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Solvability of Congruence Equations and Potential Applications in Physics

Senyue Lou [1,2,*] and Huiling Wu [3]

[1] School of Physical Science and Technology, Ningbo University, Ningbo 315211, China
[2] Institute of Fundamental Physics and Quantum Technology, Ningbo University, Ningbo 315211, China
[3] Department of Mathematics, Lishui University, Lishui 323000, China; wuhuiling@lsu.edu.cn
* Correspondence: lousenyue@nbu.edu.cn

Abstract

This paper investigates the explicit solvability of the congruence equation $10^n \equiv 1 \pmod{9n}$. By exploiting properties of repdigit Niven numbers, we introduce several infinite families of primes with the square-free property and memory, thereby providing infinitely many explicit solutions to the congruence. Furthermore, we construct infinitely many new explicit Niven numbers. The results can be extended to the most general congruence equations $a^n \pm b^n \equiv 0 \pmod{(a \pm b)n}$ for arbitrary coprime integers a and b. The results reveal a recursive structure linking the equation to prime factorizations of repunits, and we propose several conjectures concerning the square-free property of the arising prime sets. Beyond pure number-theoretic value, the derived hierarchical prime families and modular periodic structures may have broad potential physical application including discrete quantum circuit design, quasi-crystal diffraction spectrum modeling, periodic photonic grating engineering and so on.

Keywords: congruence equation; Niven numbers; primes; square-free property; memory; quantum information; quasi-crystal; integrable systems; periodic photonic structures

MSC: 11A07

1. Introduction

The study of congruence equations boasts a remarkable historical lineage, with ancient Chinese mathematicians making seminal contributions centuries prior to Gauss. The renowned "Problem of Master Sun" from the 3rd-century treatise *Sunzi Suanjing* [1] presented the earliest documented system of linear congruences:

$$x \equiv 2 \pmod 3,\ x \equiv 3 \pmod 5,\ x \equiv 2 \pmod 7$$

whose solution method ($x = 23$) established foundational principles later formalized as the Chinese Remainder Theorem [2]. This computational framework was significantly advanced by Qin Jiushao in his 1247 *Mathematical Treatise in Nine Sections* [3], which systematized modular arithmetic through the "dayan qiu yi" algorithm for solving complex congruence systems—demonstrating sophisticated number-theoretic reasoning.

The modern formalization of congruence, established by Carl Friedrich Gauss in his seminal *Disquisitiones Arithmeticae* [4] (1801), represents one of the most fundamental and powerful tools in number theory. Defined by the relation $a \equiv b \pmod m$ signifying that the integer m divides the difference $a - b$, congruence equations permeate virtually all

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branches of mathematics and underpin critical technologies in the modern digital world. At their core, these equations provide a rigorous framework for understanding periodicity, divisibility, and the intricate structure of the ring of integers modulo m ($\mathbb{Z}/m\mathbb{Z}$). While classical results like the Chinese Remainder Theorem (CRT) [5], Fermat's Little Theorem, Euler's Theorem, and quadratic reciprocity [6] provide elegant solutions and profound insights for specific classes of linear, polynomial, and exponential congruences, the field remains remarkably vibrant. Modern research continuously reveals new depths, driven by profound theoretical questions and relentless demands from applications, particularly in cryptography and algorithm design.

The paramount importance of congruence equations in contemporary science and engineering cannot be overstated. Public-key cryptography, the cornerstone of secure digital communication (e.g., TLS/SSL, digital signatures), relies intrinsically on the computational hardness of solving certain congruence equations. The RSA cryptosystem [7] hinges on the difficulty of solving $c \equiv m^e \pmod{n}$ for m when n is composite (factoring), while the security of Diffie–Hellman key exchange [8] and the Digital Signature Algorithm (DSA) [9] rests on the intractability of the discrete logarithm problem: solving $g^x \equiv a \pmod{p}$ for x in large prime fields. Beyond cryptography, congruences are essential for error-detecting codes (ISBN, credit cards, CRC checksums), pseudorandom number generation (Linear Congruential Generators), efficient hashing algorithms, computer algebra systems, combinatorial design, and scheduling problems. Their utility stems from their ability to model cyclic phenomena and operate efficiently within finite algebraic structures.

Crucially, modular congruence structures and repunit-based number theory have evolved into indispensable mathematical machinery across multiple branches of theoretical and experimental physics, including quantum computing and quantum information physics [10–12], condensed matter physics and quasi-crystal symmetry [13–15], photonics and periodic optical structures [16,17], classical and quantum integrable field theories [18], and topological quantum error correction and coding physics [11,19].

Despite centuries of study, formidable challenges persist, fueling active research. Solving general polynomial congruence equations $f(x) \equiv 0 \pmod{m}$, especially for composite moduli, remains algorithmically complex [20]. The discrete logarithm problem (DLP), while presumed hard in multiplicative groups of prime fields and elliptic curve groups, requires constant vigilance as algorithmic advances (like index calculus and number field sieves) and the looming threat of quantum computing (via Shor's algorithm [10]) necessitate the exploration of new, post-quantum secure algebraic structures and hardness assumptions [21]. Lattice-based cryptography, isogeny-based cryptography, and multivariate cryptography represent modern paradigms where the security often reduces to solving novel types of congruence systems or finding short vectors satisfying congruence relations. Furthermore, efficiently finding solutions to systems of congruences modulo distinct primes or prime powers (leveraging CRT) and understanding the distribution of solutions are central concerns in computational number theory with direct implications for primality testing, integer factorization, and coding theory [22]. Complementarily, the lack of systematic construction for infinite nested prime families satisfying repunit divisibility constraints has limited progress in the physical modeling directions outlined above—this paper fills that gap via constructive theorems and structured conjectures for the special congruence $10^n \equiv 1 \pmod{9n}$.

This paper explores some solution structures of a special congruence equation,

$$10^n \equiv 1 \pmod{9n}. \tag{1}$$

This equation is of significant interest in number theory due to its profound connection with repunit numbers $R_n = (10^n - 1)/9$, which consist of n consecutive units 1. The congruence

$10^n \equiv 1 \pmod{9n}$ is algebraically equivalent to $n \mid R_n$ (n divides the repunit number R_n), corresponding to the search for repdigit Niven numbers—numbers divisible by their digit sum [23,24]. Some important Niven numbers have been exploded by many authors. For instance, Cooper and Kennedy prove the existence of arbitrarily long sequences of consecutive Niven numbers [25]. Grundman refines Cooper–Kennedy's theorem by establishing explicit bounds for the minimal starting point of k consecutive Niven sequences [26].

The congruence Equation (1) naturally arises in the study of repeating decimal expansions, primality testing algorithms, combinatorial sequences, and physical cyclic modular systems. Although specific solution families (e.g., $n = 3^k$ for $k = 0, 1, 2, \ldots$) are well-documented, the comprehensive solution space remains elusive. To identify additional solutions of (1), the well-known Zsigmondy theorem [27], the square-free conjecture for cyclotomic polynomials [28], and the ABC conjecture [29] may prove useful. In this work, we characterize novel solution sets by introducing specialized prime collections and conducting a rigorous analysis of the divisibility constraints governing repdigit Niven numbers [30]. We review several core topics in elementary and analytic number theory, each supported by a representative reference.

