

Article

Rumor Propagation in Multilingual Environments and Stochastic Dynamics: A Modeling Study Based on Age Structure and Isolation Intervention

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Abstract

This paper investigates rumor dissemination dynamics with generalized nonlinear incidence in multilingual environments. We propose a novel stochastic modeling framework that integrates age structure, mandatory isolation mechanisms, and media intervention. Firstly, a deterministic model is constructed by coupling ordinary differential equations (ODEs) and partial differential equations (PDEs), followed by a thorough analysis of the positive invariant set of the model solutions. Secondly, we further establish a relevant stochastic differential system to account for environmental randomness in networks, and prove the existence and uniqueness of its global positive solution. Subsequently, using Itô's formula and the strong law of large numbers, several sufficient conditions for the disappearance of rumors are derived. Meanwhile, based on the Khasminskii method, the existence of a unique stationary distribution is analyzed under the continuous spread of rumors. Finally, the theoretical findings are verified through numerical simulations, which demonstrate that strengthening isolation intervention can markedly reduce the peak size of rumor spread and accelerate rumor extinction. Meanwhile, real case data from the Weibo platform further verify the favorable practical adaptability of the proposed model.

Keywords: generalized nonlinear incidence; multilingual rumor dissemination; age structure; rumor extinction; stochastic interference; stationary distribution

1. Introduction

With the sustained and rapid development of digital information technology, short-video platforms, news portals and diverse online social groups have been deeply integrated into people's daily communication, significantly lowering the barriers to information publication and sharing [1]. Although such open online communication ecosystems facilitate public information exchange, they also fuel the widespread dissemination of misinformation. Rampantly circulating rumors erode social trust and disrupt public order, creating a host of hidden hazards for cyberspace and social governance [2]. Against this backdrop, online rumors constitute an unignorable issue in modern social governance. To cultivate a transparent, credible online environment and sustain harmonious, stable social order, it is essential to conduct in-depth research on propagation mechanisms of negative information across social media platforms.

The modeling of rumor propagation can be traced back to 1964, when Daley and Kendall proposed the random rumor propagation model (DK Model) [3], which first classi-



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fied the population into three categories: susceptible, spreaders, and dropouts, laying the theoretical foundation for this field. In 1973, Maki and Thompson made key improvements to the DK model proposed in 1964, introducing the MT model. This model accurately describes the behavior of individuals who are aware of a rumor but no longer participate in its spread, further refining the dynamic characterization of rumor propagation and significantly enhancing the model's applicability to real-world scenarios [4]. A significant turning point in research occurred after 2001, with the development of complex network theory and a key breakthrough in rumor propagation modeling. Zanette's work is groundbreaking, as he first used complex networks to break through the assumption of uniform distribution of interpersonal relationships in traditional models [5]. On this basis, Moreno et al. systematically studied the propagation dynamics in small world networks and scale-free networks, revealing the strong dependence of rumor propagation on key nodes in the network [6]. However, the rapid development of social networks has given rise to many new characteristics, including the diversification of technology-driven communication scenarios, the innovation in information carriers, the emergence of super-node media, and the shifts in communication paradigms [7,8]. From private sharing among friends and family to public broadcasts to the general audience, and from interest-based community discussions to viral dissemination based on trending topics, multiple information dissemination patterns coexist and interweave within social networks. This phenomenon causes the originally singular rumor propagation path to evolve into a complex system interwoven with multiple modes [9–12]. Only by thoroughly analyzing the core principles of different communication patterns and constructing a multi-mode collaborative analytical framework can we effectively explain the complex mechanisms of rumor dissemination on new online platforms.

Research on rumor dissemination often used simple linear models [13–16]. Although these models can initially describe the unidirectional spreading process of rumors, they have obvious limitations. With the development of society and the increasingly complex communication environment, the academic community has gradually realized that linear models cannot effectively depict the true mechanism of rumor transmission. The dissemination of rumors in reality not only involves linear transmission from spreaders to receivers, but also involves complex processes such as multi-party interaction, environmental impact, and feedback loops. Reference [17] used a bilinear correlation function to analyze the steady-state distribution and probability density of the epidemic model; Fu et al. [18] studied the SVIQR model with saturation incidence rate; Liu et al. [19] established a SIVS model based on non-monotonic incidence rate. Although these specific forms of nonlinear propagation functions may highlight certain propagation characteristics of the model, it is difficult for them to fully reflect the comprehensive mechanisms of rumor dissemination. In contrast, generalized nonlinear generation functions can more comprehensively characterize the complexity of the propagation process, enhancing the authenticity and applicability of the model. Inspired by references [20–23], this paper adopts a generalized correlation function to model the rumor propagation process.

It is worth noting that although the above nonlinear models optimize the expression of transmission terms, they are all constructed under a single-language communication background and ignore the propagation differences caused by cross-linguistic interactions. Against the backdrop of globalization, linguistic and geographical barriers to information dissemination have gradually faded. The rise of cross-border population mobility and multilingual social platforms enables rumors to rapidly permeate diverse social groups and continuously mutate and evolve during cross-cultural communication. Multilingual rumors not only expand the scope of influence of rumor propagation but also pose a prominent threat to social stability in multi-ethnic and multilingual countries [24]. To thoroughly

explore the propagation mechanism of such rumors, Yu et al. constructed a heterogeneous rumor propagation model containing two types of language spreaders and introduced an impulsive isolation control strategy, proving that rumors can die out within a finite time under this control scheme [25]. Based on the reaction–diffusion theory, Xia et al. established a rumor propagation model suitable for multilingual environments and systematically analyzed the existence of equilibrium points, the rumor extinction conditions and global stability [26]. In addition, References [27,28] further investigated the dynamic characteristics of multilingual rumors from the perspectives of bilayer reaction–diffusion networks and discrete networks, respectively. Research on rumor propagation in multilingual environments can not only make up for the limitations of traditional single-language research perspectives but also accurately predict the propagation paths and risks of rumors in multilingual regions. It provides support for formulating targeted rumor governance strategies and offers scientific references for global public crisis response and international public opinion guidance.

However, although the aforementioned multilingual rumor propagation models fully incorporate multilingual communication characteristics, they are all built within a deterministic framework and neglect the stochastic interference factors inherent to real-world propagation processes. The structure of real social networks fluctuates dynamically, and user behaviors such as reposting and memory fading are inherently random. These persistent disturbances disrupt the evolution of rumor diffusion, rendering deterministic models unable to accurately characterize the actual laws of public opinion evolution. Against this backdrop, constructing stochastic dynamic systems by introducing Gaussian white noise has become a mainstream modeling approach that aligns with real communication scenarios. Relevant stochastic research has yielded fruitful outcomes to date: Huo et al. [29] developed a stochastic rumor model with a Holling Type II functional response, incorporated rumor-refuting strategies and solved suboptimal control conditions, verifying the significant impact of stochastic perturbations on rumor evolution; Zhang et al. [30] established a delayed stochastic model with generalized nonlinear incidence rates, analyzed the asymptotic behavior of the system via Lyapunov functions, and revealed the differentiated effects of noise, time delays and control parameters; Su et al. [31] built a stochastic SVI propagation system based on reaction–diffusion equations. On this basis, this paper constructs a stochastic rumor propagation model adapted to multilingual propagation scenarios, which effectively remedies the defects of traditional deterministic models and greatly improves the model’s applicability and universality in real and complex public opinion scenarios.

The impact of rumors has far exceeded that of false information itself, seriously threatening personal safety, social stability, and public interests. Therefore, it is necessary to control the spread of rumors in a timely manner and reduce their harm. Existing research has proposed various control methods, including optimal control based on mathematical models [32,33], event-triggered intermittent control [34,35], and non-stationary suppression mechanisms [36]. From the perspective of social intervention, there are also ways to suppress the spread of rumors, such as through public education [37,38], media coverage [39,40], anti-rumor campaigns [41], and competitive games [42].

In addition, substantial progress has been made in research on individual-level intervention among various control strategies. Fu et al. [43] constructed an SIQR optimal control model incorporating individual feedback and isolation interventions, and verified that the combination of isolation and vaccination can effectively block transmission, providing a theoretical basis for long-term epidemic prevention and control. Based on the Ornstein–Uhlenbeck process, Yang et al. [44] developed a stochastic epidemic dynamic model with an isolation mechanism to explore how isolation measures affect disease extinction and long-term steady states under stochastic fluctuations. Basir et al. [45] established

an extended optimal control epidemic model integrating public awareness and isolation prevention mechanisms, and analyzed the synergistic regulatory effects of isolation intervention and public perception on epidemic transmission. Although existing studies have fully confirmed that isolation intervention is an effective control method to curb transmission, they overlook the realistic dynamic characteristic that isolated individuals will gradually leave isolation over time. To address this research gap, Tong et al. introduced a duration control mechanism combined with age structure, optimized the model structure by setting refined isolation duration rules, and significantly improved the model's fitting accuracy for real control scenarios [46]. Therefore, introducing a dynamic isolation factor with duration constraints can further refine the dynamic characterization of isolation control and deepen the understanding of the internal mechanism through which isolation suppresses rumor propagation.

Online media serves as a vital channel to intervene in rumor propagation and guide public perception. Conventional viewpoints hold that authoritative information can mitigate the adverse impacts of rumors and standardize public communication behaviors [47]. Nevertheless, the popularization of diverse social platforms has made online media the primary carrier of information dissemination. Its coverage scale, publicity intensity and information credibility can all exert significant influences on the spread of rumors, while existing propagation models fail to elaborate on the dynamic impacts brought by online media. Accordingly, this paper incorporates the intervention factor of online media and explores the evolutionary laws of rumor propagation under its influence so as to provide theoretical references for the prevention and control of online public opinion.

A comprehensive review of existing literature reveals that the vast majority of current multilingual rumor propagation models adopt a deterministic framework and ignore the objective impact of stochastic environmental disturbances [24–27]. Existing stochastic rumor propagation and isolation intervention models rarely simultaneously take age-differentiated isolation mechanisms and the heterogeneous characteristics of multilingual dissemination into account [29–31,43–45]. Meanwhile, all age-structured propagation models fail to depict differentiated spreading laws and the effects of dynamic media intervention under multilingual scenarios [46]. To address the deficiencies in the above-mentioned studies, this paper constructs a stochastic age-structured dynamical model of rumor propagation with generalized nonlinear transmission rates applicable to multilingual environments. The main innovations of this work are summarized as follows:

- Current rumor propagation models [18,24] largely neglect the role of stochastic disturbances and generalized nonlinear mechanisms. Our work bridges this gap by integrating these critical factors, yielding a model that more faithfully represents online media environments and offers greater generality and practicality;
- By incorporating age structure and isolation mechanisms, this study proposes a multilingual rumor propagation model that couples ODEs and PDEs. This approach accurately depicts the restrictive controls imposed on spreaders during the dissemination process;
- The study conducts a systematic analysis of rumor propagation dynamics, focusing on two pivotal aspects. Sufficient conditions for rumor eradication are established using Itô's formula and the strong law of large numbers, while the criteria for rumor persistence are revealed via the Khasminskii method.

The organization of this research is outlined below: Section 2 establishes a stochastic age-structured rumor spreading model under a multilingual environment with generalized nonlinear transmission rates. Section 3 focuses on analyzing the existence of global solutions to this model, while Section 4 delves into sufficient conditions for rumor extinction. Section 5 studies the ergodic steady-state distribution of the stochastic model. To validate

the theoretical results, numerical simulations are conducted in Section 6. Section 7 provides a practical case to verify the practicality of the model. Section 8 provides some conclusions.

2. Model Construction

On online social platforms, users spread rumors in different languages according to their language habits. We divide them into two categories: class A users spread rumors on the platform in unofficial languages, while class B users spread rumors using only the platform's official language [22,23]. Furthermore, we categorize users into seven types based on the roles they play on the platform: rumor recognizer-a ($U_a(t)$) and rumor recognizer-b ($U_b(t)$) respectively represent users in class A and class B who are not yet aware of the rumor; rumor spreader-a ($I_a(t)$) and rumor spreader-b ($I_b(t)$) represent users in categories A and B who are spreading the rumor, respectively; rumor isolation individuals $G(t)$ and $Q(t)$ correspond to users who remain silent due to isolation in class A and class B, respectively; and $R(t)$ represents the recovery of individuals, that is, users who lose interest in the rumor and no longer spread it. The detailed rumor propagation process is illustrated in Figure 1, with the definitions of each model parameter summarized in Table 1.

Table 1. The meanings of the relevant parameters.

Symbol	Meaning
T_1	The recruitment rate of rumor recognizer-a
T_2	The recruitment rate of rumor recognizer-b
η_1	The transmission rate from rumor recognizer-a to spreader-a
η_2	The transmission rate from rumor recognizer-b to spreader-b
$\langle k \rangle$	Average degree of the network
u	The emigration rate of every group of individuals
ι_1	The probability that rumor spreader-a is isolated
ι_2	The probability that rumor spreader-b is isolated
π	The probability that rumor spreader-a transforms into rumor spreader-b
ζ_1	The recovery rate of rumor recognizer-a
ζ_2	The recovery rate of rumor recognizer-b
ε_1	The recovery probability of rumor spreader-a
ε_2	The recovery probability of rumor spreader-b
$l_i (i = 1, 2)$	The interference role of online media
$w(\theta)$	The rate of decline in isolation effectiveness in forcibly isolated individuals at isolation age θ
$v(\lambda)$	The rate at which isolation effectiveness declines for forcibly isolated individuals at isolation age λ
$G(\theta, t)$	The density of the forcibly isolated population at isolation age θ and time t
$Q(\lambda, t)$	The population density of forcibly isolated individuals with isolation age λ at time t
$\int_0^\infty G(\theta, t) d\theta$	The total number of isolated individuals of type A at time t
$\int_0^\infty Q(\lambda, t) d\lambda$	The total number of class B people isolated at time t

When rumor recognizers come into contact with rumor spreaders, the former may transition into disseminators with a certain probability, while some individuals may directly return to a non-spreading state due to a lack of interest in the rumor. Meanwhile, Class A spreaders are proficient in both official and unofficial languages, whereas Class B spreaders only use the official language. Accordingly, Class A propagators will convert to Class B when they aim to expand the reach of rumors to a broader audience. During the propagation process, certain spreaders are subject to monitoring by online regulatory authorities and may be placed under isolation measures that restrict their spreading behavior, rendering them isolated individuals. Given that users on online platforms often maintain multiple accounts, isolated individuals may cancel their original accounts and register new

ones. In addition, some disseminators may cease spreading due to diminished interest or forgetfulness, thereby becoming recovered individuals.

