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A Spectral Tau Algorithm Using Shifted Convolved Fibonacci Polynomials for Singular Functional Models

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Abstract

In this paper, for some second-order singular functional differential equations (SOSFDEs), a tau-type spectral technique is built using non-origin-symmetric polynomials called shifted convolved Fibonacci polynomials as the approximation basis. Derivative formulae, integral identities required for the tau projection, power-form and inverse representations, and several operational relations are derived for this basis. The singular term and the shifted arguments can be handled directly within an algebraic framework thanks to these formulas, eliminating the need to introduce a grid on the computing interval. For the unknown coefficients, the resultant approximation yields a finite system. The uniform convergence of the truncated shifted convolved Fibonacci expansion is established, together with error estimates and a stability result for the proposed approximation. To demonstrate the procedure's accuracy, three cases are utilized as benchmarks. As the truncation order increases, the reported errors gradually diminish, and the computed solutions correspond closely with the exact answers. Further evidence of the reliability and efficiency of the suggested shifted convolved Fibonacci tau procedure is provided by comparisons with existing numerical data.

Keywords: convolved Fibonacci polynomials; shifted polynomials; tau method; singular functional differential equations; models; error analysis



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1. Introduction

Special polynomials and classical functions form a basic part of modern approximation theory and numerical analysis; standard sources include [1–3]. Their recurrence relations, explicit forms, and approximation features make them convenient bases for constructing numerical algorithms for differential equations (DEs). During the last few years, many symmetric and non-symmetric polynomials have been used in both fractional and integer-order models. For example, shifted Legendre polynomial spectral collocation was used for nonlinear Riesz distributed-order fractional DEs in [4]. Shifted Jacobi polynomials were adopted for fractional DEs in [5], while a collocation scheme based on fractional-order Lagrange operational matrices was proposed for space–time fractional partial DEs

in [6]. Bell polynomials were employed for Ψ -fractional integro-DEs in [7]. Fractional Bernoulli polynomials were used for nonlinear systems of time-fractional partial DEs in [8]. In addition, Müntz–Legendre polynomials were applied to fractional variational problems in [9], and composite Legendre–Legendre rational collocation was designed for the Fokker–Planck equation on an unbounded interval in [10]. The Andre Jeannin basis was used for a nonlinear fractional Rosenau–Hyman equation involving the Caputo–Hadamard derivative in [11]. Shifted Legendre polynomials were also used for the Duffing equation under several boundary conditions in [12]. Pseudo-spectral differentiation matrices associated with second-kind Chebyshev polynomials were developed in [13] for high-order nonlinear DEs. Further recent polynomial-based spectral algorithms include shifted Pell–Lucas techniques for fractional and classical Rosenau–Hyman models [14], shifted Pell polynomial schemes for third-order Rosenau–Hyman and Gilson–Pickering equations [15], a shifted Dickson polynomial method for the time-fractional Huxley equation [16], and a shifted Chebyshev polynomial approach for a gas-dynamics equation involving the Caputo time-fractional derivative [17].

Fibonacci-type polynomial systems, including shifted, generalized, and wavelet versions, have become useful bases in numerical approximation. Their simple recurrences and tractable algebraic structure allow compact operational representations. These features explain their growing use in ordinary, fractional, stochastic, and nonlinear models. A Fibonacci-polynomial method for fractional DEs was proposed in [18]. Fibonacci wavelets were used in an epidemic model in [19]. Shifted Vieta–Fibonacci polynomials were used in a pseudo-spectral Galerkin method for stochastic models in [20]. Another shifted Vieta–Fibonacci approach was developed for stochastic fractional integro-DEs in [21]. Fibonacci wavelets were tested on a fractional cryosphere model in [22], and a related wavelet approach was applied to the Benjamin–Bona–Mahony equation in [23].

Nonlinear DEs arise in many scientific and engineering applications, such as fluid flow, heat transfer, biological interactions, epidemiology, chemical kinetics, control systems, optics, and wave propagation. Unlike linear equations, they can model bifurcation, pattern generation, chaotic motion, and soliton dynamics. Since exact solutions are usually unavailable, accurate computational methods are essential. Numerical solutions of nonlinear fractional DEs with applications in biological and engineering sciences were discussed in [24]. A Crank–Nicolson finite-difference method for a damped saturable nonlinear Schrödinger equation was developed in [25]. Connecting orbits and stationary states for nonlinear evolution equations on unbounded domains were analyzed in [26]. An optimal iterative algorithm for nonlinear models in one and higher dimensions was proposed in [27]. Multi-hump solitons and rogue waves for coupled nonlinear Schrödinger equations were studied in [28] and coupled fractional nonlinear Schrödinger equations in dual-core optical fibers in [29].

Spectral methods are well known for their high accuracy when the underlying solution is smooth. Collocation, Galerkin, and tau formulations often achieve accurate approximations with a small number of basis functions. Recent examples include the Haar wavelet collocation method for multi-delay fractional DEs [30], a parallel multi-domain spectral collocation algorithm for Caputo–Hadamard fractional equations [31], and a Vieta–Lucas collocation method for engineering heat-transfer applications [32]. Spectral collocation approaches were also designed for long-time transient diffusion equations [33]. Tau methods have also been successfully used for high-order boundary value problems [34]. A Chebyshev derivative-based spectral tau formulation was also developed to approximate solutions to real-life initial-boundary-value problems [35].

Singular DEs appear in fluid mechanics, astrophysics, heat transfer, and population models. Terms such as $1/t$ create a loss of regularity near the singular point, and the diffi-

culty increases when the equation contains nonlinearities and functional arguments. Hence, a useful numerical method should incorporate the singular structure without sacrificing accuracy. Recent studies have addressed both theory and computation for singular equations. In [36] Fibonacci-wavelet operational matrices were used for linear and nonlinear singular value problems. In [37] the neural network techniques were applied for Lane–Emden-type nonlinear equations. In [38] the variational methods for singular fractional p -Laplace equations were discussed. In [39] the fitted methods for singularly perturbed spatiotemporal delay DEs were considered, and in [40] the asymptotic approximations for the second-order quasi-linear singularly perturbed problems with multiple turning points were developed. In [41], spline collocation on graded meshes was used for singular fractional DEs, and in [42], the Kirchhoff parabolic problems with logarithmic nonlinearities were examined.

The main contributions of this paper are summarized below:

- New operational identities are obtained for shifted convolved Fibonacci polynomials, including power form and inverse expansions, derivative relations, and some definite integral formulas.
- A shifted convolved Fibonacci tau method is formulated for second-order singular functional DEs.
- The proposed discretization is written in matrix form, reducing the original model to a finite algebraic system.
- Convergence and error estimates are established for the corresponding truncated expansion.
- Numerical examples and comparison are given to show the accuracy and efficiency of the algorithm.

The motivation for choosing the shifted convolved Fibonacci basis stems from its easy-to-understand and tractable algebraic structure, which is both basic and recursive. Additionally, there are explicit formulas that can be used to deal with derivatives, products, and shifted functional arguments. In addition, the shifted classical Fibonacci polynomials, which are included as the special case $n = 1$, are generalized as a parameterized basis of this family. The convolution parameter provides more flexibility in constructing spectral approximations. Particularly, the numerical findings show that the generalized convolved family's approximations based on the usual Fibonacci polynomials are not necessarily the most accurate, and that using the convolution parameter appropriately can help increase accuracy. We stress that the suggested basis is not meant to supplant or assert an advantage over the standard orthogonal bases like Chebyshev or Legendre polynomials; instead, we aim to create and study a distinct parameterized tau formulation for singular functional differential equations.

The paper is organized as follows. Section 3 collects and proves the identities needed for the shifted convolved Fibonacci basis, including power representations, inverse formulas, derivative relations, connection coefficients, product relations, and definite integrals. Section 4 develops the tau approximation and its algebraic matrix form for the singular functional model. Section 5 discusses convergence analysis and error estimates. Numerical examples are presented in Section 6. Finally, Section 7 gives concluding comments and possible extensions.

2. Preliminaries on Fibonacci and Convolved Fibonacci Polynomials

This section records the definitions and identities that are required later. We first recall the classical Fibonacci polynomials and then introduce their convolved and shifted-convolved versions, along with the recurrence and explicit formulas used in the sequel.