Related to repdigit Niven numbers, several relevant foundational references are worth noting. Shanks [31] develops practical algorithms together with theoretical analysis for computing the multiplicative order of integers modulo a composite integer m, a core tool underpinning numerous number-theoretic computations. Snyder [32] studies the factorization structure of repunit numbers $R_n = (10^n - 1)/9$ and characterizes their prime divisors. Ref. [33] is a landmark contribution settling the primitive prime divisor problem for Lucas and Lehmer sequences, generalizing Zsigmondy-type results to linear recurrence sequences. Suzuki [34] investigates the coefficient behavior of cyclotomic polynomials and establishes new bounds on the magnitude of these coefficients. For Zsigmondy's theorem, the original paper [27] formulates this celebrated theorem, which asserts the existence of primitive prime divisors of $a^n - b^n$ apart from a small set of exceptional cases. For the broader class of congruences of the form $a^n \equiv b^n \pmod{mn}$, Vandiver [35] examines the solvability and structural restrictions of such congruences via arguments involving multiplicative orders and divisibility.

Before studying the solutions of the congruence Equation (1), we summarize the symbols and abbreviations in the following Table 1.

Table 1. Notations and abbreviations.

Symbol	Definition	Brief Explanation
$a \equiv b \pmod{m}$	Congruence relation	Integer $a - b$ is divisible by modulus m
$\gcd(x, y)$	Greatest common divisor	The largest positive integer dividing both x and y
$v_p(n)$	p-adic valuation of n	Largest integer $k \geq 0$ such that $p^k \mid n$; $v_p(n) = 0$ if $p \nmid n$
$a_{(n)}$	Repdigit number	n-digit repdigit formed by repeating digit a
$[367]_i$	Composite repdigit expression	$[367]_i = 3_{(3^i)}6_{(3^i-1)}7$
R_j	Quotient of repunit factorization	$R_j = \dfrac{10^{3^{j+1}} - 1}{3^{j+3}}$
a_k	Cyclotomic-derived integer	$a_k = \dfrac{1}{3}\left(10^{2\cdot 3^k} + 10^{3^k} + 1\right) = \dfrac{1}{3}\Phi_{3^{k+1}}(10)$
m_{ij}	Prime factors of $[367]_i$	$[367]_i = \prod_j m_{ij}$
$\mathcal{M}_i$	Prime set $\{m_{i1}, m_{i2}, \ldots\}$	Collecting primes from $[367]_i$

Table 1. *Cont.*

Symbol	Definition	Brief Explanation
A_{jk}	Generalized cyclotomic integer	$A_{jk} = \frac{1}{3}\left(10^{2\cdot 3^k\cdot 37^j} + 10^{3^k\cdot 37^j} + 1\right)$
R_{jk}	Generalized repunit quotient	$R_{jk} = \frac{10^{3^k\cdot 37^j} - 1}{3^{k+2}\cdot 37^j}$
$\mathcal{P}_{jk}$	Prime set for R_{jk}	Set of prime factors of R_{jk}
$\mathbb{Z}/m\mathbb{Z}$	Finite ring	Ring of integers modulo m
$\Phi_n(x)$	Cyclotomic polynomial	n-th cyclotomic polynomial in variable x
$\Phi_n(a,b)$	Homogeneous cyclotomic polynomial	Homogeneous form for binomial factorization $a^n - b^n$
$t = \mathrm{ord}_p(10)$	Multiplicative order	Multiplicative order of integer 10 modulo prime p
$p \mid q$	Divisibility	Integer p divides integer q
$p \nmid q$	Non-divisibility	Integer p does not divide integer q
$C_\pm$	Congruence class	$C_\pm : \ a^n \pm b^n \equiv 0 \pmod{(a \pm b)n}$ for coprime $a > b > 0$
Abbreviation (non-symbol)		
SFP	Square-Free Property	Integer can be written as product of distinct primes
PCT	Power Closure Theorem	Theorem 1: power closure property of solutions
PPT	Product–Power Theorem	Theorem 2: product–power property for coprime solutions
PST	Parameterized Solution Theorem	Theorem 4: multi-parameter family of congruence solutions
CFL	Cubic Factorization Lemma	Lemma 2, cubic-based factorization identity
CPCL	Cyclotomic Pair Coprimality Lemma	Lemma 3, pairwise coprimality for a_i sequence
CRT	Chinese Remainder Theorem	Chinese Remainder Theorem for system of congruences

2. Solution Theorems Related to Repdigit Niven Numbers

For notational convenience in subsequent analysis, we define $a_{(n)}$ as the n-digit repdigit formed by repeating digit a. Representative cases include: $1_{(4)} = 1111$ and $10_{(3)} = 1000$. Using this notation, the equivalent form of (1) can be written as

$$10^n - 1 = 9_{(n)} = 0 \pmod{9n}. \tag{2}$$

Equation (2) shows us that we should find possible n such that $n \mid 1_{(n)}$ and then the repdigit Niven numbers in the form $\frac{a}{9}\left(10^n - 1\right)$ for $a \in \mathcal{N}_1 \equiv \{1, 2, \ldots, 9\}$.

Theorem 1 (power closure theorem (PCT))**.** *Let n be a positive integer satisfying the congruence (1), Then for every positive integer m, n^m also satisfies*

$$10^{n^m} \equiv 1 \pmod{9n^m}.$$

Proof of Theorem 1. Proof by mathematical induction on *m*.
Base case (*m* = 1): When $m = 1$, $n^1 = n$. By assumption, $10^n \equiv 1 \pmod{9n}$. The statement holds.
Inductive hypothesis: Assume that for some integer $m \geq 1$, the statement holds, i.e.,

$$10^{n^m} \equiv 1 \pmod{9n^m}.$$

We need to prove the statement for $m+1$:

$$10^{n^{m+1}} \equiv 1 \pmod{9n^{m+1}}.$$

Inductive step: Note that $n^{m+1} = n^m \cdot n$. Write

$$10^{n^{m+1}} = \left(10^{n^m}\right)^n.$$

By the inductive hypothesis, there exists integer t such that

$$10^{n^m} = 1 + t \cdot 9n^m.$$

Raise both sides to the n-th power using binomial theorem:

$$\left(1 + 9tn^m\right)^n = 1 + n \cdot (9tn^m) + \binom{n}{2}(9tn^m)^2 + \cdots$$

All terms starting from the second term contain a factor $9n^{m+1}$. Hence

$$\left(1 + 9tn^m\right)^n \equiv 1 \pmod{9n^{m+1}}.$$

Therefore

$$10^{n^{m+1}} = \left(10^{n^m}\right)^n \equiv 1 \pmod{9n^{m+1}}.$$

By the principle of mathematical induction, the theorem is true for all integers $m \geq 1$. □

Remark 1. *The above induction argument is valid for the congruence structure. We also note that the condition* $10^n \equiv 1 \pmod{9n}$ *implies* $\gcd(10, n) = 1$, *which guarantees* 10 *is invertible modulo* n.

Corollary 1. $n = 3^k,\ k \geq 0$, *are solutions of the congruence Equation* (1).

The Corollary 1 of Theorem 1 holds because $n = 3$ is trivially a solution of (1).

To find more solutions via Theorem 1, we write down first the following well known Chinese Remainder Theorem, where $\gcd(x, y)$ means the greatest common divisor of x and y:

Lemma 1 (Chinese Remainder Theorem (CRT))**.** *Let* $m_1, m_2, \ldots, m_k$ *be pairwise coprime positive integers, namely* $\gcd(m_i, m_j) = 1$ *for all* $i \neq j$. *For any integers* $a_1, a_2, \ldots, a_k$, *the system of congruences*

$$\begin{cases} x \equiv a_1 \pmod{m_1} \\ x \equiv a_2 \pmod{m_2} \\ \vdots \\ x \equiv a_k \pmod{m_k} \end{cases}$$

admits a unique solution modulo $M = m_1 m_2 \cdots m_k$.

