

Article

Support-Based Fuzzy Density via Support Topologies and Mapping Properties

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Abstract

We study a support-based notion of fuzzy density together with fuzzy weight in Lowen fuzzy topological spaces. Two equivalent descriptions of the density invariant are considered: one based on supports of fuzzy sets meeting every nonzero fuzzy open set and the other on ordinary subsets meeting every nonzero fuzzy open set in a positive degree. The invariant is characterized by the ordinary density of the associated support topology. This representation is used to distinguish results transferred from ordinary topology from phenomena that retain membership-level information. We show that the supports of a fuzzy base form an ordinary base for the support topology, yielding a structural comparison between ordinary weight and fuzzy weight. For surjective fuzzy continuous and fuzzy open mappings, the fiber-cardinality estimate is shown to be sharp, while a counterexample demonstrates that fuzzy openness cannot in general be omitted from the reverse estimate. We also construct two Lowen fuzzy topologies with the same support topology and the same support-based fuzzy density but different fuzzy weights, with genuinely non-crisp membership functions playing an essential role in one of them. Applications to finite products, group quotients and fuzzy symmetric powers are obtained, and the density and weight of lower-semicontinuous Lowen fuzzy topologies are computed.

Keywords: Lowen fuzzy topology; support-based fuzzy density; support topology; fuzzy weight; fuzzy open mapping; group quotient; fuzzy symmetric power



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1. Introduction

The theory of fuzzy sets was initiated by Zadeh in [1] and subsequently developed in several directions. Chang introduced fuzzy topological spaces in [2], replacing ordinary open subsets by functions with values in the unit interval. Lowen later proposed a stronger version of fuzzy topology in which every constant fuzzy set is open [3]. This additional requirement makes Lowen fuzzy topological spaces particularly suitable for categorical constructions and for fuzzy analogues of classical topological results.

Fundamental constructions in Lowen fuzzy topology, including initial and final fuzzy topologies, products, quotients and compactness, were studied in [3–5]. Related product

and quotient theorems were obtained by Wong and by Pu and Liu [6,7]. Further developments concerning convergence, uniform structures, neighbourhood systems and the role of constant fuzzy sets can be found in [8–11].

Cardinal invariants such as density and weight are basic tools in classical topology. They measure, respectively, the minimum size of a dense subset and the minimum size of a base. Their properties and relations form an important part of general and set-theoretic topology; see [12–14]. Related cardinal properties have also been investigated for spaces arising from probability-measure and semiadditive-functional constructions [15,16]. It is therefore natural to study analogous cardinal characteristics in Lowen fuzzy topological spaces, where the open objects are fuzzy sets rather than ordinary subsets. In several frameworks in the broader fuzzy-topological literature, fuzzy dense sets are defined through closure conditions of the form $\text{cl}(\lambda) = 1$; such notions are used, for example, in the study of fuzzy Baire and resolvability phenomena [17]. Related closure-based dense and nowhere-dense phenomena continue to be used in more recent work on fuzzy resolvability and irreducibility properties [18]. Recent work on fuzzy metric structures likewise illustrates the continuing interaction between fuzzy structures and their associated topological properties [19].

The present paper is mainly devoted to a support-based notion of fuzzy density formulated specifically for Lowen fuzzy topological spaces. There are two natural ways to describe this invariant. One may use ordinary subsets meeting every nonzero fuzzy open set in a positive degree, or one may use supports of fuzzy sets meeting every nonzero fuzzy open set. We prove that these approaches determine the same cardinal invariant. We then associate with every Lowen fuzzy topology an ordinary support topology and show that the invariant is completely determined by the ordinary density of this associated topology. Consequently, the precise positive membership grades are deliberately forgotten by this invariant. This makes the present notion different in purpose from closure-based fuzzy-density concepts: it is not proposed as a replacement for them, but as a cardinal invariant of the ordinary topological information detected by supports. Accordingly, the proposed invariant is intentionally support-sensitive rather than membership-sensitive and should be understood as complementary to, rather than competing with, closure-based fuzzy-density concepts.

We next investigate the behavior of support-based fuzzy density under subspaces and mappings. We prove that support-based fuzzy density is monotone for open fuzzy subspaces, while it need not be hereditary for arbitrary fuzzy subspaces. We also show that it does not increase under surjective fuzzy continuous mappings. For surjective mappings which are additionally fuzzy open, we obtain a reverse estimate depending on the cardinalities of their fibers. In particular, support-based fuzzy density is preserved by such mappings with countable fibers.

These mapping results are applied to finite products and orbit-space constructions. We establish a finite-product formula for support-based fuzzy density and obtain two-sided estimates for the support-based fuzzy density of a quotient arising from a group action by fuzzy homeomorphisms. The estimates depend on the cardinalities of the orbits and yield density preservation whenever all orbits are countable. In particular, support-based fuzzy density is preserved under finite group quotients. The corresponding result for fuzzy symmetric powers follows as a consequence. In the classical setting, cardinal properties associated with finite symmetric-power constructions have also been investigated in [20]. Fuzzy symmetric power functors and fuzzy weight were previously studied in [21]; in the present paper, symmetric powers serve only as an application of the more general density results.

The final part of the paper concerns the relation between support-based fuzzy density and fuzzy weight. We prove that support-based fuzzy density does not exceed fuzzy weight. For the Lowen fuzzy topology consisting of lower semicontinuous fuzzy sets associated with an ordinary topological space, we determine both invariants in terms of their classical counterparts. We also show that fuzzy weight need not decrease under a surjective fuzzy continuous mapping. In contrast, surjective mappings which are both fuzzy continuous and fuzzy open do not increase fuzzy weight.

A central purpose of the support-topology representation is to make the source of each result explicit. Accordingly, the results of the paper fall into two classes. First, the open-subspace inequality, monotonicity under surjective fuzzy continuous mappings, and the finite-product density formula are support-determined transfers of classical cardinal-function facts once the compatibility of the support topology with subspaces, mappings and products has been established. Second, the fuzzy-base comparison, the lower-semicontinuous fuzzy-base construction, and the behavior of fuzzy weight retain membership-level information that is not encoded by supports alone. The fiber-cardinality theorem occupies an intermediate position: its cardinal mechanism is support-topological, whereas its hypotheses are formulated through fuzzy continuity, fuzzy openness, and the fuzzy direct-image operation. The non-crisp example in Section 5 makes the support-versus-membership distinction explicit by keeping the support topology and the support-based density fixed while changing the fuzzy weight.

2. Preliminaries

In this section we recall the basic notions of Lowen fuzzy topology and the construction of fuzzy products, quotient spaces and symmetric powers. Throughout the paper, we write

$$I = [0, 1].$$

For a set X , let I^X denote the family of all maps from X to I .

Definition 1 ([3]). A Lowen fuzzy topology on a set X is a family

$$\mathcal{S} \subseteq I^X$$

satisfying the following conditions:

1. every constant fuzzy set

$$c_\lambda(x) = \lambda, \quad \lambda \in I,$$

belongs to $\mathcal{S}$;

2. $\mathcal{S}$ is closed under arbitrary pointwise suprema;
3. $\mathcal{S}$ is closed under finite pointwise infima.

The pair $(X, \mathcal{S})$ is called a Lowen fuzzy topological space. The members of $\mathcal{S}$ are called fuzzy open sets.

The constant fuzzy sets c_0 and c_1 will be denoted simply by 0 and 1, respectively.

Definition 2 ([3]). Let $(X, \mathcal{S}_X)$ and $(Y, \mathcal{S}_Y)$ be Lowen fuzzy topological spaces. A mapping

$$f : X \longrightarrow Y$$

is called fuzzy continuous if

$$\mu \circ f \in \mathcal{S}_X$$

for every $\mu \in \mathcal{S}_Y$.

Definition 3 ([21]). Let $(X, \mathcal{S})$ be a Lowen fuzzy topological space and let $A \subseteq X$. The induced fuzzy topology on A is defined by

$$\mathcal{S}|_A = \{\mu|_A : \mu \in \mathcal{S}\}.$$

The fuzzy topological space

$$(A, \mathcal{S}|_A)$$

is called the fuzzy subspace of $(X, \mathcal{S})$ determined by A .

Definition 4 ([3]). Let $(X, \mathcal{S})$ be a Lowen fuzzy topological space. A subfamily

$$\mathcal{B} \subseteq \mathcal{S}$$

is called a fuzzy base for $\mathcal{S}$ if, for every $\mu \in \mathcal{S}$, there exists a subfamily

$$\mathcal{B}_\mu \subseteq \mathcal{B}$$

such that

$$\mu = \bigvee_{v \in \mathcal{B}_\mu} v.$$

For an ordinary topological space (X, τ) , we use the notation

$$d(X, \tau) = \min\{|D| : D \subseteq X \text{ is dense in } (X, \tau)\}$$

for its ordinary density, and

$$w(X, \tau) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } (X, \tau)\}$$

for its ordinary weight.

Throughout the paper, ω denotes the first infinite cardinal. All expressions of the form $\omega + \kappa$ are cardinal sums; hence

$$\omega + \kappa = \max\{\omega, \kappa\}.$$

This normalization treats finite and countable values uniformly and ensures that the fuzzy cardinal invariants considered below are infinite cardinals.

