparatively weak for Q_3 and more visible for Q_2 and Q_4 .

The s – r trajectories in Figure 7 remain in the region $s > 0$ and $r < 1$ over the displayed finite-redshift evolution and move toward the Λ CDM point $(0, 1)$ near the future boundary of the numerical interval. For Q_1 , a larger coupling shifts the present point toward $(0, 1)$, with r_0 increasing and s_0 decreasing. For Q_2 and Q_3 , the present point moves away from the fixed point as ξ increases. For Q_4 , r_0 decreases, whereas s_0 changes only slightly and is not strictly monotonic. Consequently, the statefinder plane distinguishes both the interaction prescriptions and the response of their present kinematic states to the coupling.

For quantitative comparison, flat Λ CDM with $\Omega_{m0} = 0.3153$ and $\Omega_{\Lambda 0} = 0.6847$ has $w_{\text{de}0} = -1$, $q_0 = -0.52705$, $z_t = 0.63156$, $r_0 = 1$, and $s_0 = 0$. The benchmark BHDE cases therefore remain distinguishable from Λ CDM through their transition redshifts and present-day statefinder coordinates. The approach of the numerical trajectories toward $(0, 1)$ near $z = -0.99$ is a finite-interval result and is not presented as an analytical proof of a future attractor.

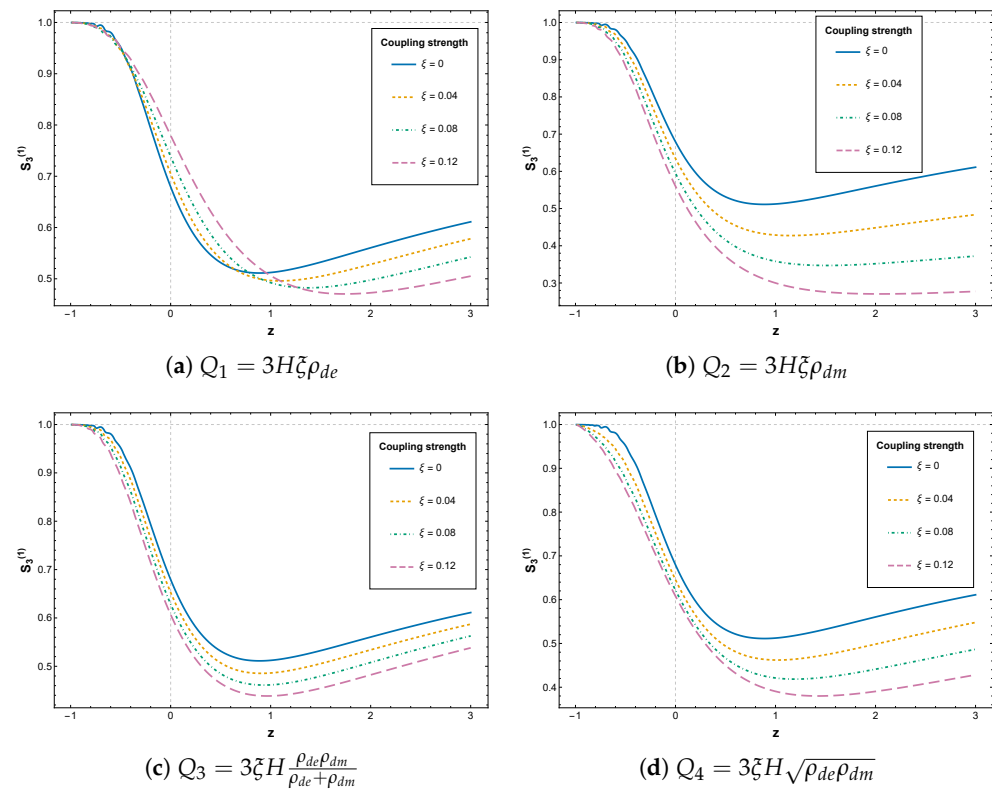


Figure 5. Evolution of $S_3^{(1)} = r$ for the four interaction forms and different coupling strengths. The horizontal reference $S_3^{(1)} = 1$ represents the flat Λ CDM value.