Theorem 2 (product-power theorem (PPT))**.** *Let positive integers* n_1, n_2 *satisfy* $\gcd(n_1, n_2) = 1$. *Suppose* $n = n_1$ *and* $n = n_1 n_2$ *are solutions of the congruence*

$$10^n \equiv 1 \pmod{9n}.$$

Then for any positive integers $m_1 \geq 1,\ m_2 \geq 0$,

$$n' = n_1^{m_1} n_2^{m_2}$$

is also a solution, i.e.,

$$10^{n'} \equiv 1 \pmod{9n'}.$$

Proof. The congruence $10^k \equiv 1 \pmod{9k}$ is equivalent to

$$10^k \equiv 1 \pmod{9} \quad \text{and} \quad 10^k \equiv 1 \pmod{k}.$$

Since $10 \equiv 1 \pmod{9}$, $10^k \equiv 1 \pmod{9}$ holds for all positive integers k. It suffices to prove $10^{n'} \equiv 1 \pmod{n'}$.

Since $\gcd(n_1, n_2) = 1$, we have $\gcd(n_1^{m_1}, n_2^{m_2}) = 1$. By the Chinese Remainder Theorem, $10^{n'} \equiv 1 \pmod{n'}$ is equivalent to

$$10^{n'} \equiv 1 \pmod{n_1^{m_1}}, \qquad 10^{n'} \equiv 1 \pmod{n_2^{m_2}}.$$

Part 1: $10^{n'} \equiv 1 \pmod{n_1^{m_1}}$. By hypothesis, n_1 satisfies $10^{n_1} \equiv 1 \pmod{9n_1}$. Applying Theorem 1 (PCT), $n_1^{m_1}$ is also a solution, so

$$10^{n_1^{m_1}} \equiv 1 \pmod{n_1^{m_1}}.$$

Because $n' = n_1^{m_1} n_2^{m_2}$ is a multiple of $n_1^{m_1}$,

$$10^{n'} = \left(10^{n_1^{m_1}}\right)^{n_2^{m_2}} \equiv 1^{n_2^{m_2}} = 1 \pmod{n_1^{m_1}}.$$

Part 2: $10^{n'} \equiv 1 \pmod{n_2^{m_2}}$. We know $n_1 n_2$ satisfies $10^{n_1 n_2} \equiv 1 \pmod{9 n_1 n_2}$. Since $\gcd(n_1, n_2) = 1$, reducing modulo n_2 gives

$$10^{n_1 n_2} \equiv 1 \pmod{n_2}.$$

We induct on m_2 to show $10^{n_1 n_2^{m_2}} \equiv 1 \pmod{n_2^{m_2}}$. Base case $m_2 = 1$: $10^{n_1 n_2} \equiv 1 \pmod{n_2}$, which holds.

Inductive step: assume $10^{n_1 n_2^t} \equiv 1 \pmod{n_2^t}$. Write

$$10^{n_1 n_2^t} = 1 + r n_2^t, \quad r \in \mathbb{Z}.$$

Raise to the n_2-th power:

$$10^{n_1 n_2^{t+1}} = \left(1 + r n_2^t\right)^{n_2} = 1 + r n_2^{t+1} + \text{terms divisible by } n_2^{2t}.$$

For $t \geq 1$, $2t \geq t + 1$, so

$$10^{n_1 n_2^{t+1}} \equiv 1 \pmod{n_2^{t+1}}.$$

Thus for all $m_2 \geq 0$, $10^{n_1 n_2^{m_2}} \equiv 1 \pmod{n_2^{m_2}}$. Since $n' = n_1^{m_1} n_2^{m_2} = n_1 n_2^{m_2} \cdot n_1^{m_1 - 1}$, n' is a multiple of $n_1 n_2^{m_2}$. Then

$$10^{n'} = \left(10^{n_1 n_2^{m_2}}\right)^{n_1^{m_1 - 1}} \equiv 1^{n_1^{m_1 - 1}} = 1 \pmod{n_2^{m_2}}.$$

Conclusion. By Chinese Remainder Theorem,

$$10^{n'} \equiv 1 \pmod{n_1^{m_1} n_2^{m_2}} = n'.$$

Combined with $10^{n'} \equiv 1 \pmod{9}$, we get

$$10^{n'} \equiv 1 \pmod{9n'}.$$

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The proof of PPT is completed. □

Corollary 2. *$n = n_{jk} = 37^j \cdot 3^k$ are solutions of the congruence Equation (1) for all $j \geq 1$ and $k \geq 1$.*

Corollary 2 of Theorem 2 is correct since $10^{3\cdot 37} = 1 \pmod{9 \cdot 3 \cdot 37}$ and $\gcd(3, 37) = 1$.

Corollary 3. *If $n = n_1$ and $n = \prod_{i=1}^{N} n_i$ are solutions of the congruence Equation (1), so are for $n = \prod_{i=1}^{N} n_i^{k_i}$ with $k_i \geq 0$, $i = 1, 2, \ldots, N$.*

The correctness of Corollary 3 comes from the repeated uses of Theorem 2 (PPT).

According to the known solutions of (1), $n = 3^k$, we can extend them to more general forms by using the following theorem:

Theorem 3. *If $n = n'$ is a solution of (1), and p is a proper factor (a prime factor, without loss of generality) of $\frac{10^{n'}-1}{9n'}$, i.e.,*

$$10^{n'} - 1 = 9_{(n')} = (9 \cdot n') \cdot (p) \cdot x, \tag{3}$$

then $n = p \cdot n'$ and $n = x \cdot n'$ are also solutions of (1).

Proof. It suffices to prove $n = p \cdot n'$ is a solution, because the proof for $n = x \cdot n'$ is identical. From (3), we have

$$\begin{aligned} 10^{p\cdot n'} - 1 &= 9_{(p\cdot n')} = 9_{(n')} \cdot \left[10_{(p-1)\cdot n'} + 10_{(p-2)\cdot n'} + \ldots + 10_{(2\cdot n')} + 10_{(n')} + 1\right] \\ &= (9 \cdot n') \cdot (p) \cdot x \cdot [y] \\ &= (9 \cdot n' \cdot p) \cdot (x \cdot y) \equiv 0 \pmod{9 \cdot n' \cdot p}. \end{aligned} \tag{4}$$

The theorem is proved. □

From Theorem 3, we know that using the known solutions, say, $n' = 3^{j+1}$ we can find various primes, p_{ji}, by factor the integers

$$R_j \equiv \frac{10^{3^{j+1}} - 1}{3^{j+3}} = \prod_{i=1}^{I_j} p_{ji},\ j = 0, 1, \ldots, \infty. \tag{5}$$

To derive the details of p_{ij} from (5), we introduce two additional lemmas: the Cubic Factorization Lemma and the Cyclotomic Pair Coprimality Lemma.