Thus, unlike the fuzzy invariants used in this paper, the classical cardinals $d(X, \tau)$ and $w(X, \tau)$ are written without adjoining the cardinal ω . Accordingly, the formulas

$$d_f(X, \mathcal{L}(\tau)) = \omega + d(X, \tau)$$

and

$$w_f(X, \mathcal{L}(\tau)) = \omega + w(X, \tau)$$

remain valid for both finite and infinite classical cardinal values.

We next recall the fuzzy product construction. Let $n \geq 1$ and let

$$(X_1, \mathcal{S}_1), \dots, (X_n, \mathcal{S}_n)$$

be Lowen fuzzy topological spaces, and let

$$X = \prod_{i=1}^n X_i.$$

For each $i \in \{1, \dots, n\}$, denote by

$$\pi_i : X \longrightarrow X_i$$

the i -th coordinate projection.

Definition 5 ([4,6,7]). The fuzzy product topology on X , denoted by

$$\bigotimes_{i=1}^n \mathcal{S}_i,$$

is the smallest Lowen fuzzy topology on X containing every coordinate pullback

$$\mu_i \circ \pi_i, \quad \mu_i \in \mathcal{S}_i, \quad 1 \leq i \leq n.$$

The resulting fuzzy topological space is denoted by

$$\prod_{i=1}^n (X_i, \mathcal{S}_i).$$

Proposition 1. The family

$$\mathcal{B}_\Pi = \left\{ \bigwedge_{i=1}^n (\mu_i \circ \pi_i) : \mu_i \in \mathcal{S}_i, \quad 1 \leq i \leq n \right\}$$

is a fuzzy base for the product topology

$$\bigotimes_{i=1}^n \mathcal{S}_i.$$

Proof. By definition, the product fuzzy topology is generated by the family of all coordinate pullbacks

$$\mu_i \circ \pi_i.$$

Therefore, every member of the product fuzzy topology is an arbitrary pointwise supremum of finite pointwise infima of such pullbacks.

Since each $\mathcal{S}_i$ is closed under finite pointwise infima, every finite infimum of coordinate pullbacks can be written in the form

$$\bigwedge_{i=1}^n (\mu_i \circ \pi_i),$$

where coordinates not occurring in the original finite infimum are represented by the constant fuzzy set 1. Hence, every fuzzy open set in the product is a supremum of members of $\mathcal{B}_\Pi$, and therefore $\mathcal{B}_\Pi$ is a fuzzy base. $\square$

We shall also use quotient fuzzy topologies.

Definition 6 ([4]). Let $(X, \mathcal{S})$ be a Lowen fuzzy topological space and let

$$q : X \longrightarrow Y$$

be a surjective mapping. The quotient fuzzy topology on Y induced by q is the family

$$\mathcal{S}_q = \{v \in I^Y : v \circ q \in \mathcal{S}\}.$$

The fuzzy topological space

$$(Y, \mathcal{S}_q)$$

is called the fuzzy quotient space of $(X, \mathcal{S})$ determined by q .

The family $\mathcal{S}_q$ is a Lowen fuzzy topology, and the mapping

$$q : (X, \mathcal{S}) \longrightarrow (Y, \mathcal{S}_q)$$

is fuzzy continuous and surjective.

We now recall the construction of fuzzy symmetric powers.

Definition 7 ([21]). Let $(X, \mathcal{S})$ be a Lowen fuzzy topological space and let $n \geq 1$. Let $\mathfrak{S}_n$ be the symmetric group on $\{1, \dots, n\}$, acting on X^n by coordinate permutations.

The n -th symmetric power of X is the orbit space

$$SP^n(X) = X^n / \mathfrak{S}_n.$$

The orbit of a point

$$(x_1, \dots, x_n) \in X^n$$

is denoted by

$$[x_1, \dots, x_n].$$

Let

$$q_n : X^n \longrightarrow SP^n(X), \quad q_n(x_1, \dots, x_n) = [x_1, \dots, x_n],$$

be the canonical orbit mapping. We endow $SP^n(X)$ with the quotient fuzzy topology

$$\mathcal{Q}_X = \{v \in I^{SP^n(X)} : v \circ q_n \in \mathcal{S}^{\otimes n}\},$$

where

$$\mathcal{S}^{\otimes n} = \underbrace{\mathcal{S} \otimes \dots \otimes \mathcal{S}}_{n \text{ times}}$$

is the n -fold fuzzy product topology on X^n .

The Lowen fuzzy topological space

$$SP_{\text{Fuzzy}}^n(X, \mathcal{S}) = (SP^n(X), \mathcal{Q}_X)$$

is called the n -th fuzzy symmetric power of $(X, \mathcal{S})$.

By the definition of the quotient fuzzy topology, the canonical mapping

$$q_n : (X^n, \mathcal{S}^{\otimes n}) \longrightarrow (SP^n(X), \mathcal{Q}_X)$$

is fuzzy continuous and surjective.

3. Support-Based Fuzzy Density and Support Topology

In this section we introduce and study support-based fuzzy density in Lowen fuzzy topological spaces. Since fuzzy open sets are functions rather than ordinary subsets, density

can be described in two natural ways: by ordinary subsets of the underlying set and by supports of fuzzy dense sets. We first prove that these two approaches determine the same cardinal invariant.

Definition 8. Let $(X, \mathcal{S})$ be a Lowen fuzzy topological space. A subset $A \subseteq X$ is called *crisp fuzzy dense* if, for every nonzero fuzzy open set

$$\mu \in \mathcal{S} \setminus \{0\},$$

there exists a point $x \in A$ such that

$$\mu(x) > 0.$$

The crisp fuzzy density of $(X, \mathcal{S})$ is defined by

$$d_{cf}(X, \mathcal{S}) = \omega + \min\{|A| : A \subseteq X \text{ is crisp fuzzy dense in } (X, \mathcal{S})\}.$$

Definition 9. Let $(X, \mathcal{S})$ be a Lowen fuzzy topological space. A fuzzy set $v \in I^X$ is called *fuzzy dense* if

$$\mu \wedge v \neq 0$$

for every nonzero fuzzy open set

$$\mu \in \mathcal{S} \setminus \{0\}.$$

The support of v is defined by

$$\text{supp}(v) = \{x \in X : v(x) > 0\}.$$

The support-based fuzzy density of $(X, \mathcal{S})$ is defined by

$$d_f(X, \mathcal{S}) = \omega + \min\{|\text{supp}(v)| : v \in I^X \text{ is fuzzy dense in } (X, \mathcal{S})\}.$$

Proposition 2. Let $(X, \mathcal{S})$ be a Lowen fuzzy topological space. Then

$$d_f(X, \mathcal{S}) = d_{cf}(X, \mathcal{S}).$$

Proof. Put

$$\kappa_{cf} = \min\{|A| : A \subseteq X \text{ is crisp fuzzy dense in } (X, \mathcal{S})\}$$

and

$$\kappa_f = \min\{|\text{supp}(v)| : v \in I^X \text{ is fuzzy dense in } (X, \mathcal{S})\}.$$

Let $A \subseteq X$ be crisp fuzzy dense. We show that its characteristic function χ_A is fuzzy dense. Let

$$\mu \in \mathcal{S} \setminus \{0\}.$$

Since A is crisp fuzzy dense, there exists $x \in A$ such that

$$\mu(x) > 0.$$

Since $\chi_A(x) = 1$, we have

$$(\mu \wedge \chi_A)(x) = \mu(x) > 0.$$

Hence

$$\mu \wedge \chi_A \neq 0,$$

and therefore χ_A is fuzzy dense. Moreover,

$$\text{supp}(\chi_A) = A.$$

Consequently,

$$\kappa_f \leq \kappa_{cf}.$$

Conversely, let $\nu \in I^X$ be fuzzy dense. We show that

$$\text{supp}(\nu)$$

is crisp fuzzy dense. Let

$$\mu \in \mathcal{S} \setminus \{0\}.$$

Since ν is fuzzy dense,

$$\mu \wedge \nu \neq 0.$$

Thus there exists $x \in X$ such that

$$\mu(x) > 0 \quad \text{and} \quad \nu(x) > 0.$$

Therefore

$$x \in \text{supp}(\nu)$$

and $\mu(x) > 0$. Hence $\text{supp}(\nu)$ is crisp fuzzy dense, and thus

$$\kappa_{cf} \leq \kappa_f.$$

It follows that

$$\kappa_f = \kappa_{cf}.$$

Therefore,

$$d_f(X, \mathcal{S}) = \omega + \kappa_f = \omega + \kappa_{cf} = d_{cf}(X, \mathcal{S}).$$

□

In view of Proposition 2, the fuzzy and crisp notions of density determine the same cardinal invariant. Therefore, in what follows we use the notation

$$d_f(X, \mathcal{S})$$

for their common value.

Definition 10. A Lowen fuzzy topological space $(X, \mathcal{S})$ is called fuzzy separable if it contains a countable crisp fuzzy dense subset. Equivalently,

$$d_f(X, \mathcal{S}) = \omega.$$

The following construction describes the ordinary topological structure detected by support-based fuzzy density.

Proposition 3. Let $(X, \mathcal{S})$ be a Lowen fuzzy topological space. Then the family

$$\tau_{\text{supp}}(\mathcal{S}) = \{\text{supp}(\mu) : \mu \in \mathcal{S}\}$$

is an ordinary topology on X .

Proof. Since

$$\text{supp}(0) = \emptyset$$

and

$$\text{supp}(1) = X,$$

the family $\tau_{\text{supp}}(\mathcal{S})$ contains both $\emptyset$ and X .