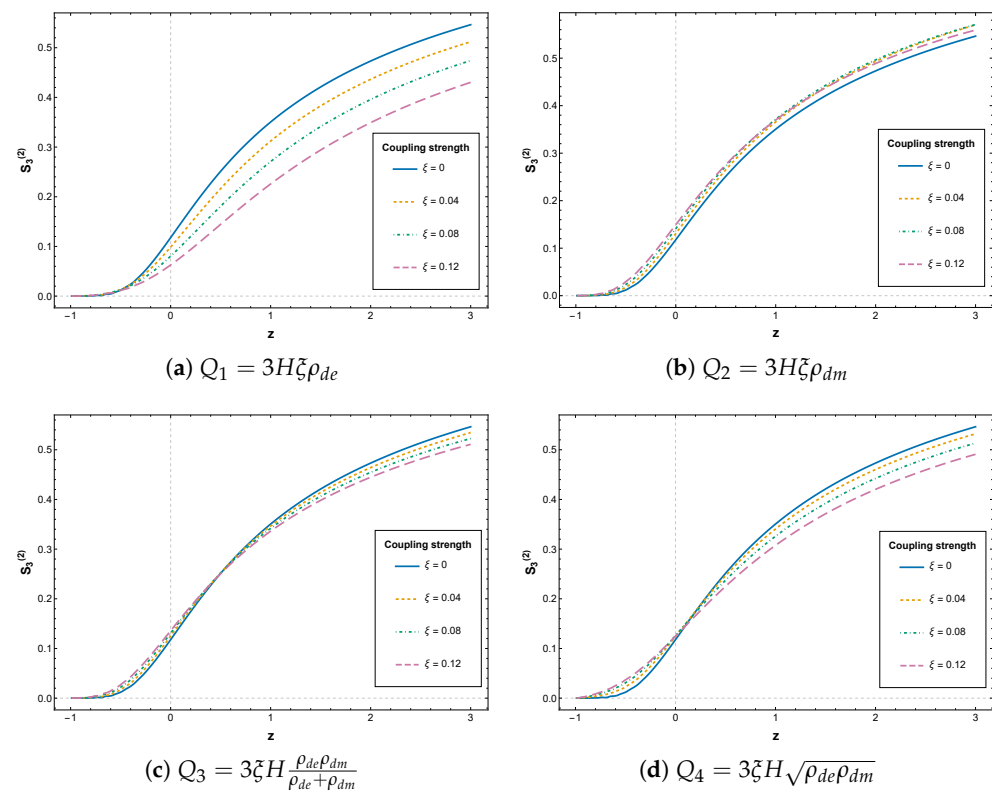


Figure 6. Evolution of $S_3^{(2)} = s$ for the four interaction forms and different coupling strengths. The horizontal reference $S_3^{(2)} = 0$ represents the flat Λ CDM value.

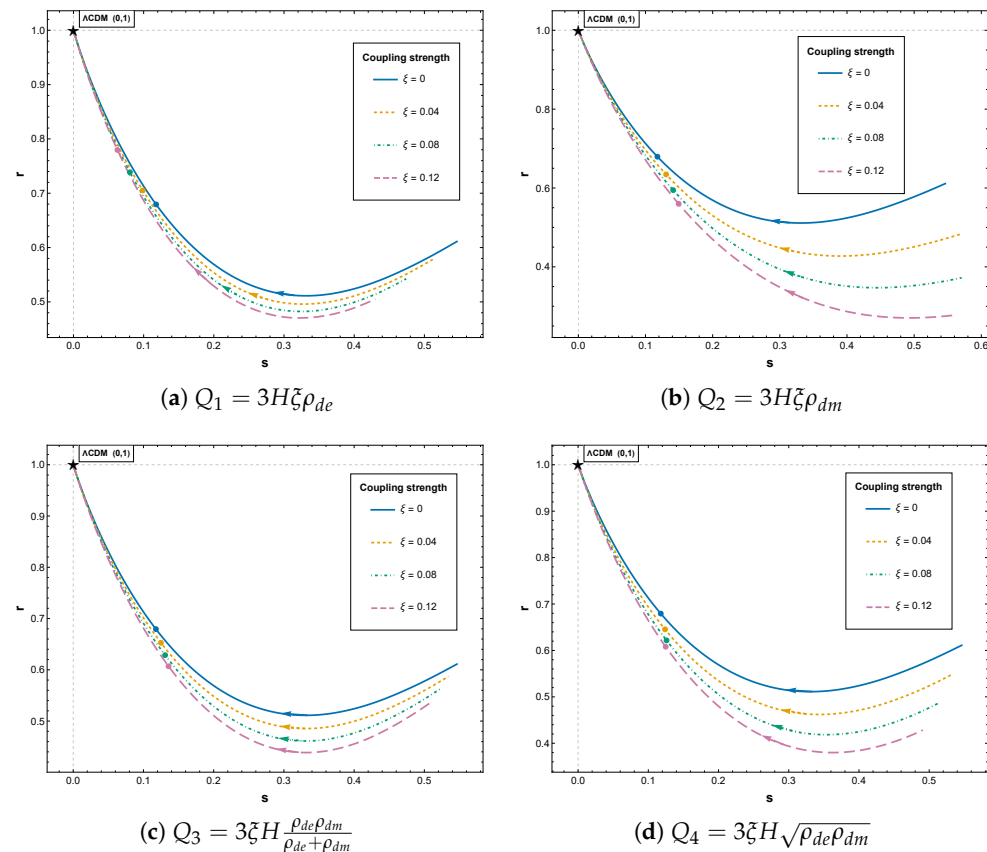


Figure 7. Statefinder trajectories in the s – r plane. The black star denotes the Λ CDM fixed point $(0, 1)$, the colored circles denote the present states at $z = 0$, and the arrows indicate the direction of evolution from higher to lower redshift.