2.1. Classical Fibonacci Polynomials

The Fibonacci polynomials $F_n(x)$ are defined through the recurrence relation

$$F_{j+2}(x) = x F_{j+1}(x) + F_j(x), \quad j \geq 0,$$

subject to the initial values

$$F_0(x) = 0, \quad F_1(x) = 1.$$

Their explicit combinatorial form is

$$F_{j+1}(x) = \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j-k}{k} x^{j-2k}, \quad j \geq 0. \quad (1)$$

2.2. Convolved Fibonacci Polynomials

Throughout this paper, $n \in \mathbb{N}$, $n \geq 1$, is regarded as a fixed convolution parameter. The convolved Fibonacci polynomials $F_j^{(n)}(x)$, discussed in [43], extend the classical Fibonacci family and satisfy the following recurrence relation:

$$j F_j^{(n)}(x) - (n+j-1) x F_{j-1}^{(n)}(x) - (2n+j-2) F_{j-2}^{(n)}(x) = 0, \quad j \geq 2, \quad (2)$$

with starting values

$$F_0^{(n)}(x) = 1, \quad F_1^{(n)}(x) = n x.$$

The corresponding finite expansion and inverse relation are given by [43]

$$F_j^{(n)}(x) = \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(n)_{j-r}}{r! (j-2r)!} x^{j-2r}, \quad (3)$$

$$x^j = j! \Gamma(n) \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^r (j+n-2r)}{\Gamma(1+j+n-r) r!} F_{j-2r}^{(n)}(x). \quad (4)$$

Remark 1. The classical Fibonacci polynomials are recovered as a special case of the convolved Fibonacci polynomials. More precisely, by setting the convolution parameter $n = 1$, one obtains

$$F_j^{(1)}(x) = F_{j+1}(x), \quad j \geq 0,$$

where $F_{j+1}(x)$ denotes the classical Fibonacci polynomial of degree j .

In the present work, the shifted convolved Fibonacci polynomials (SCFPs) are denoted consistently by $\tilde{F}_j^{(n)}$ and are introduced by

$$\tilde{F}_j^{(n)}(x) = F_j^{(n)}(2x-1).$$

Here $F_j^{(n)}$ denotes the unshifted convolved Fibonacci polynomial, whereas $\tilde{F}_j^{(n)}$ denotes its shifted counterpart. This shifted family satisfies the following recurrence relation:

$$j \tilde{F}_j^{(n)}(x) = (2x-1)(n+j-1) \tilde{F}_{j-1}^{(n)}(x) + (2n+j-2) \tilde{F}_{j-2}^{(n)}(x), \quad j \geq 2, \quad (5)$$

with the following initial values:

$$\tilde{F}_0^{(n)}(x) = 1, \quad \tilde{F}_1^{(n)}(x) = n(2x-1). \quad (6)$$

3. Auxiliary Identities for Shifted Convolved Fibonacci Polynomials

This section derives the identities required for the tau construction. We obtain a power-form representation and its inverse, and then present formulas for differentiation, products, connections, and definite integrals of the shifted convolved Fibonacci basis.

3.1. Power Form Representation and Its Inverse Expansion

We begin by expanding the SCFPs in ordinary powers of the independent variable, followed by the inverse representation.

Theorem 1. For every non-negative integer k , the shifted polynomial $\tilde{F}_k^{(n)}(x)$ admits the representation

$$\tilde{F}_k^{(n)}(x) = \sum_{\ell=0}^k \frac{(-1)^{k-\ell} 2^\ell (n)_k}{\ell! (k-\ell)!} {}_2F_1 \left(\begin{matrix} -k+\ell, & 1-k+\ell \\ & 2 \end{matrix} \middle| -4 \right) x^\ell. \quad (7)$$

Proof. By substituting $2x - 1$ into the finite series (3), we get

$$\tilde{F}_k^{(n)}(x) = \sum_{r=0}^{\lfloor k/2 \rfloor} \frac{(n)_{k-r} (2x-1)^{k-2r}}{r! (k-2r)!}. \quad (8)$$

Using the binomial expansion of $(2x - 1)^{k-2r}$ gives

$$(2x - 1)^{k-2r} = \sum_{\ell=0}^{k-2r} \binom{k-2r}{\ell} (2x)^\ell (-1)^{k-2r-\ell}. \quad (9)$$

Combining (9) with (8) yields

$$\tilde{F}_k^{(n)}(x) = \sum_{r=0}^{\lfloor k/2 \rfloor} \sum_{\ell=0}^{k-2r} \frac{(n)_{k-r} 2^\ell (-1)^{k-\ell} x^\ell}{r! \ell! (k-2r-\ell)!}. \quad (10)$$

Reordering the finite summations and grouping equal powers of x , one obtains

$$\tilde{F}_k^{(n)}(x) = \sum_{\ell=0}^k \frac{(-1)^{k-\ell} 2^\ell}{\ell!} \left(\sum_{r=0}^{\lfloor (k-\ell)/2 \rfloor} \frac{(n)_{k-r}}{r! (k-\ell-2r)!} \right) x^\ell. \quad (11)$$

The following elementary Pochhammer transformations are then used:

$$(n)_{k-r} = (n)_k \frac{(-1)^r}{(1-k-n)_r}, \quad (12)$$

$$\frac{1}{(k-\ell-2r)!} = \frac{(-k+\ell)_{2r}}{(k-\ell)!}, \quad (13)$$

and therefore (11) is converted into

$$\tilde{F}_k^{(n)}(x) = \sum_{\ell=0}^k \frac{(-1)^{k-\ell} 2^\ell (n)_k}{\ell! (k-\ell)!} \left(\sum_{r=0}^{\lfloor (k-\ell)/2 \rfloor} \frac{(-k+\ell)_{2r} (-1)^r}{(1-k-n)_r r!} \right) x^\ell. \quad (14)$$

We also use the duplication relation

$$(-k+\ell)_{2r} = 2^{2r} \left(\frac{-k+\ell}{2} \right)_r \left(\frac{1-k+\ell}{2} \right)_r. \quad (15)$$

Thus, (14) becomes

$$\tilde{F}_k^{(n)}(x) = \sum_{\ell=0}^k \frac{(-1)^{k-\ell} 2^\ell (n)_k}{\ell! (k-\ell)!} \left(\sum_{r=0}^{\lfloor (k-\ell)/2 \rfloor} \frac{\left(\frac{-k+\ell}{2}\right)_r \left(\frac{1-k+\ell}{2}\right)_r}{(1-k-n)_r r!} (-4)^r \right) x^\ell. \quad (16)$$

The inner finite sum in (16) is a terminating hypergeometric series, namely

$$\sum_{r=0}^{\lfloor (k-\ell)/2 \rfloor} \frac{\left(\frac{-k+\ell}{2}\right)_r \left(\frac{1-k+\ell}{2}\right)_r}{(1-k-n)_r r!} (-4)^r = {}_2F_1 \left(\begin{matrix} \frac{-k+\ell}{2}, \frac{1-k+\ell}{2} \\ 1-k-n \end{matrix} \middle| -4 \right). \quad (17)$$

Substituting (17) into (16) gives

$$\tilde{F}_k^{(n)}(x) = \sum_{\ell=0}^k \frac{(-1)^{k-\ell} 2^\ell (n)_k}{\ell! (k-\ell)!} {}_2F_1 \left(\begin{matrix} \frac{-k+\ell}{2}, \frac{1-k+\ell}{2} \\ 1-k-n \end{matrix} \middle| -4 \right) x^\ell. \quad (18)$$

This completes the proof. $\square$

We next establish the inverse of (7). The following auxiliary recurrence will be used.

Lemma 1. For fixed $m \geq 0$ and $0 \leq j \leq m$, define $A_{j,m}$ as

$$A_{j,m} = \frac{m!}{2^m} \frac{{}_2F_1 \left(\begin{matrix} \frac{j-m}{2}, \frac{j-m+1}{2} \\ n+j+1 \end{matrix} \middle| -4 \right)}{(m-j)!(n)_j}. \quad (19)$$

Then,

$$A_{j,m+1} = \frac{1}{2} A_{j,m} + \frac{j A_{j-1,m}}{2(n+j-1)} - \frac{(2n+j) A_{j+1,m}}{2(n+j+1)}. \quad (20)$$

Proof. Insert (19) into the right-hand side of (20). Together with

$$(n)_j = (n)_{j-1}(n+j-1), \quad (n)_{j+1} = (n)_j(n+j),$$

one obtains

$$\begin{aligned} & \frac{1}{2} A_{j,m} + \frac{j A_{j-1,m}}{2(n+j-1)} - \frac{(2n+j) A_{j+1,m}}{2(n+j+1)} \\ &= \frac{m!}{2^{m+1} (m+1-j)!(n)_j} \left[(m+1-j) {}_2F_1 \left(\begin{matrix} \frac{j-m}{2}, \frac{j-m+1}{2} \\ n+j+1 \end{matrix} \middle| -4 \right) \right. \\ & \quad \left. + j {}_2F_1 \left(\begin{matrix} \frac{j-m-1}{2}, \frac{j-m}{2} \\ n+j \end{matrix} \middle| -4 \right) \right. \\ & \quad \left. - \frac{(2n+j)(m+1-j)(m-j)}{(n+j)(n+j+1)} {}_2F_1 \left(\begin{matrix} \frac{j-m+1}{2}, \frac{j-m+2}{2} \\ n+j+2 \end{matrix} \middle| -4 \right) \right]. \end{aligned} \quad (21)$$

Expanding the hypergeometric functions term by term and simplifying the resulting finite sums give

$$\frac{1}{2}A_{j,m} + \frac{jA_{j-1,m}}{2(n+j-1)} - \frac{(2n+j)A_{j+1,m}}{2(n+j+1)} = \frac{(m+1)!}{2^{m+1}} \frac{{}_2F_1\left(\begin{matrix} \frac{j-m-1}{2}, \frac{j-m}{2} \\ n+j+1 \end{matrix} \middle| -4\right)}{(m+1-j)!(n)_j}.$$