Let $\{\mu_j : j \in J\} \subseteq \mathcal{S}$. Since $\mathcal{S}$ is closed under arbitrary pointwise suprema,

$$\bigvee_{j \in J} \mu_j \in \mathcal{S}.$$

Moreover,

$$\text{supp}\left(\bigvee_{j \in J} \mu_j\right) = \bigcup_{j \in J} \text{supp}(\mu_j).$$

Hence $\tau_{\text{supp}}(\mathcal{S})$ is closed under arbitrary unions.

Now let $\mu, \nu \in \mathcal{S}$. Since $\mathcal{S}$ is closed under finite pointwise infima,

$$\mu \wedge \nu \in \mathcal{S}.$$

Furthermore,

$$\text{supp}(\mu \wedge \nu) = \text{supp}(\mu) \cap \text{supp}(\nu).$$

Thus $\tau_{\text{supp}}(\mathcal{S})$ is closed under finite intersections.

Therefore $\tau_{\text{supp}}(\mathcal{S})$ is an ordinary topology on X . $\square$

We call $\tau_{\text{supp}}(\mathcal{S})$ the *support topology* associated with $(X, \mathcal{S})$. For brevity, in the remainder of the paper we write

$$\tau_s(\mathcal{S}) = \tau_{\text{supp}}(\mathcal{S}).$$

Remark 1. The sets $\text{supp}(\mu) = \{x : \mu(x) > 0\}$ are the strict zero-level cuts of fuzzy open sets. Thus the support-topology construction is closely related to the classical theory of level topologies in fuzzy topology; see, for example, Wuyts [22]. We do not claim that the general idea of extracting ordinary topologies from level cuts is new. The role of $\tau_s(\mathcal{S})$ here is structural: it identifies exactly the ordinary topological information detected by the support-based cardinal invariant and allows that information to be compared with membership-sensitive quantities such as fuzzy weight.

The support-topology construction is compatible with the basic operations used below. We record the relevant facts explicitly.

Proposition 4. Let $(X, \mathcal{S})$ be a Lowen fuzzy topological space and let $A \subseteq X$. Then

$$\tau_s(\mathcal{S}|_A) = \tau_s(\mathcal{S})|_A.$$

Proof. For every $\mu \in \mathcal{S}$,

$$\text{supp}(\mu|_A) = A \cap \text{supp}(\mu).$$

As μ ranges over $\mathcal{S}$, the sets on the right-hand side are precisely the members of the topology induced on A by $\tau_s(\mathcal{S})$. $\square$

Proposition 5. Let

$$f : (X, \mathcal{S}_X) \longrightarrow (Y, \mathcal{S}_Y)$$

be fuzzy continuous. Then

$$f : (X, \tau_s(\mathcal{S}_X)) \longrightarrow (Y, \tau_s(\mathcal{S}_Y))$$

is continuous.

Proof. For every $\mu \in \mathcal{S}_Y$, fuzzy continuity gives $\mu \circ f \in \mathcal{S}_X$, and

$$f^{-1}(\text{supp}(\mu)) = \text{supp}(\mu \circ f).$$

Thus the inverse image of every member of $\tau_s(\mathcal{S}_Y)$ belongs to $\tau_s(\mathcal{S}_X)$. $\square$

Proposition 6. Let $n \geq 1$ and let $(X_1, \mathcal{S}_1), \dots, (X_n, \mathcal{S}_n)$ be Lowen fuzzy topological spaces. Then

$$\tau_s\left(\bigotimes_{i=1}^n \mathcal{S}_i\right) = \prod_{i=1}^n \tau_s(\mathcal{S}_i).$$

Proof. For a basic fuzzy open set

$$B = \bigwedge_{i=1}^n (\mu_i \circ \pi_i), \quad \mu_i \in \mathcal{S}_i,$$

we have

$$\text{supp}(B) = \prod_{i=1}^n \text{supp}(\mu_i).$$

Hence the supports of basic fuzzy open sets form an ordinary base for the product of the support topologies. By Proposition 1, every fuzzy open set in the product is a supremum of such basic fuzzy open sets, and its support is the union of their supports. The asserted equality follows. $\square$

Theorem 1. Let $(X, \mathcal{S})$ be a Lowen fuzzy topological space. Then

$$d_f(X, \mathcal{S}) = \omega + d(X, \tau_s(\mathcal{S})),$$

where $d(X, \tau_s(\mathcal{S}))$ denotes the ordinary density of the topological space

$$(X, \tau_s(\mathcal{S})).$$

Proof. A subset $A \subseteq X$ is crisp fuzzy dense in $(X, \mathcal{S})$ if and only if, for every nonzero $\mu \in \mathcal{S}$, there exists $x \in A$ such that

$$\mu(x) > 0.$$

This condition is equivalent to

$$A \cap \text{supp}(\mu) \neq \emptyset$$

for every nonempty member

$$\text{supp}(\mu) \in \tau_s(\mathcal{S}).$$

Hence A is crisp fuzzy dense in $(X, \mathcal{S})$ if and only if A is dense in the ordinary topological space

$$(X, \tau_s(\mathcal{S})).$$

Therefore,

$$\min\{|A| : A \subseteq X \text{ is crisp fuzzy dense in } (X, \mathcal{S})\} = d(X, \tau_s(\mathcal{S})).$$

By Proposition 2, we conclude that

$$d_f(X, \mathcal{S}) = \omega + d(X, \tau_s(\mathcal{S})).$$

□

The representation theorem separates support-determined information from membership-level information. In the next section, whenever a hypothesis on a fuzzy construction induces the corresponding ordinary hypothesis on the support topologies, we explicitly use the relevant classical cardinal argument rather than regarding the transferred statement itself as a new classical theorem in fuzzy notation.

Corollary 1. *A Lowen fuzzy topological space $(X, \mathcal{S})$ is fuzzy separable if and only if the ordinary topological space*

$$(X, \tau_s(\mathcal{S}))$$

is separable.

4. Subspaces, Mappings, Products and Quotients

We now study the behavior of support-based fuzzy density under subspaces, mappings, products and quotient constructions.

In classical topology, density is monotone with respect to open subspaces:

$$d(A) \leq d(X)$$

whenever A is an open subspace of X . We establish the analogous property for support-based fuzzy density under the more general assumption that the underlying subset is open in the support topology. Since openness in a fuzzy topological space is described by fuzzy open sets, we first recall the corresponding notion of an open fuzzy subspace.

Definition 11. *Let $(X, \mathcal{S})$ be a Lowen fuzzy topological space and let $A \subseteq X$. The fuzzy subspace*

$$(A, \mathcal{S}|_A)$$

is called an open fuzzy subspace of $(X, \mathcal{S})$ if

$$\chi_A \in \mathcal{S},$$

where

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Theorem 2. *Let $(X, \mathcal{S})$ be a Lowen fuzzy topological space and let $A \subseteq X$. If*

$$A \in \tau_s(\mathcal{S}),$$

then

$$d_f(A, \mathcal{S}|_A) \leq d_f(X, \mathcal{S}).$$

Proof. By Proposition 4,

$$\tau_s(\mathcal{S}|_A) = \tau_s(\mathcal{S})|_A.$$

Since A is open in $\tau_s(\mathcal{S})$, the ordinary open-subspace inequality gives

$$d(A, \tau_s(\mathcal{S}|_A)) \leq d(X, \tau_s(\mathcal{S})).$$

Theorem 1 now yields

$$d_f(A, \mathcal{S}|_A) \leq d_f(X, \mathcal{S}).$$

□

Corollary 2. Every open fuzzy subspace $(A, \mathcal{S}|_A)$ of a Lowen fuzzy topological space $(X, \mathcal{S})$ satisfies

$$d_f(A, \mathcal{S}|_A) \leq d_f(X, \mathcal{S}).$$

Proof. If $\chi_A \in \mathcal{S}$, then

$$A = \text{supp}(\chi_A) \in \tau_s(\mathcal{S}).$$

The assertion follows from Theorem 2. □

The openness assumption in Theorem 2 is essential. The following example shows that support-based fuzzy density is not hereditary with respect to arbitrary fuzzy subspaces.

Example 1. Let $\lambda < \kappa$ be infinite cardinals. Consider the disjoint sets

$$C = \{b_{\xi} : \xi < \lambda\}, \quad A = \{a_{\alpha} : \alpha < \kappa\},$$

and put

$$X = A \cup C.$$

Choose a mapping

$$p : \kappa \longrightarrow \lambda.$$

For every $\alpha < \kappa$, define

$$u_{\alpha} = \chi_{\{a_{\alpha}, b_{p(\alpha)}\}},$$

and, for every $\xi < \lambda$, define

$$v_{\xi} = \chi_{\{b_{\xi}\}}.$$

Let $\mathcal{S}$ be the smallest Lowen fuzzy topology on X containing all constant fuzzy sets and all fuzzy sets

$$u_{\alpha} \quad (\alpha < \kappa) \quad \text{and} \quad v_{\xi} \quad (\xi < \lambda).$$

By the definition of the generated Lowen fuzzy topology, every member of $\mathcal{S}$ is an arbitrary pointwise supremum of finite pointwise infima of the specified generators and constant fuzzy sets. This normal form will be used below.