Lemma 2 (Cubic Factorization Lemma (CFL))**.** *Let R_j being defined by (5) then*

$$R_{j+1} = R_j \cdot 3_{(3^{j+1})}6_{(3^{j+1}-1)}7 = \prod_{i=0}^{j+1}[367]_i,\ [367]_i \equiv 3_{(3^i)}6_{(3^i-1)}7 = \frac{1}{3}\left(y_i^2 + y_i + 1\right),\ y_i = 10^{3^i}. \tag{6}$$

Proof. It is well known that

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2). \tag{7}$$

Substituting $a = 10^{j+2}$ and $b = 1$ into (7) yields immediately

$$R_{j+1} = \frac{1}{3}R_j\left[(10^{3^{j+1}})^2 + 10^{3^{j+1}} + 1\right] = R_j 3_{(3^{j+1})}6_{(3^{j+1}-1)}7 = R_j[367]_{j+1}. \tag{8}$$

The repeated application of the recursion relation (8) finishes the proof procedure of Lemma 2. □

Lemma 3 (Cyclotomic Pair Coprimality Lemma (CPCL))**.** *Let*

$$a_i = \frac{1}{3}(10^{2\cdot 3^i} + 10^{3^i} + 1).$$

Then $\gcd(a_j, a_k) = 1$.

Proof. While the Maple code for $\gcd(a_j, a_k)$ immediately yields the result 1, here we present a rigorous proof without relying on any software.

From the algebraic identity

$$x^2 + x + 1 = \frac{x^3 - 1}{x - 1},$$

substitute $x = 10^{3^i}$:

$$a_i = \frac{(10^{3^i})^2 + 10^{3^i} + 1}{3} = \frac{10^{3^{i+1}} - 1}{3\left(10^{3^i} - 1\right)}.$$

Assume $k > j$ without loss of generality. Write

$$a_j = \frac{10^{3^{j+1}} - 1}{3\left(10^{3^j} - 1\right)}, \quad a_k = \frac{10^{3^{k+1}} - 1}{3\left(10^{3^k} - 1\right)}.$$

Since $k > j$, 3^k is a multiple of 3^{j+1}. Let $M = 3^{j+1}$. Then $10^{3^k} = \left(10^M\right)^{3^{k-j-1}}$. For any integer $t \geq 1$, $10^{Mt} - 1$ is divisible by $10^M - 1$. Thus $10^{3^k} - 1$ is divisible by $10^{3^{j+1}} - 1$.

Let $\gcd(a_k, a_j) = d$. Then $d \mid a_j$, so $d \mid (10^{3^{j+1}} - 1)$. Also $d \mid a_k$. We have

$$a_k = \frac{10^{3^{k+1}} - 1}{3(10^{3^k} - 1)}.$$

Since $10^{3^{k+1}} - 1 = (10^{3^k})^3 - 1 = (10^{3^k} - 1)\left((10^{3^k})^2 + 10^{3^k} + 1\right)$, so $a_k = \frac{(10^{3^k})^2 + 10^{3^k} + 1}{3}$.

Because $d \mid (10^{3^{j+1}} - 1)$, we have $10^{3^{j+1}} \equiv 1 \pmod{d}$. Since 3^k is a multiple of 3^{j+1}, $10^{3^k} \equiv 1 \pmod{d}$. Substitute into a_k:

$$a_k \equiv \frac{1^2 + 1 + 1}{3} = 1 \pmod{d}.$$

But $d \mid a_k$, so $a_k \equiv 0 \pmod{d}$. Combining gives

$$0 \equiv 1 \pmod{d}.$$

This implies $d \mid 1$, hence $d = 1$. Therefore $\gcd(a_k, a_j) = 1$. The proof of CPCL is completed. □

Denoting the primes m_{ij}, $j = 1, 2, \ldots, J_i$ as the primes of $[367]_i$, i.e.,

$$[367]_i = \prod_{j=1}^{J_i} m_{ij}. \tag{9}$$

By using Theorems 1–3 and Lemma 2, CFL, we can obtain the following parameterized solution theorem.

Theorem 4 (Parameterized Solution Theorem (PST))**.** *The congruence Equation* (1) *admits solutions parameterized as*

$$n = 3^k \cdot \prod_{i=0}^{I} \prod_{j=1}^{J_i} m_{ij}^{k_{ij}}, \tag{10}$$
$$\text{for } k \geq i, \text{ if } \mathcal{K}_i \equiv \{k_{i1}, k_{i2}, \ldots, k_{iJ_i}\} \neq \{0\},\ i = 0, 1, \ldots, I.$$

Proof. Since the primes m_{ij} appear as the proper factors of the solution $n = 3^k$ for $k \geq i$ (Lemma 2), the conclusion (10) comes directly from Theorem 3 and Corollary 3 of Theorem 2. This completes the proof of PST. □

For concreteness, we write down some explicit primes m_{ij} for small i. Using the Maple command, ifactor(R_j), for $j \leq 3$, we have

$$\begin{aligned}
R_0 &= \text{ifactor}\left(\frac{10^3-1}{3^3}\right) = 37 = m_{01},\\
R_1 &= \text{ifactor}\left(\frac{10^{3^2}-1}{3^4}\right) = 37 \cdot 333667 = R_0 \cdot m_{11},\\
R_2 &= \text{ifactor}\left(\frac{10^{3^3}-1}{3^5}\right) = R_1 \cdot m_{21} \cdot m_{22},\\
R_3 &= \text{ifactor}\left(\frac{10^{3^4}-1}{3^6}\right) = R_2 \cdot m_{31} \cdot m_{32} \cdot m_{33} \cdot m_{34} \cdot m_{35},
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
&\mathcal{M}_0 = \{m_{01}\} = \{37\}, \mathcal{M}_1 = \{m_{11}\} = \{333667\}, \mathcal{M}_2 = \{m_{21}, m_{22}\} = \{757, 440334654777631\},\\
&\mathcal{M}_3 = \{m_{31}, m_{32}, m_{33}, m_{34}, m_{35}\}\\
&\quad = \{163, 9397, 2462401, 676421558270641, 1306548978080077784250461 17\}.
\end{aligned} \tag{12}$$

By Lemma 2 and Theorem 3, we can find infinitely many candidates for m_{ij} by factorizing $[367]_i$. However, it is quite difficult to determine explicit primes using the Maple command `ifactor` for $i \geq 4$ due to computational overflow.

Using the simple Maple code,

$$\begin{aligned}
&S := \{\text{seq}(\text{ithprime}(i),\ i = 1, \ldots, 10^8)\} :\\
&mj := \{\} : \text{for i from 1 to nops}(S) \text{ do if mod}(R_j, S[i]) = 0\\
&\qquad \text{then } mj := mj \text{ union } \{S[i]\} \text{ end if end do : } mj
\end{aligned}$$

one can find all the primes less than 10^8-th prime, $m_{cr} = 2038074743$, included in R_j. The explicit primes obtained from this simple Maple code for $j = 4, 5, 6$ and 7 read

$$\begin{aligned}
\mathcal{M}_4 &= \{\text{primes} > m_{cr}\},\\
\mathcal{M}_5 &= \{m_{51} = 313471,\ m_{52} = 2064529,\ \text{primes} > m_{cr}\},\\
\mathcal{M}_6 &= \{m_{61} = 2558791,\ m_{62} = 4448359,\ m_{63} = 127020961,\ m_{64} = 934785037,\ \text{primes} > m_{cr}\},\\
\mathcal{M}_7 &= \{m_{71} = 618846643,\ \text{primes} > m_{cr}\}.
\end{aligned} \tag{13}$$

In principle, PST yields infinitely many solutions starting from the known solution $n = 3^k$. From each solution given in the theorem, one can further generate infinitely many new solutions. We stop here and proceed to discuss some properties and conjectures related to the obtained prime sets.

