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Von Kármán's Similarity Hypothesis and Its Application to Turbulent Boundary Layer Theory

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Abstract

Von Kármán's similarity hypothesis, that the turbulent fluctuations in two-dimensional parallel mean flow are similar at all points in the flow, is extended to two-dimensional turbulent boundary layer flow with constant mainstream velocity. The Boussinesq hypothesis introducing the eddy viscosity for turbulent mean flow and Prandtl's mixing length model for the eddy viscosity for two-dimensional parallel mean flow are first briefly reviewed, and the derivation of mixing length for two-dimensional parallel mean flow based on von Kármán's similarity hypothesis is outlined. The physical interpretation of von Kármán's mixing length given by Betz is presented and reviewed. The extension of von Kármán's mixing length to two-dimensional turbulent boundary layer flow is then presented by defining characteristic scales for the mean flow in the boundary layer as well as for the fluctuations, and imposing the boundary layer approximation. Finally, dimensional analysis for finding von Kármán-like mixing lengths is presented.

Keywords: Reynolds stresses; Boussinesq hypothesis; eddy viscosity; Prandtl's mixing length model; similarity hypothesis; von Kármán mixing length; turbulent boundary layer; dimensional analysis; Buckingham Π theorem

1. Introduction

When investigating turbulence, the Boussinesq hypothesis [1] relates the components of the turbulent Reynolds stresses to the components of the mean velocity gradient tensor. The hypothesis introduces the concept of eddy viscosity, which models momentum transfer due to the turbulent eddies. Unlike shear viscosity, the eddy viscosity is a property of the flow and not of the fluid. The eddy viscosity needs to be modelled, as it is not determined by the Boussinesq hypothesis. Prandtl derived a mixing length model for the eddy viscosity of a two-dimensional or channel mean flow [2]. In Prandtl's mixing length (PML) model, the eddy viscosity is the product of the square of the mixing length and the magnitude of the mean velocity gradient. However, Prandtl's model does not determine the form of the mixing length, which again depends on the flow. Prandtl's mixing length model for the eddy viscosity works well for the majority of flows, where there is a region of rapid change in the direction transverse to the mean flow direction; this includes boundary layer flows such as those at a solid boundary, as well as turbulent free shear flows such as turbulent wake and jet flows [3–7]. However, Prandtl's mixing length model for eddy viscosity fails to accurately capture the intricacies of the flow in regions very close to the wall [8]. Additionally, Prandtl's model predicts an eddy viscosity of zero in regions with zero mean velocity gradient, such as in the centreline of wake and jet flows, where it is known experimentally that the eddy viscosity is indeed nonzero on the centreline [9].

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In 1930, von Kármán derived an expression for PML for parallel mean flow [10]. To do this, he introduced a similarity hypothesis which relates different points in the flow. Betz provided a physical interpretation of von Kármán's mixing length (VKML) concept [11].

The aim of this paper is to extend the PML model for the eddy viscosity and von Kármán's similarity hypothesis and mixing length, from a two-dimensional parallel mean flow to the two-dimensional mean flow of a turbulent boundary layer with constant mainstream velocity. Boundary layer theory applies not only to the mean flow near a wall but also to the region of sharp change in the mean flow of a turbulent jet and wake.

In 1937, Goldstein [12] investigated von Kármán's similarity hypothesis and mixing length by analysing the two-dimensional turbulent flow between parallel planes and the three-dimensional turbulent flow through pipes. Lighthill [13] commented on Goldstein's investigation:

> He (Goldstein) probed also von Kármán's 'similarity' theory of the mixture length in a paper [12] which did much to facilitate general recognition of the flaw in that theory: the crucial assumption that turbulent transport at a given point in a shear layer was influenced by eddies primarily sensing conditions near the point rather than (as we now believe) by eddies involved in and responding to conditions over most of the layer.

However, very few formulae for the mixing length have been derived and many theoretical investigations use a mixing length that is either a constant or a power of the distance along the mean flow. Also, two-dimensional boundary layer mean flows are not parallel flows. It is therefore of value to investigate further VKML and its extension to two-dimensional boundary layer mean flows while being aware that it may not describe accurately all the features of turbulent flows.

Many biographies have been written on Prandtl and von Kármán. Two good accounts of their life and work are contained in the Biographical Memoirs of Fellows of the Royal Society, London [14,15] and in the book titled A Voyage Through Turbulence [16,17]. These biographies provide background information on research in turbulent fluid flow at the time during which the PML model of eddy viscosity and the VKML were formulated.

The rest of this paper is outlined as follows: in Section 2, we review the Boussinesq hypothesis which introduced the concept of eddy viscosity and the eddy viscosity PML model; in Section 3, an outline of the derivation of the VKML for parallel mean flow is presented; in Section 4, the physical interpretation of the VKML given by Betz is described; in Section 5, the PML model for eddy viscosity and VKML model for parallel mean flow are extended to two-dimensional boundary layer mean flow; in Section 6, we review the derivation of the VKML using dimensional analysis and investigate the derivation of other mixing lengths using dimensional analysis; finally, our conclusions are summarised in Section 7.

2. Boussinesq Hypothesis and Prandtl's Mixing Length Model for Eddy Viscosity

Initially, we develop the background for the Boussinesq hypothesis which introduced the eddy viscosity of a turbulent flow, which applies in general for a three-dimensional turbulent flow. Prandtl's derivation of the mixing length model of eddy viscosity for a two-dimensional mean parallel flow, which introduced the concept of mixing length, is outlined afterwards.

In component form, the Reynolds decomposition of the fluid velocity $\mathbf{u}$ for a turbulent flow is

$$u_i(x_1, x_2, x_3, t) = \bar{u}_i(x_1, x_2, x_3) + u_i'(x_1, x_2, x_3, t), \tag{1}$$

where $i = 1, 2,$ and 3, $\bar{u}_i$ is the mean velocity, and u'_i is the fluctuation. A steady turbulent flow for which the mean velocity is independent of time is considered ([18], p. 498). The mean of the mean flow and the mean of the fluctuation satisfy $\bar{\bar{u}}_i = \bar{u}_i$ and $\overline{u'}_i = 0$, respectively, but the mean of the product of the fluctuations is non-zero. The fluid is incompressible. Then,

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0, \tag{2}$$

and by substituting Equation (1) into Equation (2) and taking the time average, it can be verified that

$$\frac{\partial \bar{u}_1}{\partial x_1} + \frac{\partial \bar{u}_2}{\partial x_2} + \frac{\partial \bar{u}_3}{\partial x_3} = 0, \tag{3}$$

$$\frac{\partial u'_1}{\partial x_1} + \frac{\partial u'_2}{\partial x_2} + \frac{\partial u'_3}{\partial x_3} = 0. \tag{4}$$

The Reynolds stresses are

$$\bar{\tau}_{ij} = -\rho\overline{u'_i u'_j}. \tag{5}$$

The Boussinesq hypothesis for an incompressible fluid states ([19], p. 93) that

$$-\rho\overline{u'_i u'_j} = -\frac{2}{3}\rho k\delta_{ij} + 2\mu_T \bar{S}_{ij}\,, \tag{6}$$

where

$$\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{7}$$

is the rate of strain tensor of the mean flow,

$$k = \frac{1}{2}\overline{u'_i u'_i} \tag{8}$$

is the mean kinetic energy per unit mass of the fluctuations, ρ is the fluid density assumed constant, and μ_T is the eddy viscosity. By using the conservation of mass of the mean flow (Equation (3)), it can be verified that the traces of each side of Equation (6) are the same. The kinematic eddy viscosity ν_T is

$$\nu_T = \frac{\mu_T}{\rho}. \tag{9}$$

The Boussinesq hypothesis does not specify the form of the eddy viscosity μ_T, which is a property of the flow.

In 1925, Prandtl [18,20] derived the mixing length model of the eddy viscosity for two-dimensional parallel mean flow:

$$\bar{u} = \bar{u}(y), \quad \bar{v} = 0, \quad \bar{w} = 0. \tag{10}$$

An example is turbulent flow between two parallel plates. The coordinate system is illustrated in Figure 1. From the Boussinesq hypothesis in Equation (6),

$$\bar{\tau}_{xy} = -\rho\overline{u'v'} = \mu_T \frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y), \tag{11}$$

where u' and v' are the fluctuations in the x- and y-directions. Prandtl assumed that the fluid particles coalesce into lumps which retain their momentum, and so their velocity, in the $x-$direction as they traverse a distance ℓ' in the y-direction. By performing a Taylor expansion of $\bar{u}(y + \ell')$ about the level y and neglecting terms of order ${\ell'}^2$, Prandtl derived the fluctuation

$$u' = \ell' \frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y). \tag{12}$$

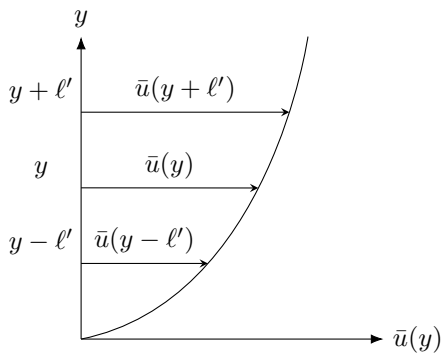


Figure 1. Coordinate system and velocity profile for two-dimensional parallel mean flow.