The expression on the right is precisely $A_{j,m+1}$, and hence (20) follows. $\square$

Theorem 2. For each non-negative integer m , the following inverse expansion holds:

$$x^m = \frac{m!}{2^m} \sum_{j=0}^m \frac{{}_2F_1\left(\begin{matrix} \frac{j-m}{2}, \frac{j-m+1}{2} \\ n+j+1 \end{matrix} \middle| -4\right)}{(m-j)!(n)_j} \tilde{F}_j^{(n)}(x). \quad (22)$$

Proof. For fixed $m \geq 0$ and $0 \leq j \leq m$, define

$$A_{j,m} = \frac{m!}{2^m} \frac{{}_2F_1\left(\begin{matrix} \frac{j-m}{2}, \frac{j-m+1}{2} \\ n+j+1 \end{matrix} \middle| -4\right)}{(m-j)!(n)_j}. \quad (23)$$

With this notation, (22) can be written as

$$x^m = \sum_{j=0}^m A_{j,m} \tilde{F}_j^{(n)}(x). \quad (24)$$

We prove (24) by induction on m . The case $m = 0$ is immediate. Assume that it is true for a fixed m ; it remains to show that

$$x^{m+1} = \sum_{j=0}^{m+1} A_{j,m+1} \tilde{F}_j^{(n)}(x). \quad (25)$$

Multiplying both sides of (24) by x , one obtains

$$x^{m+1} = \sum_{j=0}^m A_{j,m} x \tilde{F}_j^{(n)}(x). \quad (26)$$

From the shifted recurrence (5), one obtains

$$x \tilde{F}_j^{(n)}(x) = \frac{\tilde{F}_j^{(n)}(x)}{2} + \frac{(j+1)\tilde{F}_{j+1}^{(n)}(x)}{2(n+j)} - \frac{(2n+j-1)\tilde{F}_{j-1}^{(n)}(x)}{2(n+j)}. \quad (27)$$

Using (27) in (26) yields

$$x^{m+1} = \sum_{j=0}^m A_{j,m} \left(\frac{\tilde{F}_j^{(n)}(x)}{2} + \frac{(j+1)\tilde{F}_{j+1}^{(n)}(x)}{2(n+j)} - \frac{(2n+j-1)\tilde{F}_{j-1}^{(n)}(x)}{2(n+j)} \right). \quad (28)$$

Collecting the coefficients of each basis function $\tilde{F}_j^{(n)}(x)$ and shifting the indices in the second and third sums give

$$x^{m+1} = \sum_{j=0}^{m+1} \left(\frac{1}{2} A_{j,m} + \frac{j A_{j-1,m}}{2(n+j-1)} - \frac{(2n+j) A_{j+1,m}}{2(n+j+1)} \right) \tilde{F}_j^{(n)}(x), \quad (29)$$

where $A_{j,m} = 0$ whenever $j < 0$ or $j > m$.

Lemma 1 gives

$$A_{j,m+1} = \frac{1}{2} A_{j,m} + \frac{j A_{j-1,m}}{2(n+j-1)} - \frac{(2n+j) A_{j+1,m}}{2(n+j+1)}. \quad (30)$$

Consequently,

$$x^{m+1} = \sum_{j=0}^{m+1} A_{j,m+1} \tilde{F}_j^{(n)}(x). \quad (31)$$

The induction step is proved, and the result follows. $\square$

3.2. Linearization, Connection, and Differentiation Formulas

This subsection provides the operational identities used later in the discretization. We state differentiation and linearization formulas and derive a connection relation for shifted arguments. A second-derivative expansion follows as a direct consequence.

Theorem 3. Let i and q be non-negative integers such that $1 \leq q \leq i$. The derivative formula expression below was developed in [43]:

$$\frac{d^q}{dx^q} F_i^{(n)}(x) = \sum_{\ell=0}^{\lfloor \frac{i-q}{2} \rfloor} (-1)^\ell (i-2\ell+n-q) \binom{\ell+q-1}{\ell} (i-\ell+n+1-q)_{q-1} F_{i-q-2\ell}^{(n)}(x). \quad (32)$$

Corollary 1. Let i and q be non-negative integers with $1 \leq q \leq i$. Then,

$$\frac{d^q}{dx^q} \tilde{F}_i^{(n)}(x) = 2^q \sum_{\ell=0}^{\lfloor \frac{i-q}{2} \rfloor} (-1)^\ell (i-2\ell+n-q) \binom{\ell+q-1}{\ell} (i-\ell+n+1-q)_{q-1} \tilde{F}_{i-q-2\ell}^{(n)}(x). \quad (33)$$

Proof. The result follows from (32) after applying the change in variable $x \mapsto 2x-1$, and the chain rule. $\square$

Theorem 4. For non-negative integers i and j , the product $\tilde{F}_i^{(n)}(x) \tilde{F}_j^{(n)}(x)$ has the linear expansion

$$\tilde{F}_i^{(n)}(x) \tilde{F}_j^{(n)}(x) = \sum_{s=0}^{\min(i,j)} \mathcal{L}_{i,j,s}^{(n)} \tilde{F}_{i+j-2s}^{(n)}(x), \quad (34)$$

where

$$\mathcal{L}_{i,j,s}^{(n)} = \frac{(-1)^s (i+j-2s+n)(i+j-2s)! \Gamma(i-s+n) \Gamma(j-s+n) (n)_s (i+j-2s+2n)_s}{\Gamma(n)(i-s)!(j-s)!s! \Gamma(1+i+j-s+n)}. \quad (35)$$

Proof. This is obtained from the linearization formula for the unshifted convolved Fibonacci polynomials in [43] by replacing the argument with $2x-1$. $\square$

Lemma 2. For each non-negative integer m , the following shifted-argument connection relation is valid:

$$\tilde{F}_m^{(n)}(ax+\beta) = \sum_{r=0}^m B_{m,r}^{(n;a,\beta)} \tilde{F}_r^{(n)}(x), \quad (36)$$

where

$$B_{m,r}^{(n;a,\beta)} = \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{u=0}^{\lfloor \frac{m-2h-r}{2} \rfloor} \frac{(n)_{m-h} \binom{m-2h}{r+2u} a^{r+2u} (a+2\beta-1)^{m-2h-r-2u} (-1)^u (r+2u)! (r+n) \Gamma(n)}{(m-2h)! h! u! \Gamma(r+u+n+1)}. \quad (37)$$

Proof. By the definition of the shifted basis,

$$\tilde{F}_m^{(n)}(ax + \beta) = F_m^{(n)}(2ax + 2\beta - 1) = F_m^{(n)}(a(2x - 1) + a + 2\beta - 1). \quad (38)$$

Using the finite form (3), we find

$$\tilde{F}_m^{(n)}(ax + \beta) = \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(n)_{m-h}}{h! (m-2h)!} \left(a(2x - 1) + a + 2\beta - 1 \right)^{m-2h}. \quad (39)$$

Expansion by the binomial theorem gives

$$\tilde{F}_m^{(n)}(ax + \beta) = \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(n)_{m-h}}{h! (m-2h)!} \sum_{k=0}^{m-2h} \binom{m-2h}{k} a^k (a+2\beta-1)^{m-2h-k} (2x-1)^k. \quad (40)$$

We next use the inversion relation

$$(2x-1)^k = k! \Gamma(n) \sum_{u=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^u (k+n-2u)}{u! \Gamma(k+n-u+1)} \tilde{F}_{k-2u}^{(n)}(x). \quad (41)$$

Substituting (41) into (40), one obtains

$$\begin{aligned} \tilde{F}_m^{(n)}(ax + \beta) &= \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(n)_{m-h}}{h! (m-2h)!} \sum_{k=0}^{m-2h} \binom{m-2h}{k} a^k (a+2\beta-1)^{m-2h-k} \\ &\quad \times k! \Gamma(n) \sum_{u=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^u (k+n-2u)}{u! \Gamma(k+n-u+1)} \tilde{F}_{k-2u}^{(n)}(x). \end{aligned} \quad (42)$$

Finally, setting $k = r + 2u$, changing the order of summation, and collecting the coefficient of $\tilde{F}_r^{(n)}(x)$ lead to

$$\tilde{F}_m^{(n)}(ax + \beta) = \sum_{r=0}^m B_{m,r}^{(n;a,\beta)} \tilde{F}_r^{(n)}(x), \quad (43)$$

where

$$B_{m,r}^{(n;a,\beta)} = \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{u=0}^{\lfloor \frac{m-2h-r}{2} \rfloor} \frac{(n)_{m-h} \binom{m-2h}{r+2u} a^{r+2u} (a+2\beta-1)^{m-2h-r-2u} (-1)^u (r+2u)! (r+n) \Gamma(n)}{(m-2h)! h! u! \Gamma(r+u+n+1)}. \quad (44)$$