5. Conclusions

We have investigated interacting Barrow holographic dark energy with the Hubble horizon as the infrared cutoff in a fractal cosmological background. Four phenomenological interactions between BHDE and an effective pressureless matter component were considered, including the nonlinear Q_3 and Q_4 prescriptions. The autonomous equations were integrated for the benchmark parameters $(\Delta, \omega, \beta) = (0.8, 0.263, 0.123)$ and the Planck-inspired normalization $\Omega_{de0} = 0.6847$ over $-0.99 \leq z \leq 3$.

The numerical solutions remain within the required physical background domain throughout this interval: $\Omega_{de} > 0$, $\Omega_{dm} \geq 0$, the principal denominator remains positive, and the additional factors entering Q_3 and Q_4 remain real. These checks establish the consistency of the homogeneous benchmark solutions over the numerical domain, but they do not address perturbative stability.

The interaction modifies both the strength and the timing of the late-time acceleration. All four prescriptions reduce to the same non-interacting solution, with $z_t = 0.85845$. At $\tilde{\xi} = 0.12$, the transition redshifts are 2.02175, 1.97160, 1.25782, and 1.87260 for Q_1 , Q_2 , Q_3 , and Q_4 , respectively. Thus, Q_1 produces the strongest coupling-induced shift and Q_3 the weakest. The Q_4 solutions recover a decelerating branch within the plotted past domain, with a finite transition redshift for every coupling considered. Increasing the coupling also makes the present-day values of w_{de} and q more negative, although the magnitude of the response depends on the interaction structure.

The statefinder hierarchy provides an additional geometric discrimination. Increasing $\tilde{\xi}$ moves the present Q_1 state toward the Λ CDM point, while the other prescriptions show different directions and strengths of departure. Near the future boundary of the numerical

interval, the trajectories move toward the Λ CDM coordinates, but this finite-interval behavior is not an analytical demonstration of an asymptotic attractor.

A fixed-background SN Ia comparison was performed using 1046 selected Pantheon supernovae, diagonal FITRES uncertainties, and an analytically profiled additive nuisance parameter. For $\xi = 0.04$, flat Λ CDM gives the lowest χ^2 , AIC, and BIC. Among the four interacting BHDE benchmarks, Q_4 is closest to the reference model, with $\Delta\text{AIC} = \Delta\text{B} = 1.20913$. This ranking is specific to the selected parameter values and the adopted simplified likelihood; it should not be interpreted as a global model-selection result.

For the adopted normalization, the modified closure relation gives $\Omega_{\text{dm}0} = 0.191637$, below the standard cosmological matter abundance. The present construction should therefore be regarded as a reduced late-time benchmark rather than a complete precision-cosmology model. The present conclusions are specific to the Hubble-horizon infrared cutoff; alternative cutoffs require separate background equations and an independent numerical analysis. Future work should include separately evolved baryons and radiation, the full SN covariance, joint constraints from BAO, CMB, and cosmic-chronometer data, and cosmological perturbations, including the stability of the nonlinear interaction prescriptions. A dedicated analysis of $H(t)$ and $a(t)$ is also required before any conclusion can be drawn about future-time attractors or finite-time singularities.

Appendix A. Comparative Analysis with Alternative Dark Energy Frameworks

This appendix compares the fractal Barrow holographic dark energy model with standard BHDE without fractal corrections, fractal Rényi holographic dark energy, and conventional holographic dark energy. The comparison illustrates how the fractal measure and the choice of entropy deformation modify the background expansion history.

The parameter choices adopted for the four comparison models are listed below.

- (1) Fractal Barrow Holographic Dark Energy (Fractal BHDE): $\Delta = 0.8$, $\omega = 0.263$, $\beta = 0.123$
- (2) Standard Barrow Holographic Dark Energy (Standard BHDE): $\Delta = 0.8$, $\omega = 0.263$, $\beta = 0$ (fractal effects negated)
- (3) Fractal Rényi Holographic Dark Energy (Fractal RHDE): $\omega = 0.263$, $\beta = 0.123$, $\delta = -1400$ following the parameter choice of [64], where the constant C appearing in the evolution equations is related to the Rényi parameter through $C^2 \equiv 1/(1 + \delta)$.
- (4) Standard Holographic Dark Energy (Standard HDE): $c = 1.0$, a typical choice in standard holographic dark energy models [65].

For the interacting comparison, we take $\xi = 0.12$ so that the influence of the coupling is clearly visible in the trajectories. The interaction is fixed to $Q_1 = 3H\xi\rho_{\text{de}}$, consistently with the corresponding prescription studied in the main text. The evolution equations for Ω_{de} , w_{de} , and q are summarized below.