3. Solution-Related Properties and Conjectures

Before discussing the solution-related properties and conjectures for the obtained prime sets, we give some definitions.

Definition 1 (Square-free property (SFP) [5]). *An integer N is said to have SFP if it can be expressed as*

$$N = \prod_{k=1}^{K} p_k, \quad p_i \neq p_j, \quad \forall j \neq i,\ i,\ j = 1,\ 2,\ \ldots,\ K,$$

where p_k are primes. In this case N is called square-free.

Definition 2 (Memory [36]). *A sequence of sets $\mathcal{P}_k = \{p_{ki}, i = 1, 2, \ldots, P_k\}$ for $k = 1, 2, \ldots$ is said to possess memory if it satisfies*

$$\mathcal{P}_{k+1} \supset \mathcal{P}_k, \qquad \forall k = 1, 2, \ldots.$$

Remark 2. *For the sake of notational simplicity, we introduce a new arithmetic invariant termed "memory". This nomenclature is borrowed from physical concepts solely for the purpose of analogy and intuitive interpretation; it does not correspond to a standard object in classical number theory. The mathematical definition provided here is fully self-contained within the context of set sequences.*

Definition 3 (Disjoint). *A sequence of sets $\mathcal{M}_k = \{m_{ki}, i = 1, 2, \ldots, M_k\}$ for $k = 1, 2, \ldots$ is called disjoint if it satisfies*

$$\mathcal{M}_i \cap \mathcal{M}_j = \varnothing,\ \forall j \neq i.$$

Theorem 5. *Let $\mathcal{P}_k = \{p_{ki}, i = 1, 2, \ldots, P_k\}$, $\mathcal{M}_k = \{m_{ki},\ i = 1,\ 2,\ \ldots,\ M_k\}$, p_{ki} are primes of*

$$R_k \equiv \frac{10^{3^{k+1}} - 1}{3^{k+3}} = \prod_{i=1}^{I_k} p_{ki}, \tag{14}$$

and m_{ki} are primes of

$$a_k = [367]_k = 3_{(3^k)}6_{(3^k-1)}7 = \frac{1}{3}(10^{2\cdot 3^k} + 10^{3^k} + 1) = \prod_{i=1}^{M_k} m_{ki}. \tag{15}$$

Then $\mathcal{P}_k$, $k = 1,\ 2,\ \ldots,\ \infty$ possess the memory and $\mathcal{M}_k$, $k = 1,\ 2,\ \ldots,\ \infty$ are disjoint.

Proof. CFL or Equation (8) endows $\mathcal{P}_k$ with the memory property, while CPCL, i.e., $\gcd(a_i, a_j) = 1$, ensures that $\mathcal{M}_k$ is disjoint. This completes the proof of Theorem 5. □

Conjecture 1. *Both R_k and a_k defined in Theorem 5 have the square-free property.*

It is interesting that if the following well-known cyclotomic polynomial square-free conjecture is correct, then our Conjecture 1 can be proven.

Conjecture 2 (Cyclotomic Polynomial Square-Free Conjecture [29,37]). *Let $b \geq 2$ be an integer. For every positive integer n, the cyclotomic polynomial evaluated at integer b, $\Phi_n(b)$, is square-free. That is, there is no prime p such that $p^2 \mid \Phi_n(b)$.*

Theorem 6 (Conditional Theorem). *Assume that for every positive integer n, the cyclotomic polynomial value $\Phi_n(10)$ is square-free. Then for all integers $k \geq 1$, both R_k and a_k are square-free, where*

$$R_k = \frac{10^{3^{k+1}} - 1}{3^{k+3}}, \qquad a_k = \frac{1}{3}\left(10^{2\cdot 3^k} + 10^{3^k} + 1\right).$$

Proof. Because R_k is just a product of series of disjoint a_k, it is sufficient to prove the theorem only for a_k.

First, recall the cyclotomic polynomial identity

$$\Phi_{3^{k+1}}(x) = x^{2\cdot 3^k} + x^{3^k} + 1.$$

Substitute $x = 10$. We obtain

$$a_k = \frac{1}{3}\Phi_{3^{k+1}}(10).$$

By hypothesis from the cyclotomic polynomial square-free conjecture, $\Phi_{3^{k+1}}(10)$ is square-free. We check the 3-adic valuation:

$$\Phi_{3^{k+1}}(10) = 10^{2\cdot 3^k} + 10^{3^k} + 1 \equiv 1 + 1 + 1 = 3 \pmod 9,$$

so $v_3(\Phi_{3^{k+1}}(10)) = 1$. Hence $a_k = \Phi_{3^{k+1}}(10)/3$ is an integer and square-free. This completes the proof of the conditional Theorem 6. □

It is also worth mentioning that there are some other types of unsolved conjectures, say, the nonexistence conjecture of 10 Wieferich primes [38] and the ABC conjecture [29], which can be used to prove our Conjecture 1.

4. Extended Theorems and Conjectures on the Solutions of (1) and the Related Primes

The solutions in PST (Theorem 4) are inspired by the known solutions $n = 3^k$. This naturally raises the question: Can additional solutions be derived using these newly obtained solutions (10)?

Any prime appearing in the solution set of PST can be used to discover new primes that yield novel solutions to (1). Due to the similarity, we consider only a special solution involving the prime 37 to illustrate the recursive procedure for finding new solutions to (1) and new disjoint prime sets.

From PST, we know that $n = 3^k \cdot 37^j$ with $k \geq 0$ and $j \geq 1$ is a solution of the congruence Equation (1). Thus, according to Theorem 3 and Corollary 3 of Theorem 2, we can prove the following solution theorem:

Theorem 7. *Denoting*

$$R_{jk} = \frac{10^{3^k\cdot 37^j} - 1}{3^{k+2}\cdot 37^j} = \prod_{i=1}^{I_{jk}} p_{ijk}, \tag{16}$$

$$\mathcal{P}_k = \{p_{ijk},\ i = 1,\ 2,\ \ldots,\ I_{jk}\}, \tag{17}$$

$$A_{jk} = \frac{1}{3}\left(10^{2\cdot 3^k\cdot 37^j} + 10^{3^k\cdot 37^j} + 1\right) = [367]_{jk} = 3_{(3^k\cdot 37^j)}6_{(3^k\cdot 37^j-1)}7 = \prod_{l=1}^{L_{jk}} m_{ljk}, \tag{18}$$

$$\mathcal{M}_{jk} = \{m_{1jk}, m_{2jk}, \cdots, m_{(L_{jk})jk}\}, \tag{19}$$

then
(i)

$$n = 3^a \cdot 37^b \prod_{i=1}^{I_{jk}} p_{ijk}^{c_{ijk}}, \quad a \geq k,\ b \geq j,\ c_{ijk} \geq 1,\ j \geq 1,\ k \geq 1. \tag{20}$$

are solutions of (1),
(ii) $\mathcal{P}_k$ *possesses memory and*
(iii) $\mathcal{M}_{jk}$ *are disjoint.*

Proof. (i) From Theorem 4, we know $n' = 3^k \cdot 37^j$ is a solution of the congruence Equation (1). p_{ijk} are primes of $\frac{10^{n'}-1}{9n'}$, and thus according to Theorem 2 and its Corollary 3, n given by (20) is really a solution of (1).
(ii) Substituting $a = 10^{3^k \cdot 37^j}$, $b = 1$ into (7) yields

$$\left(10^{3^k \cdot 37^j}\right)^3 - 1 = \left(10^{3^k \cdot 37^j} - 1\right)\left[10^{2 \cdot 3^k \cdot 37^j} + 10^{3^k \cdot 37^j} + 1\right] = 3\left(10^{3^k \cdot 37^j} - 1\right)A_k, \tag{21}$$

Applying (21) to (16), we have

$$R_{j(k+1)} = R_{jk}A_{jk} \tag{22}$$

$$= R_{j(k-1)}A_{jk}A_{j(k-1)} = \ldots = R_{j1}\prod_{i=1}^{k} A_{ji}. \tag{23}$$

The recursion relation (22) implies that the prime sets $\mathcal{P}_k$ possess the memory property.
(iii) To prove the third conclusion of the theorem is equivalent to check $\gcd(A_{jk},\ A_{il}) = 1,\ (j,k) \neq (i,l)$. Though the Maple code, $\gcd(A_{jk},\ Ail)$, really shows the result 1, we still present a strict proof.