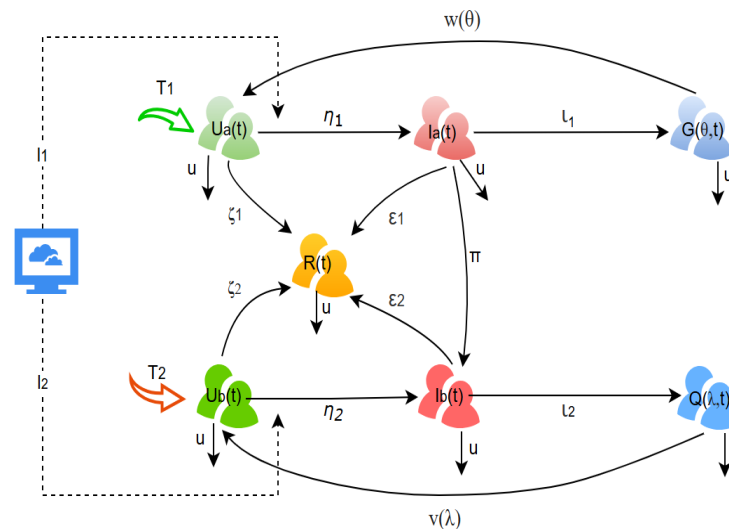


Figure 1. The state transition diagram.

Furthermore, online platforms disseminate a large volume of unvalidated trending information. To quantify the enhanced spreading effect induced by such content, we reformulate the relevant transition probabilities as follows:

$$\begin{cases} \eta_1 = \tilde{\eta}_1 + l_1, \\ \eta_2 = \tilde{\eta}_2 + l_2, \end{cases}$$

where $\tilde{\eta}_1, \tilde{\eta}_2$ represent the proportion of exposure that is not affected by online media interventions. In addition, the intervention effect of online media is given by $l_i = n_i - n_i e^{-gq}$ ($i = 1, 2$). It is worth noting that n_i corresponds to the information credibility released by online media. g indicates the intensity of online media reports, ranging from 0 to 1. And q denotes the number of online media outlets involved in information dissemination.

Based on the above propagation rules, we have developed a new rumor propagation model as follows:

$$\begin{cases} \frac{dU_a(t)}{dt} = T_1 - \eta_1 \langle k \rangle F_1(U_a(t)) H_1(I_a(t)) + \int_0^\infty w(\theta) G(\theta, t) d\theta - (u + \zeta_1) U_a(t), \\ \frac{dI_a(t)}{dt} = \eta_1 \langle k \rangle F_1(U_a(t)) H_1(I_a(t)) - (u + \iota_1 + \varepsilon_1 + \pi) I_a(t), \\ \frac{\partial G(\theta, t)}{\partial \theta} + \frac{\partial G(\theta, t)}{\partial t} = -(u + w(\theta)) G(\theta, t), \\ \frac{dU_b(t)}{dt} = T_2 - \eta_2 \langle k \rangle F_2(U_b(t)) H_2(I_b(t)) + \int_0^\infty v(\lambda) Q(\lambda, t) d\lambda - (u + \zeta_2) U_b(t), \\ \frac{dI_b(t)}{dt} = \eta_2 \langle k \rangle F_2(U_b(t)) H_2(I_b(t)) + \pi I_a(t) - (u + \iota_2 + \varepsilon_2) I_b(t), \\ \frac{\partial Q(\lambda, t)}{\partial \lambda} + \frac{\partial Q(\lambda, t)}{\partial t} = -(u + v(\lambda)) Q(\lambda, t), \\ \frac{dR(t)}{dt} = \zeta_1 U_a(t) + \zeta_2 U_b(t) + \varepsilon_1 I_a(t) + \varepsilon_2 I_b(t) - u R(t), \end{cases} \quad (1)$$

where $\langle k \rangle = \sum_{k=1}^n k p(k)$ denotes the average degree of individuals, $p(k)$ represents the probability that an individual has degree k , which satisfies $\sum_{k=1}^n p(k) = 1$.

Model (1) satisfies the following initial conditions

$$U_a(0) > 0, U_b(0) > 0, I_a(0) \geq 0, I_b(0) \geq 0, G(\theta, 0) \geq 0, Q(\lambda, 0) \geq 0, R(0) \geq 0. \quad (2)$$

Furthermore, model (1) operates under the boundary condition

$$G(0, t) = \iota_1 I_a(t), \quad Q(0, t) = \iota_2 I_b(t). \quad (3)$$

Here are some assumptions that will be used in subsequent analysis as follows.

(Δ1) $F(U)$ is a continuously differentiable and monotonically increasing function with $F(0) = 0$. Given a constant $\gamma > 0$, we define

$$d_\gamma = \inf_{0 < U \leq \gamma} \frac{F(U)}{U} < \infty, \quad D_\gamma = \sup_{0 < U \leq \gamma} \frac{F(U)}{U} < \infty.$$

(Δ2) $H(I)$ is a twice continuously differentiable function satisfying $H(0) = 0$ and $H'(0) > 0$, and $\frac{H(I)}{I}$ is monotonically decreasing on $\mathbb{R}_+$.

Remark 1. It is assumed that no individuals remain in the isolated compartment when the isolated ages θ and λ tend to infinity, that is,

$$\lim_{\theta \rightarrow \infty} G(\theta, t) = 0, \quad \lim_{\lambda \rightarrow \infty} Q(\lambda, t) = 0. \quad (4)$$

Lemma 1. For model (1), its positive invariant set is

$$\Gamma = \left\{ (U_a(t), I_a(t), G(t), U_b(t), I_b(t), Q(t), R(t)) \in \mathbb{R}_+^7 : 0 \leq U_a(t) + I_a(t) + U_b(t) + I_b(t) + \int_0^\infty G(\theta, t) d\theta + \int_0^\infty Q(\lambda, t) d\lambda + R(t) \leq \Gamma_0 \right\}.$$

Proof. Let $\Gamma(t) = U_a(t) + I_a(t) + \int_0^\infty G(\theta, t) d\theta + U_b(t) + I_b(t) + \int_0^\infty Q(\lambda, t) d\lambda + R(t)$, then

$$\begin{aligned} \frac{d\Gamma(t)}{dt} &= \frac{dU_a(t)}{dt} + \frac{dI_a(t)}{dt} + \frac{\partial \int_0^\infty G(\theta, t) d\theta}{\partial t} + \frac{dU_b(t)}{dt} + \frac{dI_b(t)}{dt} + \frac{\partial \int_0^\infty Q(\lambda, t) d\lambda}{\partial t} \\ &\quad + \frac{dR(t)}{dt} \\ &= T_1 + T_2 - u \left(U_a(t) + I_a(t) + \int_0^\infty G(\theta, t) d\theta + U_b(t) + I_b(t) + \int_0^\infty Q(\lambda, t) d\lambda + R(t) \right) \\ &\quad + u \int_0^\infty G(\theta, t) d\theta + u \int_0^\infty Q(\lambda, t) d\lambda - \iota_1 I_a(t) - \iota_2 I_b(t) \\ &\quad + \int_0^\infty w(\theta) G(\theta, t) d\theta + \int_0^\infty v(\lambda) Q(\lambda, t) d\lambda + \frac{\partial \int_0^\infty G(\theta, t) d\theta}{\partial t} + \frac{\partial \int_0^\infty Q(\lambda, t) d\lambda}{\partial t} \\ &= T_1 + T_2 - u\Gamma(t) - \iota_1 I_a(t) - \iota_2 I_b(t) + \int_0^\infty (u + w(\theta)) G(\theta, t) d\theta + \frac{\partial \int_0^\infty G(\theta, t) d\theta}{\partial t} \\ &\quad + \int_0^\infty (u + v(\lambda)) Q(\lambda, t) d\lambda + \frac{\partial \int_0^\infty Q(\lambda, t) d\lambda}{\partial t} \\ &= T_1 + T_2 - u\Gamma(t) - \iota_1 I_a(t) - \iota_2 I_b(t) - \int_0^\infty \frac{\partial G(\theta, t)}{\partial \theta} d\theta - \int_0^\infty \frac{\partial Q(\lambda, t)}{\partial \lambda} d\lambda \\ &= T_1 + T_2 - u\Gamma(t) - \iota_1 I_a(t) - \iota_2 I_b(t) - (G(\infty, t) - G(0, t)) - (Q(\infty, t) - Q(0, t)) \\ &= T_1 + T_2 - u\Gamma(t). \end{aligned}$$

Therefore, $\Gamma(t) = \frac{T_1+T_2}{u} - \left(\Gamma(0) - \frac{T_1+T_2}{u}\right)e^{-ut}$, then

$$\lim_{t \rightarrow \infty} \Gamma(t) \leq \Gamma_0 = \max\left\{\Gamma(0), \frac{T_1+T_2}{u}\right\}.$$

□

To overcome the difficulties arising from the partial differential equations in the theoretical analysis and solution of model (1), we adopt the research method proposed in reference [38,46] for subsequent analysis. Given a fixed time t , the third and sixth formulas in model (1) can be converted into the following ordinary differential equations:

$$\frac{dG(\theta)}{d\theta} = -(u + w(\theta))G(\theta), \quad (5)$$

$$\frac{dQ(\lambda)}{d\lambda} = -(u + v(\lambda))Q(\lambda). \quad (6)$$

The solutions are

$$G(\theta) = G(0)e^{-\int_0^\theta (u+w(\tau))d\tau}, Q(\lambda) = Q(0)e^{-\int_0^\lambda (u+v(m))dm}.$$

For notational convenience, let $\chi_0(\theta) = e^{-\int_0^\theta (u+w(\tau))d\tau}$, $\psi_0(\lambda) = e^{-\int_0^\lambda (u+v(m))dm}$. As the application of Volterra formulation, integrating the third equation in model (1) along the characteristic line $t - \theta = c$, where c is a constant (for more details, see Refs. [48,49]), we obtain

$$G(\theta, t) = \begin{cases} \iota_1 I_a(t - \theta)\chi_0(\theta), & t > \theta \geq 0, \\ G(\theta - t, 0) \frac{\chi_0(\theta)}{\chi_0(\theta - t)}, & \theta \geq t > 0. \end{cases}$$

In the case of $t \geq 0$, it holds that

$$\int_0^\infty w(\theta)G(\theta, t)d\theta = \int_0^t w(\theta)\iota_1 I_a(t - \theta)\chi_0(\theta)d\theta + \int_t^\infty w(\theta)G(\theta - t, 0) \frac{\chi_0(\theta)}{\chi_0(\theta - t)}d\theta, \quad (7)$$

among them

$$\int_t^\infty w(\theta)G(\theta - t, 0) \frac{\chi_0(\theta)}{\chi_0(\theta - t)}d\theta \leq e^{-ut} \int_t^\infty w(\theta)G(\theta - t, 0)d\theta.$$

Let $\tilde{\theta} = \theta - t$, then

$$e^{-ut} \int_t^\infty w(\theta)G(\theta - t, 0)d\theta \leq e^{-ut} \int_0^\infty G(\tilde{\theta}, 0)d\tilde{\theta} \leq e^{-ut}\Gamma_0 \rightarrow 0, \text{ as } t \rightarrow \infty.$$

Therefore, Equation (7) can be simplified to

$$\int_0^\infty w(\theta)G(\theta, t)d\theta = \int_0^\infty w(\theta)\iota_1 I_a(t - \theta)\chi_0(\theta)d\theta. \quad (8)$$

Similarly,

$$Q(\lambda, t) = \begin{cases} \iota_2 I_b(t - \lambda)\psi_0(\lambda), & t > \lambda \geq 0, \\ Q(\lambda - t, 0) \frac{\psi_0(\lambda)}{\psi_0(\lambda - t)}, & \lambda \geq t > 0. \end{cases}$$

For $t \geq 0$, utilizing similar analysis, it follows that

$$e^{-ut} \int_t^\infty v(\lambda)Q(\lambda - t, 0)d\lambda \leq e^{-ut} \int_0^\infty Q(\tilde{\lambda}, 0)d\tilde{\lambda} \leq e^{-ut}\Gamma_0 \rightarrow 0, \quad (9)$$

Therefore

$$\int_0^\infty v(\lambda)Q(\lambda, t)d\lambda = \int_0^\infty v(\lambda)\iota_2 I_b(t - \lambda)\psi_0(\lambda)d\lambda. \quad (10)$$

By (8) and (10), the first and third formulas of model (1) can be represented as

$$\frac{dU_a(t)}{dt} = T_1 - \eta_1 \langle k \rangle F_1(U_a(t))H_1(I_a(t)) + \int_0^\infty I_a(t - \theta)\chi(\theta)d\theta - (u + \varsigma_1)U_a(t), \quad (11)$$

$$\frac{dU_b(t)}{dt} = T_2 - \eta_2 \langle k \rangle F_2(U_b(t))H_2(I_b(t)) + \int_0^\infty I_b(t - \lambda)\psi(\lambda)d\lambda - (u + \varsigma_2)U_b(t), \quad (12)$$

where $\chi(\theta) = w(\theta)\iota_1\chi_0(\theta)$, $\psi(\lambda) = v(\lambda)\iota_2\psi_0(\lambda)$.