We first show that

$$d_f(X, \mathcal{S}) = \lambda.$$

Every nonzero fuzzy set obtained as a finite pointwise infimum of the generators is positive at some point of C . Indeed, for $\alpha, \beta < \kappa$ and $\xi, \eta < \lambda$, each of

$$u_{\alpha} \wedge u_{\beta}, \quad u_{\alpha} \wedge v_{\xi}, \quad \text{and} \quad v_{\xi} \wedge v_{\eta}$$

is either zero or is positive at a point of C . Taking the infimum with finitely many positive constant fuzzy sets preserves this property, whereas the presence of the constant zero gives the zero fuzzy set.

Consequently, every nonzero member of $\mathcal{S}$, being a pointwise supremum of such finite infima, is positive at some point of C . Hence C is crisp fuzzy dense in $(X, \mathcal{S})$. Since λ is infinite,

$$d_f(X, \mathcal{S}) \leq \omega + |C| = \lambda.$$

Conversely, let $D \subseteq X$ be crisp fuzzy dense. Since

$$v_{\xi} = \chi_{\{b_{\xi}\}} \in \mathcal{S} \setminus \{0\}$$

for every $\xi < \lambda$, there must exist a point $x \in D$ such that

$$v_{\xi}(x) > 0.$$

This is possible only when

$$x = b_{\xi}.$$

Therefore

$$C \subseteq D,$$

and hence

$$|D| \geq \lambda.$$

It follows that

$$d_f(X, \mathcal{S}) \geq \lambda.$$

Thus

$$d_f(X, \mathcal{S}) = \lambda.$$

Now consider the fuzzy subspace

$$(A, \mathcal{S}|_A).$$

For every $\alpha < \kappa$, we have

$$u_{\alpha}|_A = \chi_{\{a_{\alpha}\}}.$$

Therefore every crisp fuzzy dense subset of $(A, \mathcal{S}|_A)$ must contain each point a_{α} . Hence the only crisp fuzzy dense subset of minimal cardinality is A itself, and

$$d_f(A, \mathcal{S}|_A) = \omega + |A| = \kappa.$$

Consequently,

$$d_f(A, \mathcal{S}|_A) = \kappa > \lambda = d_f(X, \mathcal{S}).$$

Moreover, $A \notin \tau_s(\mathcal{S})$. Indeed, every nonzero member of $\mathcal{S}$ is positive at some point of C , and hence no nonzero member of $\mathcal{S}$ has support equal to A . Thus support-based fuzzy density is not hereditary with respect to arbitrary fuzzy subspaces, and the support-openness assumption in Theorem 2 cannot simply be omitted.

Theorem 3. Let

$$f : (X, \mathcal{S}_X) \longrightarrow (Y, \mathcal{S}_Y)$$

be a surjective fuzzy continuous mapping between Lowen fuzzy topological spaces. Then

$$d_f(Y, \mathcal{S}_Y) \leq d_f(X, \mathcal{S}_X).$$

Proof. By Proposition 5, f induces a continuous surjection between the associated support topologies. Hence the ordinary density inequality for continuous surjections gives

$$d(Y, \tau_s(\mathcal{S}_Y)) \leq d(X, \tau_s(\mathcal{S}_X)).$$

Applying Theorem 1 to both spaces yields

$$d_f(Y, \mathcal{S}_Y) \leq d_f(X, \mathcal{S}_X).$$

□

We now obtain a reverse estimate under an additional fuzzy openness assumption.

Let

$$f : (X, \mathcal{S}_X) \longrightarrow (Y, \mathcal{S}_Y)$$

be a mapping. We adopt the standard sup-based direct-image convention for fuzzy sets, i.e., the usual direct image associated with Zadeh's extension principle [1]. Thus, for every $\mu \in I^X$, its direct image

$$f_*(\mu) \in I^Y$$

is defined by

$$f_*(\mu)(y) = \bigvee \{\mu(x) : x \in f^{-1}(y)\}, \quad y \in Y.$$

If $f^{-1}(y) = \emptyset$, the supremum is understood to be 0.

Definition 12. A mapping

$$f : (X, \mathcal{S}_X) \longrightarrow (Y, \mathcal{S}_Y)$$

between Lowen fuzzy topological spaces is called **fuzzy open** if

$$f_*(\mu) \in \mathcal{S}_Y$$

for every $\mu \in \mathcal{S}_X$.

Remark 2. Throughout the paper, “fuzzy open” refers to the preceding direct-image condition. Other open-mapping notions occur in the fuzzy-topology literature, for example formulations expressed through level sets or related cut structures; no equivalence with those alternative notions is asserted here. The feature needed for the present support-based arguments is the identity

$$\text{supp}(f_*(\mu)) = f(\text{supp}(\mu)),$$

which makes the adopted notion compatible with ordinary openness after passing to support topologies.

Proposition 7. Let

$$f : (X, \mathcal{S}_X) \longrightarrow (Y, \mathcal{S}_Y)$$

be fuzzy open. Then

$$f : (X, \tau_s(\mathcal{S}_X)) \longrightarrow (Y, \tau_s(\mathcal{S}_Y))$$

is an open mapping.

Proof. For every $\mu \in \mathcal{S}_X$, fuzzy openness gives $f_*(\mu) \in \mathcal{S}_Y$. Moreover,

$$\text{supp}(f_*(\mu)) = f(\text{supp}(\mu)).$$

Indeed, the supremum defining $f_*(\mu)(y)$ is positive if and only if $\mu(x) > 0$ for some $x \in f^{-1}(y)$. Hence the image under f of every member of $\tau_s(\mathcal{S}_X)$ belongs to $\tau_s(\mathcal{S}_Y)$. $\square$

Corollary 3. *Let*

$$f : (X, \mathcal{S}_X) \longrightarrow (Y, \mathcal{S}_Y)$$

be surjective, fuzzy continuous and fuzzy open. Then

$$f : (X, \tau_s(\mathcal{S}_X)) \longrightarrow (Y, \tau_s(\mathcal{S}_Y))$$

is a continuous open surjection and therefore a quotient mapping in the ordinary topological sense.

Proof. Continuity follows from Proposition 5, openness from Proposition 7, and surjectivity is part of the hypothesis. Every continuous open surjection is a quotient mapping. $\square$

Theorem 4. *Let*

$$f : (X, \mathcal{S}_X) \longrightarrow (Y, \mathcal{S}_Y)$$

be a surjective fuzzy continuous and fuzzy open mapping between nonempty Lowen fuzzy topological spaces. Put

$$\kappa(f) = \sup\{|f^{-1}(y)| : y \in Y\}.$$

Then

$$d_f(Y, \mathcal{S}_Y) \leq d_f(X, \mathcal{S}_X) \leq \max\{d_f(Y, \mathcal{S}_Y), \kappa(f)\}.$$

Proof. The first inequality follows immediately from Theorem 3.

By Proposition 7, the underlying mapping between the support topologies is open. We give the corresponding density argument explicitly in order to retain the fiber-cardinality estimate.

To prove the second inequality, put

$$\lambda = \min\{|D| : D \subseteq Y \text{ is crisp fuzzy dense in } (Y, \mathcal{S}_Y)\}.$$

Choose a crisp fuzzy dense subset $D \subseteq Y$ such that

$$|D| = \lambda.$$

Thus

$$d_f(Y, \mathcal{S}_Y) = \omega + \lambda.$$

We show that

$$f^{-1}(D)$$

is crisp fuzzy dense in $(X, \mathcal{S}_X)$. Let

$$\mu \in \mathcal{S}_X \setminus \{0\}.$$

Since f is fuzzy open,

$$f_*(\mu) \in \mathcal{S}_Y.$$

Moreover, $f_*(\mu) \neq 0$. Indeed, there exists $x_0 \in X$ such that

$$\mu(x_0) > 0.$$

Putting

$$y_0 = f(x_0),$$

we obtain

$$f_*(\mu)(y_0) \geq \mu(x_0) > 0.$$

Since D is crisp fuzzy dense in $(Y, \mathcal{S}_Y)$, there exists $y \in D$ such that

$$f_*(\mu)(y) > 0.$$

If $\mu(x) = 0$ for every $x \in f^{-1}(y)$, then the supremum defining $f_*(\mu)(y)$ would be zero. Therefore there exists $x \in f^{-1}(y)$ such that

$$\mu(x) > 0.$$

Since $y \in D$, we have

$$x \in f^{-1}(D).$$

Thus $f^{-1}(D)$ is crisp fuzzy dense in $(X, \mathcal{S}_X)$.

Furthermore,

$$|f^{-1}(D)| \leq |D| \cdot \kappa(f) = \lambda \cdot \kappa(f).$$

Consequently,

$$\begin{aligned} d_f(X, \mathcal{S}_X) &\leq \omega + |f^{-1}(D)| \\ &\leq \omega + \lambda \cdot \kappa(f) \\ &\leq \max\{\omega, \lambda, \kappa(f)\} \\ &= \max\{d_f(Y, \mathcal{S}_Y), \kappa(f)\}. \end{aligned}$$

This completes the proof. $\square$

The following example shows that the fiber term in the preceding estimate is genuinely necessary and that the upper bound can be attained when the fibers are larger than the density of the range.