Let $f(x) = \frac{x^2+x+1}{3}$. Use the identity

$$x^2 + x + 1 = \frac{x^3 - 1}{x - 1}, \quad x \neq 1.$$

Substitute $x = 10^{3^k 37^j}$:

$$A_{jk} = f\left(10^{3^k 37^j}\right) = \frac{10^{3^{k+1} 37^j} - 1}{3\left(10^{3^k 37^j} - 1\right)}.$$

Let $M = 3^k 37^j$, $N = 3^l 37^i$. Then

$$A_{jk} = \frac{10^{3M} - 1}{3(10^M - 1)}, \qquad A_{il} = \frac{10^{3N} - 1}{3(10^N - 1)}.$$

Take any prime $p \mid \gcd(A_{jk}, A_{il})$. Then

$$p \mid (10^{3M} - 1), \quad p \mid (10^{3N} - 1), \quad p \nmid (10^M - 1), \quad p \nmid (10^N - 1).$$

Let $t = \mathrm{ord}_p(10)$, the multiplicative order of 10 modulo p. By definition of order:

$$t \mid 3M, \quad t \mid 3N, \quad t \nmid M, \quad t \nmid N.$$

Since $t \mid 3M$ and $t \nmid M$, we get $3 \mid t$.

Because $p \mid A_{jk}$, we have

$$10^{2M} + 10^M + 1 \equiv 0 \pmod{p}.$$

This implies $(10^M)^3 \equiv 1 \pmod{p}$ and $10^M \not\equiv 1 \pmod{p}$. Thus 10^M is a primitive 3-rd root of unity modulo p. The order of 10 modulo p is exactly $t = 3M$. Similarly, $p \mid A_{il}$ implies $\mathrm{ord}_p(10) = 3N$.

Since the order is uniquely defined,

$$3M = 3N \implies M = N.$$

$$3^k 37^j = 3^l 37^i.$$

By unique prime factorization, $k = l$ and $j = i$. This contradicts $(j,k) \neq (i,l)$. Therefore no common prime divisor exists, so

$$\gcd(A_{jk}, A_{il}) = 1.$$

The proof of Theorem is completed. □

Conjecture 3. *$\mathcal{M}_{jk}$ defined in (19) (and then R_{jk} defined in (16)) for $k \geq 1$ possess SFP.*

While Conjecture 3 could be proven assuming other unsolved conjectures mentioned in the proof of Theorem 6, such as the cyclotomic polynomial square-free conjecture and the ABC conjecture, we do not pursue this further. Instead, we provide concrete information for the special case where $j = 1$.

For $j = 1$ of Theorem 7, we have

$$R_{1k} = \frac{10^{37\cdot 3^k} - 1}{37 \cdot 3^{k+2}} = \prod_{i=1}^{I_k} p_{ik} = R_{11} \prod_{i=1}^{k} A_{1i}, \quad \text{(SFP, Conjecture 3)}, \tag{24}$$

$$\mathcal{P}_{1k} = \{p_{ik},\ i = 1,\ 2,\ \ldots,\ I_k\}, \quad \text{(Memory, Theorem 7)}, \tag{25}$$

$$A_{1k} = \frac{1}{3}\left(10^{74\cdot 3^k} + 10^{37\cdot 3^k} + 1\right) = \prod_{l=1}^{L_k} m_{lk}, \quad [\text{SFP}, \gcd(A_{1k}, A_{1l}) = 1], \tag{26}$$

$$\mathcal{M}_{1k} = \{m_{1k}, m_{2k}, \cdots, m_{L_k k}\}, \quad \text{(disjoint, Theorem 7)}, \tag{27}$$

where $R_{11} = p_{11} \cdot p_{21} \cdot p_{31} \cdot p_{41} \cdot p_{51} \cdot p_{61} \cdot p_{71}$, $p_{11} = 37$, $p_{21} = 2028119$, $p_{31} = 247629013$, $p_{41} = 30557051518647307$, $p_{51} = 2212394296770203368013$, $p_{61} = 884598117086562911927 1997$, $p_{71} = 900778143960550179382572 37117$.

According to Theorem 3, the primes appearing in Theorems 4 and 7 can be used to generate infinitely many new solutions of the congruence Equation (1). This procedure can be applied recursively.

5. Potential Applications in Physics

The number theoretic structures uncovered in this work, hierarchical square-free prime families with memory properties, recursive solution families of the congruence

$$10^n \equiv 1 \pmod{9n}, \tag{28}$$

and infinite repdigit Niven numbers, provide novel parametric building blocks for multiple branches of theoretical and applied physics. These mathematical objects encode modular periodicity, nested divisibility constraints and disjoint prime factor decompositions, which map naturally onto cyclic discrete symmetries, aperiodic long-range order, periodic boundary value problems and stabilizer group structures in physical models. Below we elaborate five major application directions, starting with periodic photonic grating engineering which is treated in full detail, followed by enriched discussion for discrete integrable lattice field theories.

5.1. Periodic Photonic Grating Engineering

Repunit numbers $R_n = (10^n - 1)/9$ admit a direct physical interpretation for one-dimensional multilayer dielectric stacks and sampled fiber Bragg gratings [16]. The divisibility condition $n \mid R_n$, algebraically equivalent to our target congruence $10^n \equiv 1 \pmod{9n}$, defines self-consistent super-period parameters for composite photonic gratings. For a sampled fiber Bragg grating or integrated silicon photonic grating device, the fundamental Bragg resonance condition reads

$$\lambda_B = 2n_{\text{eff}}\Lambda, \tag{29}$$

where Λ denotes the physical grating period and n_{eff} the waveguide effective refractive index [39]. When constructing multi-resonance gratings by superposing several periodic components, parasitic higher-order reflections and spurious sidelobes emerge if the composite super-period lacks appropriate divisibility properties; these artifacts degrade filter performance for optical-communication filters, frequency-comb generators and integrated photonic resonators [17].

The infinite explicit solution families for n derived in Theorems 4 and 7 deliver a new reservoir of analytic super-period candidates for multi-resonance composite photonic gratings. Each solution n corresponds to a composite super-cycle for cascaded dielectric multilayers. Since these n factor into the square-free nested primes obeying the memory property formulated in Theorem 5, one may systematically select pairwise coprime prime divisors to independently control distinct resonant wavelengths. Concretely, suppose $n = p_1 p_2 \dots p_k$ is a square-free solution; each prime divisor p_i can be assigned to one independent grating subperiod. The resulting device yields k well separated resonant bands, suppressing crosstalk originating from commensurate higher-order harmonics.