- (1) Fractal Barrow Holographic Dark Energy
The evolution equations within the fractal cosmology framework are given by Equations (33) and (34).
- (2) Standard Barrow Holographic Dark Energy
Setting $\beta = 0$ eliminates the fractal contributions, yielding

$$\Omega'_{\text{de}} = -\Delta\Omega_{\text{de}} \frac{3\xi\Omega_{\text{de}} - 3(1 - \Omega_{\text{de}})}{2 - \Omega_{\text{de}}(2 - \Delta)}, \quad (\text{A1})$$

$$w_{\text{de}} = \xi + \frac{(2 - \Delta)}{3} \frac{3\xi\Omega_{\text{de}} - 3(1 - \Omega_{\text{de}})}{2 - \Omega_{\text{de}}(2 - \Delta)} - 1. \quad (\text{A2})$$

(3) Fractal Rényi Holographic Dark Energy

The Rényi entropy modification within fractal cosmology leads to

$$\Omega'_{de} = \frac{(C^2 - \Omega_{de})[2\beta(\gamma + \beta) + 3\zeta\Omega_{de} + (\beta - 3)(1 + \gamma - \Omega_{de})]}{1 + \gamma - C^2}, \quad (\text{A3})$$

$$w_{de} = \frac{3\zeta}{(\beta - 3)} + \frac{C^2}{(\beta - 3)\Omega_{de}} \frac{2\beta(\gamma + \beta) + 3\zeta\Omega_{de} + (\beta - 3)(1 + \gamma - \Omega_{de})}{1 + \gamma - C^2} - 1. \quad (\text{A4})$$

where C is determined by the Rényi parameter δ .

(4) Standard Holographic Dark Energy

The conventional holographic dark energy model follows [66]:

$$\Omega'_{de} = \Omega_{de}(1 - \Omega_{de}) \left(1 + \frac{2\sqrt{\Omega_{de}}}{c} \right) - 3\zeta\Omega_{de}(1 - \Omega_{de}), \quad (\text{A5})$$

$$w_{de} = -\frac{1}{3} - \frac{2}{3c} \sqrt{\Omega_{de}} - \frac{\zeta}{\Omega_{de}}. \quad (\text{A6})$$

The evolutionary trajectories of Ω_{de} and w_{de} for these four models are presented in Figure A1, for both non-interacting ($\zeta = 0$) and interacting ($\zeta = 0.12$) cases.

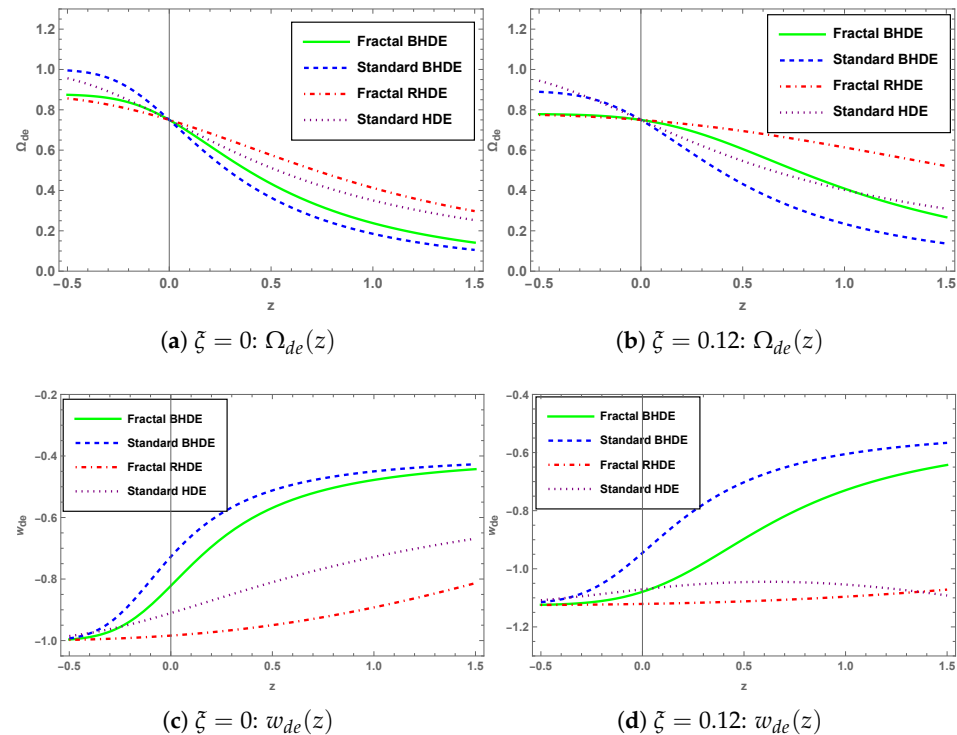


Figure A1. Comparison of the background evolution for four dark-energy scenarios: fractal BHDE, standard BHDE ($\beta = 0$), fractal Rényi HDE, and standard HDE. Results are shown for the benchmark interaction $Q_1 = 3\zeta H\rho_{de}$ with $\zeta = 0$ and $\zeta = 0.12$.