A significant assumption in the PML model is that the fluctuation in the y-direction, denoted v', is on the same order of magnitude as u':

$$v' = -cu' \tag{13}$$

where c is a constant. Prandtl showed that u' and v' have opposite signs, meaning that c is a positive constant; hence,

$$\bar{\tau}_{xy} = -\rho\overline{u'v'} = \rho\ell^2\left|\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y)\right|\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y), \tag{14}$$

where

$$\ell^2 = c\overline{\ell'\ell'}, \tag{15}$$

and the absolute value of $\mathrm{d}\bar{u}/\mathrm{d}y$ is taken in Equation (14) to ensure that the sign of $\bar{\tau}_{xy}$ changes with the sign of $\mathrm{d}\bar{u}/\mathrm{d}y$. Comparing Equations (11) and (14) yields

$$\mu_T = \rho\ell^2\left|\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y)\right|. \tag{16}$$

Equation (16) is the PML model for the eddy viscosity of a two-dimensional parallel mean flow, while the length ℓ is Prandtl's mixing length; this is the length that a lump of fluid is displaced in the transverse y-direction while retaining its momentum in the x-direction. For many turbulent flows, the mixing length can be related in a simple way to a characteristic length of the flow.

3. Von Kármán's Mixing Length

In 1930, von Kármán derived an expression for PML of two-dimensional incompressible parallel mean flow based on a similarity hypothesis [10]. His similarity theory was published in the Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, which was Göttingen's Academy of Sciences journal which published groundbreaking research. The 1930 issue in which von Kármán submitted his research also featured submissions by other notable researchers, including Kolmogoroff (presented at the sitting of the society by Alexandroff) and Weyl. In this section, the derivation of the VKML provided by Schlichting [18] is outlined. Other derivations of the VKML have been made; for example, Doshi and Gill [21] derived the VKML as a special case when investigating the extension of the PML model for the eddy viscosity to higher-order derivatives of $\bar{u}(y)$.

A picture of a young von Kármán can be seen in Figure 2. This photograph was taken two years prior to his derivation of his mixing length.



Figure 2. Photograph of Theodore von Kármán taken in 1928. Credit: AIP Emilio Segré Visual Archives.

Consider the incompressible two-dimensional parallel mean flow

$$u(x,y,t) = \bar{u}(y) + u'(x,y,t), \quad v(x,y,t) = 0 + v'(x,y,t). \tag{17}$$

Since mass is conserved, Equations (3) and (4) are satisfied for the two-dimensional flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0. \tag{18}$$

The kinematic viscosity ν is neglected because $\nu \ll \nu_T$. The Navier–Stokes equation with conservative body force

$$\mathbf{F} = -\nabla(\chi) \tag{19}$$

is

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\left(\frac{p}{\rho} + \chi\right). \tag{20}$$

The pressure is eliminated by operating on Equation (20) with the curl operator, which gives the vorticity propagation equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{v}, \tag{21}$$

where

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} \tag{22}$$

is the vorticity vector. Since the fluctuations satisfy the equation for conservation of mass (18), the stream function for the fluctuations $\psi'(x,y,t)$ is introduced, defined by

$$u'(x,y,t) = \frac{\partial \psi'}{\partial y}, \quad v'(x,y,t) = -\frac{\partial \psi'}{\partial x}. \tag{23}$$

Then, the x-, y-, and z-components of Equation (21) are

$$\omega_x = 0, \quad \omega_y = 0, \quad \omega_z = -\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y) - \nabla^2\psi', \tag{24}$$

where

$$\nabla^2\psi' = \frac{\partial^2\psi'}{\partial x^2} + \frac{\partial^2\psi'}{\partial y^2}. \tag{25}$$

The only non-zero component of (21) is the z-component, which in terms of the stream function (23) is

$$\frac{\partial}{\partial t}(\nabla^2\psi') + \left(\bar{u}(y) + \frac{\partial\psi'}{\partial y}\right)\frac{\partial}{\partial x}(\nabla^2\psi') - \frac{\partial\psi'}{\partial x}\left(\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}y^2}(y) + \frac{\partial}{\partial y}(\nabla^2\psi')\right) = 0. \tag{26}$$

Consider now the neighbourhood of a point (x_0, y_0). In order to transform to a coordinate system moving with mean velocity $\bar{u}(y_0)$ in which the flow is assumed to be steady, von Kármán first expanded the mean velocity $\bar{u}(y)$ in a Taylor series in $y - y_0$ about y_0:

$$\bar{u}(y) = \bar{u}(y_0) + (y - y_0)\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y_0) + \frac{1}{2}(y - y_0)^2\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}y^2}(y_0) + O\big((y - y_0)^3\big). \tag{27}$$

Substituting (27) into Equation (26) yields

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla^2\psi') + \left[\bar{u}(y_0) + (y - y_0)\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y_0) + O\big((y - y_0)^2\big) + \frac{\partial\psi'}{\partial y}\right]\frac{\partial}{\partial x}(\nabla^2\psi') \\ - \frac{\partial\psi'}{\partial x}\left[\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}y^2}(y_0) + O(y - y_0) + \frac{\partial}{\partial y}(\nabla^2\psi')\right] = 0.\end{aligned} \tag{28}$$

Von Kármán then made the transformation

$$x^* = x - \bar{u}(y_0)t, \quad y^* = y \tag{29}$$

and assumed that the fluctuation is steady with respect to the coordinate system (x^*, y^*), that is,

$$u'(x, y, t) = u'(x^*, y^*), \quad v'(x, y, t) = v'(x^*, y^*), \quad \psi'(x, y, t) = \psi'(x^*, y^*). \tag{30}$$

Using (29), Equation (28) becomes

$$(y^* - y_0)\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y_0)\frac{\partial}{\partial x^*}(\nabla^2\psi') - \frac{\mathrm{d}^2\bar{u}}{\mathrm{d}y^2}(y_0)\frac{\partial\psi'}{\partial x^*} + \frac{\partial\psi'}{\partial y^*}\frac{\partial}{\partial x^*}(\nabla^2\psi') - \frac{\partial\psi'}{\partial x^*}\frac{\partial}{\partial y^*}(\nabla^2\psi') = 0. \tag{31}$$

Von Kármán proposed the similarity hypothesis that turbulent fluctuations are similar at all points of the flow; therefore, they differ at different points only by factors of length and time scales. The units of length and time can be replaced by units of length ℓ and velocity B. By von Kármán's similarity hypothesis, the stream function of the fluctuations must depend only on the length scale and velocity scale at the point (x_0, y_0). Von Kármán made the transformation from x^*, y^*, ψ' to ξ, η, f, defined by

$$\xi = \frac{x^* - x_0}{\ell}, \quad \eta = \frac{y^* - y_0}{\ell}, \quad \text{and} \quad f'(\xi, \eta) = \frac{\psi'(x^*, y^*)}{B\ell}. \tag{32}$$

Equation (31) then becomes

$$\frac{\ell}{B}\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y_0)\eta\frac{\partial}{\partial\xi}(\nabla^2 f') - \frac{\ell^2}{B}\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}y^2}(y_0)\frac{\partial f'}{\partial\xi} + \frac{\partial f'}{\partial\eta}\frac{\partial}{\partial\xi}(\nabla^2 f') - \frac{\partial f'}{\partial\xi}\frac{\partial}{\partial\eta}(\nabla^2 f') = 0, \tag{33}$$

where

$$\nabla^2 f' = \frac{\partial^2 f'}{\partial\xi^2} + \frac{\partial^2 f'}{\partial\eta^2}. \tag{34}$$

By the similarity hypothesis, Equation (33) must be independent of ℓ, B, $\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y_0)$ and $\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}y^2}(y_0)$, since the variables ξ, η, and f' are scaled by ℓ and B. Hence,

$$\frac{\ell}{B}\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y_0) = c_1, \quad \frac{\ell^2}{B}\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}y^2}(y_0) = c_2, \tag{35}$$

where c_1 and c_2 are constants. Thus,

$$B = \frac{\ell}{c_1}\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y_0), \quad \ell = \frac{c_2}{c_1}\frac{\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y_0)}{\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}y^2}(y_0)}. \tag{36}$$

Equation (33), the partial differential equation for the stream function of the fluctuations $f'(\xi, \eta)$, then becomes

$$\left(c_1\eta + \frac{\partial f'}{\partial \eta}\right)\frac{\partial}{\partial \xi}(\nabla^2 f') - \frac{\partial f'}{\partial \xi}\frac{\partial}{\partial \eta}(\nabla^2 f') - c_2\frac{\partial f'}{\partial \xi} = 0. \tag{37}$$

Now, B and ℓ are chosen as the characteristic velocity and characteristic length of the fluctuations. Von Kármán assumed that the mixing length for two-dimensional parallel mean flow is

$$\ell = \kappa\left|\frac{\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y_0)}{\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}y^2}(y_0)}\right|, \tag{38}$$

where $\kappa = c_2/c_1$ is a dimensionless constant determined by experiment.

Basic to Prandtl's derivation of his mixing length model for the eddy viscosity is the assumption that $v' = -cu'$. This assumption was not made by von Kármán in the derivation of the mixing length (38); however, if the VKML is used in Prandtl's model for the eddy viscosity, then it needs to be verified that it applies when $v' = -cu'$.