Denoting $\chi = \int_0^\infty \chi(\theta)d\theta$, it has

$$\begin{aligned} \chi &= \int_0^\infty w(\theta)\iota_1\chi_0(\theta)d\theta \\ &= \int_0^\infty w(\theta)\iota_1 e^{-\int_0^\theta (u+w(\tau))d\tau}d\theta \\ &\leq \int_0^\infty \iota_1 e^{-u\theta}d\theta = \frac{\iota_1}{u}, \end{aligned}$$

thus,

$$\int_0^\infty w(\theta)\iota_1 I_a(t - \theta)\chi_0(\theta)d\theta \leq \chi\Gamma_0 \leq \frac{\iota_1}{u}\Gamma_0. \quad (13)$$

Similarly,

$$\int_0^\infty v(\lambda)\iota_2 I_b(t - \lambda)\psi_0(\lambda)d\lambda \leq \frac{\iota_2}{u}\Gamma_0. \quad (14)$$

Through the above analysis, model (1) can be transformed into

$$\begin{cases} \frac{dU_a(t)}{dt} = T_1 - \eta_1 \langle k \rangle F_1(U_a(t))H_1(I_a(t)) + \int_0^\infty I_a(t - \theta)\chi(\theta)d\theta - (u + \varsigma_1)U_a(t), \\ \frac{dI_a(t)}{dt} = \eta_1 \langle k \rangle F_1(U_a(t))H_1(I_a(t)) - (u + \iota_1 + \varepsilon_1 + \pi)I_a(t), \\ \frac{dU_b(t)}{dt} = T_2 - \eta_2 \langle k \rangle F_2(U_b(t))H_2(I_b(t)) + \int_0^\infty I_b(t - \lambda)\psi(\lambda)d\lambda - (u + \varsigma_2)U_b(t), \\ \frac{dI_b(t)}{dt} = \eta_2 \langle k \rangle F_2(U_b(t))H_2(I_b(t)) + \pi I_a(t) - (u + \iota_2 + \varepsilon_2)I_b(t), \\ \frac{dR(t)}{dt} = \varsigma_1 U_a(t) + \varsigma_2 U_b(t) + \varepsilon_1 I_a(t) + \varepsilon_2 I_b(t) - uR(t). \end{cases} \quad (15)$$

Since the states of ignorant individuals, spreaders, and isolated individuals are independent of that of recovered individuals, model (15) can be simplified to

$$\begin{cases} \frac{dU_a(t)}{dt} = T_1 - \eta_1 \langle k \rangle F_1(U_a(t))H_1(I_a(t)) + \int_0^\infty I_a(t - \theta)\chi(\theta)d\theta - (u + \varsigma_1)U_a(t), \\ \frac{dI_a(t)}{dt} = \eta_1 \langle k \rangle F_1(U_a(t))H_1(I_a(t)) - (u + \iota_1 + \varepsilon_1 + \pi)I_a(t), \\ \frac{dU_b(t)}{dt} = T_2 - \eta_2 \langle k \rangle F_2(U_b(t))H_2(I_b(t)) + \int_0^\infty I_b(t - \lambda)\psi(\lambda)d\lambda - (u + \varsigma_2)U_b(t), \\ \frac{dI_b(t)}{dt} = \eta_2 \langle k \rangle F_2(U_b(t))H_2(I_b(t)) + \pi I_a(t) - (u + \iota_2 + \varepsilon_2)I_b(t). \end{cases} \quad (16)$$

To analyze the impact of environmental random fluctuations on the dynamics of rumor propagation, we consider the influence coefficient η_i ($i = 1, 2$) to be continuously disturbed by white noise. Specifically, assuming that each η_i fluctuates around its average value, it is modeled as $\eta_i + \sigma_j dB_j(t)$, where $j = 1, 2$ correspond to different noise sources.

On this basis, the original deterministic model (16) can be extended to the following form of stochastic differential equations:

$$\left\{ \begin{array}{l} dU_a(t) = \left(T_1 - \eta_1 \langle k \rangle F_1(U_a(t)) H_1(I_a(t)) + \int_0^\infty I_a(t - \theta) \chi(\theta) d\theta - (u + \varsigma_1) U_a(t) \right) dt \\ \quad - \sigma_1 \langle k \rangle F_1(U_a(t)) H_1(I_a(t)) dB_1(t), \\ dI_a(t) = \left(\eta_1 \langle k \rangle F_1(U_a(t)) H_1(I_a(t)) - (u + \iota_1 + \varepsilon_1 + \pi) I_a(t) \right) dt \\ \quad + \sigma_1 \langle k \rangle F_1(U_a(t)) H_1(I_a(t)) dB_1(t), \\ dU_b(t) = \left(T_2 - \eta_2 \langle k \rangle F_2(U_b(t)) H_2(I_b(t)) + \int_0^\infty I_b(t - \lambda) \psi(\lambda) d\lambda - (u + \varsigma_2) U_b(t) \right) dt \\ \quad - \sigma_2 \langle k \rangle F_2(U_b(t)) H_2(I_b(t)) dB_2(t), \\ dI_b(t) = \left(\eta_2 \langle k \rangle F_2(U_b(t)) H_2(I_b(t)) + \pi I_a(t) - (u + \iota_2 + \varepsilon_2) I_b(t) \right) dt \\ \quad + \sigma_2 \langle k \rangle F_2(U_b(t)) H_2(I_b(t)) dB_2(t). \end{array} \right. \quad (17)$$

The process $B(t)$ is a standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$. Herein, Ω denotes the sample space; $\mathcal{F}$ is defined as the σ -algebra of events; the sequence $\{\mathcal{F}_t\}_{t \geq 0}$ refers to a right-continuous filtration with $\mathcal{F}_0$ containing all P -null sets; and P represents the probability measure governing the system. In addition, σ_j^2 ($j = 1, 2$) denotes the magnitude of the white noise intensity.

3. Existence and Uniqueness of the Global Positive Solution

Theorem 1. For any given initial value $(U_a(0), I_a(0), U_b(0), I_b(0)) \in \mathbb{R}_4^+$, model (17) has a unique solution $(U_a(t), I_a(t), U_b(t), I_b(t))$ remaining in $\mathbb{R}_4^+$ with probability one for all $t > 0$.

Proof. It is evident that all coefficients in model (17) satisfy the local Lipschitz condition. Therefore, for any initial condition $(U_a(0), I_a(0), U_b(0), I_b(0)) \in \mathbb{R}_4^+$, there exists a unique local solution to model (17) when $t \in [0, \rho_e]$, where ρ_e denotes the explosion time. By virtue of Itô's formula, it follows that the unique local solution of model (17) is positive. Next, we proceed to prove that this solution is global, i.e., $\rho_e = \infty$ almost surely.

Let m_0 be sufficiently large such that $U_a(0), I_a(0), U_b(0), I_b(0)$ lie within $[1/m_0, m_0]$. For any given $m > m_0$, define the stopping time as follows

$$\rho_m = \inf \left\{ t \in [0, \rho_e] : U_a(t) \notin \left(\frac{1}{m}, m \right) \text{ or } I_a(t) \notin \left(\frac{1}{m}, m \right) \text{ or } U_b(t) \notin \left(\frac{1}{m}, m \right) \right. \\ \left. \text{or } I_b(t) \notin \left(\frac{1}{m}, m \right) \right\}.$$

Set $\inf \emptyset = \infty$, where $\emptyset$ stands for an empty set. As $m \rightarrow \infty$, ρ_m is increasing. Denote $\rho_\infty = \lim_{m \rightarrow \infty} \rho_m$, then $\rho_\infty \leq \rho_e$ a.s.

Next, we need to prove that $\rho_\infty = \infty$ a.s. We proceed by contradiction. If not, then there exists a constant $K > 0$ such that for any $\omega \in (0, 1)$, $\mathbb{P}\{\rho_\infty \leq K\} \geq \omega$. Therefore, there exists a positive number $m_1 \geq m_0$ such that when $m \geq m_1$,

$$\mathbb{P}\{\rho_m \leq K\} \geq \omega. \quad (18)$$

The following function $V : \mathbb{R}_4^+ \rightarrow \mathbb{R}^+$ is defined by

$$V(U_a(t), I_a(t), U_b(t), I_b(t)) = U_a(t) - c_1 - c_1 \ln \frac{U_a(t)}{c_1} + I_a(t) - 1 - \ln I_a(t) \\ + U_b(t) - c_2 - c_2 \ln \frac{U_b(t)}{c_2} + I_b(t) - 1 - \ln I_b(t).$$

Clearly, V is a non-negative function, where c_1, c_2 denote a positive constant. By virtue of Itô's formula, we derive

$$\begin{aligned} dV(U_a(t), I_a(t), U_b(t), I_b(t)) = & \mathcal{L}V(U_a(t), I_a(t), U_b(t), I_b(t))dt \\ & - \left(1 - \frac{c_1}{U_a(t)}\right) \sigma_1 \langle k \rangle F_1(U_a(t)) H_1(I_a(t)) dB_1(t) \\ & + \left(1 - \frac{1}{I_a(t)}\right) \sigma_1 \langle k \rangle F_1(U_a(t)) H_1(I_a(t)) dB_1(t) \\ & - \left(1 - \frac{c_2}{U_b(t)}\right) \sigma_2 \langle k \rangle F_2(U_b(t)) H_2(I_b(t)) dB_2(t) \\ & + \left(1 - \frac{1}{I_b(t)}\right) \sigma_2 \langle k \rangle F_2(U_b(t)) H_2(I_b(t)) dB_2(t), \end{aligned} \quad (19)$$

Choose

$$c_1 = \frac{u + \iota_1 + \varepsilon_1}{\eta_1 \langle k \rangle D_{\Gamma_0} H'(0)}, \quad c_2 = \frac{u + \iota_2 + \varepsilon_2}{\eta_2 \langle k \rangle D_{\Gamma_0} H'(0)},$$

Under this choice, we can bound the generator uniformly by a constant W . The full tedious algebraic expansion of the generator $\mathcal{L}V$, detailed simplification processes and bounding estimates required to obtain this uniform bound are provided in Appendix A. Substituting this uniform bound into the above stochastic differential form yields

$$\begin{aligned} dV \leq & Wdt + \left(\frac{c_1}{U_a(t)} - \frac{1}{I_a(t)}\right) \sigma_1 \langle k \rangle F_1(U_a(t)) H_1(I_a(t)) dB_1(t) \\ & + \left(\frac{c_2}{U_b(t)} - \frac{1}{I_b(t)}\right) \sigma_2 \langle k \rangle \\ & \times F_2(U_b(t)) H_2(I_b(t)) dB_2(t). \end{aligned}$$

Integrating both sides from 0 to $\rho_m \wedge K$ and taking the mathematical expectation, we obtain

$$\begin{aligned} E[V(U_a(\rho_m \wedge K), I_a(\rho_m \wedge K), U_b(\rho_m \wedge K), I_b(\rho_m \wedge K))] \leq & V(U_a(0), I_a(0), U_b(0), I_b(0)) \\ & + WK. \end{aligned} \quad (20)$$

When $m \geq m_1$, we denote $\Omega_m = \{\rho_m \leq K\}$. From (18), we have $\mathbb{P}(\Omega_m) \geq \omega$. Notice that for any $\omega \in \Omega_m$, at least one component reaches the boundary m or $1/m$, and hence

$$V(U_a(\rho_m, \omega), I_a(\rho_m, \omega), U_b(\rho_m, \omega), I_b(\rho_m, \omega)) \geq (m - 1 - \ln m) \wedge \left(\frac{1}{m} - 1 + \ln m\right).$$

Taking the expectation over Ω_m and letting $m \rightarrow \infty$ leads to a contradiction. Thus $\rho_\infty = \infty$ almost surely, which completes the proof of global existence and uniqueness. $\square$

4. Extinction of the Rumor

This section will delve into the sufficient conditions for the disappearance of rumors. These thresholds and criteria can predict public opinion trends and define the effective scope of control measures, providing quantitative support for multilingual online rumor governance and mitigating relevant social risks.

Lemma 2 ([50]). *Let M_t be a real-valued continuous local martingale with $M_0 = 0$. If $\lim_{t \rightarrow \infty} \frac{\langle M, M \rangle_t}{t} < \infty$ a.s., then $\lim_{t \rightarrow \infty} \frac{M(t)}{t} = 0$, a.s.*

Theorem 2. *Let $(U_a(t), I_a(t), U_b(t), I_b(t))$ be the solution of model (17) with any initial value $(U_a(0), I_a(0), U_b(0), I_b(0)) \in R_4^+$. The following conclusions are obtained:*

(i) If

$$\frac{\eta_1^2}{\sigma_1^2} + \frac{\eta_2^2}{\sigma_2^2} < 2 \min\{u + \iota_1 + \varepsilon_1, u + \iota_2 + \varepsilon_2\},$$

then

$$\limsup_{t \rightarrow \infty} \frac{\ln(I_a(t) + I_b(t))}{t} \leq \frac{1}{2} \left(\frac{\eta_1^2}{\sigma_1^2} + \frac{\eta_2^2}{\sigma_2^2} \right) - \min\{u + \iota_1 + \varepsilon_1, u + \iota_2 + \varepsilon_2\} < 0.$$

(ii) If

$$R_0^e = \frac{H}{\min\{u + \iota_1 + \varepsilon_1, u + \iota_2 + \varepsilon_2\}} - \frac{J}{2 \min\{u + \iota_1 + \varepsilon_1, u + \iota_2 + \varepsilon_2\}} < 1,$$

and

$$\sigma_1^2 \langle k \rangle < \frac{\eta_1}{F_1(\Gamma_0)H'(0)}, \quad \sigma_2^2 \langle k \rangle < \frac{\eta_2}{F_2(\Gamma_0)H'(0)},$$

then

$$\limsup_{t \rightarrow \infty} \frac{\ln(I_a(t) + I_b(t))}{t} < 0,$$

$$\text{where } H = \sum_{i=1}^2 \eta_i \langle k \rangle F_i(\Gamma_0)H'(0), \quad J = \sum_{i=1}^2 \sigma_i^2 \langle k \rangle^2 (F_i(\Gamma_0)H'(0))^2.$$