Figure A1a,b display the evolution of the dark-energy density parameter Ω_{de} for the non-interacting ($\zeta = 0$) and interacting ($\zeta = 0.12$) cases, respectively. Across the displayed interval, all four models evolve from a smaller dark-energy contribution at positive redshift toward dark-energy domination at late times. For the chosen parameters, fractal BHDE varies most gradually, particularly for $z > 1$, whereas standard HDE evolves more steeply and approaches $\Omega_{de} = 1$ more rapidly for $z < 0$ in the interacting case. Standard BHDE and fractal Rényi HDE exhibit intermediate behavior. Thus, the fractal correction changes the rate at which the dark-energy fraction evolves in this benchmark comparison.

Figure A1c,d show the equation-of-state evolution for the non-interacting ($\zeta = 0$) and interacting ($\zeta = 0.12$) cases. For the non-interacting benchmark, fractal BHDE remains in the quintessence-like region $w_{de} > -1$ over the displayed redshift interval [Figure A1c]. With interaction [Figure A1d], w_{de} can cross the phantom divide while remaining above approximately -1.2 over the plotted domain. The finite-redshift curves describe the background evolution shown in the figure; the asymptotic future behavior requires a separate analysis. Standard HDE gives more negative w_{de} values at low redshift, reaching approximately $w_{de} = -1.1$ at $z = 0$ for $\zeta = 0.12$. Standard BHDE and fractal Rényi HDE show intermediate behavior.

The comparative analysis highlights characteristic features of fractal BHDE. For the considered parameters, the non-interacting solution remains quintessence-like over the plotted interval, the dark-energy fraction evolves comparatively smoothly, and the response to the selected coupling is milder than in the standard HDE benchmark.

Appendix B. Metric Variation and the Modified Friedmann Equations

The metric variation of the fractal action used in Section 2 is summarized below. The measure weight $v(x)$ is treated as a prescribed scalar profile when the action is varied with respect to $g^{\mu\nu}$. Up to a boundary term, the variation of the weighted Einstein–Hilbert term is

$$\delta(\sqrt{-g} v R) = \sqrt{-g} [v G_{\mu\nu} + g_{\mu\nu} \square v - \nabla_\mu \nabla_\nu v] \delta g^{\mu\nu}. \quad (\text{A7})$$

For the kinetic term of the measure weight,

$$\delta[-\sqrt{-g} v \omega \partial_\alpha v \partial^\alpha v] = \sqrt{-g} v \omega \left[\frac{1}{2} g_{\mu\nu} \partial_\alpha v \partial^\alpha v - \partial_\mu v \partial_\nu v \right] \delta g^{\mu\nu}. \quad (\text{A8})$$

Defining the matter energy–momentum tensor through

$$\delta S_m = -\frac{1}{2} \int d^D x \sqrt{-g} v T_{\mu\nu} \delta g^{\mu\nu}, \quad (\text{A9})$$

the metric field equation becomes

$$G_{\mu\nu} + \frac{1}{v} (g_{\mu\nu} \square v - \nabla_\mu \nabla_\nu v) + \omega \left[\frac{1}{2} g_{\mu\nu} \partial_\alpha v \partial^\alpha v - \partial_\mu v \partial_\nu v \right] = \frac{1}{M_p^2} T_{\mu\nu}. \quad (\text{A10})$$

For the spatially flat FRW metric and a homogeneous measure weight $v = v(t)$,

$$\partial_\alpha v \partial^\alpha v = -\dot{v}^2, \quad \square v = -(\ddot{v} + 3H\dot{v}). \quad (\text{A11})$$

The 00 component of Equation (A10) is

$$3H^2 + 3H \frac{\dot{v}}{v} - \frac{\omega}{2} \dot{v}^2 = \frac{\rho}{M_p^2}, \quad (\text{A12})$$

which, after division by three, gives Equation (8). The spatial diagonal component can be written as

$$2\dot{H} + 3H^2 + \frac{\ddot{v} + 2H\dot{v}}{v} + \frac{\omega}{2} \dot{v}^2 = -\frac{p}{M_p^2}. \quad (\text{A13})$$

Combining Equations (A12) and (A13) and using Equation (A11) gives

$$\dot{H} + H^2 - H \frac{\dot{v}}{v} + \frac{\omega}{3} \dot{v}^2 - \frac{\square v}{2v} = -\frac{\rho + 3p}{6M_p^2}, \quad (\text{A14})$$

which is Equation (9). This derivation also shows explicitly that the conventional flat-FRW equations are recovered when $v = 1$ and $\dot{v} = 0$.