Now,

$$u' = \frac{\partial \psi'}{\partial y} = B\frac{\partial f'}{\partial \eta}, \quad v' = -\frac{\partial \psi'}{\partial x} = -B\frac{\partial f'}{\partial \xi}, \tag{39}$$

and if $v' = -cu'$, then

$$\frac{\partial f'}{\partial \xi} = c\frac{\partial f'}{\partial \eta}. \tag{40}$$

When Equation (40) is satisfied,

$$\frac{\partial f'}{\partial \eta}\frac{\partial}{\partial \xi}(\nabla^2 f') = \frac{\partial f'}{\partial \xi}\frac{\partial}{\partial \eta}(\nabla^2 f') \tag{41}$$

and the PDE (33) reduces to

$$\frac{\ell}{B}\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y_0)\eta\frac{\partial}{\partial \xi}(\nabla^2 f') - \frac{\ell^2}{B}\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}y^2}(y_0)\frac{\partial f'}{\partial \xi} = 0. \tag{42}$$

The similarity hypothesis does not require that each coefficient in Equation (42) are constant, only that the coefficients in Equation (42) are proportional. Equation (42) can be rewritten as

$$\eta\frac{\partial}{\partial \xi}(\nabla^2 f') - \ell\frac{\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}y^2}(y_0)}{\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y_0)}\frac{\partial f'}{\partial \xi} = 0 \tag{43}$$

where the coefficient of $\partial f'/\partial \xi$ must now be a constant. Hence,

$$\ell \frac{\frac{\mathrm{d}^2 \bar{u}}{\mathrm{d}y^2}(y_0)}{\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y_0)} = c_3, \tag{44}$$

where c_3 is a constant and so

$$\ell = c_3 \frac{\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}(y_0)}{\frac{\mathrm{d}^2 \bar{u}}{\mathrm{d}y^2}(y_0)}. \tag{45}$$

Therefore, Von Kármán's mixing length remains valid when $v' = -cu'$ and Equation (37) reduces to the linear PDE

$$\eta \frac{\partial}{\partial \xi}\left(\nabla^2 f'\right) - c_3 \frac{\partial f'}{\partial \xi} = 0. \tag{46}$$

The stream function $f'(\xi, \eta)$ satisfies the system of differential equations in (40) and (46). It is worth noting that the assumption $v' = -cu'$ is a modelling idealisation rather than a general property of turbulent fluctuations. Prandtl criticised the assumption as being a rough approximation; generally, u' and v' are not pointwise proportional. The relation is adopted here as a closure device, following Prandtl, and is used in this paper only to check compatibility with Prandtl's eddy viscosity form.

4. Physical Interpretation and Derivation by Betz

Albert Betz was an associate of Prandtl for about forty years and a construction engineer for his wind turbines. He is well known today for his derivation of the Betz limit of about 59% for the maximum efficiency of a horizontal-axis wind turbine [22]. Betz provided a physical interpretation of the VKML concept [11]. Betz was able to expand the concept graphically and deduce the same result (Equation (38)) for the mixing length. In Figure 3, Prandtl and Betz observe eddies in a shallow water tank. From 1937 to 1945, Betz was the successor to Prandtl as Head of the Kaiser Wilhelm Society's Aerodynamics Laboratory. Later, from 1947 to 1956, Betz was the Director at the Max Planck Institute for Fluid Dynamics.

It is unknown to the authors if a translation from German to English of Betz's paper exists. The physical insight of the paper is both intuitive and significant. His argument is summarised here, and we refer the reader to the original for the full development. The original order, equations, and notation used by Betz are preserved. Betz denoted the mean flow by a capital letter and the fluctuation by a small letter. Betz also used rot and ζ instead of the more familiar curl and w for the curl operator and angular velocity, respectively. A prime denotes ordinary differentiation and not a fluctuation. Betz considered parallel mean flow in the x-direction with mean flow depending on y. Betz's note contains no equation numbers or section divisions. For ease of reference, his equations are given in the order they appear in his note.

Betz builds a physical picture of turbulent transport by following fluid parcels displaced across a mean shear layer. Consider a reference layer of the mean flow with local mean velocity U_0, as illustrated in Figure 4, and let the transverse turbulent motion exchange parcels between this layer and its neighbours. The exchange makes one side of the layer momentarily slower and the other momentarily faster, producing a fluctuating disturbance about U_0. For a parcel originating a distance ℓ_1 from the reference layer, a linear expansion of the mean profile gives the streamwise velocity it carries as

$$U = U_0 \pm \ell_1 U_0', \tag{47}$$

so that the corresponding streamwise disturbance velocity at the reference layer is

$$u = \pm \ell_1 U_0'. \tag{48}$$

The step that distinguishes Betz's account from Prandtl's is the observation that a displaced parcel transports not only its momentum but also its mean rotation:

$$\zeta = \operatorname{rot} U = U'. \tag{49}$$

Writing ζ_0 for the mean rotation at the reference layer, the same first-order expansion applied to Equation (49) gives the rotation carried by a parcel from a distance ℓ_1 as

$$\zeta = \zeta_0 \pm \ell_1 U_0'', \tag{50}$$

and so gives a rotational disturbance about the reference layer of

$$\zeta - \zeta_0 = \pm \ell_1 U_0''. \tag{51}$$



Figure 3. Ludwig Prandtl (left) and Albert Betz observing the formation of vortices behind objects in flow at the shallow water tank, Max Planck Institute for Fluid Dynamics, July 1950. Reproduced with permission: Archives of the Max Planck Society, Berlin, VI. Abt., Rep. 1, Prandtl, Ludwig III/3.

Betz then interprets the displaced parcels as an ensemble of counter-rotating vortices embedded in the otherwise undisturbed flow, so that the turbulent fluctuation field is reconstructed as a field of such eddies. Let ℓ_2 denote the characteristic distance between the centres of oppositely rotating parcels, which Betz takes to be of the same order as their diameter. The transverse velocity induced between neighbouring counter-rotating eddies then scales as

$$v \sim (\zeta - \zeta_0)\ell_2 = \pm \ell_1 \ell_2 U_0''. \tag{52}$$

Invoking von Kármán's similarity hypothesis that the turbulent processes at different points of the flow are mechanically similar forces the two length scales and two velocity components to be proportional:

$$\ell_2 \sim \ell_1, \qquad v \sim u. \tag{53}$$

Introducing a common length scale ℓ for both ℓ_1 and ℓ_2 and combining Equations (48) and (52) yields

$$\ell^2 U'' \sim \ell U', \qquad \text{or equivalently} \qquad \ell \sim \frac{U'}{U''}, \tag{54}$$

which recovers von Kármán's expression for the mixing length.

It should be noted that Betz used the mean rotation, "mittlere Drehung", and not the mean rate of rotation, "mittlere Drehgeschwindigkeit". The fluid parts to which Betz refers are the fluid lumps of Prandtl. Prandtl assumed that the fluid lumps retain their linear momentum, and so their linear velocity, as they move from one layer to another in a turbulent flow [2]; Betz went one step further and assumed that the fluid lumps also retain their angular momentum, and so their angular velocity. In the analysis that follows, we return to modern notation for clarity, where $\bar{\boldsymbol{\omega}}$ is the mean angular velocity vector and $\bar{u}$ is the mean velocity. As a prime is reserved for fluctuations, we use Leibniz's notation for derivatives.

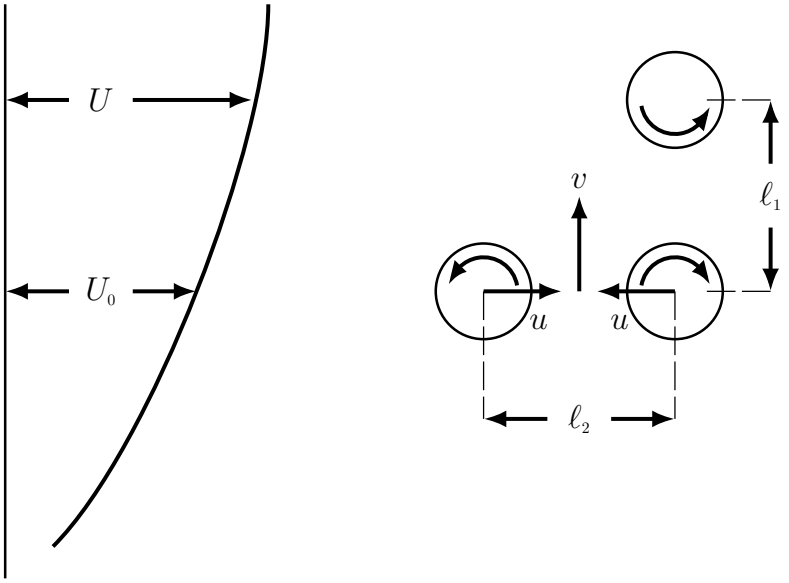


Figure 4. Displacement of liquid parts and the resulting disturbance velocities. Diagram adapted from [11] "Die v. Kármánsche Ähnlichkeitsüberlegung für turbulente Vorgänge in physikalischer Auffassung".