Proof. Letting $V_0 = I_a(t) + I_b(t)$, utilizing Itô's formula, we derive the differential form of $\ln V_0$ as shown below:

$$\begin{aligned} d \ln V_0 = & \frac{1}{I_a(t) + I_b(t)} \left(\eta_1 \langle k \rangle F_1(U_a(t)) H_1(I_a(t)) - (u + \iota_1 + \varepsilon_1 + \pi) I_a(t) + \eta_2 \langle k \rangle F_2(U_b(t)) \right. \\ & \times H_2(I_b(t)) + \pi I_a(t) - (u + \iota_2 + \varepsilon_2) I_b(t) \Big) dt - \frac{1}{2(I_a(t) + I_b(t))^2} \left((\sigma_1 \langle k \rangle F_1(U_a(t)) \right. \\ & \times H_1(I_a(t)))^2 + (\sigma_2 \langle k \rangle F_2(U_b(t)) H_2(I_b(t)))^2 \Big) dt + \frac{1}{I_a(t) + I_b(t)} \left(\sigma_1 \langle k \rangle F_1(U_a(t)) \right. \\ & \times H_1(I_a(t)) dB_1(t) + \sigma_2 \langle k \rangle F_2(U_b(t)) H_2(I_b(t)) dB_2(t) \Big). \end{aligned}$$

Integrating and dividing by time t , we get an upper bound of $\frac{\ln(I_a + I_b)}{t}$. The quadratic variation of local martingales $M_1(t), M_2(t)$ is uniformly bounded, so $\frac{M_i(t)}{t} \rightarrow 0$ as $t \rightarrow \infty$ by Lemma 2. Define two auxiliary functions:

$$\begin{aligned} \Xi_1(x) &= -\frac{\sigma_1^2 \langle k \rangle^2}{2} x^2 + (\eta_1 \langle k \rangle + \epsilon)x - \min\{u + \iota_1 + \varepsilon_1, u + \iota_2 + \varepsilon_2\}, \\ \Xi_2(x) &= -\frac{\sigma_2^2 \langle k \rangle^2}{2} x^2 + (\eta_2 \langle k \rangle + \epsilon)x, \end{aligned}$$

The cumbersome integral expansion, extremum computation of the above auxiliary functions and multi-layer inequality deductions required for this theorem are all supplied in Appendix B. The maximum values of $\Xi_1(x)$ and $\Xi_2(x)$ are calculated therein. Taking $\epsilon \rightarrow 0$ yields the criterion (i). For condition (ii), we utilize the monotonicity of $\Xi_i(x)$ and the bound $F_i(\Gamma_0)H'(0)$. Complete piecewise estimates involving threshold T_0 are also provided in Appendix B, which ultimately shows that $R_0^e < 1$ leads to rumor extinction. The proof is complete. $\square$

5. Stationary Distribution of the Rumor

In this section, we shall investigate the stochastic persistence of model (17). The stationary distribution intuitively reflects the long-term scale of rumor spreaders among multilingual groups, predicts long-term public opinion risks, and offers quantitative references for regular supervision and multilingual rumor refutation strategies. First, we present some useful lemmas, which will facilitate our derivation of the main results. It should be clarified that the stochastic model (17) has no equilibrium state. Hence, there arises a compelling need to examine whether this model admits a stationary distribution, that is, the phenomenon of stochastic persistence.

Let $X(t)$ be a homogeneous Markov process in E_d (E_d denotes a d -dimensional Euclidean space), described by the following equation:

$$dX(t) = b(X)dt + \sum_{r=1}^k g_r(X)dB_r(t).$$

The diffusion matrix of $X(t)$ is defined as follows

$$\tilde{A}(X) = (a_{ij}(X)), \quad a_{ij}(X) = \sum_{r=1}^k g_r^i(X)g_r^j(X).$$

Note that

$$R_0^s = \frac{T_1 T_2 (u + \varsigma_1)(u + \varsigma_2)u}{\left(\eta_1 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + u + \varsigma_1 + \frac{1}{2} \sigma_1^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 \right) \iota_2 \Gamma_0} \times \frac{1}{(T_1 + \frac{\iota_1 \Gamma_0}{u}) \left(\eta_2 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + u + \varsigma_2 + \frac{1}{2} \sigma_2^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 \right)} > 1,$$

and inequality is

$$\delta \langle k \rangle^2 < \frac{2u}{\sigma_1^2 D_{\Gamma_0}^2 (H'(0))^2 \Gamma_0^2 \vee \sigma_1^2 (F_1(\Gamma_0) H'(0))^2 \vee \sigma_2^2 D_{\Gamma_0}^2 (H'(0))^2 \Gamma_0^2 \vee \sigma_2^2 (F_2(\Gamma_0) H'(0))^2}. \quad (21)$$

Lemma 3 ([51]). Suppose there exists a bounded domain $\mathbb{D} \subset \mathbb{E}_d$ with a regular boundary Θ , and the Markov process $X(t)$ satisfies the following two conditions:

- (i) There exists a positive constant $\tilde{M}$ such that $\sum_{i,j=1}^d a_{ij} \zeta_i \zeta_j \geq \tilde{M} |\zeta|^2$, $x \in \mathbb{D}$, $\zeta \in \mathbb{E}_d$;
- (ii) There exists a non-negative C^2 -function V such that $\mathcal{L}V$ is negative for any $\mathbb{E}_d \setminus \mathbb{D}$. Then, for every $x \in \mathbb{E}_d$, we have

$$\mathbb{P}_x \left\{ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T h(X(t)) dt = \int_{\mathbb{E}_d} h(x) \pi(dx) \right\} = 1,$$

where $h(\cdot)$ is an integrable function with respect to the measure π .

Theorem 3. If $R_0^s < 1$, and the inequality (18) holds, there exists a unique ergodic stationary distribution within model (17).

Proof. It suffices to verify conditions (i) and (ii) in Lemma 3. The proof is carried out in two steps.

Step 1. We construct the diffusion matrix and derive its uniform ellipticity on bounded closed sets, which verifies the first condition of Lemma 3.

$$\tilde{A} = \begin{pmatrix} A_1 & 0 & 0 & \\ 0 & A_1 & 0 & \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{pmatrix},$$

where $A_1 = \sigma_1^2 \langle k \rangle^2 (F_1(U_a)H_1(I_a))^2$, $A_2 = \sigma_2^2 \langle k \rangle^2 (F_2(U_b)H_2(I_b))^2$. A positive lower bound $\tilde{M}$ exists such that the quadratic form of the diffusion matrix is bounded below by $\tilde{M}\|\zeta\|^2$ over $\overline{D}_k$.

Step 2. We construct the composite C^2 Lyapunov function $\tilde{V} = \Theta V_1 + V_2 + V_3$, with properly chosen constants Θ, c_1, c_2, c_3 to satisfy the required negative drift condition outside a compact set. All lengthy generator expansions, piecewise domain estimates and multi-case bounding computations involved in this stationary distribution verification, together with the full expansion of generator $\mathcal{L}\tilde{V}$ and eight subdomain case analysis, are provided in Appendix C. After verifying $\mathcal{L}\tilde{V} \leq -1$ on $\mathbb{R}_4^+ \setminus D_\epsilon$, the second condition of Lemma 3 holds. In summary, all assumptions of Lemma 3 are satisfied, so the model admits a unique ergodic stationary distribution. $\square$

6. Numerical Simulations

In this section, we will use mathematical tools to conduct numerical simulations to verify the validity of the theoretical results. For the convenience of research, we assume that the generalized nonlinear incidence rate is a special case of the linear incidence rate, i.e., $F_1(U_a(t)) = U_a(t)$, $H_1(I_a(t)) = I_a(t)$, $F_2(U_b(t)) = U_b(t)$, $H_2(I_b(t)) = I_b(t)$.

First, based on the discrete methods in References [27,28], we discretize the stochastic differential equation with age structure, i.e., model (17), as follows:

$$\begin{aligned} U_a(j+1) &= U_a(j) + \left[T_1 - \eta_1 \langle k \rangle U_a(j) I_a(j) + D_1 - (u + \varsigma_1) U_a(j) \right] \Delta t - \sigma_1 \langle k \rangle U_a(j) I_a(j) \\ &\quad \times \sqrt{\Delta t} \xi_{1j} - \frac{\sigma_1^2}{2} U_a(j) I_a(j) (\xi_{1j}^2 - 1) \Delta t, \\ I_a(j+1) &= I_a(j) + \left[\eta_1 \langle k \rangle U_a(j) I_a(j) - (u + \iota_1 + \varepsilon_1 + \pi) I_a(j) \right] \Delta t + \sigma_1 \langle k \rangle U_a(j) I_a(j) \\ &\quad \times \sqrt{\Delta t} \xi_{1j} + \frac{\sigma_1^2}{2} U_a(j) I_a(j) (\xi_{1j}^2 - 1) \Delta t, \\ U_b(j+1) &= U_b(j) + \left[T_2 - \eta_2 \langle k \rangle U_b(j) I_b(j) + D_2 - (u + \varsigma_2) U_b(j) \right] \Delta t - \sigma_2 \langle k \rangle U_b(j) I_b(j) \\ &\quad \times \sqrt{\Delta t} \xi_{2j} - \frac{\sigma_2^2}{2} U_b(j) I_b(j) (\xi_{2j}^2 - 1) \Delta t, \\ I_b(j+1) &= I_b(j) + \left[\eta_2 \langle k \rangle U_b(j) I_b(j) + \pi I_a(j) - (u + \iota_2 + \varepsilon_2) I_b(j) \right] \Delta t + \sigma_2 \langle k \rangle U_b(j) I_b(j) \\ &\quad \times \sqrt{\Delta t} \xi_{2j} + \frac{\sigma_2^2}{2} U_b(j) I_b(j) (\xi_{2j}^2 - 1) \Delta t, \end{aligned}$$

where $D_1 = \int_0^\infty I_a(t - \theta) \chi(\theta) d\theta$, $D_2 = \int_0^\infty I_b(t - \lambda) \psi(\lambda) d\lambda$, ξ_{1j} and ξ_{2j} are independent random variables that follow the standard normal distribution $N(0, 1)$.

6.1. Sensitivity Analysis

In this section, the normalized local sensitivity index method is adopted to conduct local sensitivity analysis on the stochastic persistence threshold $\mathcal{R}_0^s$ in the stochastic rumor propagation model. The normalized sensitivity index is defined as

$$Y_W^{\mathcal{R}_0^s} = \frac{\partial \mathcal{R}_0^s}{\partial W} \cdot \frac{W}{\mathcal{R}_0^s}$$

where W denotes any parameter to be analyzed in the model. A larger absolute value of $Y_W^{\mathcal{R}_0^s}$ indicates a stronger regulatory effect of the corresponding parameter on the threshold $\mathcal{R}_0^s$. A positive index implies a positive correlation between the parameter and $\mathcal{R}_0^s$, while a negative index represents a negative correlation. As shown in Figure 2, the sensitivity indices of parameters $T_1, T_2, u, \varsigma_1, \varsigma_2$ satisfy $Y > 0$, which are positively correlated with $\mathcal{R}_0^s$. In contrast, the indices of $\eta_1, \eta_2, \langle k \rangle, \iota_1, \iota_2, \sigma_1, \sigma_2$ satisfy $Y < 0$, showing negative correlations with $\mathcal{R}_0^s$. Among all parameters, u possesses the maximum absolute value of positive sensitivity, and η_2 has the largest absolute value of negative sensitivity, which are the core parameters dominating the stochastic persistence threshold $\mathcal{R}_0^s$.

From the perspective of public opinion control, increasing the positive sensitive parameters will raise $\mathcal{R}_0^s$ and facilitate the long-term persistence of rumors. Conversely, raising the values of negative sensitive parameters can effectively reduce the stochastic persistence threshold $\mathcal{R}_0^s$ and accelerate the dissipation of rumors. In particular, improving the transmission coefficient η_2 , isolation rate ι_2 and network average degree $\langle k \rangle$ achieves the most remarkable regulatory effect on restraining rumor spread.

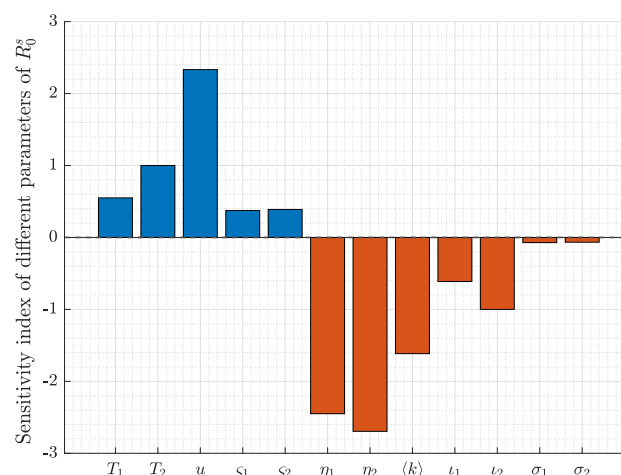


Figure 2. The fitting of model (17) with real data.

6.2. Extinction of Rumor

Take $T_1 = 0.3$, $T_2 = 0.25$, $\tilde{\eta}_1 = 0.3$, $\tilde{\eta}_2 = 0.31$, $\varsigma_1 = 0.23$, $\varsigma_2 = 0.2$, $l_1 = 0.2 - 0.2e^{(-0.1*5)}$, $l_2 = 0.2 - 0.2e^{(-0.1*5)}$, $\langle k \rangle = 3$, $u = 0.75$, $\iota_1 = 0.15$, $\iota_2 = 0.2$, $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.01$, $\pi = 0.001$, $\sigma_1 = 0.41$, $\sigma_2 = 0.46$.

$$w(\theta) = \begin{cases} 0, & \text{if } 0 < \theta \leq 10; \\ 0.6667(\theta - 10)^2 e^{-0.6(\theta - 10)}, & \text{if } 10 < \theta \leq 30; \\ 0.0185, & \text{if } 30 < \theta. \end{cases}$$

$$v(\lambda) = \begin{cases} 0, & \text{if } 0 < \lambda \leq 10; \\ 0.5(\lambda - 10)^2 e^{-0.6(\lambda - 10)}, & \text{if } 10 < \lambda \leq 30; \\ 0.02, & \text{if } 30 < \lambda. \end{cases}$$

The piecewise forms of $w(\theta)$ and $v(\lambda)$ used herein are taken from Reference [38]. We specify the parameters such that the conditions of Theorem 2 are satisfied, namely $R_0^e = 0.9897 < 1$, $\sigma_1^2 \langle k \rangle < \frac{\eta_1}{F_1(I_0)H'(0)}$ and $\sigma_2^2 \langle k \rangle < \frac{\eta_2}{F_2(I_0)H'(0)}$. Numerical simulations comparing deterministic and stochastic environments (Figures 3–5) show that both types of rumor spreaders will eventually disappear. Figure 3 depicts the density dynamics

of the two spreader types in deterministic and stochastic environments: both gradually vanish, with type A declining faster than type B. Figure 4 compares the state density changes of users under these two environments, which further verifies that rumors will eventually die out regardless of whether the environment is deterministic or stochastic. Additionally, Figure 5 contrasts the state evolution of different users under deterministic and stochastic conditions.