The stated connection coefficients are therefore obtained. $\square$

Lemma 3. For $i \geq 2$, one has the expansion

$$\begin{aligned} \frac{d^2}{dx^2} \tilde{F}_i^{(n)}(ax + \beta) &= 4 \sum_{\ell=0}^{\lfloor \frac{i-2}{2} \rfloor} (-1)^\ell (i-2\ell+n-2)(\ell+1)(i-\ell+n-1) \\ &\quad \times \sum_{r=0}^{i-2-2\ell} B_{i-2-2\ell,r}^{(n;a,\beta)} \tilde{F}_r^{(n)}(x). \end{aligned} \quad (45)$$

Proof. From the derivative formula (33) with $q = 2$, and then evaluating the resulting derivative at $ax + \beta$, one obtains

$$\frac{d^2}{dx^2} \tilde{F}_i^{(n)}(ax + \beta) = 4 \sum_{\ell=0}^{\lfloor \frac{i-2}{2} \rfloor} (-1)^\ell (i - 2\ell + n - 2)(\ell + 1)(i - \ell + n - 1) \tilde{F}_{i-2-2\ell}^{(n)}(ax + \beta). \quad (46)$$

Applying the connection formula (36) to $\tilde{F}_{i-2-2\ell}^{(n)}(ax + \beta)$, one obtains

$$\tilde{F}_{i-2-2\ell}^{(n)}(ax + \beta) = \sum_{r=0}^{i-2-2\ell} B_{i-2-2\ell,r}^{(n;a,\beta)} \tilde{F}_r^{(n)}(x). \quad (47)$$

Substitution of (47) into (46) proves (45). $\square$

3.3. Some Useful Integral Identities

We now derive integral relations involving the shifted convolved Fibonacci basis. These include single integrals, products, triple products, and derivative-related integrals required in the tau formulation.

Lemma 4. For every non-negative integer m , we have

$$J_m^{(n)} := \int_0^1 \tilde{F}_m^{(n)}(x) dx = \frac{1}{2} \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(n)_{m-\ell} (1 + (-1)^m)}{(m-2\ell)! \ell! (m-2\ell+1)}. \quad (48)$$

Proof. Since $\tilde{F}_m^{(n)}(x) = F_m^{(n)}(2x - 1)$, the substitution $y = 2x - 1$ gives

$$\int_0^1 \tilde{F}_m^{(n)}(x) dx = \frac{1}{2} \int_{-1}^1 F_m^{(n)}(y) dy. \quad (49)$$

Using the finite power representation of $F_m^{(n)}$, we can write

$$F_m^{(n)}(y) = \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(n)_{m-\ell}}{(m-2\ell)! \ell!} y^{m-2\ell}, \quad (50)$$

and

$$\int_{-1}^1 y^q dy = \frac{1 + (-1)^q}{q + 1}, \quad (51)$$

one obtains (48). $\square$

Lemma 5. For non-negative integers r and k , set

$$I_{r,k}^{(n)} := \int_0^1 \tilde{F}_r^{(n)}(x) \tilde{F}_k^{(n)}(x) dx. \quad (52)$$

Then,

$$I_{r,k}^{(n)} = \frac{1}{2} \sum_{s=0}^{\min(r,k)} \mathcal{L}_{r,k,s}^{(n)} \sum_{\ell=0}^{\lfloor \frac{r+k-2s}{2} \rfloor} \frac{(n)_{r+k-2s-\ell} (1 + (-1)^{r+k-2s-2\ell})}{(r+k-2s-2\ell)! \ell! (r+k-2s-2\ell+1)}. \quad (53)$$

Note that the symmetry of the integrand implies that

$$I_{r,k}^{(n)} = I_{k,r}^{(n)}, \quad r, k \geq 0.$$

Proof. Integrating the linearization formula (34) on $[0, 1]$ yields

$$I_{r,k}^{(n)} = \sum_{s=0}^{\min(r,k)} \mathcal{L}_{r,k,s}^{(n)} \int_0^1 \tilde{F}_{r+k-2s}^{(n)}(x) dx. \quad (54)$$

Then, (48), with $m = r + k - 2s$, gives (53). $\square$

Theorem 5. Let $i \geq 2$, $j \geq 0$, and $r \geq 0$. Then, the following integral formulas are valid:

$$\begin{aligned} \int_0^1 x \frac{d^2}{dx^2} \tilde{F}_i^{(n)}(ax + \beta) \tilde{F}_j^{(n)}(x) dx &= 4 \sum_{\ell=0}^{\lfloor \frac{i-2}{2} \rfloor} (-1)^\ell (i - 2\ell + n - 2)(\ell + 1)(i - \ell + n - 1) \\ &\times \sum_{s=0}^{i-2-2\ell} B_{i-2-2\ell,s}^{(n;a,\beta)} \left[\frac{1}{2} I_{s,j}^{(n)} + \frac{j+1}{2(n+j)} I_{s,j+1}^{(n)} - \frac{2n+j-1}{2(n+j)} I_{s,j-1}^{(n)} \right], \end{aligned} \quad (55)$$

$$\int_0^1 \frac{d}{dx} \tilde{F}_i^{(n)}(bx + \rho) \tilde{F}_j^{(n)}(x) dx = 2 \sum_{\ell=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^\ell (i - 2\ell + n - 1) \sum_{s=0}^{i-1-2\ell} B_{i-1-2\ell,s}^{(n;b,\rho)} I_{s,j}^{(n)}, \quad (56)$$

$$\int_0^1 \tilde{F}_r^{(n)}(x) \tilde{F}_i^{(n)}(cx + \xi) \tilde{F}_j^{(n)}(x) dx = \sum_{s=0}^i B_{i,s}^{(n;c,\xi)} \sum_{\ell=0}^{\min(r,j)} \mathcal{L}_{r,j,\ell}^{(n)} I_{s,r+j-2\ell}^{(n)} \quad (57)$$

where $I_{r,k}^{(n)}$ is given by (53), $B_{m,r}^{(n;a,\beta)}$ are defined by (37), $\mathcal{L}_{r,j,\ell}^{(n)}$ are the linearization coefficients given in (35), and $I_{r,-1}^{(n)} = 0$.

Proof. For (55), multiply (45) by $x \tilde{F}_j^{(n)}(x)$, integrate over $[0, 1]$, and then use (27).

For the second formula, taking the derivative first and then evaluating the resulting derivative at $bx + \rho$, the first-derivative relation gives

$$\frac{d}{dx} \tilde{F}_i^{(n)}(bx + \rho) = 2 \sum_{\ell=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^\ell (i - 2\ell + n - 1) \sum_{s=0}^{i-1-2\ell} B_{i-1-2\ell,s}^{(n;b,\rho)} \tilde{F}_s^{(n)}(x). \quad (58)$$

Multiplying (58) by $\tilde{F}_j^{(n)}(x)$ and integrating over $[0, 1]$, making use of the integral in (52), one obtains the second identity in (56).

For the last relation, the connection formula gives

$$\tilde{F}_i^{(n)}(cx + \xi) = \sum_{s=0}^i B_{i,s}^{(n;c,\xi)} \tilde{F}_s^{(n)}(x). \quad (59)$$

Hence,

$$\int_0^1 \tilde{F}_r^{(n)}(x) \tilde{F}_i^{(n)}(cx + \xi) \tilde{F}_j^{(n)}(x) dx = \sum_{s=0}^i B_{i,s}^{(n;c,\xi)} \int_0^1 \tilde{F}_s^{(n)}(x) \tilde{F}_r^{(n)}(x) \tilde{F}_j^{(n)}(x) dx. \quad (60)$$

Using the product expansion (34),

$$\tilde{F}_r^{(n)}(x) \tilde{F}_j^{(n)}(x) = \sum_{\ell=0}^{\min(r,j)} \mathcal{L}_{r,j,\ell}^{(n)} \tilde{F}_{r+j-2\ell}^{(n)}(x). \quad (61)$$

Inserting (61) into (60) yields

$$\int_0^1 \tilde{F}_r^{(n)}(x) \tilde{F}_i^{(n)}(cx + \xi) \tilde{F}_j^{(n)}(x) dx = \sum_{s=0}^i B_{i,s}^{(n;c,\xi)} \sum_{\ell=0}^{\min(r,j)} \mathcal{L}_{r,j,\ell}^{(n)} I_{s,r+j-2\ell}^{(n)}.$$

This proves the required identity. $\square$

Remark 2. It is worth emphasizing that the linearization formula in Theorem 4 is used twice in deriving the preceding integral identities: first, to evaluate the product integrals of two shifted convolved Fibonacci polynomials, and second, to derive the corresponding triple-product integrals. Its use is particularly relevant here since the adopted shifted convolved Fibonacci basis is non-orthogonal. In contrast, for orthogonal polynomial bases equipped with their associated weight functions, the corresponding weighted product integrals can usually be evaluated directly using the orthogonality relation.

4. Tau Formulation for Singular Functional DEs

This section constructs the shifted convolved Fibonacci tau approximation for second-order singular functional differential equations.