Since the parallel flow is of the form $\mathbf{U} = (\bar{\mathbf{u}} = (\bar{u}_x(y), 0, 0)$, the mean rate of rotation vector or mean angular velocity vector is

$$\bar{\boldsymbol{\omega}} = \frac{1}{2}\text{curl}\ \bar{\mathbf{u}} = -\frac{1}{2}\frac{\mathrm{d}\bar{u}_x}{\mathrm{d}y}\hat{\mathbf{z}}, \tag{55}$$

where $\hat{\mathbf{z}}$ is a unit vector in the z-direction. Only the z-component w_z of the mean rate of rotation vector is non-zero, and

$$w_z = -\frac{1}{2}\frac{\mathrm{d}\bar{u}_x}{\mathrm{d}y}. \tag{56}$$

In Equation (51), the $\pm$ sign should be replaced by $\mp$ and include a factor of $1/2$; $\pm$ is used to distinguish between a fluid lump coming from a layer above layer $y = y_0$ and a fluid lump coming from a layer below layer $y = y_0$. In particular, the mean rate of rotation ζ at a distance ℓ_1 from layer $y = y_0$ is given by a Taylor series

$$\begin{aligned}\bar{w}_z = \bar{w}_z(y_0 \pm \ell_1) &= \bar{w}_z(y_0) \pm \ell_1 \frac{\mathrm{d}\bar{w}_z}{\mathrm{d}y}(y_0) + O(\ell_1^2) \\ &= \bar{w}_z(y_0) \pm \ell_1 \frac{\mathrm{d}}{\mathrm{d}y}\left(-\frac{1}{2}\frac{\mathrm{d}\bar{u}_x}{\mathrm{d}y}(y_0)\right) + O(\ell_1^2) \\ &= \bar{w}_z(y_0) \mp \frac{1}{2}\ell_1 \frac{\mathrm{d}^2\bar{u}_x}{\mathrm{d}y^2}(y_0) + O(\ell_1^2). \end{aligned} \tag{57}$$

Thus, to first order

$$\bar{w}_z(y) - \bar{w}_z(y_0) = \mp\frac{1}{2}\ell_1 \frac{\mathrm{d}^2\bar{u}_x}{\mathrm{d}y^2}(y_0), \quad \bar{w}_z(y_0) = -\frac{1}{2}\frac{\mathrm{d}\bar{u}_x}{\mathrm{d}y}(y_0), \tag{58}$$

and the fluctuation v' is given by

$$v' = (\bar{w}_z(y) - \bar{w}_z(y_0))\frac{1}{2}\ell_2 = \mp\frac{1}{4}\ell_1\ell_2 \frac{\mathrm{d}^2\bar{u}_x}{\mathrm{d}y^2}(y_0). \tag{59}$$

In the Betz model, the fluctuation velocity v is obtained from the rate of rotation of the fluid lump, while in the Prandtl mixing length model v is assumed to be the same order of magnitude as u, with its sign determined by the direction of motion to layer $y = y_0$. Correction to the sign of ζ does not affect the final result (Equation (54)) for the VKML.

5. Prandtl's Eddy Viscosity and Von Kármán's Mixing Length Extended to Boundary Layer Flow

Prandtl's model for the eddy viscosity and von Kármán's mixing length model apply to two-dimensional parallel mean flow. Next, the results are extended to two-dimensional boundary layer flow, which applies not only to a boundary layer at a solid boundary but also to free shear flows such as jet and wake flows. This extends the application of these models to a wider range of physical scenarios, such as applying the VKML model to turbulent flow in the wake behind wind turbines.

Consider first Prandtl's model for the eddy viscosity and the two-dimensional turbulent boundary layer flow

$$u(t,x,y) = \bar{u}(x,y) + u'(t,x,y), \quad v(t,x,y) = \bar{v}(x,y) + v'(t,x,y). \tag{60}$$

From Boussinesq's hypothesis,

$$\bar{\tau}_{xy} = -\rho\overline{u'v'} = \mu_T\left(\frac{\partial\bar{u}}{\partial y}(x,y) + \frac{\partial\bar{v}}{\partial x}(x,y)\right). \tag{61}$$

Introducing the dimensionless variables of turbulent boundary layer theory,

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{\delta}, \quad \bar{u}^* = \frac{\bar{u}}{U_0}, \quad \bar{v}^* = \frac{\bar{v}}{V}, \quad \bar{\tau}^*_{xy} = \frac{\bar{\tau}_{xy}}{\rho U_0^2}, \tag{62}$$

where

$$V = \frac{\delta}{L}U_0, \quad \delta = \frac{L}{\sqrt{Re_T}}, \quad Re_T = \frac{U_0 L}{\nu_T} \tag{63}$$

and where L and δ are the characteristic lengths along and perpendicular to the boundary layer, respectively, with U_0 as the mainstream velocity. The turbulent Reynolds number Re_T is defined in terms of the kinematic eddy viscosity in (9). Expressed in terms of the dimensionless variables, Equation (61) becomes

$$\bar{\tau}^*_{xy} = \frac{1}{\sqrt{Re_T}}\left(\frac{\partial\bar{u}^*}{\partial y^*} + \frac{1}{Re_T}\frac{\partial\bar{v}^*}{\partial x^*}\right). \tag{64}$$

In order for a turbulent boundary layer to exist, $\sqrt{Re_T} \gg 1$; however, in the boundary layer approximation, terms of order $1/\sqrt{Re_T}$ are retained while terms of order $1/Re_T$ and smaller are neglected. Imposing the boundary layer approximation on Equation (64) gives

$$\bar{\tau}^*_{xy} = \frac{1}{\sqrt{Re_T}} \frac{\partial \bar{u}^*}{\partial y^*}, \tag{65}$$

which expressed in dimensional variables is

$$\bar{\tau}_{xy} = \mu_T \frac{\partial \bar{u}}{\partial y}(x, y). \tag{66}$$

By proceeding as for parallel mean flow and considering fluid lumps that transverse distance ℓ' at position x along the boundary layer while assuming that the fluctuations u' and v' are the same order of magnitude, it is found that

$$\bar{\tau}_{xy} = \rho \ell^2 \left| \frac{\partial \bar{u}}{\partial y}(x, y) \right| \frac{\partial \bar{u}}{\partial y}(x, y), \tag{67}$$

where ℓ^2 is defined by Equation (15). Equating (67) and (66) for τ_{xy} gives

$$\mu_T = \rho \ell^2 \left| \frac{\partial \bar{u}}{\partial y}(x, y) \right|. \tag{68}$$

Equation (68) is the extension of Prandtl's mixing length model for the eddy viscosity from parallel mean flow to the mean flow of a two-dimensional boundary layer.

Next, we consider the extension of von Kármán's similarity hypothesis and mixing length model to two-dimensional boundary layer flows. Prandtl's two-dimensional boundary layer equations are

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\nu \frac{\partial u}{\partial y} \right) + U_0 \frac{\mathrm{d} U_0}{\mathrm{d} x}, \tag{69}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{70}$$

where U_0 is the mainstream velocity. In boundary layer theory,

$$U_0 \frac{\mathrm{d} U_0}{\mathrm{d} x} = -\frac{1}{\rho} \frac{\mathrm{d} p}{\mathrm{d} x}. \tag{71}$$

Take $U_0(x)$ as a constant which applies for boundary layer flow along a flat plate as well as for jet and wake flows. The pressure gradient is zero in the boundary layer, unlike in parallel mean flow. We do not need to operate on Equation (69) with the curl operator. The order of the PDE for the stream function of the fluctuations will then be one less than in Equation (26) for parallel mean flow. Von Kármán neglected the kinematic viscosity ν in the parallel mean flow in Equations (20) and (21) as $\nu \ll \nu_T$. Therefore, setting $\nu = 0$, Equation (69) reduces to

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = 0. \tag{72}$$

The velocity components are separated into a mean velocity and a fluctuation given by Equation (60). Substituting Equation (60) into Equations (70) and (72) yields

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial v'}{\partial y} = 0, \tag{73}$$

$$\frac{\partial u'}{\partial t} + \overline{u} \frac{\partial \overline{u}}{\partial x} + \overline{u} \frac{\partial u'}{\partial x} + u' \frac{\partial \overline{u}}{\partial x} + u' \frac{\partial u'}{\partial x} + \overline{v} \frac{\partial \overline{u}}{\partial y} + \overline{v} \frac{\partial u'}{\partial y} + v' \frac{\partial \overline{u}}{\partial y} + v' \frac{\partial u'}{\partial y} = 0. \tag{74}$$

By taking the mean value of Equation (73), this separates into two conservation of mass equations:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0, \tag{75}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0. \tag{76}$$

Equation (74) is simplified by defining characteristic scales for the mean flow in the boundary layer and for the fluctuations, then determining the magnitude of each term. Terms of small magnitude will be neglected.

The scales for the mean velocity in the x and y directions are defined by Equations (62) and (63), and are the mainstream velocity U_0 and $V = (\delta/L)U_0$ derived from the conservation of mass for the mean flow, Equation (75). The characteristic time for the mean flow is

$$T = \frac{L}{U_0} = \frac{\delta}{V}. \tag{77}$$

When estimating the magnitude of the derivatives of $\bar{u}$ and $\bar{v}$ with respect to x, y, and t, the scales L, δ, and L/U_0 are used for $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial t$, respectively.

For the fluctuations, assume that the length scales in the x and y directions are the same; these are denoted by ℓ. Then, from the conservation of mass for the fluctuations (Equation (76)), the velocity scales in the x and y directions must be the same and are denoted by B. The time scale T' for a fluctuation is

$$T' = \frac{\ell}{B}. \tag{78}$$

When estimating the magnitude of the derivatives of u' and v' with respect to x, y, and t, the scales ℓ, ℓ, and ℓ/B are used for $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial t$, respectively.

Because the time scale for a fluctuation is much less than the time scale for the mean flow,

$$\frac{\ell}{B} \ll \frac{L}{U_0} \quad \text{or} \quad \frac{\ell}{L}\frac{U_0}{B} \ll 1. \tag{79}$$

Assume that ℓ and δ are the same order of magnitude. Condition (79) then becomes

$$\frac{\delta}{L}\frac{U_0}{B} \ll 1. \tag{80}$$

However, from the condition for a boundary layer to exist,

$$\frac{\delta}{L} \ll 1. \tag{81}$$

Hence,

$$\frac{U_0}{B} \sim 1. \tag{82}$$

Therefore, the mainstream velocity U_0 and the velocity scale for the fluctuations B are the same order of magnitude.