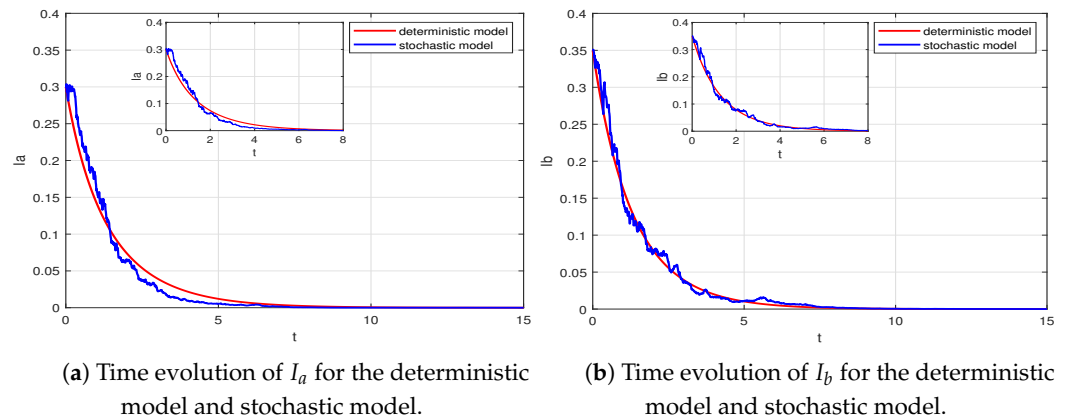


Figure 3. Dynamic behaviors of rumor extinction.

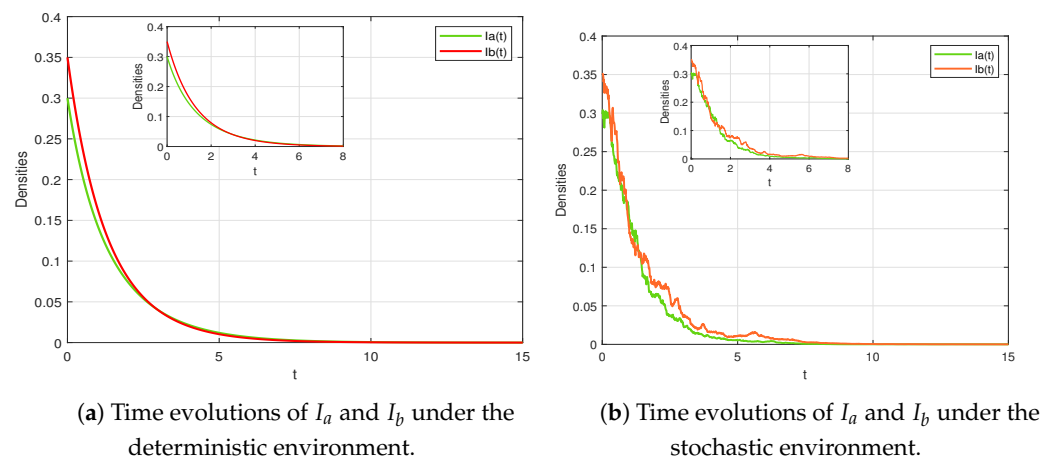


Figure 4. Dynamics of two rumor-spreading populations under deterministic and stochastic environments.

6.3. Stationary Distribution of the Rumor

Given $\sigma_1 = 0.04$, $\sigma_2 = 0.04$, $T_1 = 0.5$, $T_2 = 0.48$, $\tilde{\eta}_1 = 0.13$, $\tilde{\eta}_2 = 0.15$, $\langle k \rangle = 5$, $u = 0.4$, $\varsigma_1 = 0.38$, $\varsigma_2 = 0.4$, $l_1 = 0.2 - 0.2e^{(-0.1*5)}$, $l_2 = 0.2 - 0.2e^{(-0.1*5)}$, $\iota_1 = 0.1$, $\iota_2 = 0.12$, $\varepsilon_1 = 0.13$, $\varepsilon_2 = 0.1$, $\pi = 0.01$. $w(\theta)$ and $v(\lambda)$ are chosen as those in Section 6.1. We have $R_0^s = 1.2899 > 1$, satisfying the criteria in Theorem 3. This implies rumor persistence: the density of spreaders will stabilize at a non-zero steady state. As depicted in panels (a) and (b) of Figure 6, which present the temporal evolution of state densities for the two categories of spreaders under the deterministic and stochastic models, respectively, their densities will eventually stabilize at non-zero levels, thereby verifying the sustained propagation of the rumor. Similarly, as shown in Figure 7a,b, we further illustrate the state density evolution of rumor propagation in both deterministic and stochastic models. It can be observed that although the overall trend of rumor spreader density in a random environment is consistent with that in a deterministic environment, due to the interference of white noise, the random model no longer has a stable equilibrium point, and its spreader density instead fluctuates around the equilibrium point of the deterministic model. Figure 8 further

illustrates the temporal density evolution of recognizers and spreaders across distinct user subgroups. In the deterministic scenario (Figure 8a), the density of recognizers rises rapidly to a high stable level, while the density of spreaders drops sharply to a low stable level for a fixed state distribution. In the stochastic scenario with fluctuations (Figure 8b), both densities oscillate significantly, yet the core pattern—recognizers dominance and spreader minority—persists throughout evolution.

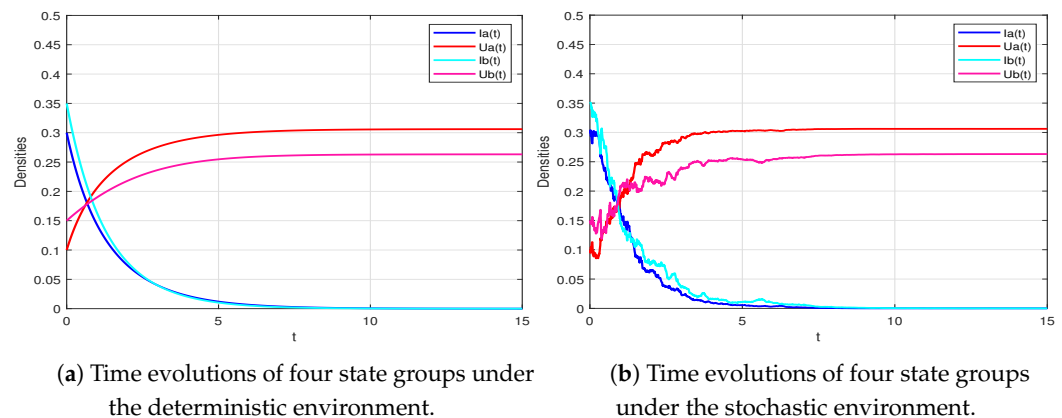


Figure 5. Comparison on dynamical evolutions of four state groups under the rumor extinction scenario.

Additionally, we take $\sigma_1 = 0.02$, $\sigma_2 = 0.02$, $T_1 = 0.6$, $T_2 = 0.58$, $\tilde{\eta}_1 = 0.1$, $\tilde{\eta}_2 = 0.12$, $\langle k \rangle = 5$, $u = 0.6$, $\varsigma_1 = 0.58$, $\varsigma_2 = 0.6$, $l_1 = 0.2 - 0.2e^{(-0.1 \cdot 5)}$, $l_2 = 0.2 - 0.2e^{(-0.1 \cdot 5)}$, $\iota_1 = 0.2$, $\iota_2 = 0.22$, $\varepsilon_1 = 0.13$, $\varepsilon_2 = 0.12$, $\pi = 0.001$. The choice of $w(\theta)$ and $v(\lambda)$ follows the same approach as outlined above. We obtain $R_0^s = 2.6444 > 1$, which ensures the existence of a stationary distribution. Therefore, based on these parameters, we conducted 5000 numerical simulations and collected the density distribution of individuals infected with rumors, as shown in Figure 9a,b. This chart displays the frequency histograms and corresponding marginal density function curves of two types of rumor-infected individuals. We can also clearly observe from the graph that, over time, the density of the two types of rumor-infected individuals does not approach zero, which is consistent with the conclusion of Theorem 3.

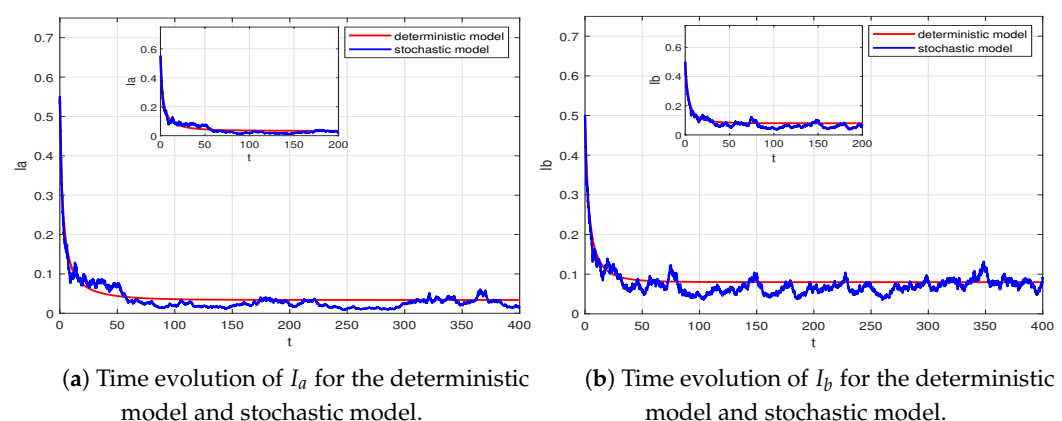


Figure 6. Comparative analysis of rumor persistence between the deterministic model and stochastic model. (a) Time evolution of I_a for the deterministic model and stochastic model; (b) Time evolution of I_b for the deterministic model and stochastic model.

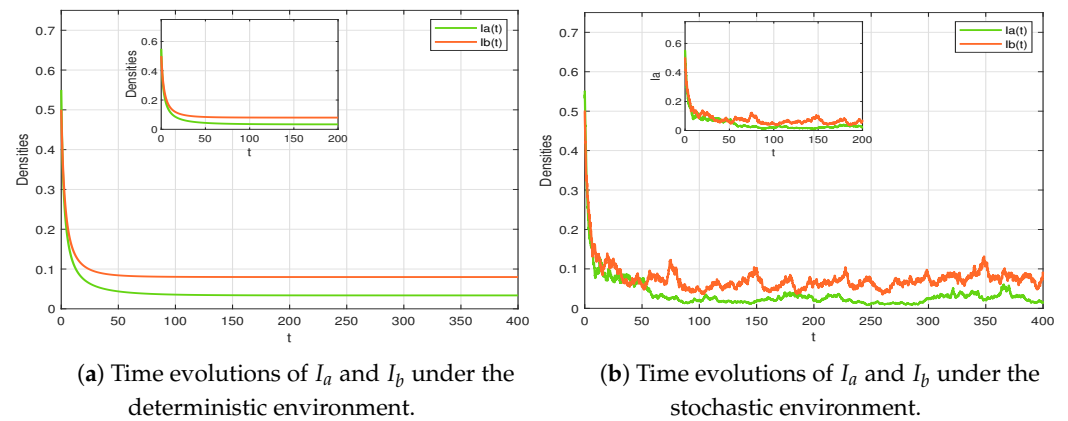


Figure 7. Dynamics of two rumor-spreading populations under deterministic and stochastic environments for rumor persistence.

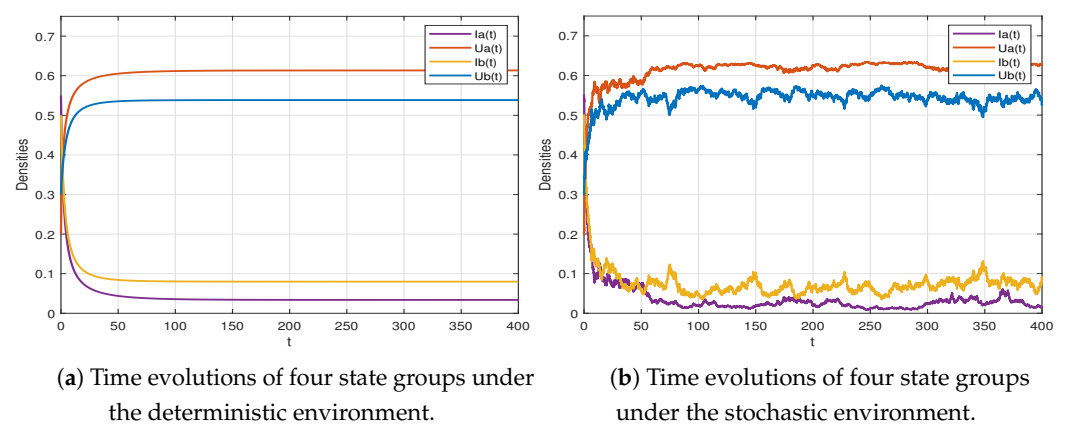


Figure 8. Comparison on dynamical evolutions of four state groups under the rumor persistence scenario.