Example 2. Let $\kappa > \omega$ be an infinite cardinal, let D_κ be a discrete space of cardinality κ , and put

$$Y = \mathbb{R}, \quad X = \mathbb{R} \times D_\kappa.$$

Let τ_Y be the usual topology on $\mathbb{R}$ and let $\tau_X = \tau_Y \times \delta_\kappa$, where δ_κ is the discrete topology on D_κ . Denote by $\mathcal{L}(\tau_X)$ and $\mathcal{L}(\tau_Y)$ the Lowen fuzzy topologies of all lower semicontinuous I -valued functions with respect to these ordinary topologies (the Lowen-topology property of this construction is verified in Proposition 11 below), and consider the projection

$$p : (X, \mathcal{L}(\tau_X)) \longrightarrow (Y, \mathcal{L}(\tau_Y)), \quad p(t, d) = t.$$

The ordinary projection is continuous, open and surjective. Hence $v \circ p$ is lower semicontinuous whenever $v \in \mathcal{L}(\tau_Y)$, so p is fuzzy continuous. Moreover, if $\mu \in \mathcal{L}(\tau_X)$ and $r \in [0, 1]$, then

$$\{y : p_*(\mu)(y) > r\} = p(\{x : \mu(x) > r\}),$$

which is open because p is an open mapping. Thus p is fuzzy open.

For a topology τ , the support topology of $\mathcal{L}(\tau)$ equals τ : supports of lower semicontinuous fuzzy sets are open, while every $U \in \tau$ is the support of χ_U . Therefore

$$d_f(Y, \mathcal{L}(\tau_Y)) = \omega + d(\mathbb{R}) = \omega.$$

On the other hand, every slice $\mathbb{R} \times \{d\}$ is a nonempty open subset of X , so every dense subset of X must meet all κ slices. The set $\mathbb{Q} \times D_\kappa$ is dense, and hence

$$d(X, \tau_X) = \kappa, \quad d_f(X, \mathcal{L}(\tau_X)) = \kappa.$$

Every fiber of p has cardinality κ , so $\kappa(p) = \kappa$. We therefore obtain

$$d_f(X, \mathcal{L}(\tau_X)) = \kappa = \max\{d_f(Y, \mathcal{L}(\tau_Y)), \kappa(p)\},$$

with

$$\kappa(p) > d_f(Y, \mathcal{L}(\tau_Y)).$$

Thus the upper estimate in Theorem 4 is sharp.

Remark 3. The fuzzy-openness assumption is also essential for a reverse estimate of this form. Let $|X| = \kappa > \omega$, let $\mathcal{S}_X = I^X$, and let

$$\mathcal{S}_Y = \{c_t : t \in I\}$$

be the Lowen fuzzy topology consisting only of constant fuzzy sets. The identity map

$$\text{id}_X : (X, \mathcal{S}_X) \longrightarrow (X, \mathcal{S}_Y)$$

is surjective and fuzzy continuous, but it is not fuzzy open. Its fibers are singletons, whereas

$$d_f(X, \mathcal{S}_X) = \kappa \quad \text{and} \quad d_f(X, \mathcal{S}_Y) = \omega.$$

Hence

$$d_f(X, \mathcal{S}_X) \not\leq \max\{d_f(X, \mathcal{S}_Y), \kappa(\text{id}_X)\} = \omega.$$

Thus fuzzy continuity alone cannot yield the reverse inequality from Theorem 4.

Corollary 4. Let

$$f : (X, \mathcal{S}_X) \longrightarrow (Y, \mathcal{S}_Y)$$

be a surjective fuzzy continuous and fuzzy open mapping. If

$$\kappa(f) \leq d_f(Y, \mathcal{S}_Y),$$

then

$$d_f(X, \mathcal{S}_X) = d_f(Y, \mathcal{S}_Y).$$

Proof. By Theorem 4,

$$d_f(Y, \mathcal{S}_Y) \leq d_f(X, \mathcal{S}_X) \leq \max\{d_f(Y, \mathcal{S}_Y), \kappa(f)\}.$$

Since

$$\kappa(f) \leq d_f(Y, \mathcal{S}_Y),$$

the right-hand side is equal to $d_f(Y, \mathcal{S}_Y)$. $\square$

Corollary 5. Let

$$f : (X, \mathcal{S}_X) \longrightarrow (Y, \mathcal{S}_Y)$$

be a surjective fuzzy continuous and fuzzy open mapping whose fibers are at most countable, that is,

$$|f^{-1}(y)| \leq \omega \quad (y \in Y).$$

Then

$$d_f(X, \mathcal{S}_X) = d_f(Y, \mathcal{S}_Y).$$

In particular, the equality holds whenever f is finite-to-one.

Proof. Since d_f is ω -normalized, the assumption

$$|f^{-1}(y)| \leq \omega \quad (y \in Y)$$

implies

$$\kappa(f) \leq \omega \leq d_f(Y, \mathcal{S}_Y).$$

The result now follows from Corollary 4. $\square$

Theorem 5. Let $n \geq 1$, and let

$$(X_1, \mathcal{S}_1), \dots, (X_n, \mathcal{S}_n)$$

be nonempty Lowen fuzzy topological spaces. Then

$$d_f\left(\prod_{i=1}^n (X_i, \mathcal{S}_i)\right) = \max_{1 \leq i \leq n} d_f(X_i, \mathcal{S}_i).$$

Proof. By Proposition 6, the support topology of the fuzzy product is the ordinary finite product of the support topologies.

Put

$$\delta_i = d(X_i, \tau_s(\mathcal{S}_i)), \quad i = 1, \dots, n.$$

For nonempty finite products, the standard density estimates give

$$\max_{1 \leq i \leq n} \delta_i \leq d\left(\prod_{i=1}^n X_i, \prod_{i=1}^n \tau_s(\mathcal{S}_i)\right) \leq \prod_{i=1}^n \delta_i.$$

where the lower bound follows from the coordinate projections and the upper bound from the product of dense subsets. After adjoining ω , finite cardinal products collapse to the maximum of the normalized factors. Hence, by Theorem 1,

$$\begin{aligned} d_f\left(\prod_{i=1}^n (X_i, \mathcal{S}_i)\right) &= \omega + d\left(\prod_{i=1}^n X_i, \prod_{i=1}^n \tau_s(\mathcal{S}_i)\right) \\ &= \max_{1 \leq i \leq n} \{\omega + \delta_i\} \\ &= \max_{1 \leq i \leq n} d_f(X_i, \mathcal{S}_i). \end{aligned}$$

Thus the formula is a support-topological transfer of the classical finite-product density estimates together with the ω -normalization. $\square$

Corollary 6. Let $n \geq 1$, and let

$$(X_1, \mathcal{S}_1), \dots, (X_n, \mathcal{S}_n)$$

be nonempty Lowen fuzzy topological spaces. Then the fuzzy product

$$\prod_{i=1}^n (X_i, \mathcal{S}_i)$$

is fuzzy separable if and only if every factor

$$(X_i, \mathcal{S}_i)$$

is fuzzy separable.

Proof. By Theorem 5,

$$d_f \left(\prod_{i=1}^n (X_i, \mathcal{S}_i) \right) = \max_{1 \leq i \leq n} d_f(X_i, \mathcal{S}_i).$$

Therefore the density of the product is equal to ω if and only if

$$d_f(X_i, \mathcal{S}_i) = \omega$$

for every $i \in \{1, \dots, n\}$. This is equivalent to fuzzy separability of all factors. $\square$

Group Quotients and Symmetric Powers

We now apply the preceding mapping results to orbit spaces arising from group actions by fuzzy homeomorphisms.

Definition 13. Let $(X, \mathcal{S})$ be a Lowen fuzzy topological space. A bijection

$$h : X \longrightarrow X$$

is called a fuzzy homeomorphism if both h and h^{-1} are fuzzy continuous.

Let G be a group acting on $(X, \mathcal{S})$ by fuzzy homeomorphisms. Denote the orbit of $x \in X$ by

$$Gx = \{g(x) : g \in G\},$$

and let

$$q_G : X \longrightarrow X/G, \quad q_G(x) = Gx,$$

be the canonical orbit mapping.

We endow X/G with the quotient fuzzy topology

$$\mathcal{Q}_G = \{v \in I^{X/G} : v \circ q_G \in \mathcal{S}\}.$$

Proposition 8. The orbit mapping

$$q_G : (X, \mathcal{S}) \longrightarrow (X/G, \mathcal{Q}_G)$$

is surjective, fuzzy continuous and fuzzy open.

Proof. Surjectivity is immediate, and fuzzy continuity follows from the definition of the quotient fuzzy topology.

Let

$$\mu \in \mathcal{S}.$$

For every $x \in X$, we have

$$\begin{aligned} ((q_G)_*(\mu) \circ q_G)(x) &= (q_G)_*(\mu)(Gx) \\ &= \bigvee \{\mu(z) : z \in Gx\} \\ &= \bigvee_{g \in G} \mu(g(x)). \end{aligned}$$

Therefore,

$$(q_G)_*(\mu) \circ q_G = \bigvee_{g \in G} (\mu \circ g).$$

Since every $g \in G$ is a fuzzy homeomorphism,

$$\mu \circ g \in \mathcal{S}.$$

Because $\mathcal{S}$ is closed under arbitrary pointwise suprema, it follows that

$$(q_G)_*(\mu) \circ q_G \in \mathcal{S}.$$

Hence

$$(q_G)_*(\mu) \in \mathcal{Q}_G.$$

Thus q_G is fuzzy open. $\square$

For the action of G on X , put

$$\kappa_G(X) = \sup\{|Gx| : x \in X\}.$$

Theorem 6. Let a group G act on a nonempty Lowen fuzzy topological space $(X, \mathcal{S})$ by fuzzy homeomorphisms. Then

$$d_f(X/G, \mathcal{Q}_G) \leq d_f(X, \mathcal{S}) \leq \max\{d_f(X/G, \mathcal{Q}_G), \kappa_G(X)\}.$$

Proof. By Proposition 8, the orbit mapping

$$q_G : (X, \mathcal{S}) \longrightarrow (X/G, \mathcal{Q}_G)$$

is surjective, fuzzy continuous and fuzzy open. Hence, by Corollary 3, the induced map between the support topologies is an ordinary quotient mapping.