This algebraic construction alleviates a well-known practical bottleneck of conventional sampled grating design: existing approaches usually rely on heuristic numerical optimization to separate multiple pass bands, whereas our method provides algebraically constrained super-period parameters without brute-force scanning [39]. These candidate integer super-periods can be directly imported into standard electromagnetic simulation workflows, including rigorous coupled wave analysis (RCWA) and finite-difference time domain (FDTD), to compute reflection and transmission spectra for silicon photonic resonators, fiber Bragg grating filters and optical frequency comb hardware [40].

Furthermore, the recursive construction from Theorem 3 enables hierarchical grating engineering: starting with a base solution n', multiplying by prime factors of $(10^{n'} - 1)/(9n')$ yields larger composite super-periods that introduce additional independent resonances. The memory property of Theorem 5 further suggests that newly appearing prime factors are disjoint from earlier primes, helping to avoid accidental harmonic degeneracy among different resonant bands. Several practical open questions follow from this mathematical physical correspondence: Quantitatively, how do the square-free nesting and disjoint prime constraints modify band gap width, resonance isolation and device insertion loss? Can such number-theoretic prior information reduce the computational overhead of inverse design pipelines for multi-resonance photonic gratings [17]?

5.2. Classical and Quantum Integrable Lattice Field Theories

A broad class of discrete integrable lattice models, including the discrete nonlinear Schrödinger equation, Toda lattices, and discrete KP hierarchies, are governed by periodic boundary conditions encoded in exponential congruences of the form $k^n \equiv 1 \pmod{m}$ [18]. Our thorough investigation of the prototype congruence $10^n \equiv 1 \pmod{9n}$ establishes a constructive mathematical template for generating exact periodic boundary condition data for such integrable systems.

For integrable lattices formulated via the Lax pair zero-curvature formalism, periodic boundary conditions impose strong constraints on admissible spectral parameters of Lax matrices [30]. The infinite solution set $\{n\}$ of Equation (1) furnishes a large family of candidate lattice periods for multi-periodic soliton boundary conditions. The prime factors of these moduli correspond to discrete quantum numbers associated with local and nonlocal conserved charges of the integrable hierarchy, and the square-free factorizations substantially simplify spectral decomposition of Lax operators [18].

The recursive procedure given in Theorem 3 provides an algorithmic route to construct multi-periodic boundary configurations: given a known solution n', extract prime divisors of $(10^{n'}-1)/(9n')$ and multiply n' by each such prime to obtain new valid periods. Compared to exhaustive brute-force enumeration over integer lattice sizes, this recursive generation reduces the search space for physically meaningful periodic boundary conditions. Such multi-periodic boundary data can feed into Darboux transformation algorithms to build multi-soliton and breather solutions for discrete integrable optical lattices [41]. An open direction is to clarify to what extent the memory and the square-free theorem and conjectures constrain the spectrum of transfer matrices and the set of conserved quantities for these periodic integrable lattices.

5.3. Quantum Computing and Quantum Information Processing

The infinite nested square-free prime sets constructed in Theorems 1–7 supply tunable composite moduli for benchmarking Shor's quantum factoring algorithm and designing discrete modular quantum phase gates [10]. Each solution n of $10^n \equiv 1 \pmod{9n}$ defines a finite ring $\mathbb{Z}/9n\mathbb{Z}$; the square-free factorization simplifies gate decomposition and error mitigation benchmarking for noisy intermediate-scale quantum (NISQ) hardware [11]. Moreover, repdigit-based linear feedback shift registers built from repunit Niven numbers produce certified long-period pseudo-random sequences. These sequences serve as controllable synthetic noise sources for quantum noise simulation and blind quantum state tomography, where high-entropy deterministic pseudo-randomness is required to characterize measurement imperfections without full reliance on hardware-level true random number generators [12].

For topological quantum error correction codes such as surface codes and color codes, the memory property of prime sets supports hierarchical multi-level stabilizer decomposition [19]. Nested prime factorizations of modulus $9n$ allow one to build scalable multi-scale block codes where logical qubits are encoded across nested finite ring sub-modules, offering alternative pathways toward fault-tolerant quantum architectures.

5.4. Condensed-Matter Physics and Quasicrystal Modeling

Quasicrystals produce sharp diffraction peaks originating from long-range aperiodic order which cannot be indexed using a single primitive unit cell of ordinary periodic crystals [13]. Their reciprocal space wave vector sets can be parameterized by algebraic modular divisibility conditions closely related to repunit divisibility. The square-free, disjoint prime families from Theorems 4 and 7 provide novel parameter sequences for constructing Fibonacci-type quasi-crystal super-lattices, eliminating redundant squared prime factors in symmetry order parameters [14].

For cyclic spin chain models with periodic boundary conditions, our congruence's solution values n deliver new system size parameters for exact diagonalization. Working over finite rings $\mathbb{Z}/n\mathbb{Z}$ built from these special integers may simplify spectral computation of spin excitation spectra and topological edge states, opening possibilities for finite-size scaling studies for spin lattice Hamiltonians subject to built-in modular constraints [15].

5.5. Computational Physics Simulation Algorithms

The hierarchical prime memory and the square-free properties of our solution set benefit large-scale lattice simulations via FFT decomposition. Square-free composite moduli mitigate aliasing artifacts in periodic discrete grid models during parallelized fast Fourier transform computations, as they avoid high powers of small primes that induce large convolution-related artifacts. The constructive Maple compatible generation procedure from Theorems 1–7 yields reproducible large prime test sets for validating quantum Monte Carlo pseudorandom number generators. This supplies a complementary test suite beyond conventional statistical randomness tests, by probing simulation outputs under modular constraints originating from repunit divisibility.

To conclude this section, we summarize the potential physical applications in Table 2 below:

Table 2. Mapping from a number-theoretic solution $n = 111$ ($10^{111} \equiv 1 \pmod{9 \cdot 111}$) to physical model parameters, corresponding to the potential applications.

Physics Application Domain	Mathematical Quantity	Derived Physical Parameter	Brief Interpretation (Based on This Section and Cited References)
Periodic photonic grating engineering (Section 5.1)	$n = 111$ (super-period integer)	Composite super-cycle integer $N_{\text{super}} = 111$; grating physical period $\Lambda = N_{\text{super}} \cdot \Lambda_0$	Λ_0: base grating unit period. $N_{\text{super}} = 111 = 3 \times 37$ (square-free factors) gives two independent resonant bands, suppressing harmonic crosstalk in sampled fiber Bragg gratings [16,17]. Bragg condition: $\lambda_B = 2n_{\text{eff}}\Lambda$.
Classical & quantum integrable lattice field theories (Section 5.2)	Modulus $n = 111$	Lattice size $L = 111$ for periodic boundary conditions	$L = 111$ serves as multi-periodic boundary for discrete integrable hierarchies (Toda, discrete nonlinear Schrödinger). Prime factors $\{3, 37\}$ decompose spectral structure of Lax operators [18].
Quantum computing & quantum information (Section 5.3)	Finite ring $\mathbb{Z}/(9n)\mathbb{Z} = \mathbb{Z}/999\mathbb{Z}$	Modulus $M = 999$ for modular quantum phase gates; test modulus for Shor's factoring algorithm	Square-free decomposition $999 = 3^3 \times 37$. Used for benchmarking Shor's algorithm and noisy intermediate-scale quantum (NISQ) gate-decomposition benchmarks [10,11]. Supports stabilizer group construction for topological error-correction codes.
Condensed-matter physics & quasicrystal modeling (Section 5.4)	Prime factor set $\{3, 37\}$ of $n = 111$	Reciprocal-space wave-vector indices $k \propto \{3, 37\}$	Integer indices parameterize aperiodic super-lattice diffraction peaks for Fibonacci-type quasicrystals, avoiding squared-prime redundant symmetry terms [13,14]. Also gives system-size $L = 111$ for exact diagonalization of cyclic spin chains.
Computational physics simulation algorithms (Section 5.5)	Square-free prime factors $\{3, 37\}$	FFT grid dimension $N_{\text{grid}} = 111$	Using $N_{\text{grid}} = 111$ (square-free prime divisors) mitigates aliasing artifacts in parallel FFT lattice simulations. $n = 111$ is also a test integer for quantum-Monte-Carlo pseudorandom-number validation [22].