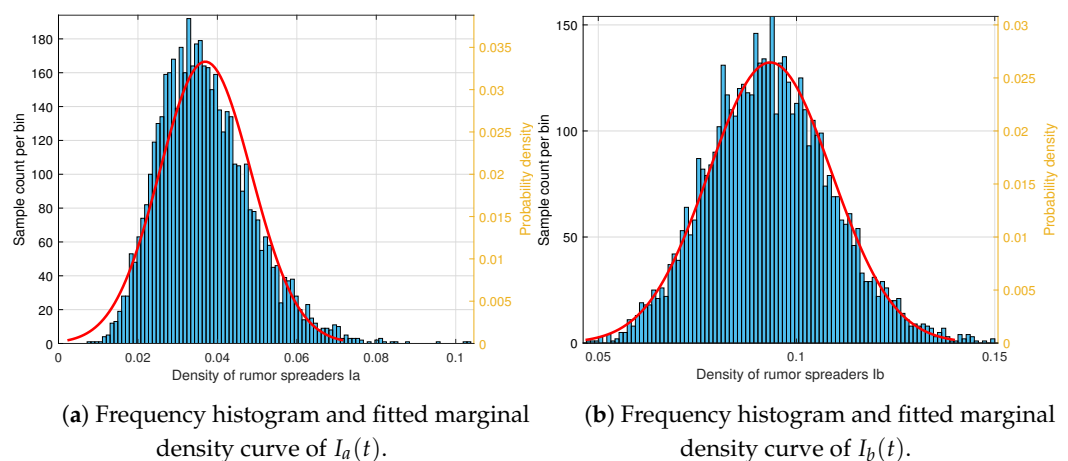


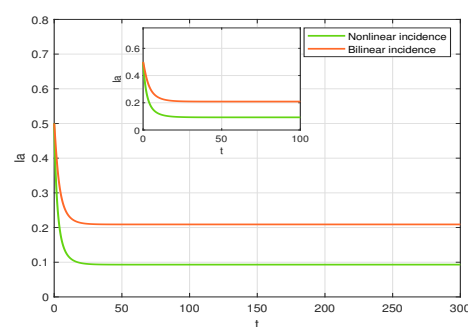
Figure 9. Frequency histograms and corresponding marginal density function curves.

6.4. The Influence of Several Crucial Parameters on the Dynamics of Rumor Transmission

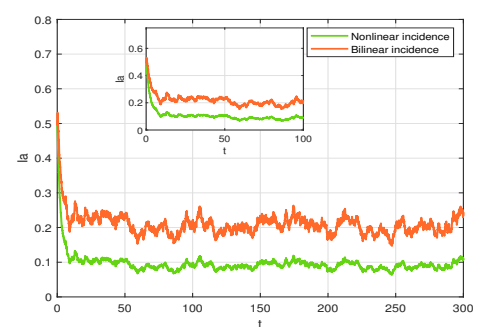
(1) The impact of nonlinear incidence on the rumor

Parameters in this section are consistent with Figure 6, except for the incidence rate function. We compare the bilinear and nonlinear propagation functions in Figure 10, in which while their overall trends are identical, the steady-state density of rumor spreaders is lower under the nonlinear function, indicating a smaller propagation scale. This aligns with the saturation effect, where the rumor spread rate decreases as the number of spreaders increases, which the nonlinear function effectively captures.

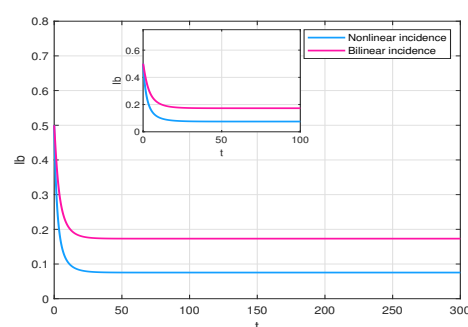
Additionally, we analyze four nonlinear propagation functions in Figure 11 (a) Holling-II type $\frac{I_i(t)}{1+\alpha_j I_i(t)}$, (b) Holling-III type $\frac{I_i(t)^2}{1+\alpha_j I_i(t)^2}$, (c) Holling-IV type $\frac{I_i(t)}{\frac{I_i(t)^2}{\alpha_j} + I_i(t) + d_j}$, and (d) Monod-Haldane type $\frac{I_i(t)}{1+\alpha_j I_i(t)^2}$ ($i = a, b; j = 1, 2$). Here, $\alpha_1 = d_1 = 0.12$, $\alpha_2 = d_2 = 0.13$, and all other parameters match Figure 6. As shown in the figure, we can clearly see that the rumor propagation trends in Figure 11a,d are roughly the same, with only slight differences in the scale of rumor propagation. However, Figure 11b,c exhibit rumor propagation trends that are markedly different from those in the other two figures, and these differences have a considerable impact on the prediction of rumor propagation. Therefore, under different incidence rate functions, there may be obvious differences in the mode and scale of rumor propagation. This motivates us to conduct further investigations into the impacts of various nonlinear factors in future research. Such efforts will enable a more thorough understanding of the roles these nonlinear factors play in rumor propagation and more accurate predictions of the evolutionary direction of public opinion.



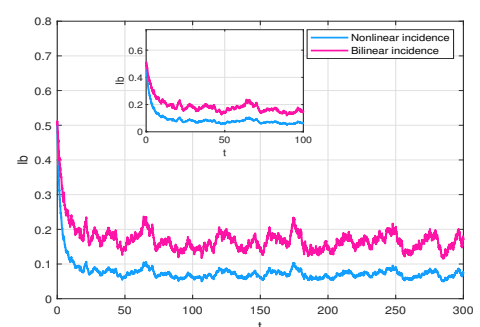
(a) Time evolution of I_a under bilinear and nonlinear incidences in the deterministic environment.



(b) Time evolution of I_a under bilinear and nonlinear incidences in the stochastic environment.



(c) Time evolution of I_b under bilinear and nonlinear incidences in the deterministic environment.



(d) Time evolution of I_b under bilinear and nonlinear incidences in the stochastic environment.

Figure 10. Comparison of rumor propagation dynamics under bilinear incidence and nonlinear incidence.

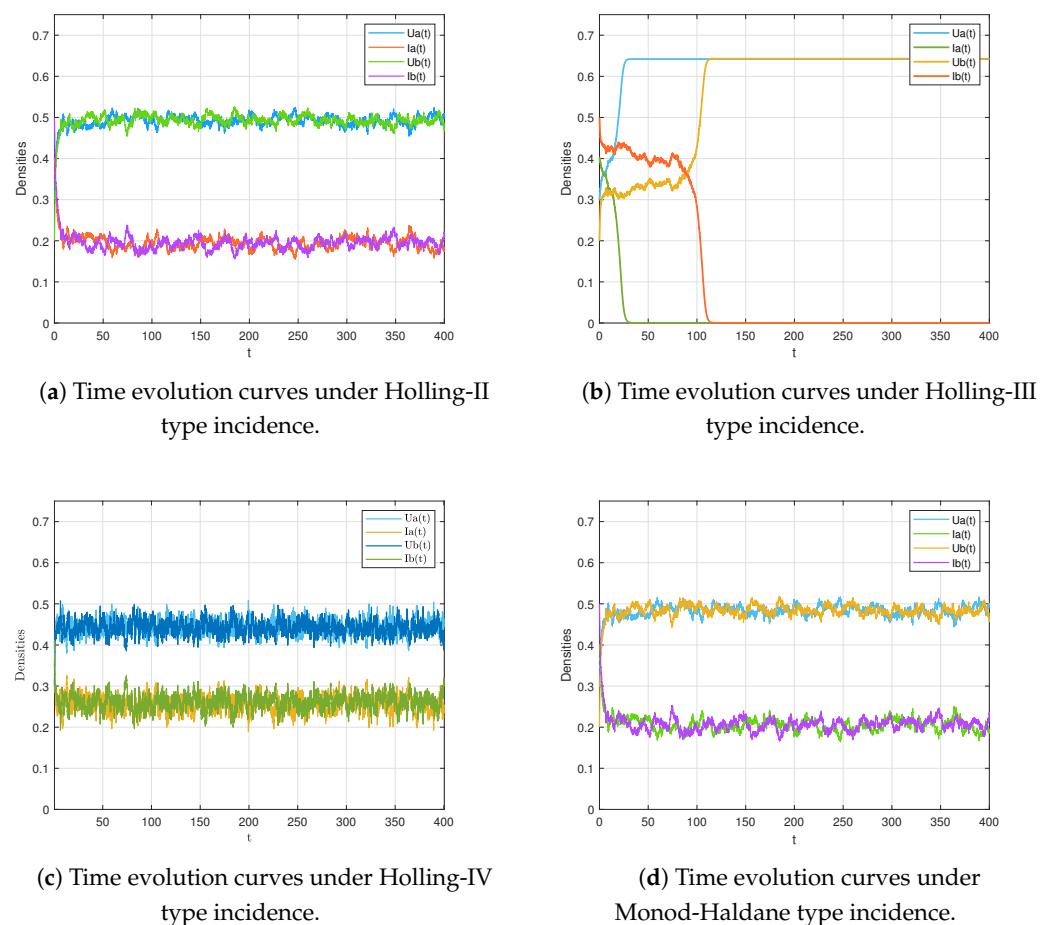


Figure 11. Comparison of rumor propagation dynamics under four different nonlinear incidence functions.

(2) The influence of isolation level on rumor propagation

We set $\iota_1 = 0.05, 0.1, 0.15, 0.2, 0.25$ and $\iota_2 = 0.1$, with other parameters consistent with those in Figure 6. The values of $w(\theta)$ and $v(\lambda)$ also remain the same as in Figure 6. In addition, the isolation rate ι_1 has a negligible impact on the spreader $I_b(t)$; therefore, this result is omitted from the figure, and we only present the density evolution process of $I_a(t)$. Based on the above parameters, the influence of the isolation rate ι_1 on $I_a(t)$ is shown in Figure 12. As shown in the figure, as the isolation rate increases, the stable density of $I_a(t)$ gradually decreases, indicating a smaller scale of rumor spread.

Similarly, we set $\iota_2 = 0.05, 0.1, 0.15, 0.2, 0.25$ and $\iota_1 = 0.1$, with other parameters being the same as those selected above. Likewise, we only present the influence of ι_2 on the density of spreaders $I_b(t)$ as shown in Figure 13. As shown in the figure, as the isolation rate ι_2 increases, the stable density of $I_b(t)$ gradually decreases, indicating a smaller scale of rumor propagation. The above numerical simulations show that adopting certain prevention and control measures is important for inhibiting the spread of rumors. Therefore, when rumors spread unchecked, appropriate isolation and speech restriction measures should be taken to prevent further rumor spread.

The above numerical simulation results consistently indicate that implementing the timely prevention and control measures is crucial for suppressing the spread of rumors. Therefore, when rumors spread on a large scale, it is necessary to take appropriate measures such as isolation and speech guidance to effectively block their transmission chain.

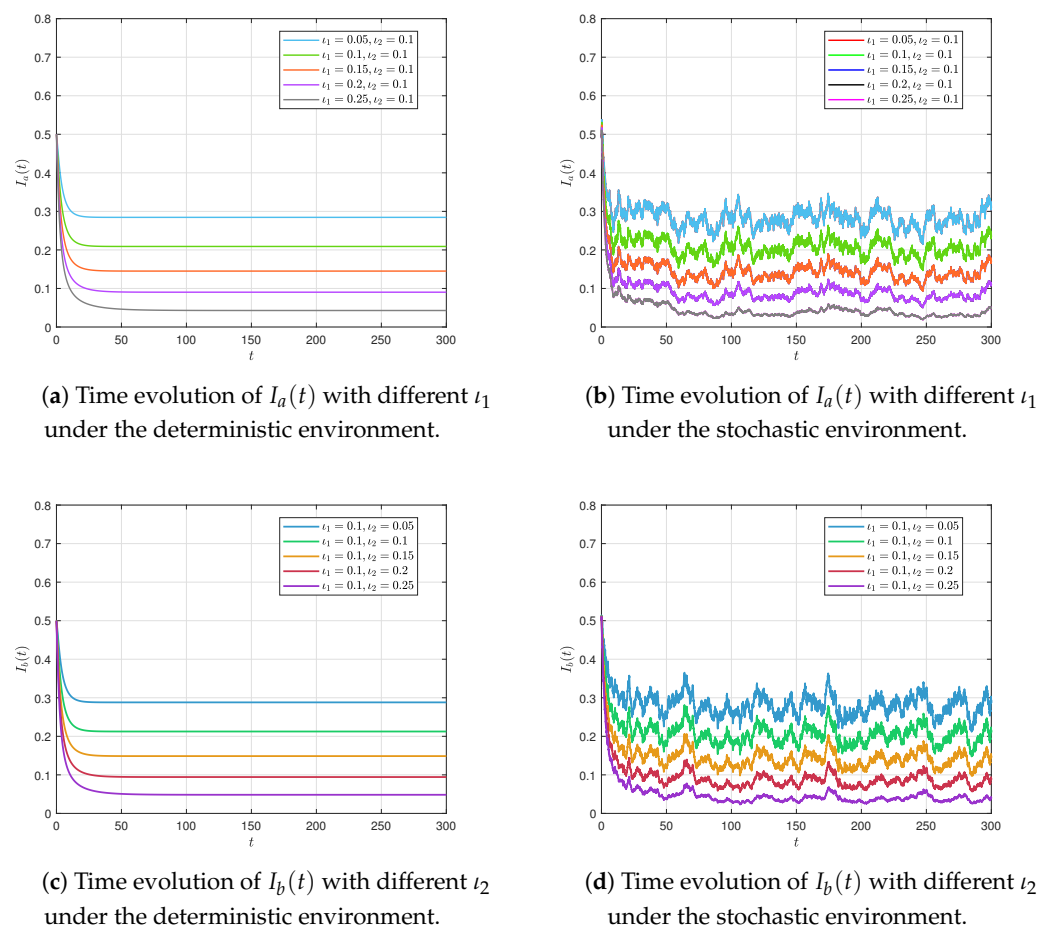


Figure 12. Comparison of the impact of isolation level parameters on rumor spreader density.