In Table 1, the magnitude of each term in Equation (74) is evaluated using the boundary layer approximation. In this approach, it can be determined which terms should be neglected. The magnitude of each term in Equation (74) is compared with the magnitude of the largest term $\partial u'/\partial t$. Therefore, Equation (74) is divided by B^2/ℓ.

Table 1. Table containing the magnitude of the terms in Equation (74).

Term	Magnitude	Multiplied by $\frac{\ell}{B^2}$	Outcome
$\frac{\partial u'}{\partial t}$	$\frac{B^2}{\ell}$	1	Retain
$\overline{u}\frac{\partial \overline{u}}{\partial x}$	$\frac{U_0^2}{L}$	$\frac{\ell}{L}\left(\frac{U_0}{B}\right)^2 \sim \frac{\ell}{L}$	Neglect
$\overline{u}\frac{\partial u'}{\partial x}$	$\frac{U_0 B}{\ell}$	$\frac{U_0}{B} \sim 1$	Retain
$u'\frac{\partial \overline{u}}{\partial x}$	$\frac{U_0 B}{L}$	$\frac{\ell}{L}\frac{U_0}{B} \sim \frac{\ell}{L}$	Neglect
$u'\frac{\partial u'}{\partial x}$	$\frac{B^2}{\ell}$	1	Retain
$\overline{v}\frac{\partial \overline{u}}{\partial y}$	$\frac{U_0^2}{L}$	$\frac{\ell}{L}\left(\frac{U_0}{B}\right)^2 \sim \frac{\ell}{L}$	Neglect
$\overline{v}\frac{\partial u'}{\partial y}$	$\frac{\delta U_0 B}{L\ell}$	$\frac{\delta U_0}{LB} \sim \frac{\delta}{L}$	Neglect
$v'\frac{\partial \overline{u}}{\partial y}$	$\frac{U_0 B}{\delta}$	$\frac{\ell}{\delta}\frac{U_0}{B} \sim 1$	Retain
$v'\frac{\partial u'}{\partial y}$	$\frac{B^2}{\ell}$	1	Retain

After neglecting terms of order ℓ/L and δ/L, Equation (74) is reduced to

$$\frac{\partial u'}{\partial t} + (\overline{u}(x,y) + u')\frac{\partial u'}{\partial x} + \left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial u'}{\partial y}\right)v' = 0. \tag{83}$$

Since the fluctuations u' and v' satisfy the continuity equation (Equation (76)), a stream function for the fluctuations $\psi'(t,x,y)$ can be introduced, and is defined by

$$u' = \frac{\partial \psi'}{\partial y} \quad , \quad v' = -\frac{\partial \psi'}{\partial x}. \tag{84}$$

Equation (83) in terms of the stream function is now

$$\frac{\partial^2 \psi'}{\partial t \partial y} + \left(\overline{u}(x,y) + \frac{\partial \psi'}{\partial y}\right)\frac{\partial^2 \psi'}{\partial x \partial y} - \left(\frac{\partial \overline{u}}{\partial y}(x,y) + \frac{\partial^2 \psi'}{\partial y^2}\right)\frac{\partial \psi'}{\partial x} = 0. \tag{85}$$

Consider the neighbourhood of a point (x_0, y_0). Generalising the transformation in (29) introduced by von Kármán for parallel mean flow, a transformation is made to a coordinate system moving with mean velocity components $\overline{u}(x_0, y_0)$ and $\overline{v}(x_0, y_0)$, in which the flow is assumed to be steady. First, expand $\overline{u}(x,y)$ and $\frac{\partial \overline{u}}{\partial y}(x,y)$ in Taylor series in $x - x_0$ and $y - y_0$ about the point (x_0, y_0). Neglecting terms of order $(x - x_0)^2$ and $(y - y_0)^2$,

$$\overline{u}(x,y) = \overline{u}(x_0,y_0) + (x - x_0)\frac{\partial \overline{u}}{\partial x}(x_0,y_0) + (y - y_0)\frac{\partial \overline{u}}{\partial y}(x_0,y_0), \tag{86}$$

$$\frac{\partial \overline{u}}{\partial y}(x,y) = \frac{\partial \overline{u}}{\partial y}(x_0,y_0) + (x - x_0)\frac{\partial^2 \overline{u}}{\partial x \partial y}(x_0,y_0) + (y - y_0)\frac{\partial^2 \overline{u}}{\partial y^2}(x_0,y_0). \tag{87}$$

Under this expansion, Equation (85) now becomes

$$\frac{\partial^2 \psi'}{\partial t \partial y} + \left[\overline{u}(x_0,y_0) + (x - x_0)\frac{\partial \overline{u}}{\partial x}(x_0,y_0) + (y - y_0)\frac{\partial \overline{u}}{\partial y}(x_0,y_0) + \frac{\partial \psi'}{\partial y}\right]\frac{\partial^2 \psi'}{\partial x \partial y}$$

$$-\left[\frac{\partial \overline{u}}{\partial y}(x_0,y_0)+(x-x_0)\frac{\partial^2 \overline{u}}{\partial x\partial y}(x_0,y_0)+(y-y_0)\frac{\partial^2 \overline{u}}{\partial y^2}(x_0,y_0)+\frac{\partial^2 \psi'}{\partial y^2}\right]\frac{\partial \psi'}{\partial x}=0. \quad (88)$$

Transform to a coordinate system (x^*,y^*) moving with mean velocity $(\overline{u}(x_0,y_0),\overline{v}(x_0,y_0))$ defined by

$$x^*=x-\overline{u}(x_0,y_0)t \quad , \quad y^*=y-\overline{v}(x_0,y_0)t.$$

Then,

$$x^*-x_0^*=x-x_0 \quad , \quad y^*-y_0^*=y-y_0. \quad (89)$$

It is assumed that the fluctuations are steady with respect to the transformed coordinate, that is,

$$u'(x,y,t)=u'(x^*,y^*) \quad , \quad v'(x,y,t)=v'(x^*,y^*) \quad , \quad \psi'(x,y,t)=\psi'(x^*,y^*). \quad (90)$$

Equation (88) in the transformed coordinate system is now

$$-\overline{v}(x_0,y_0)\frac{\partial^2 \psi'}{\partial {y^*}^2}+\left[({x^*}-{x_0}^*)\frac{\partial \overline{u}}{\partial x}(x_0,y_0)+({y^*}-{y_0}^*)\frac{\partial \overline{u}}{\partial y}(x_0,y_0)+\frac{\partial \psi'}{\partial y^*}\right]\frac{\partial^2 \psi'}{\partial x^*\partial y^*}$$

$$-\left[\frac{\partial \overline{u}}{\partial y}(x_0,y_0)+({x^*}-{x_0}^*)\frac{\partial^2 \overline{u}}{\partial x\partial y}(x_0,y_0)+({y^*}-{y_0}^*)\frac{\partial^2 \overline{u}}{\partial y^2}(x_0,y_0)+\frac{\partial^2 \psi'}{\partial {y^*}^2}\right]\frac{\partial \psi'}{\partial x^*}=0. \quad (91)$$

The terms in Equation (91) are again evaluated using the boundary layer approximation. The characteristic stream function for the fluctuations is ℓB and the length scale for $x^*-x_0^*$ and $y^*-y_0^*$ is ℓ. In Table 2, the magnitude of each term in Equation (91) is determined and compared with the magnitude B^2/ℓ of the largest terms.

Table 2. Table containing the magnitude of the terms in Equation (91).

Term	Magnitude	Multiplied by $\frac{\ell}{B^2}$	Outcome
$\overline{v}(x_0,y_0)\frac{\partial^2 \psi'}{\partial {y^*}^2}$	$\frac{BU\delta}{L\ell}$	$\frac{\delta}{L}\frac{U}{B}\sim\frac{\delta}{L}$	Neglect
$({x^*}-{x_0}^*)\frac{\partial^2 \psi'}{\partial x^*\partial y^*}\frac{\partial \overline{u}}{\partial x}(x_0,y_0)$	$\frac{BU}{L}$	$\frac{\ell}{L}\frac{U}{B}\sim\frac{\ell}{L}$	Neglect
$({y^*}-{y_0}^*)\frac{\partial^2 \psi'}{\partial x^*\partial y^*}\frac{\partial \overline{u}}{\partial y}(x_0,y_0)$	$\frac{BU}{\delta}$	$\frac{\ell}{\delta}\frac{U}{B}\sim 1$	Retain
$\frac{\partial \psi'}{\partial y^*}\frac{\partial^2 \psi'}{\partial x^*\partial y^*}$	$\frac{B^2}{\ell}$	1	Retain
$\frac{\partial \psi'}{\partial x^*}\frac{\partial \overline{u}(x_0,y_0)}{\partial y}$	$\frac{UB}{\delta}$	$\frac{\ell}{\delta}\frac{U}{B}\sim 1$	Retain
$({x^*}-{x_0}^*)\frac{\partial \psi'}{\partial x^*}\frac{\partial^2 \overline{u}}{\partial x\partial y}(x_0,y_0)$	$\frac{\ell BU}{L\delta}$	$\frac{\ell}{L}\frac{\ell}{\delta}\frac{U}{B}\sim\frac{\ell}{L}$	Neglect
$({y^*}-{y_0}^*)\frac{\partial \psi'}{\partial x^*}\frac{\partial^2 \overline{u}}{\partial y^2}(x_0,y_0)$	$\frac{\ell BU}{\delta^2}$	$\left(\frac{\ell}{\delta}\right)^2\frac{U}{B}\sim 1$	Retain
$\frac{\partial \psi'}{\partial x^*}\frac{\partial^2 \psi'}{\partial {y^*}^2}$	$\frac{B^2}{\ell}$	1	Retain