6. Summary and Discussions

In this paper, we systematically investigate the exponential congruence $10^n \equiv 1 \pmod{9n}$, which is equivalent to the divisibility condition $n \mid R_n$, where $R_n = (10^n - 1)/9$ is the n-digit repunit number. Solutions of this congruence correspond to repdigit Niven numbers, i.e., integers divisible by their digit sum. Starting from the trivial seed solution $n = 3$, we establish a hierarchy of constructive theorems to generate infinite families of solutions.

The *Power Closure Theorem (Theorem 1, PCT)* shows that if a positive integer n satisfies the congruence, then every positive-integer power n^m is also a solution. Based on PCT, we further prove the *Product Power Theorem (Theorem 2, PPT)*: given two coprime integers n_1, n_2 such that both n_1 and $n_1 n_2$ are solutions, all numbers of the form $n' = n_1^{m_1} n_2^{m_2}$ ($m_1 \geq 1$, $m_2 \geq 0$) are valid solutions. Repeated application of PPT yields Corollary 3, which extends the result to multiple pairwise coprime factors with arbitrary non-negative exponents.

The key recursive construction is given in Theorem 3: suppose n' solves $10^{n'} \equiv 1 \pmod{9n'}$ and p is a prime divisor of $(10^{n'} - 1)/(9n')$, then both $p \cdot n'$ and its complemen-

tary co-factor multiplied by n' are new solutions. Combined with the cubic factorization lemma (Lemma 2, CFL) and the cyclotomic pairwise coprimality lemma (Lemma 3, CPCL), we derive the *Parameterized Solution Theorem (Theorem 4, PST)*. PST explicitly constructs multi-parameter infinite solution families

$$n = 3^k \prod_{i=0}^{I} \prod_{j=1}^{J_i} m_{ij}^{k_{ij}},$$

where m_{ij} are prime factors of the composite repdigit objects $[367]_i$. These primes come from factorizations of repunit quotients R_j. Lemma 2 reveals the recursive multiplicative structure of R_j, while Lemma 3 proves that the cyclotomic-derived integers a_i are pairwise coprime, guaranteeing disjointness among their prime divisor sets. Theorem 5 establishes two central properties for these prime collections: the prime sets $\mathcal{P}_k$ inherit the *memory property* (nested superset relation), whereas the prime sets $\mathcal{M}_k$ are mutually disjoint.

We formulate two core number-theoretic conjectures. Conjecture 1 asserts that both R_k and a_k possess the square-free property (SFP). Under the assumption that cyclotomic polynomial values $\Phi_n(10)$ are square-free (Conjecture 2), Theorem 6 conditionally proves Conjecture 1. Generalizing the construction starting from the seed $3 \cdot 37$, Theorem 7 produces further infinite solution families $n = 3^a \cdot 37^b \prod p_{ijk}^{c_{ijk}}$. Theorem 7 also extends the memory property and disjointness result to the generalized prime sets $\mathcal{P}_{jk}$ and $\mathcal{M}_{jk}$, and we state Conjecture 3 for the square-free property of these generalized repunit-related integers. Direct complete factorization for large indices is computationally prohibitive, so numerical verification is limited to small values of indices.

The solution families and their associated hierarchical prime structures are not only of intrinsic number-theoretic interest but also admit concrete potential physical applications, as elaborated in Section 5. Each solution n of the congruence provides candidate parameters for several branches of physics:

1. *Periodic photonic grating engineering.* Solutions serve as composite super-periods for sampled fiber Bragg gratings and multi-resonance dielectric stacks. Disjoint prime factors allow independent tuning of resonant bands and help suppress harmonic crosstalk, offering an algebraic alternative to purely heuristic numerical optimization.
2. *Classical and quantum integrable lattice field theories.* The integers n furnish admissible periodic boundary-condition sizes for discrete integrable hierarchies. Their prime-factor decomposition simplifies spectral analysis of Lax operators and conserved charges, and can be used to construct multi-soliton and breather solutions via Darboux transformations.
3. *Quantum computing and quantum information processing.* Moduli $9n$ define finite rings suitable for benchmarking Shor's factoring algorithm and designing modular quantum phase gates. Nested prime factorizations support hierarchical stabilizer decompositions for topological quantum error-correction codes. Repunit-derived sequences can act as certified long-period pseudo-random sequences for quantum noise simulation.
4. *Condensed-matter physics and quasicrystal modeling.* Disjoint prime families parameterize reciprocal-space wave-vectors for aperiodic quasi-crystal superlattices. Solution values n can also be adopted as system sizes for exact diagonalization of cyclic spin-chain models.
5. *Computational physics simulation algorithms.* Square-free solution moduli mitigate aliasing artifacts in parallel FFT-based lattice simulations. The constructive generation procedure yields reproducible test integer sets for validating quantum Monte-Carlo pseudo-random-number generators.

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Several open mathematical and physics-oriented problems remain to be investigated. From the number-theoretic perspective, we may ask whether every solution can be generated recursively from a finite set of primitive seeds via Theorem 3; whether the prime sets $\mathcal{M}_k$ capture all primes that can multiply existing solutions; how to characterize primitive prime divisors from cyclotomic polynomials and Zsigmondy's theorem for repunits; and what can be said about the asymptotic density and growth rate of solution integers. Another important direction concerns the square-free and memory properties for generalized repunit-like sequences.

From a physics viewpoint, interesting open questions include how the memory and disjointness of prime sets modify spectral features such as photonic band-gap width and resonance isolation; whether these number-theoretic constraints can reduce inverse-design computational cost for multi-band photonic devices; and whether the hierarchical prime structures can inspire new fault-tolerant quantum gate architectures. For integrable systems, it is worth exploring whether the constructed solution parameters produce novel analytical multi-soliton solutions.

Finally, the congruence $10^n \equiv 1 \pmod{9n}$ represents a special instance of the broader family of generalized exponential congruences

$$\mathcal{C}_+ : a^n + b^n \equiv 0 \pmod{(a+b)n}, \qquad \mathcal{C}_- : a^n - b^n \equiv 0 \pmod{(a-b)n},$$

for coprime integers $a > b > 0$. Many structural questions studied in this paper for $a = 10, b = 1$ can be extended to the general pairs (a, b). Future work may investigate the existence of primitive solutions, generalized memory and square-free properties, connections with cyclotomic polynomials and Zsigmondy-type primitive divisors, asymptotic distribution of solutions, as well as their corresponding physical applications for integrable systems, quasi-crystal modeling, photonic grating design, quantum information and large-scale numerical simulations.

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