Consider the SOSFDE model

$$\begin{cases} G''(at + \beta) + \frac{\gamma}{t} G'(bt + \rho) + p(t) G(ct + \xi) = q(t), \\ G(0) = g_0, \quad G'(0) = g_1, \end{cases} \quad (62)$$

where $a, \beta, \gamma, b, \rho, c, \xi, g_0$, and g_1 are prescribed real parameters, while $q(t)$ denotes a given continuous function.

Assume that $G(t)$ is square integrable on $(0, 1)$ and admits an expansion in the linearly independent basis $\{\tilde{F}_i^{(n)}(t)\}_{i \geq 0}$:

$$G(t) = \sum_{i=0}^{\infty} \hat{c}_i \tilde{F}_i^{(n)}(t). \quad (63)$$

The truncated approximation is taken in the form

$$G(t) \approx G_{\mathcal{N}}(t) = \sum_{i=0}^{\mathcal{N}} \hat{c}_i \tilde{F}_i^{(n)}(t) = \mathbf{C}^T \tilde{\mathbf{F}}^{(n)}(t), \quad (64)$$

where $\mathbf{C} = [\hat{c}_0, \hat{c}_1, \hat{c}_2, \dots, \hat{c}_{\mathcal{N}}]^T$ and $\tilde{\mathbf{F}}^{(n)}(t) = [\tilde{F}_0^{(n)}(t), \tilde{F}_1^{(n)}(t), \dots, \tilde{F}_{\mathcal{N}}^{(n)}(t)]^T$.

The corresponding residual $R_{\mathcal{N}}(t)$ is

$$R_{\mathcal{N}}(t) = G_{\mathcal{N}}''(at + \beta) + \frac{\gamma}{t} G_{\mathcal{N}}'(bt + \rho) + p(t) G_{\mathcal{N}}(ct + \xi) - q(t). \quad (65)$$

Multiplying by t removes the explicit singular denominator and gives

$$t R_{\mathcal{N}}(t) = t G_{\mathcal{N}}''(at + \beta) + \gamma G_{\mathcal{N}}'(bt + \rho) + t p(t) G_{\mathcal{N}}(ct + \xi) - t q(t). \quad (66)$$

Equivalently, we can use the following finite expansion:

$$\begin{aligned} t R_{\mathcal{N}}(t) &= \sum_{i=0}^{\mathcal{N}} \hat{c}_i t \frac{d^2 \tilde{F}_i^{(n)}(at + \beta)}{dt^2} + \gamma \sum_{i=0}^{\mathcal{N}} \hat{c}_i \frac{d \tilde{F}_i^{(n)}(bt + \rho)}{dt} \\ &\quad + \sum_{r=0}^{\mathcal{N}} \nu_r \tilde{F}_r^{(n)}(t) \sum_{i=0}^{\mathcal{N}} \hat{c}_i \tilde{F}_i^{(n)}(ct + \xi) - t q(t). \end{aligned} \quad (67)$$

The tau conditions are imposed by requiring

$$\left\langle t R_{\mathcal{N}}(t), \tilde{F}_j^{(n)}(t) \right\rangle = \int_0^1 t R_{\mathcal{N}}(t) \tilde{F}_j^{(n)}(t) dt = 0, \quad j = 0, 1, \dots, \mathcal{N} - 2. \quad (68)$$

This condition becomes

$$\begin{aligned} \sum_{i=0}^{\mathcal{N}} \hat{c}_i \left\langle t \frac{d^2 \tilde{F}_i^{(n)}(a t + \beta)}{d t^2}, \tilde{F}_j^{(n)}(t) \right\rangle + \gamma \sum_{i=0}^{\mathcal{N}} \hat{c}_i \left\langle \frac{d \tilde{F}_i^{(n)}(b t + \rho)}{d t}, \tilde{F}_j^{(n)}(t) \right\rangle \\ + \sum_{r=0}^{\mathcal{N}} \sum_{i=0}^{\mathcal{N}} \hat{c}_i v_r \left\langle \tilde{F}_r^{(n)}(t) \tilde{F}_i^{(n)}(c t + \xi), \tilde{F}_j^{(n)}(t) \right\rangle \\ = \left\langle t q(t), \tilde{F}_j^{(n)}(t) \right\rangle, \quad j = 0, 1, \dots, \mathcal{N} - 2. \end{aligned} \quad (69)$$

or, after introducing the following coefficients,

$$\sum_{i=0}^{\mathcal{N}} \hat{c}_i h_{i,j} + \gamma \sum_{i=0}^{\mathcal{N}} \hat{c}_i w_{i,j} + \sum_{i=0}^{\mathcal{N}} \sum_{r=0}^{\mathcal{N}} \hat{c}_i v_r y_{i,r}^j = q_j, \quad j = 0, 1, \dots, \mathcal{N} - 2. \quad (70)$$

The algebraic equations can then be arranged in the matrix form

$$\mathbf{H}^T \mathbf{C} + \gamma \mathbf{W}^T \mathbf{C} + \begin{bmatrix} \mathbf{C}^T \mathbf{Y}^0 \mathbf{v} \\ \mathbf{C}^T \mathbf{Y}^1 \mathbf{v} \\ \vdots \\ \mathbf{C}^T \mathbf{Y}^{\mathcal{N}-2} \mathbf{v} \end{bmatrix} = \mathbf{Q}. \quad (71)$$

$$\mathbf{Q} = (q_j)_{(\mathcal{N}-1) \times 1}, \quad q_j = \left\langle t q(t), \tilde{F}_j^{(n)}(t) \right\rangle, \quad (72)$$

$$\mathbf{H} = (h_{i,j})_{(\mathcal{N}+1) \times (\mathcal{N}-1)}, \quad h_{i,j} = \left\langle t \frac{d^2 \tilde{F}_i^{(n)}(a t + \beta)}{d t^2}, \tilde{F}_j^{(n)}(t) \right\rangle, \quad (73)$$

$$\mathbf{W} = (w_{i,j})_{(\mathcal{N}+1) \times (\mathcal{N}-1)}, \quad w_{i,j} = \left\langle \frac{d \tilde{F}_i^{(n)}(b t + \rho)}{d t}, \tilde{F}_j^{(n)}(t) \right\rangle, \quad (74)$$

$$\mathbf{Y}^j = (y_{i,r}^j)_{(\mathcal{N}+1) \times (\mathcal{N}+1)}, \quad y_{i,r}^j = \left\langle \tilde{F}_r^{(n)}(t) \tilde{F}_i^{(n)}(c t + \xi), \tilde{F}_j^{(n)}(t) \right\rangle. \quad (75)$$

Moreover, $\mathbf{v} = [v_0, v_1, v_2, \dots, v_{\mathcal{N}}]^T$.

Also, the initial conditions (62) yield

$$\mathbf{C}^T \tilde{\mathbf{F}}^{(n)}(0) = g_0, \quad \mathbf{C}^T \left. \frac{d \tilde{\mathbf{F}}^{(n)}(t)}{d t} \right|_{t=0} = g_1. \quad (76)$$

Equations (71) and (76) form an algebraic system of size $(\mathcal{N} + 1)$ for the unknown coefficients $\hat{c}_i$. This system is solved numerically by Gaussian elimination.

Remark 3. With respect to the singular term in Equation (62), we assume $G \in C^2[0, 1]$ with $G'(b t + \rho) = \mathcal{O}(t)$ as $t \rightarrow 0^+$, ensuring the singular term is well-defined in the limit. Multiplying by t yields an equivalent system for $t > 0$ and reduces to $\gamma G'_{\mathcal{N}}(\rho) = 0$ at $t = 0$, naturally enforcing boundary regularity and preserving spectral accuracy near the origin.

Remark 4. The entries of $\mathbf{Y}^j$ are organized as

$$\mathbf{Y}^j = (y_{i,r}^j)_{i,r=0}^{\mathcal{N}} = \begin{pmatrix} y_{0,0}^j & y_{0,1}^j & \cdots & y_{0,\mathcal{N}}^j \\ y_{1,0}^j & y_{1,1}^j & \cdots & y_{1,\mathcal{N}}^j \\ \vdots & \vdots & \ddots & \vdots \\ y_{\mathcal{N},0}^j & y_{\mathcal{N},1}^j & \cdots & y_{\mathcal{N},\mathcal{N}}^j \end{pmatrix}. \quad (77)$$

Remark 5. The entries of $\mathbf{H}$, $\mathbf{W}$, and $\mathbf{Y}^j$ are evaluated by the integral identities in Theorem 5.