By dividing Equation (91) by B^2/ℓ and neglecting terms of order ℓ/L and δ/L, Equation (91) reduces to

$$\left[(y^* - {y_0}^*)\frac{\partial \overline{u}}{\partial y}(x_0, y_0) + \frac{\partial \psi'}{\partial y^*}\right]\frac{\partial^2 \psi'}{\partial x^* \partial y^*} - \left[\frac{\partial \overline{u}(x_0, y_0)}{\partial y} + (y^* - {y_0}^*)\frac{\partial^2 \overline{u}}{\partial y^2}(x_0, y_0) + \frac{\partial^2 \psi'}{\partial {y^*}^2}\right]\frac{\partial \psi'}{\partial x^*} = 0. \tag{92}$$

By von Kármán's similarity hypothesis, the stream function of the fluctuation ψ' must depend only on the length scale ℓ and the velocity scale B at the point (x_0, y_0). Therefore, the variables ξ, η, f' are introduced for the fluctuation, defined by

$$\xi = \frac{x^* - x_0^*}{\ell}, \quad , \quad \eta = \frac{y^* - y_0^*}{\ell}, \quad , \quad f'(\xi, \eta) = \frac{\psi'(x^*, y^*)}{\ell B}. \tag{93}$$

Under the transformation in (93), Equation (92) becomes

$$\left[\frac{\ell}{B}\frac{\partial \overline{u}}{\partial y}(x_0, y_0)\eta + \frac{\partial f'}{\partial \eta}\right]\frac{\partial^2 f'}{\partial \xi \partial \eta} - \left[\frac{\ell}{B}\frac{\partial \overline{u}}{\partial y}(x_0, y_0) + \frac{\ell^2}{B}\frac{\partial^2 \overline{u}}{\partial y^2}(x_0, y_0)\eta + \frac{\partial^2 f'}{\partial \eta^2}\right]\frac{\partial f'}{\partial \xi} = 0. \tag{94}$$

By von Kármán's similarity hypothesis, Equation (94) must be independent of $\ell, B, \dfrac{\partial \overline{u}}{\partial y}(x_0, y_0)$ and $\dfrac{\partial^2 \overline{u}}{\partial y^2}(x_0, y_0)$. Thus, it is necessary that

$$\frac{\ell}{B}\frac{\partial \overline{u}}{\partial y}(x_0, y_0) = c_1, \tag{95}$$

where c_1 is a constant, and that

$$\frac{\ell^2}{B}\frac{\partial^2 \overline{u}}{\partial y^2}(x_0, y_0) = c_2, \tag{96}$$

where c_2 is a constant. Hence,

$$B = \frac{\ell}{c_1}\frac{\partial \overline{u}}{\partial y} \quad , \quad \ell = \frac{c_2}{c_1}\frac{\dfrac{\partial \overline{u}}{\partial y}}{\dfrac{\partial^2 \overline{u}}{\partial y^2}}, \tag{97}$$

where the derivatives are evaluated at (x_0, y_0). The partial differential equation in (94) for the stream function of the fluctuations becomes

$$\left(c_1\eta + \frac{\partial f'}{\partial \eta}\right)\frac{\partial^2 f'}{\partial \xi \partial \eta} - \frac{\partial f'}{\partial \xi}\frac{\partial^2 f'}{\partial \eta^2} - (c_1 + c_2\eta)\frac{\partial f'}{\partial \xi} = 0. \tag{98}$$

In a turbulent boundary layer flow, a second order PDE for the stream function of the fluctuations is obtained. For parallel mean flow, the third-order PDE (37) is obtained for $f'(\xi, \eta)$.

Therefore, the extension of VKML to turbulent boundary layer flow is as follows:

$$\ell = \kappa\left|\frac{\dfrac{\partial \overline{u}}{\partial y}(x, y)}{\dfrac{\partial^2 \overline{u}}{\partial y^2}(x, y)}\right| \tag{99}$$

where $\kappa = c_2/c_1$ is a dimensionless empirical constant. Note that in free shear boundary layer flows such as jets and wakes, Equation (99) develops a formal singularity where the second derivative vanishes, in particular at the inflection points of symmetric jet and wake profiles. The associated closure equation obtained by substituting the eddy viscosity into the boundary layer momentum equation may be rendered regular at such points by clearing

the second derivative to the numerator. The physical adequacy of the resulting closure at inflection points requires empirical assessment beyond the scope of the present work.

If the VKML is to be used in the extension of Prandtl's eddy viscosity to two-dimensional turbulent boundary layers (Equation (68)), then it remains to verify that (99) is valid when $v' = -cu'$. If $v' = -cu'$, then $f'(\xi, \eta)$ satisfies Equation (40); therefore,

$$\frac{\partial f'}{\partial \eta}\frac{\partial^2 f'}{\partial \xi \partial \eta} = \frac{\partial f'}{\partial \xi}\frac{\partial^2 f'}{\partial \eta^2}. \tag{100}$$

Equation (94) then reduces to

$$\eta\frac{\partial^2 f'}{\partial \xi \partial \eta} - \frac{\partial f'}{\partial \xi} - \ell\frac{\frac{\partial^2 \overline{u}}{\partial y^2}(x_0, y_0)}{\frac{\partial \overline{u}}{\partial y}(x_0, y_0)}\eta\frac{\partial f'}{\partial \xi} = 0. \tag{101}$$

However, by von Kármán's similarity hypothesis, Equation (101) must be independent of ℓ, $\frac{\partial \overline{u}}{\partial y}(x_0, y_0)$, and $\frac{\partial^2 \overline{u}}{\partial y^2}(x_0, y_0)$. Hence, it is necessary that

$$\ell\frac{\frac{\partial^2 \overline{u}}{\partial y^2}(x_0, y_0)}{\frac{\partial \overline{u}}{\partial y}(x_0, y_0)} = c_3, \tag{102}$$

where c_3 is a constant; therefore,

$$\ell = c_3\frac{\frac{\partial \overline{u}}{\partial y}(x_0, y_0)}{\frac{\partial^2 \overline{u}}{\partial y^2}(x_0, y_0)}. \tag{103}$$

Thus, the VKML (99) remains valid when $u' = -cv'$. The PDE (101) reduces to

$$\eta\frac{\partial^2 f'}{\partial \xi \partial \eta} - (1 + c_3\eta)\frac{\partial f'}{\partial \xi} = 0. \tag{104}$$

The stream function $f'(\xi, \eta)$ satisfies the system of differential equations in (40) and (104).

This extended VKML (99) applies not only to the two-dimensional turbulent boundary layer on a flat plate but also to jet and wake flows.

6. Application of Dimensional Analysis to Turbulent Mixing

Gersten and Herwig showed that von Kármán's mixing length can be derived using dimensional analysis [23].

The basic result in dimensional analysis is the Buckingham Π theorem [24]:

Theorem 1. *If a physical equation contains n physical quantities which depend on k independent dimensions where $n \geq k + 1$, then the equation can be rewritten in terms of $p = n - k$ dimensionless products $\Pi_1, \Pi_2, \ldots \Pi_p$ constructed from the original n quantities, as follows:*

$$F\left(\Pi_1, \Pi_2, \ldots, \Pi_p\right) = 0 \tag{105}$$

where F is an arbitrary function.

Dimensional analysis can be used to derive a range of mixing lengths. Consider first the three physical quantities: length ℓ, the spatial derivatives of the mean velocity $\mathrm{d}u/\mathrm{d}y$,

and $\mathrm{d}^2u/\mathrm{d}y^2$. It will be demonstrated that the resulting mixing length is von Kármán's mixing length. The dimensions of the three quantities are

$$[\ell] = L, \quad \left[\frac{\mathrm{d}u}{\mathrm{d}y}\right] = \frac{1}{T}, \quad \left[\frac{\mathrm{d}^2u}{\mathrm{d}y^2}\right] = \frac{1}{LT}. \tag{106}$$

There are two independent dimensions, the length L and time T; thus $n = 3$, $k = 2$, and $n = k + 1$. The number of dimensionless products is $p = n - k = 1$. One dimensionless product can be constructed from the original three physical quantities. The dimensionless product is of the form

$$\Pi = \ell^{\mathrm{a}}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^{\mathrm{b}}\left(\frac{\mathrm{d}^2u}{\mathrm{d}y^2}\right)^{\mathrm{c}}. \tag{107}$$

For Π to be dimensionless, it must satisfy

$$\left[\ell^{\mathrm{a}}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^{\mathrm{b}}\left(\frac{\mathrm{d}^2u}{\mathrm{d}y^2}\right)^{\mathrm{c}}\right] = L^0T^0. \tag{108}$$

Hence,

$$[\ell]^{\mathrm{a}}\left[\frac{\mathrm{d}u}{\mathrm{d}y}\right]^{\mathrm{b}}\left[\frac{\mathrm{d}^2u}{\mathrm{d}y^2}\right]^{\mathrm{c}} = L^0T^0. \tag{109}$$

Equation (109) is satisfied provided that

$$L^{\mathrm{a}}\frac{1}{T^{\mathrm{b}}}\frac{1}{L^cT^{\mathrm{c}}} = L^0T^0. \tag{110}$$

Equating the powers of L and T on the left-hand side of (110) to zero gives

$$L: \quad a - c = 0, \tag{111}$$

$$T: \quad b + c = 0, \tag{112}$$

which consists of two equations for three unknowns, a, b, and c. Unknowns b and c are solved in terms of a because an expression for the mixing length ℓ is required. In terms of a, the relationship is

$$b = -a \quad , \quad c = a, \tag{113}$$

which can be written in the form of column matrices as

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}. \tag{114}$$

The column matrix on the right-hand side provides a basis for the column matrix on the left-hand side. The basis consists of one column matrix. Therefore, there is only one dimensionless product which is in agreement with the Buckingham Π theorem. Since Π is given by (107), it follows that

$$\Pi_1 = \ell\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^{-1}\frac{\mathrm{d}^2u}{\mathrm{d}y^2}. \tag{115}$$