Author Contributions: Conceptualization, H.Z.; Methodology, W.Y.; Formal analysis, H.Z.; Investigation, W.Y.; Writing—original draft, H.Z.; Writing—review & editing, W.Y.; Supervision, W.Y.; Funding acquisition, W.Y. All authors have read and agreed to the published version of the manuscript.

Funding: W. Yang’s work is supported by the National Natural Science Foundation of China under Grants Nos. 12547110, 12175096.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: The authors acknowledge the assistance of DeepSeek-V3 for unifying and revising the reference formatting throughout this manuscript. All theoretical derivations, analytical work and original conclusions of this research were independently completed by the authors.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Riess, A.G.; Filippenko, A.V.; Challis, P.; Clocchiatti, A.; Diercks, A.; Garnavich, P.M.; Gilliland, R.L.; Hogan, C.J.; Jha, S.; Kirshner, R.P.; et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *Astron. J.* **1998**, *116*, 1009–1038.
2. Perlmutter, S.; Aldering, G.; Goldhaber, G.; Knop, R.A.; Nugent, P.; Castro, P.G.; Deustua, S.; Fabbro, S.; Goobar, A.; Groom, D.E.; et al. Measurements of Ω and Λ from 42 high-redshift supernovae. *Astrophys. J.* **1999**, *517*, 565–586.
3. Planck Collaboration. Planck 2018 results. VI. Cosmological parameters. *Astron. Astrophys.* **2020**, *641*, A6. Erratum in *Astron. Astrophys.* **2021**, *652*, C4.
4. Einstein, A. Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie. In *Das Relativitätsprinzip: Eine Sammlung von Abhandlungen*; Springer: Berlin/Heidelberg, Germany, 1917; pp. 142–152.
5. Friedmann, A. Über die Krümmung des Raumes. *Z. Phys.* **1922**, *10*, 377–386.
6. Lemaître, G. Un univers homogène de masse constante et de rayon croissant, rendant compte de la vitesse radiale des nébuleuses extra-galactiques. *Ann. Soc. Sci. Brux. A* **1927**, *47*, 49–59.
7. Hubble, E. A relation between distance and radial velocity among extra-galactic nebulae. *Proc. Natl. Acad. Sci. USA* **1929**, *15*, 168–173.
8. Alpher, R.A.; Bethe, H.; Gamow, G. The origin of chemical elements. *Phys. Rev.* **1948**, *73*, 803–804.
9. Penzias, A.A.; Wilson, R.W. A measurement of excess antenna temperature at 4080 Mc/s. *Astrophys. J.* **1965**, *142*, 419–421.
10. Guth, A.H. Inflationary universe: A possible solution to the horizon and flatness problems. *Phys. Rev. D* **1981**, *23*, 347–356.
11. Weinberg, S. The cosmological constant problem. *Rev. Mod. Phys.* **1989**, *61*, 1–23.
12. Wang, B.; Gong, Y.G.; Abdalla, E. Transition of the dark energy equation of state in an interacting holographic dark energy model. *Phys. Lett. B* **2005**, *624*, 141–146.
13. Freedman, W.L. Measurements of the Hubble constant: Tensions in perspective. *Astrophys. J.* **2021**, *919*, 16.
14. Peebles, P.J.E.; Vilenkin, A. Quintessential inflation. *Phys. Rev. D* **1999**, *59*, 063505.
15. Chiba, T.; Okabe, T.; Yamaguchi, M. Kinetically driven quintessence. *Phys. Rev. D* **2000**, *62*, 023511.
16. Wang, L.; Steinhardt, P.J. Cluster abundance constraints on quintessence models. *Astrophys. J.* **1998**, *508*, 483–490.
17. Kamenshchik, A.Y.; Moschella, U.; Pasquier, V. An alternative to quintessence. *Phys. Lett. B* **2001**, *511*, 265–268.
18. Bento, M.C.; Bertolami, O.; Sen, A.A. Revival of the unified dark energy–dark matter model? *Phys. Rev. D* **2004**, *70*, 083519.
19. Wu, Y.B.; Yang, W.Q.; Wang, C.; Wang, L. Modified Chaplygin gas as an interacting holographic dark energy model. *Sci. China Phys. Mech. Astron.* **2010**, *53*, 598–606.
20. Li, M. A model of holographic dark energy. *Phys. Lett. B* **2004**, *603*, 1–5.
21. Pavon, D.; Zimdahl, W. Holographic dark energy and cosmic coincidence. *Phys. Lett. B* **2005**, *628*, 206–210.
22. Barrow, J.D. The area of a rough black hole. *Phys. Lett. B* **2020**, *808*, 135643.
23. Saridakis, E.N. Barrow holographic dark energy. *Phys. Rev. D* **2020**, *102*, 123525.