Remark 6. The coefficients v_r in the approximation $t \mathbf{p}(t) \approx \sum_{r=0}^{\mathcal{N}} v_r \tilde{F}_r^{(n)}(t)$ are obtained from the projection equations

$$\langle t \mathbf{p}(t), \tilde{F}_j^{(n)}(t) \rangle \approx \sum_{r=0}^{\mathcal{N}} v_r \langle \tilde{F}_r^{(n)}(t), \tilde{F}_j^{(n)}(t) \rangle, \quad j = 0, 1, \dots, \mathcal{N}. \quad (78)$$

For instance, when $t \mathbf{p}(t) = t e^t$ and $\mathcal{N} = 3$, the resulting approximation is

$$t \mathbf{p}(t) \approx 1.0527 t^3 + 0.53374 t^2 + 1.1412 t - 0.012182, \quad (79)$$

which follows from solving the linear system

$$\begin{pmatrix} 0.50000 & 3.16666 & 41.1666 & 196.333 \\ 3.16666 & 60.1666 & 421.166 & 3730.33 \\ 41.1666 & 421.166 & 4993.83 & 29000.3 \\ 196.333 & 3730.33 & 29000.3 & 251496 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0.71828 \\ 7.76321 \\ 74.9570 \\ 491.350 \end{pmatrix}. \quad (80)$$

5. Convergence and Error Bounds

This section analyzes the convergence of the shifted convolved Fibonacci expansion used in the approximation. A uniform estimate for $|\tilde{F}_i^{(n)}(t)|$ is first obtained. Bounds for the expansion coefficients and for the truncation error $|\mathbf{G}(t) - \mathbf{G}_{\mathcal{N}}(t)|$ are then derived, followed by a perturbation estimate for $|\mathbf{G}_{\mathcal{N}}(t) - \tilde{\mathbf{G}}_{\mathcal{N}}(t)|$.

Lemma 6. The shifted convolved Fibonacci polynomial satisfies

$$|\tilde{F}_i^{(n)}(t)| < 2(n+1)^i, \quad t \in [0, 1]. \quad (81)$$

Proof. Assume that (81) is valid for the two preceding indices $i-1$ and $i-2$. From (5), together with $|2t-1| \leq 1$ for $t \in [0, 1]$,

$$\begin{aligned} |\tilde{F}_i^{(n)}(t)| &= \left| (2t-1) \frac{(n+i-1)}{i} \tilde{F}_{i-1}^{(n)}(t) + \frac{(2n+i-2)}{i} \tilde{F}_{i-2}^{(n)}(t) \right| \\ &\leq \left| \frac{(n+i-1)}{i} \tilde{F}_{i-1}^{(n)}(t) \right| + \left| \frac{(2n+i-2)}{i} \tilde{F}_{i-2}^{(n)}(t) \right| \\ &< \frac{(n+1)^{i-2}}{i} [(n+i-1)(n+1) + (2n+i-2)]. \end{aligned} \quad (82)$$

Using the elementary inequality

$$[(n+i-1)(n+1) + (2n+i-2)] < 2i(n+1)^2, \quad \forall i, n \geq 1, \quad (83)$$

we arrive at

$$|\tilde{F}_i^{(n)}(t)| < 2(n+1)^i, \quad t \in [0, 1], \quad (84)$$

which proves (81). $\square$

Lemma 7. If $G(t)$ is infinitely differentiable at $t = 0$, then it can be expanded as

$$G(t) = \sum_{i=0}^{\infty} \sum_{s=i}^{\infty} \frac{G^{(s)}(0) A_{i,s}}{s!} \tilde{F}_i^{(n)}(t), \quad (85)$$

where $A_{i,s}$ is defined in (23).

Proof. The argument follows the same expansion procedure used for Lemma 2 in [14]. $\square$

Theorem 6. If $G(t)$ is defined on $[0, 1]$, and $|G^{(i)}(0)| \leq \lambda^i$, $i > 0$, where $\lambda > 0$, and $G(t) = \sum_{i=0}^{\infty} \hat{c}_i \tilde{F}_i^{(n)}(t)$, then

$$|\hat{c}_i| < \frac{e^{\lambda(n+1)} (\lambda(n+1))^i}{i!}. \quad (86)$$

Moreover, the associated series is absolutely convergent.

Proof. From Lemma 7, we can write

$$|\hat{c}_i| = \left| \sum_{s=i}^{\infty} \frac{G^{(s)}(0) A_{i,s}}{s!} \right| \leq \sum_{s=i}^{\infty} \frac{|G^{(s)}(0)| |A_{i,s}|}{s!}. \quad (87)$$

From (23), for $n \geq 1$ and $i \leq s < \infty$,

$$|A_{i,s}| < (n+1)^s. \quad (88)$$

Applying the assumption $|G^{(i)}(0)| \leq \lambda^i$, $i > 0$ enables us to write

$$|\hat{c}_i| \leq \sum_{s=i}^{\infty} \frac{(n+1)^s \lambda^s}{s!} = \frac{e^{\lambda(n+1)} (\Gamma(i) - \Gamma(i, (n+1)\lambda))}{\Gamma(i)}, \quad (89)$$

where $\Gamma(.,.)$ represents the incomplete gamma function.

We use the estimate

$$\frac{e^{\lambda(n+1)} (\Gamma(i) - \Gamma(i, (n+1)\lambda))}{\Gamma(i)} < \frac{e^{\lambda(n+1)} (\lambda(n+1))^i}{i!}, \quad \forall i > 0, \quad (90)$$

and therefore inequality (89) implies

$$|\hat{c}_i| < \frac{e^{\lambda(n+1)} (\lambda(n+1))^i}{i!}. \quad (91)$$

The absolute convergence follows directly from (81) and (86), since

$$\begin{aligned} \left| \sum_{i=0}^{\infty} \hat{c}_i \tilde{F}_i^{(n)}(t) \right| &\leq \sum_{i=0}^{\infty} |\hat{c}_i| |\tilde{F}_i^{(n)}(t)| \\ &< 2e^{\lambda(n+1)(n+2)}. \end{aligned} \quad (92)$$

Thus, the expansion is absolutely convergent. $\square$

Theorem 7. Let $G(t)$ meet the hypothesis of Theorem 6, and let $G_{\mathcal{N}}(t) = \sum_{i=0}^{\mathcal{N}} \hat{c}_i \tilde{F}_i^{(n)}(t)$ be the approximate solution. Then, the following estimation is satisfied:

$$|G(t) - G_{\mathcal{N}}(t)| < \frac{2e^{\lambda(n+1)(n+2)} (\lambda(n+1)^2)^{\mathcal{N}}}{\mathcal{N}!}.$$

Proof. We write

$$G(t) - G_{\mathcal{N}}(t) = \sum_{i=\mathcal{N}+1}^{\infty} \hat{c}_i \tilde{F}_i^{(n)}(t). \quad (93)$$

Applying Theorem 6, together with Lemma 6, enables us to write $|G(t) - G_{\mathcal{N}}(t)|$, as

$$\begin{aligned} |G(t) - G_{\mathcal{N}}(t)| &< \sum_{i=\mathcal{N}+1}^{\infty} \frac{e^{\lambda(n+1)} 2(n+1)^{2i} \lambda^i}{i!} \\ &= \frac{2e^{\lambda(n+1)(n+2)} (\mathcal{N}! - \Gamma(\mathcal{N}+1, (n+1)^2 \lambda))}{\mathcal{N}!}. \end{aligned} \quad (94)$$

Finally, the inequality

$$\frac{2e^{\lambda(n+1)(n+2)} (\mathcal{N}! - \Gamma(\mathcal{N}+1, (n+1)^2 \lambda))}{\mathcal{N}!} < \frac{2e^{\lambda(n+1)(n+2)} (\lambda(n+1)^2)^{\mathcal{N}}}{\mathcal{N}!}, \quad \forall \mathcal{N} \geq 0,$$

and we get the following desired result:

$$|G(t) - G_{\mathcal{N}}(t)| < \frac{2e^{\lambda(n+1)(n+2)} (\lambda(n+1)^2)^{\mathcal{N}}}{\mathcal{N}!}. \quad (95)$$

This proves the theorem. $\square$

Theorem 8 (Stability). Let $G(t)$ satisfy the assumptions of Theorem 6 and let $\tilde{G}(t)$ be a perturbed function with $|\tilde{G}^{(i)}(0) - G^{(i)}(0)| \leq \delta \lambda^i$ for some $\delta > 0$ and their truncated expansions

$$G_{\mathcal{N}}(t) = \sum_{i=0}^{\mathcal{N}} \hat{c}_i \tilde{F}_i^{(n)}(t), \quad (96)$$

$$\tilde{G}_{\mathcal{N}}(t) = \sum_{i=0}^{\mathcal{N}} \tilde{c}_i \tilde{F}_i^{(n)}(t). \quad (97)$$

Then, the following bound holds:

$$|G_{\mathcal{N}}(t) - \tilde{G}_{\mathcal{N}}(t)| < 2\delta e^{\lambda(n+1)(n+2)}. \quad (98)$$

Proof. Based on definitions of $G_{\mathcal{N}}(t)$ and $\tilde{G}_{\mathcal{N}}(t)$ defined in (96) and (97), we can write

$$|G_{\mathcal{N}}(t) - \tilde{G}_{\mathcal{N}}(t)| \leq \sum_{i=0}^{\mathcal{N}} |\hat{c}_i - \tilde{c}_i| |\tilde{F}_i^{(n)}(t)|. \quad (99)$$

To find the estimation of $|\hat{c}_i - \tilde{c}_i|$, set $Z(t) = G(t) - \tilde{G}(t)$ for which $|Z^{(i)}(0)| \leq \delta \lambda^i$, $i > 0$. Applying the same coefficient estimate as in Theorem 6 to this difference function gives

$$|\hat{c}_i - \tilde{c}_i| < \delta \frac{e^{\lambda(n+1)} (\lambda(n+1))^i}{i!}. \quad (100)$$