For every orbit $Gx \in X/G$, we have

$$q_G^{-1}(Gx) = Gx.$$

Consequently,

$$\kappa(q_G) = \sup\{|q_G^{-1}(Gx)| : Gx \in X/G\} = \kappa_G(X).$$

The required inequalities now follow from Theorem 4. $\square$

Corollary 7. Let a group G act on a nonempty Lowen fuzzy topological space $(X, \mathcal{S})$ by fuzzy homeomorphisms. If

$$\kappa_G(X) \leq d_f(X/G, \mathcal{Q}_G),$$

then

$$d_f(X/G, \mathcal{Q}_G) = d_f(X, \mathcal{S}).$$

Proof. This follows immediately from Theorem 6. $\square$

Corollary 8. *Let a group G act on a nonempty Lowen fuzzy topological space $(X, \mathcal{S})$ by fuzzy homeomorphisms. If every orbit of the action is countable, then*

$$d_f(X/G, \mathcal{Q}_G) = d_f(X, \mathcal{S}).$$

Proof. Since every orbit is countable,

$$\kappa_G(X) \leq \omega.$$

On the other hand,

$$d_f(X/G, \mathcal{Q}_G) \geq \omega.$$

Therefore

$$\kappa_G(X) \leq d_f(X/G, \mathcal{Q}_G),$$

and the result follows from Corollary 7. $\square$

Corollary 9. *Let a finite group G act on a nonempty Lowen fuzzy topological space $(X, \mathcal{S})$ by fuzzy homeomorphisms. Then*

$$d_f(X/G, \mathcal{Q}_G) = d_f(X, \mathcal{S}).$$

Proof. Every orbit of a finite group action is finite. Hence, the result follows from Corollary 8. $\square$

As an application, we obtain the density preservation theorem for fuzzy symmetric powers.

Corollary 10. *Let $(X, \mathcal{S})$ be a nonempty Lowen fuzzy topological space and let $n \geq 1$. Then*

$$d_f(SP_{\text{Fuzzy}}^n(X, \mathcal{S})) = d_f(X, \mathcal{S}).$$

Proof. The symmetric group $\mathfrak{S}_n$ acts on

$$(X^n, \mathcal{S}^{\otimes n})$$

by coordinate permutations. Every coordinate permutation is a fuzzy homeomorphism, since it merely permutes the generators

$$\mu \circ \pi_i$$

of the product fuzzy topology.

Therefore, Corollary 9 gives

$$d_f(SP_{\text{Fuzzy}}^n(X, \mathcal{S})) = d_f(X^n, \mathcal{S}^{\otimes n}).$$

By Theorem 5,

$$d_f(X^n, \mathcal{S}^{\otimes n}) = d_f(X, \mathcal{S}).$$

Combining these equalities yields the result. $\square$

Corollary 11. *Let $(X, \mathcal{S})$ be a nonempty Lowen fuzzy topological space and let $n \geq 1$. Then $(X, \mathcal{S})$ is fuzzy separable if and only if*

$$SP_{\text{Fuzzy}}^n(X, \mathcal{S})$$

is fuzzy separable.

Proof. This follows immediately from Corollary 10. $\square$

5. Fuzzy Weight and Mapping Properties

We now turn to fuzzy weight and its relation to support-based fuzzy density. The version of fuzzy weight used below was studied in [21].

Definition 14. Let $(X, \mathcal{S})$ be a Lowen fuzzy topological space. The fuzzy weight of $(X, \mathcal{S})$ is defined by

$$w_f(X, \mathcal{S}) = \omega + \min\{|\mathcal{B}| : \mathcal{B} \text{ is a fuzzy base for } (X, \mathcal{S})\}.$$

Since the definition contains the term ω , we always have

$$w_f(X, \mathcal{S}) \geq \omega.$$

Proposition 9. Let $(X, \mathcal{S})$ be a Lowen fuzzy topological space and let $\mathcal{B}$ be a fuzzy base for $\mathcal{S}$. Then

$$\mathcal{B}_s = \{\text{supp}(v) : v \in \mathcal{B}\}$$

is an ordinary base for $\tau_s(\mathcal{S})$. Consequently,

$$\omega + w(X, \tau_s(\mathcal{S})) \leq w_f(X, \mathcal{S}).$$

Proof. For every $\mu \in \mathcal{S}$, choose $\mathcal{B}_\mu \subseteq \mathcal{B}$ such that

$$\mu = \bigvee_{v \in \mathcal{B}_\mu} v.$$

Taking supports gives

$$\text{supp}(\mu) = \bigcup_{v \in \mathcal{B}_\mu} \text{supp}(v).$$

Thus every member of $\tau_s(\mathcal{S})$ is a union of members of $\mathcal{B}_s$, so $\mathcal{B}_s$ is an ordinary base. Since $|\mathcal{B}_s| \leq |\mathcal{B}|$, minimizing over fuzzy bases yields the stated inequality after adjoining ω . $\square$

This proposition records precisely the information retained from a fuzzy base after passing to supports: the ordinary support topology cannot require a larger normalized base than the original fuzzy topology.

Definition 15. A Lowen fuzzy topological space $(X, \mathcal{S})$ is called fuzzy second countable if it has a countable fuzzy base. Equivalently,

$$w_f(X, \mathcal{S}) = \omega.$$

The following hereditary property of fuzzy weight was established in [21].

Proposition 10. Let $(X, \mathcal{S})$ be a Lowen fuzzy topological space and let $A \subseteq X$. Then

$$w_f(A, \mathcal{S}|_A) \leq w_f(X, \mathcal{S}).$$

We next establish the standard relation between support-based fuzzy density and fuzzy weight.

Theorem 7. Let $(X, \mathcal{S})$ be a nonempty Lowen fuzzy topological space. Then

$$d_f(X, \mathcal{S}) \leq w_f(X, \mathcal{S}).$$

Proof. By Theorem 1, the classical inequality $d \leq w$, and Proposition 9,

$$\begin{aligned} d_f(X, \mathcal{S}) &= \omega + d(X, \tau_s(\mathcal{S})) \\ &\leq \omega + w(X, \tau_s(\mathcal{S})) \\ &\leq w_f(X, \mathcal{S}). \end{aligned}$$

□

Corollary 12. Every nonempty fuzzy second countable Lowen fuzzy topological space is fuzzy separable.

Proof. If $(X, \mathcal{S})$ is fuzzy second countable, then

$$w_f(X, \mathcal{S}) = \omega.$$

By Theorem 7,

$$\omega \leq d_f(X, \mathcal{S}) \leq w_f(X, \mathcal{S}) = \omega.$$

Hence

$$d_f(X, \mathcal{S}) = \omega,$$

so $(X, \mathcal{S})$ is fuzzy separable. □

The next example shows that support-based density can completely ignore membership-level complexity that is nevertheless detected by fuzzy weight. It also shows explicitly that two Lowen fuzzy topologies may have the same support topology and the same support-based fuzzy density while being cardinally different as fuzzy structures.

Example 3. Let X be a set of cardinality $\kappa > \omega$. Define

$$\mathcal{S}_0 = \{c_t : t \in I\},$$

the Lowen fuzzy topology of all constant fuzzy sets. For each $x \in X$, define the genuinely non-crisp fuzzy set $u_x \in I^X$ by

$$u_x(y) = \begin{cases} 1, & y = x, \\ \frac{1}{2}, & y \neq x. \end{cases}$$

Let $\mathcal{S}_\kappa$ be the smallest Lowen fuzzy topology containing all constant fuzzy sets and all u_x , $x \in X$.

Every nonzero member of either $\mathcal{S}_0$ or $\mathcal{S}_\kappa$ has full support X . Indeed, every nonzero finite infimum of positive constants and the generators u_x is positive at every point, and arbitrary suprema preserve this property. Therefore

$$\tau_s(\mathcal{S}_0) = \tau_s(\mathcal{S}_\kappa) = \{\emptyset, X\}.$$

By Theorem 1,

$$d_f(X, \mathcal{S}_0) = d_f(X, \mathcal{S}_\kappa) = \omega.$$

The rational constants form a countable fuzzy base for $\mathcal{S}_0$ (by approximation from below), so

$$w_f(X, \mathcal{S}_0) = \omega.$$

For $\mathcal{S}_\kappa$, consider

$$\mathcal{B}_\kappa = \{c_q : q \in \mathbb{Q} \cap [0, 1]\} \cup \{c_q \wedge u_x : x \in X, q \in \mathbb{Q} \cap (1/2, 1]\}.$$

Finite infima of two distinct generators u_x and u_y equal the constant $c_{1/2}$, while a finite infimum involving only one such generator has the form $c_t \wedge u_x$. Rational approximation from below therefore shows that every member of $\mathcal{S}_\kappa$ is a supremum of members of $\mathcal{B}_\kappa$. Hence

$$w_f(X, \mathcal{S}_\kappa) \leq \kappa.$$

Conversely, let $\mathcal{B}$ be any fuzzy base for $\mathcal{S}_\kappa$. For each $x \in X$, the representation of u_x as a supremum of members of $\mathcal{B}$ yields some $b_x \in \mathcal{B}$ such that

$$b_x \leq u_x \quad \text{and} \quad b_x(x) > \frac{1}{2}.$$

If $x \neq y$ and $b_x = b_y$, then

$$b_x \leq u_x \wedge u_y = c_{1/2},$$

contradicting $b_x(x) > 1/2$. Thus $x \mapsto b_x$ is injective, so $|\mathcal{B}| \geq \kappa$. Consequently,

$$w_f(X, \mathcal{S}_\kappa) = \kappa.$$

We have therefore constructed two Lowen fuzzy topologies with the same support topology and the same support-based fuzzy density but with

$$w_f(X, \mathcal{S}_0) = \omega < \kappa = w_f(X, \mathcal{S}_\kappa).$$

This demonstrates directly that fuzzy weight can retain membership-level information that is invisible to support-based density.