24. Jawad, A.; Butt, S.; Rani, S.; Asif, K. Cosmological aspects of sound speed parameterizations in fractal universe. *Eur. Phys. J. C* **2019**, *79*, 926.
25. Calcagni, G. Fractal universe and quantum gravity. *Phys. Rev. Lett.* **2010**, *104*, 251301.
26. Abdollahi Zadeh, M. New Tsallis agegraphic dark energy in fractal cosmology. *arXiv* **2019**, arXiv:1912.04959.
27. Al Mamon, A. Study of Tsallis holographic dark energy model in the framework of fractal cosmology. *Mod. Phys. Lett. A* **2020**, *35*, 2050251.
28. Al Mamon, A.; Mishra, A.K.; Sharma, U.K. Barrow holographic dark energy in fractal cosmology. *Int. J. Geom. Methods Mod. Phys.* **2022**, *19*, 2250231.
29. Pawar, D.D.; Raut, D.K.; Patil, W.D. FRW model with two-fluid source in fractal cosmology. *Pramana* **2022**, *96*, 133.
30. Chakraborty, G.; Chattopadhyay, S.; Güdekli, E.; Radinschi, I. Thermodynamics of Barrow holographic dark energy with specific cut-off. *Symmetry* **2021**, *13*, 562.
31. Chakraborty, G.; Chattopadhyay, S.; Güdekli, E. Viscous holographic $f(Q)$ cosmology with some versions of holographic dark energy with generalized cut-offs. *Res. Astron. Astrophys.* **2021**, *21*, 317.
32. Zubair, M.; Muneer, Q.; Güdekli, E. Thermodynamics and stability analysis of Tsallis holographic dark energy (THDE) models in $F(R, T)$ gravity. *Ann. Phys.* **2022**, *445*, 169068.
33. Saha, S.; Güdekli, E.; Chattopadhyay, S.; Acan Yıldız, G.D. Thermodynamics of the most generalized form of holographic dark energy and some particular cases with corrected entropies. *Nucl. Phys. B* **2025**, *1014*, 116867.
34. Joshi, P.; Güdekli, E.; Chattopadhyay, S. Cosmological and thermodynamic insights into Barrow–Ricci–Gauss–Bonnet holographic dark energy in the symmetric teleparallel framework of $f(Q)$ gravity. *Nucl. Phys. B* **2026**, *1027*, 117470.
35. Sultana, S.; Ranjit, C.; Chattopadhyay, S.; Güdekli, E. Cosmology of the interacting Tsallis holographic dark energy in $f(R, T)$ gravity framework. *Int. J. Mod. Phys. D* **2024**, *33*, 2450035.
36. Sarkar, A.; Chattopadhyay, S.; Güdekli, E. A statistical analysis of observational Hubble parameter data to discuss the cosmology of holographic Chaplygin gas. *Symmetry* **2021**, *13*, 701.
37. Myrzakulov, Y.; Koussour, M.; Çalışkan, A.; Güdekli, E.; Muminov, S.; Rayimbaev, J. Observational constraints on freezing quintessence in a nonlinear $f(R, L_m)$ gravity. *Int. J. Geom. Methods Mod. Phys.* **2025**, *22*, 2550092.
38. Yousaf, Z. Viscous fluid cosmology: A path to cosmic acceleration. *Phys. Dark Universe* **2025**, *48*, 101884.
39. Alshammari, M.; Rizwan, M.; Almatroud, O.A.; Bhatti, M.Z.; Alshammari, S.; Yousaf, Z. Imprints of holographic dark energy and minimally deformed wormholes in general relativity. *Phys. Dark Universe* **2026**, *51*, 102219.
40. Alblowy, A.H.; Rizwan, M.; Iqbal, N.; Mohammed, W.W.; Asad, H. Testing complexity to diagnose wormholes existence: Static and spherically symmetric case. *Eur. Phys. J. C* **2026**, *86*, 45.
41. Yousaf, Z.; Alshammari, M.; Rizwan, M.; Rayimbaev, J.; Akhmedov, M.; Bhatti, M.Z. Physical viability of general relativistic wormholes sourced by dark matter distributions. *Phys. Dark Universe* **2026**, *52*, 102269.
42. Nashed, G.G.L.; Eid, A. Holographic dark energy as a source for wormholes in modified gravity. *Phys. Lett. B* **2026**, *873*, 140159.
43. Hsu, S.D.H. Entropy bounds and dark energy. *Phys. Lett. B* **2004**, *594*, 13–16.
44. Jiang, Y.; Han, Z.X.; Zhang, Q.; Yang, W.; Wu, Y.B.; Li, J.; Lou, H.; Zhao, C.; Wang, Y. Model-independent diagnostics in interacting dark energy models. *Universe* **2020**, *6*, 49.
45. Yang, W.; Pan, S.; Nunes, R.C.; Mota, D.F. Dark calling dark: Interaction in the dark sector in the presence of neutrino properties after the Planck CMB final release. *J. Cosmol. Astropart. Phys.* **2020**, *04*, 008.
46. Pan, S.; Yang, W.; Di Valentino, E.; Saridakis, E.N.; Chakraborty, S. Interacting scenarios with dynamical dark energy: Observational constraints and alleviation of the H_0 tension. *Phys. Rev. D* **2019**, *100*, 103520.
47. Pan, S.; Yang, W.; Singha, C.; Saridakis, E.N. Observational constraints on sign-changeable interaction models and alleviation of the H_0 tension. *Phys. Rev. D* **2019**, *100*, 083539.
48. Yang, W.; Mena, O.; Pan, S.; Di Valentino, E. Dark sectors with dynamical coupling. *Phys. Rev. D* **2019**, *100*, 083509.
49. Yang, W.; Vagnozzi, S.; Di Valentino, E.; Nunes, R.C.; Pan, S.; Mota, D.F. Listening to the sound of dark sector interactions with gravitational wave standard sirens. *J. Cosmol. Astropart. Phys.* **2019**, *7*, 037.
50. Yang, W.; Banerjee, N.; Paliathanasis, A.; Pan, S. Reconstructing the dark matter and dark energy interaction scenarios from observations. *Phys. Dark Universe* **2019**, *26*, 100383.
51. Yang, W.; Mukherjee, A.; Di Valentino, E.; Pan, S. Interacting dark energy with time-varying equation of state and the H_0 tension. *Phys. Rev. D* **2018**, *98*, 123527.
52. Yang, W.; Pan, S.; Herrera, R.; Chakraborty, S. Large-scale (in)stability analysis of an exactly solved coupled dark-energy model. *Phys. Rev. D* **2018**, *98*, 043517.
53. Yang, W.; Pan, S.; Di Valentino, E.; Nunes, R.C.; Vagnozzi, S.; Mota, D.F. Tale of stable interacting dark energy, observational signatures, and the H_0 tension. *J. Cosmol. Astropart. Phys.* **2018**, *9*, 019.
54. Yang, W.; Pan, S.; Barrow, J.D. Large-scale stability and astronomical constraints for coupled dark-energy models. *Phys. Rev. D* **2018**, *97*, 043529.