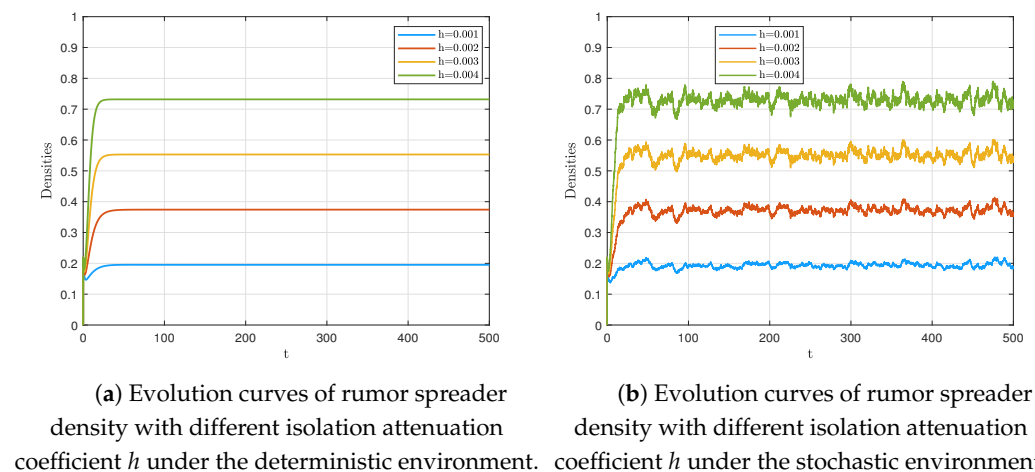


Figure 13. Comparison of the impact of isolation attenuation coefficient h on rumor spreader density under deterministic and stochastic environments.

(3) Effects of isolation attenuation rate on rumor transmission dynamics

The decay rate of isolation effect is also a key factor affecting the spread of rumors. The parameters are set as follows: $\sigma_1 = \sigma_2 = 0.01$; the isolation function is defined as $w(\theta) = h(1 + 8e^{(-0.1\theta)})^{-1}$, $v(\lambda) = h(1 + 8e^{(-0.1\lambda)})^{-1}$, where $h \in [0, 1]$ is a constant; all other parameters remain consistent with those in Figure 6.

Figure 13 shows the evolution of the density of rumor spreaders under different values of h . It can be seen from the figure that the higher the decay rate of the isolation effect, the greater the peak density of rumor spreaders. This also affects the final spread scale, leading to a larger overall spread. Therefore, when implementing isolation measures, the information dissemination channels of the isolated individuals (such as social media accounts and speaking privileges) can be restricted at different levels. High-risk individuals can have their active posting functions suspended, retaining only essential communication channels to increase the effectiveness of isolation. Moreover, isolation is not better the longer it lasts; a reasonable duration should be set according to the transmission decay principle to avoid wasting resources on ineffective isolation.

(4) The impact of stochastic noise on the rumor

To investigate the impact of white noise on rumor propagation, we select different noise intensity combinations: fix $\sigma_2 = 0.03$ and set $\sigma_1 = 0.01, 0.02, 0.03, 0.04, 0.05$; fix $\sigma_1 = 0.03$ and set $\sigma_2 = 0.01, 0.02, 0.03, 0.04, 0.05$. All other parameters are kept consistent with those used in Figure 6. Figures 14 and 15 illustrate the effects of σ_1 and σ_2 on the temporal density evolution of the two types of rumor spreaders ($I_a(t)$ and $I_b(t)$). When σ_2 is fixed and σ_1 varies, both $I_a(t)$ and $I_b(t)$ first undergo rapid initial changes and then fluctuate around specific ranges. Different values of σ_1 lead to distinct fluctuation frequencies and amplitudes, indicating that σ_1 affects the stability and variation characteristics of the densities of both spreader types. When σ_1 is fixed and σ_2 varies, $I_a(t)$ and $I_b(t)$ exhibit similar evolutionary trends: initial dynamic adjustments followed by persistent fluctuations. Specifically, different σ_2 values result in varying fluctuation behaviors for the two spreader types. In summary, stochastic noise introduces variability into the temporal evolution of $I_a(t)$ and $I_b(t)$, thereby influencing their fluctuation patterns, stability, and the ranges of density variation over time.

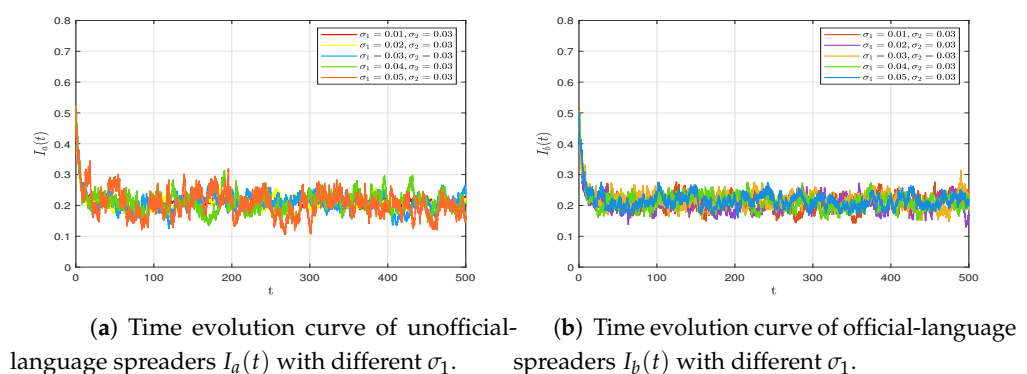


Figure 14. Comparison of the impact of parameter σ_1 on density evolutions of two types of rumor spreaders.

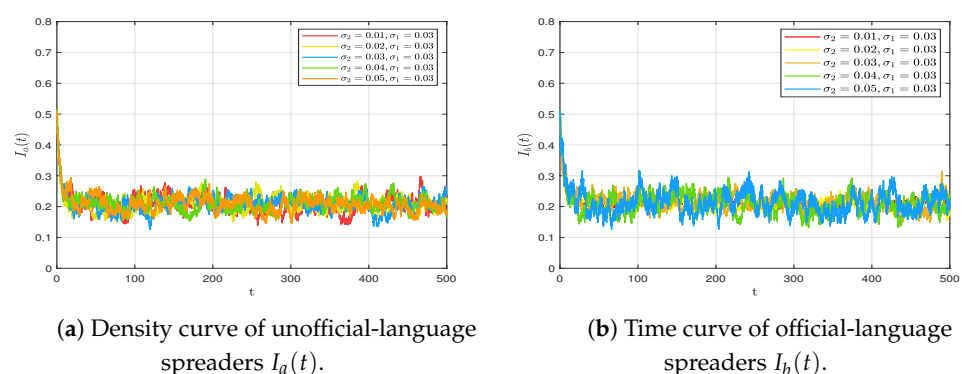


Figure 15. Time evolution of two types of propagators under σ_2 variation.

(5) Network media intervention's influence on rumor transmission

At present, the Internet is full of various forms of media. These media have broken the boundaries between communities, making information dissemination more extensive and faster, and playing an important role in the process of Internet public opinion topic dissemination. Next, we will explore the impact of online media on the spread of rumors. Firstly, we compare the values of $l_1 = l_2 = 0.2 - 0.2e^{(-0.1 \cdot 5)}$ and $l_1 = l_2 = 0$, while keeping all other parameters constant. As shown in Figure 16a,b, it can be seen that with media interference, the density of rumor spreaders has increased. Figure 17a,b compare l_1 . From the figure, it can be seen that as l_1 increases, it has a greater impact on $I_a(t)$ and a smaller impact on $I_b(t)$, while both types of propagators eventually reach a stable density. Furthermore, Figure 18a,b simulate the impact of l_2 on rumor propagation. Similarly, an increase in l_2 has a greater impact on $I_a(t)$ and a smaller impact on $I_b(t)$, but at the same time, it increases the eventual stable density of both types of propagators.

Therefore, the interference of online media will increase the ultimate scale of rumor dissemination, and relevant departments should standardize community rules, speak in a civilized manner, and take certain regulatory measures.

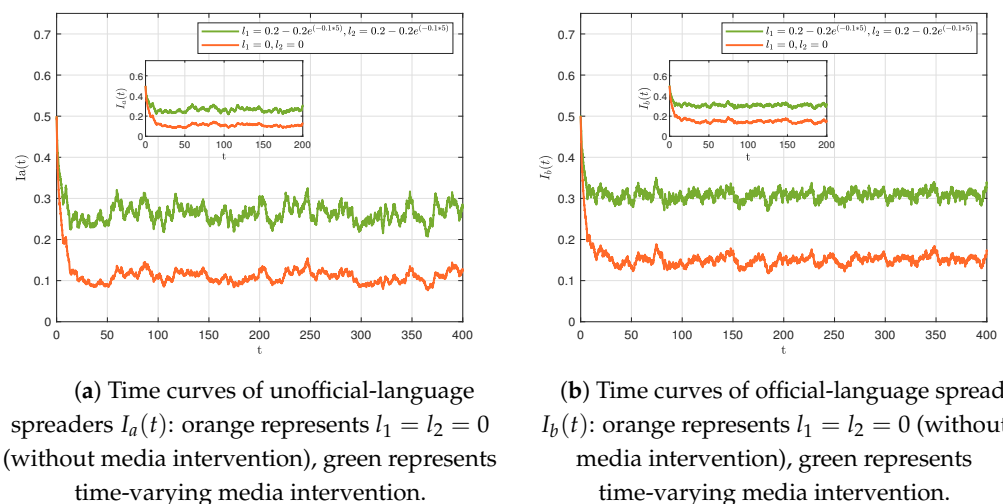


Figure 16. Comparing the impact of online media interference on the density of rumor spreaders.

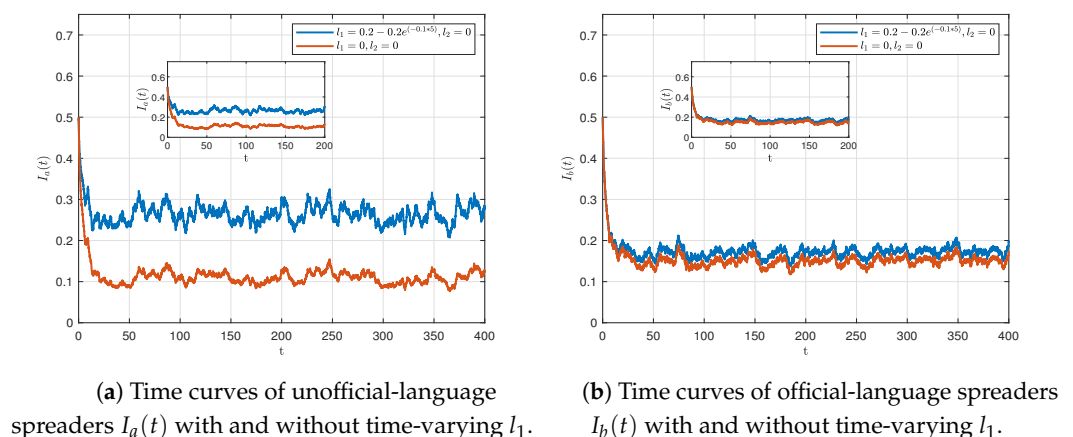


Figure 17. The density changes of two types of propagators only under the l_1 .

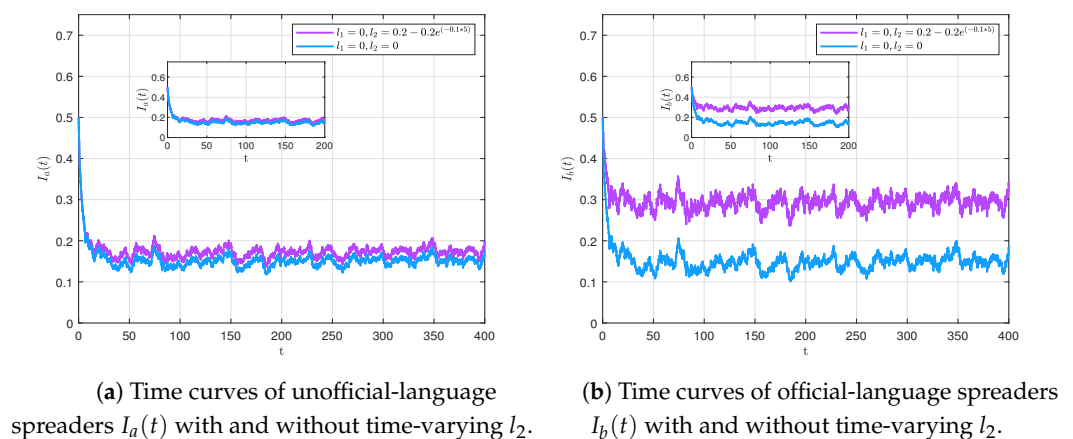


Figure 18. The density changes of two types of propagators only under the l_2 .

7. Model Application

In this section, we validate the applicability and effectiveness of the proposed model through an actual Weibo case.

On the afternoon of 22 December 2017, at 14:05, People’s Daily posted a news story on Weibo titled “Chinese tourists in Thailand pull an elephant’s tail, enraging it; tour guide crushed to death while trying to save people.” Once reported, it sparked widespread discussion among netizens. Leveraging online platforms, the topic quickly went viral. Using statistical data from the literature [25], detailed data are shown in Table 2. To more intuitively display changes in the data, a histogram of the data was drawn, as shown in Figure 19a. Additionally, taking two-hour intervals as nodes, we collected the number of comments and reposts by disseminators within 42 h after the rumor occurred and plotted a line chart to more clearly observe the trend of changes among the disseminators as shown in Figure 19b. Analysis of the actual data shows that the number of spreaders initially increased and then decreased, reaching a peak at the 6th hour. As the dissemination’s popularity waned and the timeliness of the information declined, the scale of propagation gradually reduced, along with the number of spreaders. It was not until the 42nd hour that the Thai police officially issued a debunking statement, bringing an end to the rumor’s spread.

Table 2. The real data about the event of “Thai elephant trampling”.