Now, using Theorem 6 along with Lemma 6, we get

$$\begin{aligned} |G_{\mathcal{N}}(t) - \tilde{G}_{\mathcal{N}}(t)| &\leq \sum_{i=0}^{\mathcal{N}} |\hat{c}_i - \tilde{c}_i| \left| \tilde{F}_i^{(n)}(t) \right| \\ &< \delta \sum_{i=0}^{\infty} \frac{e^{\lambda(n+1)} 2(n+1)^{2i} \lambda^i}{i!} \\ &= 2\delta e^{\lambda(n+1)(n+2)}, \end{aligned} \quad (101)$$

where $2e^{\lambda(n+1)(n+2)}$ is the constant of stability. $\square$

Remark 7. The theoretical stability established in Theorem 8 should be distinguished from the numerical conditioning of the resulting algebraic systems:

1. Owing to the non-orthogonality of the adopted basis, the resulting systems may become ill-conditioned as the truncation order increases, which can amplify round-off errors.
2. To mitigate the effects of ill-conditioning and round-off errors, all numerical computations were carried out using 50-digit working precision in Wolfram Mathematica.
3. For sufficiently smooth, and particularly analytic, solutions, the rapid decay of the expansion coefficients further contributes to the high accuracy of the spectral approximation. The numerical results confirm that high accuracy is maintained even at relatively large truncation orders.

6. Numerical Illustrations

This section presents three benchmark problems to assess the accuracy and computational performance of the shifted-convolved Fibonacci tau approximation. In all numerical tables and figures, n denotes the convolution parameter, whereas $\mathcal{N}$ denotes the truncation order.

Test Problem 1. Consider the equation,

$$\begin{cases} G''(3t-1) + \frac{2}{t} G'(2t) + t G(t+1) = t^4 + 3t^3 + 3t^2 + 44t - 6, \\ G(0) = 1, \quad G'(0) = 0, \end{cases} \quad (102)$$

whose closed-form solution is $G(t) = t^3 + 1$.

The application of the proposed method described in Section 4 for $n = 3$ and $\mathcal{N} = 3$ yields the following system:

$$\begin{pmatrix} 0.33333 & 14.500 & 85.800 & 513.67 \\ 0.50000 & 4.2000 & 144.90 & 965.37 \\ 1 & -3 & 9 & -22 \\ 0 & 6 & -24 & 84 \end{pmatrix} \begin{pmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{pmatrix} = \begin{pmatrix} 13.183 \\ 21.907 \\ 1 \\ 0 \end{pmatrix}, \quad (103)$$

which gives $\{\hat{c}_0 = 0.93750, \hat{c}_1 = 0.075000, \hat{c}_2 = 0.062500, \hat{c}_3 = 0.012500\}$, and hence the approximate solution $G_{\mathcal{N}}(t)$ can be obtained as shown in Figure 1 that present a comparison between the exact and approximate solutions. Also, Table 1 presents the L_{∞} errors at various n .

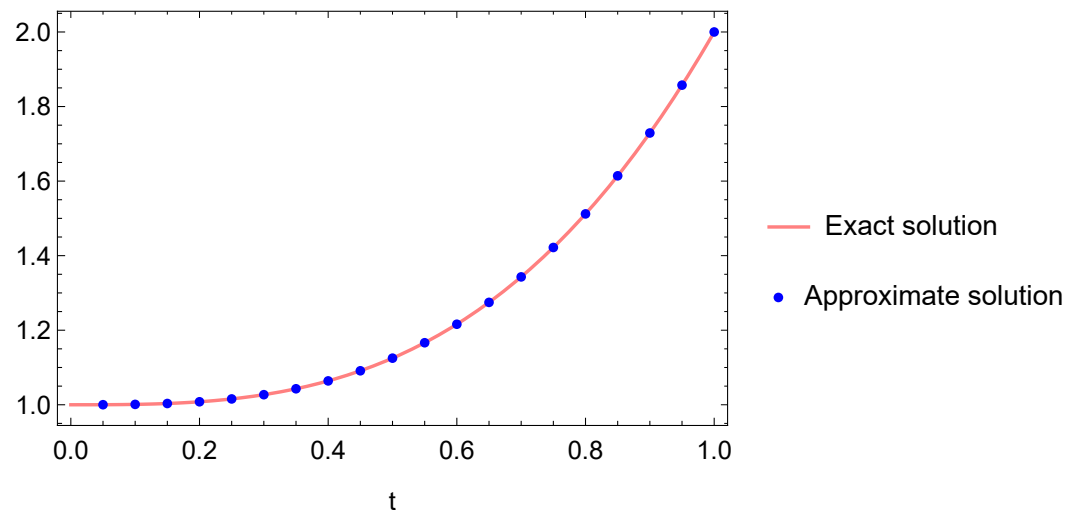


Figure 1. The exact and approximate solutions of Example 1 at $\mathcal{N} = 3$ and $n = 3$.

Table 1. Errors of Example 1 at $\mathcal{N} = 3$.

n	1	6	11	19
L_∞	3.33067×10^{-16}	2.498×10^{-16}	4.44089×10^{-16}	2.77556×10^{-16}
t	0.997298	0.957993	0.999922	0.841008

Test Problem 2. Consider the equation,

$$\begin{cases} G''(2t-1) + \frac{2}{t}G'(3t) + tG(t+1) = e^t \left(et + e^{t-1} + \frac{2e^{2t}}{t} \right), \\ G(0) = 1, \quad G'(0) = 1, \end{cases} \quad (104)$$

whose closed-form solution is $G(t) = e^t$.

Table 2 presents the CPU time used, L_∞ and L_2 errors at various $\mathcal{N}$ when $n = 5$. Table 3 presents the absolute errors at various n when $\mathcal{N} = 22$. Figure 2 shows the absolute errors at various $\mathcal{N}$ when $n = 20$. Figure 3 illustrates the L_2 and L_∞ errors at various $\mathcal{N}$ when $n = 25$.

Table 2. Errors and CPU time used of Example 2 at $n = 5$.

$\mathcal{N}$	3	7	11	15	19	22
L_∞	8.9601×10^{-2}	2.4239×10^{-3}	3.41882×10^{-5}	1.6257×10^{-7}	6.42634×10^{-10}	1.25588×10^{-11}
L_2	6.1277×10^{-2}	1.1033×10^{-3}	1.6456×10^{-5}	6.63244×10^{-8}	4.4376×10^{-10}	5.5238×10^{-12}
CPU time	1.297	2.079	4.203	9.155	20.017	33.765

Table 3. The absolute errors of Example 2 at $\mathcal{N} = 22$.

t	$n = 1$	$n = 7$	$n = 13$
0.1	1.20859×10^{-12}	1.40332×10^{-13}	1.53877×10^{-13}
0.2	3.74901×10^{-12}	3.21965×10^{-13}	3.61489×10^{-13}
0.3	6.16329×10^{-12}	2.51132×10^{-13}	3.10196×10^{-13}
0.4	7.29439×10^{-12}	2.72893×10^{-13}	2.15383×10^{-13}
0.5	6.34159×10^{-12}	1.35669×10^{-12}	1.33116×10^{-12}
0.6	2.8717×10^{-12}	3.01847×10^{-12}	3.05933×10^{-12}
0.7	3.20188×10^{-12}	5.20473×10^{-12}	5.34639×10^{-12}
0.8	1.16653×10^{-11}	7.80265×10^{-12}	8.07399×10^{-12}
0.9	2.20677×10^{-11}	1.06604×10^{-11}	1.10845×10^{-11}

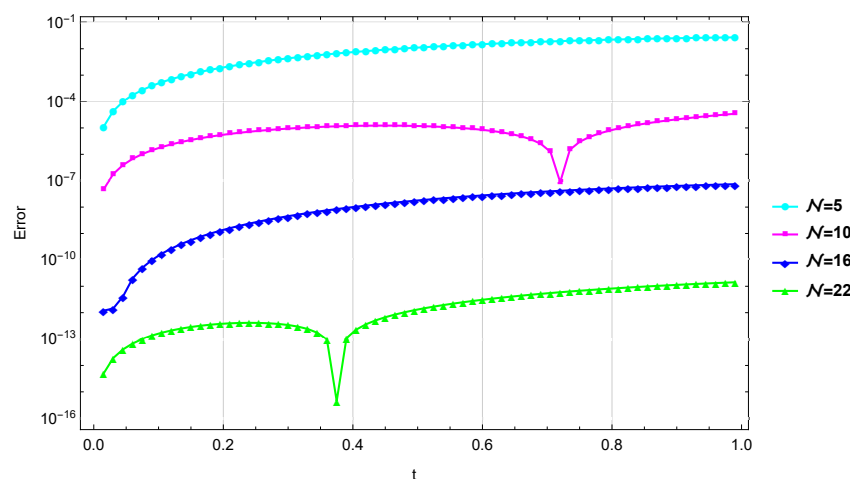


Figure 2. The absolute errors of Example 2 at various N when $n = 20$.