55. Yang, W.; Xu, L. Cosmological constraints on interacting dark energy with redshift-space distortion after Planck data. *Phys. Rev. D* **2014**, *89*, 083517.
56. Yang, W.; Xu, L. Testing coupled dark energy with large-scale structure observations. *J. Cosmol. Astropart. Phys.* **2014**, *8*, 034.
57. Nunes, R.C.; Pan, S.; Saridakis, E.N. New constraints on interacting dark energy from cosmic chronometers. *Phys. Rev. D* **2016**, *94*, 023508.
58. Rani, N.; Jain, D.; Mahajan, S.; Mukherjee, A.; Pires, N. Transition redshift: New constraints from parametric and nonparametric methods. *J. Cosmol. Astropart. Phys.* **2015**, *12*, 045.
59. Scolnic, D.M.; Jones, D.O.; Rest, A.; Pan, Y.C.; Chornock, R.; Foley, R.J.; Huber, M.E.; Kessler, R.; Narayan, G.; Riess, A.G.; et al. The complete light-curve sample of spectroscopically confirmed SNe Ia from Pan-STARRS1 and cosmological constraints from the combined Pantheon sample. *Astrophys. J.* **2018**, *859*, 101.
60. Sahni, V.; Saini, T.D.; Starobinsky, A.A.; Alam, U. Statefinder—A new geometrical diagnostic of dark energy. *JETP Lett.* **2003**, *77*, 201–206.
61. Alam, U.; Sahni, V.; Saini, T.D.; Starobinsky, A.A. Exploring the expanding Universe and dark energy using the statefinder diagnostic. *Mon. Not. R. Astron. Soc.* **2003**, *344*, 1057–1074.
62. Yu, F.; Cui, J.L.; Zhang, J.F.; Zhang, X. Statefinder hierarchy exploration of the extended Ricci dark energy. *Eur. Phys. J. C* **2015**, *75*, 274.
63. Zhao, Z.; Wang, S. Diagnosing holographic-type dark-energy models with the statefinder hierarchy, composite null diagnostic, and w – w' pair. *Sci. China Phys. Mech. Astron.* **2018**, *61*, 039811.
64. Zhang, S.; Qi, Y.; Yang, W. Theoretical analysis of Rényi holographic dark energy in fractal cosmology. *Mod. Phys. Lett. A* **2022**, *37*, 2250250.
65. Yadav, V.; Yadav, S.K.; Yadav, A.K. Observational constraints on generalized dark matter properties in the presence of neutrinos with the final Planck release. *Phys. Dark Universe* **2023**, *42*, 101363.
66. Zhang, X. Holographic Ricci dark energy: Current observational constraints, quintom feature, and the reconstruction of scalar-field dark energy. *Phys. Rev. D* **2009**, *79*, 103509.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.