We next relate support-based fuzzy density and fuzzy weight to the corresponding classical cardinal invariants.

Let (X, τ) be an ordinary topological space. Denote by

$$\mathcal{L}(\tau)$$

the family of all lower semicontinuous fuzzy sets on X , that is,

$$\mathcal{L}(\tau) = \left\{ \mu \in I^X : \{x \in X : \mu(x) > r\} \in \tau \text{ for every } r \in I \right\}.$$

Proposition 11. For every topological space (X, τ) , the family $\mathcal{L}(\tau)$ is a Lowen fuzzy topology on X .

Proof. Every constant fuzzy set belongs to $\mathcal{L}(\tau)$, since each of its strict upper level sets is either X or $\emptyset$.

Let $\{\mu_j : j \in J\} \subseteq \mathcal{L}(\tau)$. For every $r \in I$, we have

$$\left\{ x \in X : \left(\bigvee_{j \in J} \mu_j \right)(x) > r \right\} = \bigcup_{j \in J} \{x \in X : \mu_j(x) > r\}.$$

The right-hand side is open in (X, τ) . Hence

$$\bigvee_{j \in J} \mu_j \in \mathcal{L}(\tau).$$

Now let $\mu, \nu \in \mathcal{L}(\tau)$. For every $r \in I$,

$$\{x \in X : (\mu \wedge \nu)(x) > r\} = \{x \in X : \mu(x) > r\} \cap \{x \in X : \nu(x) > r\}.$$

Thus

$$\mu \wedge \nu \in \mathcal{L}(\tau).$$

Therefore $\mathcal{L}(\tau)$ is a Lowen fuzzy topology on X . $\square$

Proposition 12. Let (X, τ) be a topological space. Then

$$\tau_s(\mathcal{L}(\tau)) = \tau.$$

Proof. Let $\mu \in \mathcal{L}(\tau)$. Since

$$\text{supp}(\mu) = \{x \in X : \mu(x) > 0\},$$

we have

$$\text{supp}(\mu) \in \tau.$$

Therefore,

$$\tau_s(\mathcal{L}(\tau)) \subseteq \tau.$$

Conversely, let $U \in \tau$. Its characteristic function χ_U belongs to $\mathcal{L}(\tau)$. Indeed, for every $r \in I$,

$$\{x \in X : \chi_U(x) > r\}$$

is either U or $\emptyset$. Moreover,

$$\text{supp}(\chi_U) = U.$$

Hence

$$\tau \subseteq \tau_s(\mathcal{L}(\tau)).$$

Thus

$$\tau_s(\mathcal{L}(\tau)) = \tau.$$

$\square$

Theorem 8. Let (X, τ) be a topological space. Then

$$d_f(X, \mathcal{L}(\tau)) = \omega + d(X, \tau),$$

where $d(X, \tau)$ denotes the ordinary density of (X, τ) .

Proof. By Proposition 12,

$$\tau_s(\mathcal{L}(\tau)) = \tau.$$

Therefore Theorem 1 gives

$$\begin{aligned} d_f(X, \mathcal{L}(\tau)) &= \omega + d(X, \tau_s(\mathcal{L}(\tau))) \\ &= \omega + d(X, \tau). \end{aligned}$$

$\square$

Theorem 9. Let (X, τ) be a topological space. Then

$$w_f(X, \mathcal{L}(\tau)) = \omega + w(X, \tau),$$

where $w(X, \tau)$ denotes the ordinary weight of (X, τ) .

Proof. Put

$$\kappa = \min\{|\mathcal{B}| : \mathcal{B} \text{ is an ordinary base for } (X, \tau)\}.$$

Choose an ordinary base $\mathcal{B}$ for (X, τ) such that

$$|\mathcal{B}| = \kappa.$$

Thus

$$w(X, \tau) = \kappa.$$

Consider the family

$$\mathcal{B}_f = \{q\chi_U : U \in \mathcal{B}, q \in \mathbb{Q} \cap (0, 1)\}.$$

We show that $\mathcal{B}_f$ is a fuzzy base for

$$(X, \mathcal{L}(\tau)).$$

Let $\mu \in \mathcal{L}(\tau)$ and $x \in X$. It is enough to show that

$$\mu(x) = \bigvee \{v(x) : v \in \mathcal{B}_f, v \leq \mu\}.$$

The inequality

$$\bigvee \{v(x) : v \in \mathcal{B}_f, v \leq \mu\} \leq \mu(x)$$

is immediate.

Let $0 \leq r < \mu(x)$. Choose

$$q \in \mathbb{Q} \cap (0, 1)$$

such that

$$r < q < \mu(x).$$

Since μ is lower semicontinuous, the set

$$U_q = \{y \in X : \mu(y) > q\}$$

is open and contains x . Choose $U \in \mathcal{B}$ such that

$$x \in U \subseteq U_q.$$

Then

$$q\chi_U \leq \mu$$

and

$$(q\chi_U)(x) = q > r.$$

Since $r < \mu(x)$ was arbitrary, the required equality follows. Hence $\mathcal{B}_f$ is a fuzzy base.

Moreover,

$$|\mathcal{B}_f| \leq \omega \cdot |\mathcal{B}| = \omega \cdot \kappa.$$

Consequently,

$$\begin{aligned} w_f(X, \mathcal{L}(\tau)) &\leq \omega + |\mathcal{B}_f| \\ &\leq \omega + \omega \cdot \kappa \\ &= \omega + \kappa \\ &= \omega + w(X, \tau). \end{aligned}$$

For the reverse inequality, let $\mathcal{C}$ be a fuzzy base for

$$(X, \mathcal{L}(\tau)).$$

For every $\nu \in \mathcal{C}$ and every

$$q \in \mathbb{Q} \cap (0, 1),$$

put

$$V(\nu, q) = \{x \in X : \nu(x) > q\}.$$

Since $\nu \in \mathcal{L}(\tau)$, each set $V(\nu, q)$ is open in (X, τ) .

We show that

$$\mathcal{V} = \{V(\nu, q) : \nu \in \mathcal{C}, q \in \mathbb{Q} \cap (0, 1)\}$$

is an ordinary base for (X, τ) .

Let $U \in \tau$ and let $x \in U$. Since

$$\chi_U \in \mathcal{L}(\tau)$$

and $\mathcal{C}$ is a fuzzy base,

$$\chi_U = \bigvee \{\nu \in \mathcal{C} : \nu \leq \chi_U\}.$$

Thus

$$1 = \chi_U(x) = \bigvee \{\nu(x) : \nu \in \mathcal{C}, \nu \leq \chi_U\}.$$

Hence there exists $\nu \in \mathcal{C}$ such that

$$\nu \leq \chi_U \quad \text{and} \quad \nu(x) > 0.$$

Choose

$$q \in \mathbb{Q} \cap (0, 1)$$

such that

$$0 < q < \nu(x).$$

Then

$$x \in V(\nu, q).$$

Since $\nu \leq \chi_U$ and $q > 0$, we also have

$$V(\nu, q) \subseteq U.$$

Therefore $\mathcal{V}$ is an ordinary base for (X, τ) .

It follows that

$$w(X, \tau) \leq |\mathcal{V}| \leq \omega \cdot |\mathcal{C}|.$$

Hence

$$\omega + w(X, \tau) \leq \omega + |\mathcal{C}|.$$

Taking a fuzzy base $\mathcal{C}$ of minimum cardinality, we obtain

$$\omega + w(X, \tau) \leq w_f(X, \mathcal{L}(\tau)).$$

Combining the two inequalities yields

$$w_f(X, \mathcal{L}(\tau)) = \omega + w(X, \tau).$$

□

Remark 4. The rational levels in the preceding proof are essential for the cardinal comparison. They provide a countable level skeleton: an ordinary base $\mathcal{B}$ gives the fuzzy base $\{q\chi_U : U \in \mathcal{B}, q \in \mathbb{Q} \cap (0, 1)\}$ without increasing its cardinality after the ω -normalization, while rational strict level sets extracted from an arbitrary fuzzy base recover an ordinary base. This is the mechanism behind the equality of normalized fuzzy and classical weights in Theorem 9.

Corollary 13. The inequality

$$d_f(X, \mathcal{S}) \leq w_f(X, \mathcal{S})$$

may be strict.

Proof. Let (X, τ) be a separable topological space which is not second countable. For example, one may take the Sorgenfrey line, which is separable but not second countable; equivalently, it has countable density and uncountable weight.