By the Buckingham Π theorem, the physical content of the original relation is captured by

$$F(\Pi_1) = 0, \tag{116}$$

where F is a function with a form that is not determined by dimensional analysis. This constraint fixes Π_1 to a single numerical value on every admissible realisation of the flow. Since Π_1 is the only dimensionless group formed from the chosen variables, any invertible

function of Π_1 is an equally valid dimensionless group generated by the same variable set, and each such choice yields an equivalent statement with a correspondingly transformed constant. Denoting the value of Π_1 by κ, we have

$$\ell\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^{-1}\frac{\mathrm{d}^2u}{\mathrm{d}y^2}=\kappa, \tag{117}$$

and so

$$\ell=\kappa\frac{\dfrac{\mathrm{d}u}{\mathrm{d}y}}{\dfrac{\mathrm{d}^2u}{\mathrm{d}y^2}}, \tag{118}$$

which is the VKML in which κ is a constant to be determined experimentally. A form for ℓ as a product of powers of $\mathrm{d}u/\mathrm{d}y$ and $\mathrm{d}^2u/\mathrm{d}y^2$ was not assumed. The form in (118) follows from the Buckingham Π theorem. It does not contain an arbitrary function; in general, when $n=k+1$, any one of the n physical quantities can be expressed as a product of the other $n-1$ physical quantities raised to the appropriate powers to give the correct dimensions.

Dimensional analysis can be used to derive mixing lengths involving higher spatial derivatives of the mean velocity u. Consider four physical quantities:

$$\ell,\quad \frac{\mathrm{d}u}{\mathrm{d}y},\quad \frac{\mathrm{d}^2u}{\mathrm{d}y^2},\quad \text{and}\quad \frac{\mathrm{d}^3u}{\mathrm{d}y^3}.$$

Then,

$$[\ell]=L,\quad \left[\frac{\mathrm{d}u}{\mathrm{d}y}\right]=\frac{1}{T},\quad \left[\frac{\mathrm{d}^2u}{\mathrm{d}y^2}\right]=\frac{1}{LT},\quad \left[\frac{\mathrm{d}^3u}{\mathrm{d}y^3}\right]=\frac{1}{L^2T}, \tag{119}$$

so $n=4$, $k=2$, and the number of dimensionless products $p=n-k=2$. Consider a product of the form

$$\Pi=\ell^{\mathrm{a}}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^b\left(\frac{\mathrm{d}^2u}{\mathrm{d}y^2}\right)^c\left(\frac{\mathrm{d}^3u}{\mathrm{d}y^3}\right)^d. \tag{120}$$

In order for Π to be dimensionless,

$$\left[\ell^{\mathrm{a}}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^b\left(\frac{\mathrm{d}^2u}{\mathrm{d}y^2}\right)^c\left(\frac{\mathrm{d}^3u}{\mathrm{d}y^3}\right)^d\right]=L^0T^0, \tag{121}$$

which is satisfied provided that

$$L^{\mathrm{a}}\frac{1}{T^b}\frac{1}{L^cT^c}\frac{1}{L^{2d}T^d}=L^0T^0. \tag{122}$$

Equating the exponents of L and T on the left-hand side of (122) to zero gives

$$L:\quad a-c-2d=0, \tag{123}$$

$$T:\quad b+c+d=0, \tag{124}$$

which consists of two equations for the four unknowns a, b, c, and d. Unknowns b and c are solved in terms of a and d, allowing us to determine how the mixing length in (118) is altered by the inclusion of the physical quantity $\mathrm{d}^3u/\mathrm{d}y^3$. It is found that

$$b=-a+d\quad,\quad c=a-2d, \tag{125}$$

which can be expressed in matrix form as

$$\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=a\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}+d\begin{bmatrix}0\\1\\-2\\1\end{bmatrix}. \tag{126}$$

The right-hand side forms a basis of two independent column matrices for the column matrix on the left-hand side. Therefore, there are two dimensionless products in agreement with the Buckingham Π theorem. Since the dimensionless products are of the form in (120), it follows from (126) that

$$\Pi_1 = \ell\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^{-1}\frac{\mathrm{d}^2u}{\mathrm{d}y^2}, \quad \Pi_2 = \frac{\mathrm{d}u}{\mathrm{d}y}\left(\frac{\mathrm{d}^2u}{\mathrm{d}y^2}\right)^{-2}\frac{\mathrm{d}^3u}{\mathrm{d}y^3}. \tag{127}$$

By the Buckingham Π theorem, the original physical equation can be written as

$$F(\Pi_1, \Pi_2) = 0, \tag{128}$$

or equivalently as

$$\Pi_1 = f(\Pi_2), \tag{129}$$

where f is an arbitrary function. Hence,

$$\ell = \frac{\dfrac{\mathrm{d}u}{\mathrm{d}y}}{\dfrac{\mathrm{d}^2u}{\mathrm{d}y^2}} f\left(\frac{\mathrm{d}u}{\mathrm{d}y}\left(\frac{\mathrm{d}^2u}{\mathrm{d}y^2}\right)^{-2}\frac{\mathrm{d}^3u}{\mathrm{d}y^3}\right). \tag{130}$$

Equation (130) shows how the von Kármán mixing length is altered by including the fourth physical quantity $\mathrm{d}^3u/\mathrm{d}y^3$. Since f is arbitrary, consider a special case where

$$f(\Pi_2) = \kappa\Pi_2, \tag{131}$$

where κ is a constant. Equation (130) becomes

$$\ell = \kappa\frac{\left(\dfrac{\mathrm{d}u}{\mathrm{d}y}\right)^2}{\left(\dfrac{\mathrm{d}^2u}{\mathrm{d}y^2}\right)^3}\frac{\mathrm{d}^3u}{\mathrm{d}y^3}, \tag{132}$$

where the right-hand side must be positive because ℓ is a length.

Von Kármán considered the first and second spatial derivatives of the mean velocity u. If a general case for any two arbitrary order spatial derivatives of u is considered, say r and s, then the physical quantities are

$$\ell, \quad \frac{\mathrm{d}^\mathrm{r}u}{\mathrm{d}y^\mathrm{r}}, \quad \frac{\mathrm{d}^\mathrm{s}u}{\mathrm{d}y^\mathrm{s}}, \tag{133}$$

where $n = 3$, $k = 2$, and $p = n - k = 1$. It can be verified that the dimensionless product is

$$\Pi_1 = \ell\left(\frac{\mathrm{d}^\mathrm{r}u}{\mathrm{d}y^\mathrm{r}}\right)^{\frac{1}{r-s}}\left(\frac{\mathrm{d}^\mathrm{s}u}{\mathrm{d}y^\mathrm{s}}\right)^{-\frac{1}{r-s}}, \tag{134}$$

and therefore

$$\ell = \kappa\left(\frac{\dfrac{\mathrm{d}^\mathrm{r}u}{\mathrm{d}y^\mathrm{r}}}{\dfrac{\mathrm{d}^\mathrm{s}u}{\mathrm{d}y^\mathrm{s}}}\right)^{\frac{1}{s-r}}, \tag{135}$$

where κ is a constant (not necessarily von Kármán's constant). When $r = 1$ and $s = 2$, the result is von Kármán's mixing length (118), which has one odd and one even derivative.

Special von Kármán-type mixing lengths with two odd derivatives when $r = 1$ and $s = 3$, or two even derivatives when $r = 2$ and $s = 4$, are

$$\ell = \kappa \left(\frac{\dfrac{\mathrm{d}u}{\mathrm{d}y}}{\dfrac{\mathrm{d}^3 u}{\mathrm{d}y^3}} \right)^{\frac{1}{2}} \quad \text{and} \quad \ell = \kappa \left(\frac{\dfrac{\mathrm{d}^2 u}{\mathrm{d}y^2}}{\dfrac{\mathrm{d}^4 u}{\mathrm{d}y^4}} \right)^{\frac{1}{2}}, \tag{136}$$

respectively. The higher derivative always remains in the denominator.