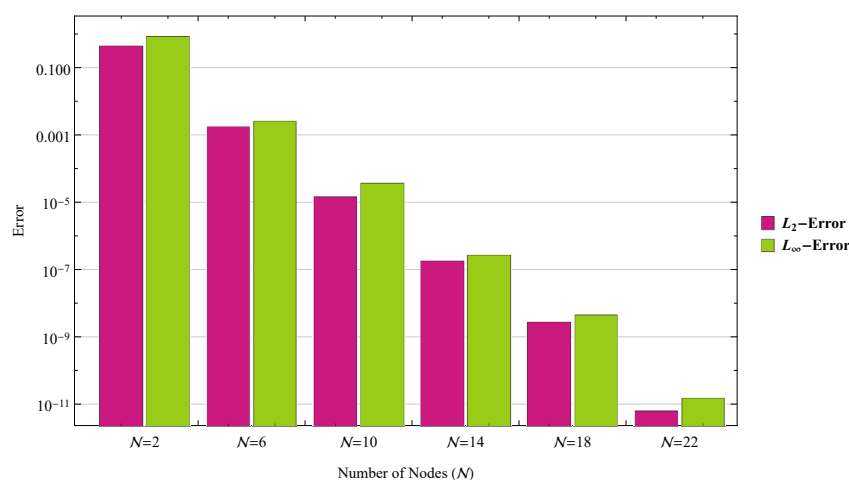


Figure 3. The L_2 and L_∞ errors of Example 2 at $n = 25$.

Test Problem 3. Consider the equation,

$$\begin{cases} G''(3t-1) + \frac{2}{t} G'(3t) + t G(t+1) = t \sin(t+1) + \sin(1-3t) + \frac{2 \cos(3t)}{t}, \\ G(0) = 0, \quad G'(0) = 1, \end{cases} \quad (105)$$

whose closed-form solution is $G(t) = \sin(t)$.

Figure 4 presents the absolute errors at various N when $n = 5$. Table 4 presents the CPU time used, L_∞ and ℓ_2 errors at various N when $n = 7$. Table 5 presents the absolute errors at various n when $N = 22$. Figure 5 shows the L_2 and L_∞ errors at various N when $n = 1$. Table 6 presents the absolute errors at $t = 10^{-1}, 10^{-3}, 10^{-5}, 10^{-7}$. As shown, the absolute error decreases continuously down to $\mathcal{O}(10^{-19})$ as $t \rightarrow 0^+$, confirming numerical stability near $t = 0$.

Remark 8. It is important to point out that the shifted classical Fibonacci polynomials, corresponding to the special case $n = 1$, do not always provide the best numerical approximation. As demonstrated in Table 3, other choices of the convolution parameter n can produce smaller errors for the same truncation order. This provides an important motivation for employing the shifted convolved Fibonacci polynomials, which generalize the shifted classical Fibonacci polynomials and offer the additional parameter n to improve the performance of the spectral approximation.

Remark 9. In our tested problems, large values of n yield smoother solutions, thereby enhancing the performance of the spectral method and reducing error. However, this behavior is problem-dependent and not guaranteed in all cases; the key factor is the regularity of the exact solution, which directly affects the convergence rate.

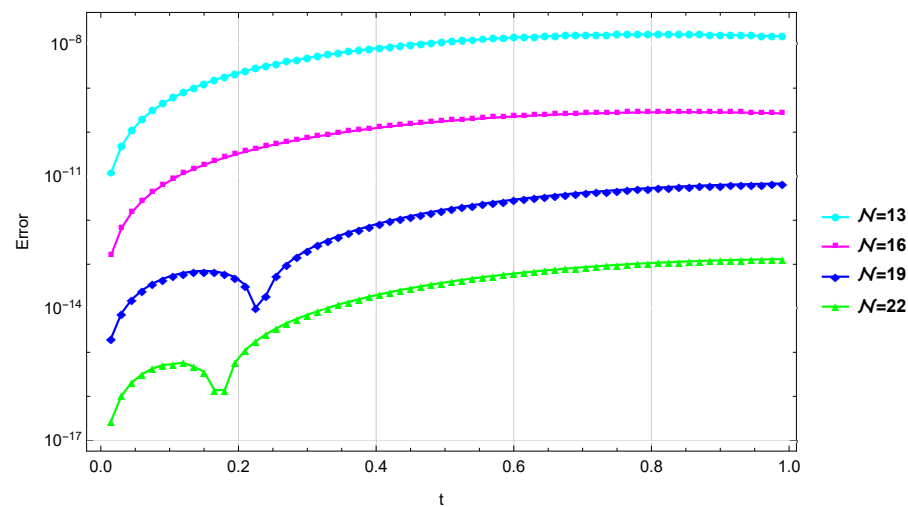


Figure 4. The absolute errors of Example 3 at various N when $n = 5$.

Table 4. Errors and CPU time used of Example 3 at $n = 7$.

N	2	6	10	14	18	22
L_∞	1.5275×10^{-1}	2.0912×10^{-4}	4.5598×10^{-7}	4.3417×10^{-9}	1.3118×10^{-11}	1.3289×10^{-13}
L_2	8.2620×10^{-2}	8.4632×10^{-5}	1.9504×10^{-7}	2.5765×10^{-9}	8.9368×10^{-12}	7.2485×10^{-14}
CPU time	1.234	1.782	3.156	8.14	17.126	33.453

Table 5. The absolute errors of Example 3 at $N = 22$.

t	$n = 2$	$n = 8$	$n = 14$
0.1	4.85723×10^{-16}	5.55112×10^{-16}	5.41234×10^{-16}
0.2	1.02696×10^{-15}	8.04912×10^{-16}	8.32667×10^{-16}
0.3	7.66054×10^{-15}	7.16094×10^{-15}	7.21645×10^{-15}
0.4	2.08722×10^{-14}	1.99840×10^{-14}	2.0095×10^{-14}
0.5	4.01901×10^{-14}	3.89133×10^{-14}	3.90243×10^{-14}
0.6	6.35048×10^{-14}	6.17284×10^{-14}	6.18394×10^{-14}
0.7	8.75966×10^{-14}	8.55982×10^{-14}	8.57092×10^{-14}
0.8	1.09246×10^{-13}	1.07137×10^{-13}	1.07248×10^{-13}
0.9	1.25344×10^{-13}	1.23124×10^{-13}	1.23457×10^{-13}

Table 6. The absolute errors of Example 3 at $N = 22$ and $n = 5$.

t	10^{-1}	10^{-2}	10^{-3}
Error	5.55112×10^{-16}	1.21431×10^{-17}	2.1684×10^{-19}

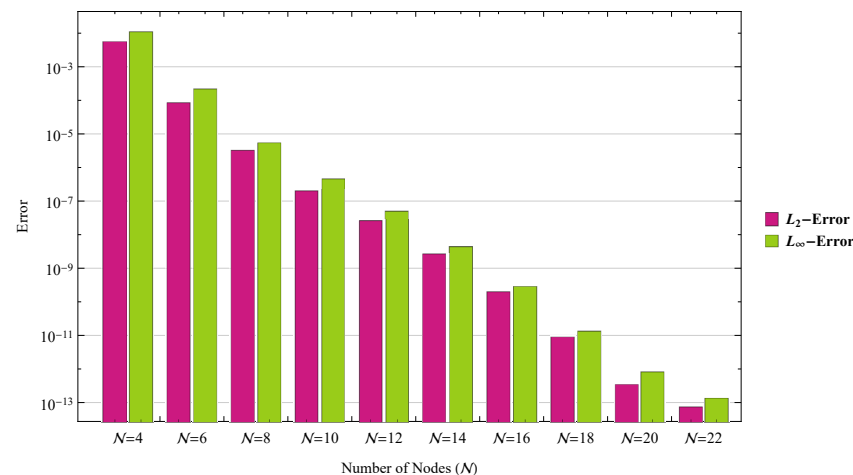


Figure 5. The L_2 and L_∞ errors of Example 3 at $n = 1$.

7. Concluding Remarks

A shifted convolved Fibonacci tau algorithm was developed for a class of second-order singular functional differential equations. The construction relied on several new identities for the proposed basis, including power expansions, inverse formulas, derivative relations, connection coefficients, linearization identities, and integral formulas. These relations enabled the singular coefficient and the functional arguments to be incorporated directly into the tau projection. As a result, the differential model was transformed into a finite algebraic system for the unknown expansion coefficients.

The convergence analysis showed that the truncated shifted convolved Fibonacci expansion converges under suitable smoothness assumptions, and explicit estimates for the coefficients and the approximation error were derived. A stability estimate was also established for perturbed data. The numerical experiments confirmed the high accuracy of the method; the errors decreased rapidly as the truncation parameter increased, and the comparisons with available results demonstrated the efficiency and stability of the scheme. The proposed framework can be extended in several directions. It may be applied to higher-order, fractional, integro-differential, and systems of functional differential equations. It can also be combined with collocation, Galerkin, or Petrov–Galerkin strategies, or adapted to other shifted convolved polynomial bases for broader classes of nonlinear mathematical models.

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