By Theorem 8,

$$d_f(X, \mathcal{L}(\tau)) = \omega + d(X, \tau) = \omega.$$

On the other hand, Theorem 9 gives

$$w_f(X, \mathcal{L}(\tau)) = \omega + w(X, \tau) > \omega.$$

Therefore,

$$d_f(X, \mathcal{L}(\tau)) < w_f(X, \mathcal{L}(\tau)).$$

□

The following example isolates the mapping phenomenon that fuzzy continuity alone does not control membership-sensitive base complexity.

Example 4. A surjective fuzzy continuous mapping need not decrease fuzzy weight.

Let

$$Z = (\mathbb{N} \times \mathbb{N}) \cup \{\infty\}.$$

Define a topology τ on Z as follows. Every subset of $\mathbb{N} \times \mathbb{N}$ is open, and a set $U \subseteq Z$ containing ∞ is open if, for every $m \in \mathbb{N}$, the set

$$\{n \in \mathbb{N} : (m, n) \notin U\}$$

is finite. Thus every neighbourhood of ∞ contains all but finitely many points of each vertical row

$$\{m\} \times \mathbb{N}.$$

This indeed defines a topology. Finite intersections of neighbourhoods of ∞ omit only finitely many points from each row, while any union that contains ∞ contains at least one such neighbourhood and therefore has the same property. The space is a countable fan-like example with one

non-isolated point: all points of $\mathbb{N} \times \mathbb{N}$ are isolated, whereas neighborhoods of ∞ are cofinite in each vertical row. We use the direct diagonal argument below, so no special identification of the space is required.

We first show that (Z, τ) is not second countable. Suppose that

$$\{U_k : k \in \mathbb{N}\}$$

is a countable local base at ∞ . For every $k \in \mathbb{N}$, choose $n_k \in \mathbb{N}$ such that

$$(k, n_k) \in U_k.$$

Put

$$V = Z \setminus \{(k, n_k) : k \in \mathbb{N}\}.$$

The set V is a neighbourhood of ∞ , since it omits exactly one point from each vertical row. However,

$$(k, n_k) \in U_k \setminus V$$

for every $k \in \mathbb{N}$. Hence

$$U_k \not\subseteq V \quad (k \in \mathbb{N}),$$

contrary to the assumption that $\{U_k : k \in \mathbb{N}\}$ is a local base at ∞ . Therefore (Z, τ) is not second countable.

Consider the discrete Lowen fuzzy topology

$$\mathcal{S}_X = I^Z.$$

Since Z is countable, the family

$$\mathcal{B}_X = \{q\chi_{\{z\}} : z \in Z, q \in \mathbb{Q} \cap (0, 1]\}$$

is countable. Moreover, for every $\mu \in I^Z$,

$$\mu = \bigvee \{q\chi_{\{z\}} : z \in Z, q \in \mathbb{Q} \cap (0, 1], q \leq \mu(z)\}.$$

Thus $\mathcal{B}_X$ is a countable fuzzy base for $\mathcal{S}_X$, and hence

$$w_f(Z, \mathcal{S}_X) = \omega.$$

Now put

$$\mathcal{S}_Y = \mathcal{L}(\tau).$$

By Theorem 9,

$$w_f(Z, \mathcal{S}_Y) = \omega + w(Z, \tau).$$

Since (Z, τ) is not second countable,

$$w(Z, \tau) > \omega,$$

and therefore

$$w_f(Z, \mathcal{S}_Y) > \omega.$$

Finally, consider the identity mapping

$$\text{id}_Z : (Z, \mathcal{S}_X) \longrightarrow (Z, \mathcal{S}_Y).$$

It is surjective and fuzzy continuous because

$$\mathcal{S}_Y \subseteq I^Z = \mathcal{S}_X.$$

Nevertheless,

$$w_f(Z, \mathcal{S}_Y) > \omega = w_f(Z, \mathcal{S}_X).$$

Hence fuzzy weight need not decrease under surjective fuzzy continuous mappings.

The preceding example shows that fuzzy continuity alone does not guarantee the monotonicity of fuzzy weight. We now show that monotonicity holds when the mapping is also fuzzy open in the sense of Definition 12.

We shall use the following elementary properties of the direct-image operation.

Lemma 1. Let $f : X \rightarrow Y$ be a mapping.

1. For every family $\{v_j : j \in J\} \subseteq I^X$,

$$f_* \left(\bigvee_{j \in J} v_j \right) = \bigvee_{j \in J} f_*(v_j).$$

2. If f is surjective, then, for every $\mu \in I^Y$,

$$f_*(\mu \circ f) = \mu.$$

Proof. For every $y \in Y$, we have

$$\begin{aligned} f_* \left(\bigvee_{j \in J} v_j \right) (y) &= \bigvee_{x \in f^{-1}(y)} \left(\bigvee_{j \in J} v_j(x) \right) \\ &= \bigvee_{j \in J} \left(\bigvee_{x \in f^{-1}(y)} v_j(x) \right) \\ &= \bigvee_{j \in J} f_*(v_j)(y). \end{aligned}$$

This proves the first assertion.

Now suppose that f is surjective. For every $y \in Y$, the fiber $f^{-1}(y)$ is nonempty, and therefore

$$\begin{aligned} f_*(\mu \circ f)(y) &= \bigvee_{x \in f^{-1}(y)} \mu(f(x)) \\ &= \bigvee_{x \in f^{-1}(y)} \mu(y) \\ &= \mu(y). \end{aligned}$$

Hence

$$f_*(\mu \circ f) = \mu.$$

□

Theorem 10. Let

$$f : (X, \mathcal{S}_X) \longrightarrow (Y, \mathcal{S}_Y)$$

be a surjective fuzzy continuous and fuzzy open mapping between Lowen fuzzy topological spaces. Then

$$w_f(Y, \mathcal{S}_Y) \leq w_f(X, \mathcal{S}_X).$$

Proof. Put

$$\kappa = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a fuzzy base for } (X, \mathcal{S}_X)\}.$$

Choose a fuzzy base $\mathcal{B}$ for $\mathcal{S}_X$ such that

$$|\mathcal{B}| = \kappa.$$

Thus

$$w_f(X, \mathcal{S}_X) = \omega + \kappa.$$

Define

$$\mathcal{C} = \{f_*(v) : v \in \mathcal{B}\}.$$

Since f is fuzzy open,

$$\mathcal{C} \subseteq \mathcal{S}_Y.$$

Moreover,

$$|\mathcal{C}| \leq |\mathcal{B}| = \kappa.$$

We show that $\mathcal{C}$ is a fuzzy base for $\mathcal{S}_Y$. Let

$$\mu \in \mathcal{S}_Y.$$

Since f is fuzzy continuous,

$$\mu \circ f \in \mathcal{S}_X.$$

As $\mathcal{B}$ is a fuzzy base, there exists a subfamily

$$\mathcal{B}_\mu \subseteq \mathcal{B}$$

such that

$$\mu \circ f = \bigvee_{v \in \mathcal{B}_\mu} v.$$

Applying f_* to both sides and using Lemma 1, we obtain

$$\begin{aligned} \mu &= f_*(\mu \circ f) \\ &= f_*\left(\bigvee_{v \in \mathcal{B}_\mu} v\right) \\ &= \bigvee_{v \in \mathcal{B}_\mu} f_*(v). \end{aligned}$$

Thus every member of $\mathcal{S}_Y$ is a pointwise supremum of members of $\mathcal{C}$. Hence $\mathcal{C}$ is a fuzzy base for $\mathcal{S}_Y$.

Consequently,

$$\begin{aligned} w_f(Y, \mathcal{S}_Y) &\leq \omega + |\mathcal{C}| \\ &\leq \omega + \kappa \\ &= w_f(X, \mathcal{S}_X). \end{aligned}$$

□

Combining Example 4 and Theorem 10, we conclude that fuzzy openness is a natural additional condition ensuring that fuzzy weight does not increase under surjective fuzzy continuous mappings.

6. Conclusions

In this paper, support-based fuzzy density in Lowen fuzzy topological spaces was studied from both fuzzy and ordinary set-theoretic viewpoints. The definition based on supports of fuzzy dense sets coincides with the definition based on crisp subsets meeting every nonzero fuzzy open set, and the resulting invariant is exactly the ω -normalized density of the support topology. This representation clarifies both the strength and the limitation of the invariant: it detects support information but deliberately forgets the precise positive membership grades.

The subspace, surjection and finite-product results were therefore organized as support-topological transfers of classical density facts. The fiber-cardinality theorem was presented as an intermediate result: its cardinal mechanism is support-topological, while its assumptions are expressed through fuzzy continuity, fuzzy openness, and the fuzzy direct-image operation. It was supplemented by a sharpness example and by a counterexample showing that fuzzy openness is essential for the reverse estimate. The quotient-map structure induced on support topologies was made explicit and then applied to group quotients and fuzzy symmetric powers.

Finally, the supports of a fuzzy base were shown to form an ordinary base for the support topology, providing a structural route from density to weight. A genuinely non-crisp example exhibits two Lowen fuzzy topologies with the same support topology and the same support-based density but different fuzzy weights, demonstrating that fuzzy weight can retain membership-level complexity invisible to supports. The lower-semicontinuous construction further identifies the normalized density and weight with their classical counterparts, with rational levels providing the essential countable skeleton in the weight calculation.

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