Dimensional analysis can be used to derive mixing lengths involving higher order spatial derivatives of the mean velocity u. Consider a general case including u and three arbitrary order spatial derivatives of u, say, r, s, and t, where $r \neq s \neq t$. The physical quantities are

$$\ell, \quad u, \quad \frac{\mathrm{d}^{\mathrm{r}} u}{\mathrm{d}y^{\mathrm{r}}}, \quad \frac{\mathrm{d}^{\mathrm{s}} u}{\mathrm{d}y^{\mathrm{s}}}, \quad \frac{\mathrm{d}^{\mathrm{t}} u}{\mathrm{d}y^{\mathrm{t}}}, \tag{137}$$

where $n = 5$, $k = 2$, and $p = n - k = 3$. These quantities have dimensions

$$[\ell] = L, \quad [u] = \frac{L}{T}, \quad \left[\frac{\mathrm{d}^{\mathrm{r}} u}{\mathrm{d}y^{\mathrm{r}}} \right] = \frac{1}{TL^{r-1}}, \quad \left[\frac{\mathrm{d}^{\mathrm{s}} u}{\mathrm{d}y^{\mathrm{s}}} \right] = \frac{1}{TL^{s-1}} \quad \left[\frac{\mathrm{d}^{\mathrm{t}} u}{\mathrm{d}y^{\mathrm{t}}} \right] = \frac{1}{TL^{t-1}} \tag{138}$$

and

$$\Pi = \ell^{\mathrm{a}} u^{\mathrm{b}} \left(\frac{\mathrm{d}^{\mathrm{r}} u}{\mathrm{d}y^{\mathrm{r}}} \right)^{\mathrm{c}} \left(\frac{\mathrm{d}^{\mathrm{s}} u}{\mathrm{d}y^{\mathrm{s}}} \right)^{\mathrm{d}} \left(\frac{\mathrm{d}^{\mathrm{t}} u}{\mathrm{d}y^{\mathrm{t}}} \right)^{\mathrm{e}}. \tag{139}$$

In order for Π to be dimensionless,

$$\left[\ell^{\mathrm{a}} u^{\mathrm{b}} \left(\frac{\mathrm{d}^{\mathrm{r}} u}{\mathrm{d}y^{\mathrm{r}}} \right)^{\mathrm{c}} \left(\frac{\mathrm{d}^{\mathrm{s}} u}{\mathrm{d}y^{\mathrm{s}}} \right)^{\mathrm{d}} \left(\frac{\mathrm{d}^{\mathrm{t}} u}{\mathrm{d}y^{\mathrm{t}}} \right)^{\mathrm{e}} \right] = L^0 T^0, \tag{140}$$

hence

$$L^{\mathrm{a}} \left(\frac{L}{T} \right)^{\mathrm{b}} \left(\frac{1}{TL^{r-1}} \right)^{\mathrm{c}} \left(\frac{1}{TL^{s-1}} \right)^{\mathrm{d}} \left(\frac{1}{TL^{t-1}} \right)^{\mathrm{e}} = L^0 T^0. \tag{141}$$

Equating the exponents of L and T on the left-hand side of (141) to zero gives

$$L: \quad a + b + (1 - r)c + (1 - s)d + (1 - t)e = 0, \tag{142}$$

$$T: \quad b + c + d + e = 0, \tag{143}$$

which when solving for b and c in terms of a, d, and e gives

$$b = \frac{(s - r)d + (t - r)e - a}{r}, \quad c = \frac{a - sd - te}{r}. \tag{144}$$

Expressed as column matrices, (144) is as follows:

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = a \begin{bmatrix} 1 \\ -\dfrac{1}{r} \\ \dfrac{1}{r} \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ \dfrac{s - r}{r} \\ -\dfrac{s}{r} \\ 1 \\ 0 \end{bmatrix} + e \begin{bmatrix} 0 \\ \dfrac{t - r}{r} \\ -\dfrac{t}{r} \\ 0 \\ 1 \end{bmatrix} \tag{145}$$

which verifies that there are three dimensionless products; hence,

$$\Pi_1 = \ell u^{-\frac{1}{r}} \left(\frac{\mathrm{d}^{\mathrm{r}} u}{\mathrm{d}y^{\mathrm{r}}} \right)^{\frac{1}{r}}, \tag{146}$$

$$\Pi_2 = u^{\frac{s-r}{r}} \left(\frac{\mathrm{d}^{\mathrm{r}} u}{\mathrm{d}y^{\mathrm{r}}}\right)^{\frac{-s}{r}} \left(\frac{\mathrm{d}^{\mathrm{s}} u}{\mathrm{d}y^{\mathrm{s}}}\right), \tag{147}$$

$$\Pi_3 = u^{\frac{t-r}{r}} \left(\frac{\mathrm{d}^{\mathrm{r}} u}{\mathrm{d}y^{\mathrm{r}}}\right)^{\frac{-t}{r}} \left(\frac{\mathrm{d}^{\mathrm{t}} u}{\mathrm{d}y^{\mathrm{t}}}\right) \tag{148}$$

and by the Buckingham Pi theorem,

$$\Pi_1 = G(\Pi_2, \Pi_3). \tag{149}$$

Choosing $G(X, Y) = KX^{\alpha}Y^{\beta}$, where K, α, and β are arbitrary constants, gives

$$\ell u^{-\frac{1}{r}} \left(\frac{\mathrm{d}^{\mathrm{r}} u}{\mathrm{d}y^{\mathrm{r}}}\right)^{\frac{1}{r}} = K \left(u^{\frac{s-r}{r}} \left(\frac{\mathrm{d}^{\mathrm{r}} u}{\mathrm{d}y^{\mathrm{r}}}\right)^{\frac{-s}{r}} \left(\frac{\mathrm{d}^{\mathrm{s}} u}{\mathrm{d}y^{\mathrm{s}}}\right) \right)^{\alpha} \left(u^{\frac{t-r}{r}} \left(\frac{\mathrm{d}^{\mathrm{r}} u}{\mathrm{d}y^{\mathrm{r}}}\right)^{\frac{-t}{r}} \left(\frac{\mathrm{d}^{\mathrm{t}} u}{\mathrm{d}y^{\mathrm{t}}}\right) \right)^{\beta}, \tag{150}$$

and therefore

$$\ell = K u^{\frac{1}{r}(1+\alpha(s-r)+\beta(t-r))} \left(\frac{\mathrm{d}^{\mathrm{r}} u}{\mathrm{d}y^{\mathrm{r}}}\right)^{-\frac{1}{r}(1+s\alpha+t\beta)} \left(\frac{\mathrm{d}^{\mathrm{s}} u}{\mathrm{d}y^{\mathrm{s}}}\right)^{\alpha} \left(\frac{\mathrm{d}^{\mathrm{t}} u}{\mathrm{d}y^{\mathrm{t}}}\right)^{\beta}. \tag{151}$$

The general result of Equation (151) has powerful consequences when determining mixing length forms. Setting $r = 1$, $s = 2$, $\alpha = -1$, and $\beta = 0$ returns von Kármán's result for the mixing length. By including higher-order derivatives, the mixing length captures the flow physics from the flow field rather than a local point, which is in agreement with comments made by Lighthill.

7. Conclusions

The mixing length depends on the parallel mean flow under investigation. In many theoretical and experimental studies, the mixing length is taken to either be constant or a power law of the distance along the mean flow. Von Kármán's mixing length is more general and is based on a similarity hypothesis. Although Lighthill has pointed out that Goldstein's investigations of turbulent flow between parallel planes and through pipes may indicate a flaw in the theory, the VKML merits further study because it applies to all parallel mean flows.

In his derivation of the mixing length, von Kármán started from the vorticity propagation equation. This eliminated the fluid pressure from the analysis and also highlighted the role of vorticity in the derivation. Since the fluctuations satisfy conservation of mass, a stream function for the fluctuations can be introduced which satisfies a third order partial differential equation. The mixing length was derived by imposing the similarity hypothesis on this partial differential equation, showing the central role played by the fluctuations in deriving the mixing length. Von Kármán did not require the assumption that the fluctuation v' in the y-direction is proportional to the fluctuation u' in the x-direction, that is, that $v' = -cu'$, which was made in the derivation of Prandtl's mixing length model for the eddy viscosity. Since the VKML is used in Prandtl's model for the eddy viscosity, it is important to verify that VKML remains valid when $v' = -cu'$. A system of two partial differential equations, one of third-order and the other of first-order, is obtained for the stream function for the fluctuations.

The physical interpretation by Betz provides a good illustration of the physical processes in deriving the VKML. The vortices used by Betz reflect the vorticity propagation equation used by von Kármán. Betz also provided a model for the source of the transverse fluctuations as the interaction of opposing vortices. He imposed von Kármán's similarity

hypothesis by insisting that the distance displaced by the turbulent transverse movements is on the same order as the distance between the centres of opposing rotating vortices, and that the fluctuation in the transverse direction is of the same order as that in the direction of the flow.

The extension of the VKML from parallel mean flow to two-dimensional boundary layer flow is significant because it includes free shear flows such as jet and wake flow as well as boundary layer flow past a flat plate. Since Prandtl's two-dimensional boundary layer equation with constant mainstream flow does not contain the pressure gradient term, it was not necessary to introduce the vorticity propagation equation, which simplifies the analysis. For this reason, the partial differential equation for the stream function of the fluctuations is only of second order. The VKML was extended to two-dimensional boundary layer flow by introducing characteristic scales and dimensionless quantities for the mean flow and for the fluctuations, and by making a boundary layer approximation neglecting terms of order ℓ/L, where ℓ is the order of the thickness of the boundary layer and L is the characteristic length in the direction of flow. The mixing length was obtained by imposing von Kármán's hypothesis on the second-order partial differential equation for the stream function for the fluctuations. It was important to again show that the extension of the derivation of the VKML to two-dimensional boundary layer flow remains valid when $v' = -cu'$. A system of two partial differential equations, now with one second-order and one first-order, was obtained for the stream function for the fluctuations.

Dimensional analysis with the Buckingham's Π theorem provides an efficient and systematic way to generate mixing lengths in mixing length theory, and also explains why mixing lengths can sometimes be in the form of power laws. Mixing lengths derived in this way need to be tested experimentally in order to determine whether they are physically sound and whether they give improved results in the centre axis or at the edge of a wake or jet. The arbitrary constant in the mixing length must also be determined experimentally, and may depend on the flow. A disadvantage is that, in general, the derivation does not explain the physical assumptions on which the result is based. In addition, the higher-order derivatives in the mixing length may require additional boundary conditions for the differential equation, which may not be immediately obvious.

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Abbreviations

The following abbreviations are used in this manuscript:

PML	Prandtl's Mixing Length
VKML	Von Kármán's Mixing Length
PDE	Partial Differential Equation

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