Time	1 h	2 h	3 h	4 h	5 h	6 h	7 h	8 h	9 h	10 h
Number	359	218	5573	4573	1178	9240	7508	1727	1328	566
Time	11 h	12 h	13 h	14 h	15 h	16 h	17 h	18 h	19 h	20 h
Number	115	5391	372	4065	6895	863	1850	521	35	116
Time	21 h	22 h	23 h	24 h	25 h	26 h	27 h	28 h	29 h	30 h
Number	25	1070	604	442	1976	246	82	389	166	379
Time	31 h	32 h	33 h	34 h	35 h	36 h	37 h	38 h	39 h	40 h
Number	655	7	59	138	694	5	43	154	2	2
Time	41 h	42 h	43 h	44 h	45 h	46 h	47 h	48 h	49 h	50 h
Number	82	0	0	0	0	0	0	0	0	0

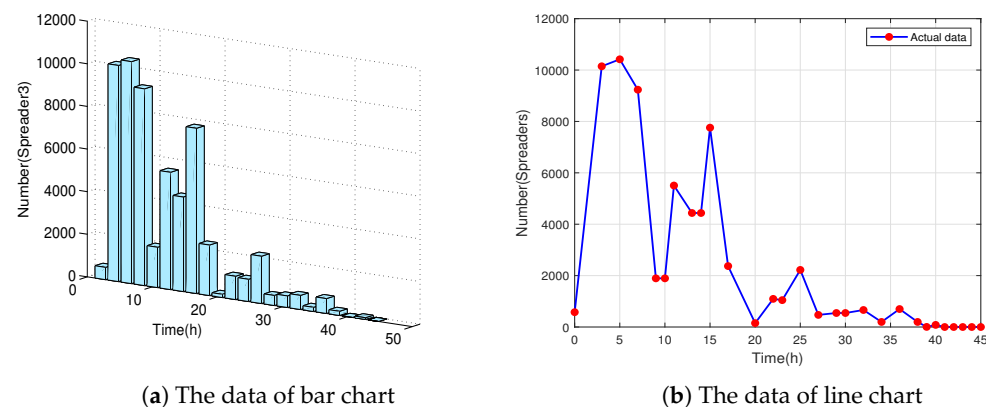


Figure 19. The number of spreaders. (a) 3D bar chart of the number of rumor spreaders at discrete time points; (b) Time series line chart of rumor spreader quantity.

Since this incident was only disseminated in official languages, we set the initial values as $U_a(0) = I_a(0) = 0$. Based on the data collected in Table 2, we standardized the data by taking half of the total number of spreaders. Thus, $I_b(0) = \frac{577}{32900} \approx 0.015$, and $U_a(0) = 1 - 0.014 = 0.985$. The other parameter values are set as follows: $T_2 = 0.045$, $\tilde{\eta}_2 = 0.32$, $\langle k \rangle = 5$, $u = 0.07$, $\zeta_2 = 0.45$, $l_2 = 0.2 - 0.2e^{(-0.1 \cdot 5)}$, $l_2 = 0.015$, $\varepsilon_2 = 0.148$, $\sigma_2 = 0.2$. The fitting results are shown in Figure 20. In the figure, the spreader density curve of the data from the Weibo platform is represented by a solid red line, while the spreader density curve of the proposed model (17) is represented by a solid blue line. It can be observed from the figure that the trend of the actual data curve is roughly consistent with that of the model (17) curve, showing a trend of first increasing, then decreasing, and finally approaching zero. This indicates that the proposed model has practical value.

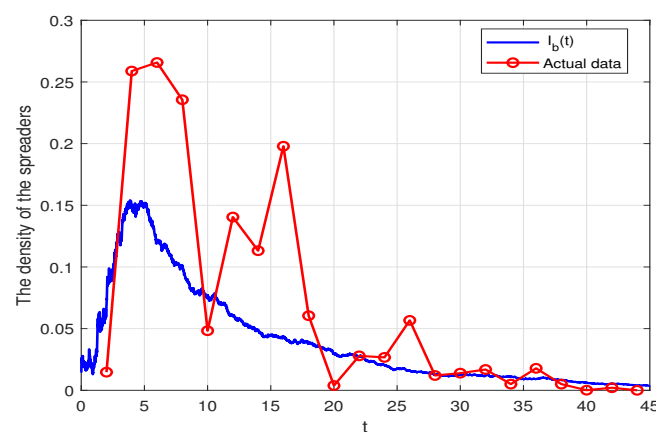


Figure 20. The fitting of model (17) with real data.

8. Conclusions

This paper proposed a novel multilingual stochastic rumor propagation model, which incorporated isolation age-related control through a generalized nonlinear incidence combined with media intervention. A deterministic rumor propagation model coupling ODEs and PDEs was constructed under the age-dependent isolation control mechanism. Subsequently, we applied the method of characteristics to transform the PDEs to ODEs, and further constructed a stochastic rumor propagation model by incorporating the interference of stochastic factors. Through a systematic analysis of the stochastic model, it was found that the constructed model has a unique global positive solution. Sufficient conditions for rumor extinction and the existence of a steady-state distribution were obtained. Finally,

through numerical simulations, we not only validated the key research findings but also investigated how diverse factors influence rumor transmission.

Internet media interventions, random disturbances, isolation control rates, and the attenuation rate of isolation effects all exert an influence on rumor propagation. Media intervention had a significant impact on rumor diffusion, as it amplified the scale of rumor dissemination, leading to greater negative effects. Random disturbances increased the uncertainty of rumor spread, making rumor management more difficult, while forcibly isolating individuals who spread negative information was an effective strategy to curb rumor propagation. Moreover, the greater the attenuation rate of isolation effects, the less significant the management effectiveness. Therefore, strengthening cyberspace management and promptly implementing isolation and control measures could effectively curb the spread of rumors. Therefore, regulators can implement hierarchical and precise governance based on the quantitative evolutionary laws derived from the numerical simulations in this paper. For rumor spreaders communicating via dialects and minority languages, authorities shall appropriately intensify isolation measures and slow the attenuation of isolation efficacy by reasonably extending effective control cycles. Meanwhile, stringent standards shall be enforced on information released by online self-media and news platforms to mitigate the rumor amplification effect triggered by media intervention. Furthermore, flexible response buffer windows can be preallocated according to the fluctuation range of public opinion under stochastic disturbances, which facilitates the construction of a long-term rumor-refutation and dynamic intervention framework tailored to multilingual social platforms and mitigates diverse social risks stemming from persistent cross-linguistic rumor propagation at the source. This fully demonstrates that strengthening cyberspace governance and rolling out targeted isolation and regulatory measures in a timely fashion can effectively sever rumor transmission chains and curb the sustained escalation of adverse public opinion.

To facilitate theoretical analysis, constant transition parameters were adopted in this study, which may fail to fully reproduce the dynamic propagation rules of real social networks. In addition, only general discrete algorithms were utilized for simulation verification, without systematic derivation of convergence and error bounds for the numerical scheme. Inspired by Refs. [52–57], in our future work, we will construct time-varying group transition functions to compare the evolutionary discrepancies of rumor systems under static and time-varying parameters. Meanwhile, we will adopt the structure-preserving numerical algorithms proposed in these references to systematically analyze the convergence properties of the discrete scheme and evaluate the comprehensive performance of various algorithms from multiple dimensions. Furthermore, we will also leverage the dynamic model established in this paper to design refined quantitative intervention and governance strategies tailored to multilingual network scenarios.

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Data Availability Statement: The authors confirm that the data supporting the findings of this study are available within the article.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

Appendix A

This appendix collects all lengthy algebraic expansions, generator $\mathcal{LV}$ expansions, and tedious bounding estimates removed from the proof of Theorem 1 in Section 3 for readability. All core theoretical frameworks are retained in the main text, and only routine computational details are supplemented here.

$$\begin{aligned} \mathcal{LV} = & \left(1 - \frac{c_1}{U_a(t)}\right) \left(T_1 - \eta_1 \langle k \rangle F_1(U_a(t)) H_1(I_a(t)) + \int_0^\infty I_a(t - \theta) \chi(\theta) d\theta - (u + \varsigma_1) \right. \\ & \times U_a(t) \left. \right) + \frac{1}{2} \frac{c_1}{U_a(t)^2} \left(-\sigma_1 \langle k \rangle F_1(U_a(t)) H_1(I_a(t)) \right)^2 + \left(1 - \frac{1}{I_a(t)}\right) \left(\eta_1 \langle k \rangle F_1(U_a(t)) \right. \\ & \times H_1(I_a(t)) - (u + \iota_1 + \varepsilon_1 + \pi) I_a(t) \left. \right) + \frac{1}{2} \frac{1}{I_a(t)^2} \left(\sigma_1 \langle k \rangle F_1(U_a(t)) H_1(I_a(t)) \right)^2 \\ & + \left(1 - \frac{c_2}{U_b(t)}\right) \left(T_2 - \eta_2 \langle k \rangle F_2(U_b(t)) H_2(I_b(t)) + \int_0^\infty I_b(t - \lambda) \psi(\lambda) d\lambda - (u + \varsigma_2) \right. \\ & \times U_b(t) \left. \right) + \frac{1}{2} \frac{c_2}{U_b(t)^2} \left(-\sigma_2 \langle k \rangle F_2(U_b(t)) H_2(I_b(t)) \right)^2 + \left(1 - \frac{1}{I_b(t)}\right) \left(\eta_2 \langle k \rangle F_2(U_b(t)) \right. \\ & \times H_2(I_b(t)) + \pi I_a(t) - (u + \iota_2 + \varepsilon_2) I_b(t) \left. \right) + \frac{1}{2} \frac{1}{I_b(t)^2} \left(\sigma_2 \langle k \rangle F_2(U_b(t)) H_2(I_b(t)) \right)^2. \end{aligned}$$

Moreover, the above equation can be simplified to

$$\begin{aligned} LV = & T_1 + \int_0^\infty I_a(t - \theta) \chi(\theta) d\theta - (u + \varsigma_1) U_a(t) - \frac{c_1 T_1}{U_a(t)} + c_1 \eta_1 \langle k \rangle \frac{F_1(U_a(t))}{U_a(t)} H_1(I_a(t)) \\ & - \frac{c_1}{U_a(t)} \int_0^\infty I_a(t - \theta) \chi(\theta) d\theta + c_1 (u + \varsigma_1) + \frac{1}{2} c_1 \sigma_1^2 \langle k \rangle^2 \left(\frac{F_1(U_a(t))}{U_a(t)} \right)^2 (H_1(I_a(t)))^2 \\ & - (u + \iota_1 + \varepsilon_1) I_a(t) - \eta_1 \langle k \rangle F_1(U_a(t)) \frac{H_1(I_a(t))}{I_a(t)} + u + \iota_1 + \varepsilon_1 + \pi + \frac{1}{2} \sigma_1^2 \langle k \rangle^2 \\ & \times F_1(U_a(t))^2 \left(\frac{H_1(I_a(t))}{I_a(t)} \right)^2 + T_2 + \int_0^\infty I_b(t - \lambda) \psi(\lambda) d\lambda - (u + \varsigma_2) U_b(t) - \frac{c_2 T_2}{U_b(t)} \\ & + c_2 \eta_2 \langle k \rangle \frac{F_2(U_b(t))}{U_b(t)} H_2(I_b(t)) - \frac{c_2}{U_b(t)} \int_0^\infty I_b(t - \lambda) \psi(\lambda) d\lambda + c_2 (u + \varsigma_2) + \frac{1}{2} c_2 \sigma_2^2 \\ & \times \langle k \rangle^2 \left(\frac{F_2(U_b(t))}{U_b(t)} \right)^2 H_2(I_b(t))^2 - (u + \iota_2 + \varepsilon_2) I_b(t) - \eta_2 \langle k \rangle F_2(U_b(t)) \frac{H_2(I_b(t))}{I_b(t)} \\ & - \frac{\pi I_a(t)}{I_b(t)} + u + \iota_2 + \varepsilon_2 + \frac{1}{2} \sigma_2^2 \langle k \rangle^2 F_2(U_b(t))^2 \left(\frac{H_2(I_b(t))}{I_b(t)} \right)^2 \\ \leq & T_1 + \frac{\iota_1 \Gamma_0}{u} + c_1 \eta_1 \langle k \rangle D_{\Gamma_0} H'(0) I_a(t) + c_1 (u + \varsigma_1) + \frac{1}{2} c_1 \sigma_1^2 \langle k \rangle^2 D_{\Gamma_0}^2 H'(0)^2 \Gamma_0^2 \\ & - (u + \iota_1 + \varepsilon_1) I_a(t) + u + \iota_1 + \varepsilon_1 + \pi + \frac{1}{2} \sigma_1^2 \langle k \rangle^2 (F_1(\Gamma_0))^2 H'(0)^2 + T_2 + \frac{\iota_2 \Gamma_0}{u} \\ & + c_2 \eta_2 \langle k \rangle D_{\Gamma_0} H'(0) I_b(t) + c_2 (u + \varsigma_2) + \frac{1}{2} c_2 \sigma_2^2 \langle k \rangle^2 D_{\Gamma_0}^2 H'(0)^2 \Gamma_0^2 + u + \iota_2 + \varepsilon_2 \\ & - (u + \iota_2 + \varepsilon_2) I_b(t) + \frac{1}{2} \sigma_2^2 \langle k \rangle^2 (F_2(\Gamma_0))^2 H'(0)^2. \end{aligned}$$

Substitute the constant coefficients

$$c_1 = \frac{u + \iota_1 + \varepsilon_1}{\eta_1 \langle k \rangle D_{\Gamma_0} H'(0)}, \quad c_2 = \frac{u + \iota_2 + \varepsilon_2}{\eta_2 \langle k \rangle D_{\Gamma_0} H'(0)},$$

and we arrive at the uniform upper bound

$$\begin{aligned} LV \leq & T_1 + T_2 + \frac{(\iota_1 + \iota_2)\Gamma_0}{u} + \frac{1}{2}\sigma_1^2\langle k \rangle^2 \left(\frac{u + \iota_1 + \epsilon_1}{\eta_1\langle k \rangle} D_{\Gamma_0} H'(0) \Gamma_0^2 + (F_1(\Gamma_0))^2 H'(0)^2 \right) \\ & + c_1(u + \varsigma_1) + c_2(u + \varsigma_2) + \frac{1}{2}\sigma_2^2\langle k \rangle^2 \left(\frac{u + \iota_2 + \epsilon_2}{\eta_2\langle k \rangle} D_{\Gamma_0} H'(0) \Gamma_0^2 + (F_2(\Gamma_0))^2 H'(0)^2 \right) \\ & + 2u + \iota_1 + \epsilon_1 + \pi + \iota_2 + \epsilon_2 = W. \end{aligned}$$

Appendix B

This appendix contains all tedious integral splitting, auxiliary function maximum computation and multi-layer bounding inequalities removed from the proof of Theorem 2 in Section 4.

Integrate over $[0, t]$ and divide by t :

$$\begin{aligned} \frac{\ln(I_a(t) + I_b(t))}{t} \leq & \frac{\ln(I_a(0) + I_b(0))}{t} + \frac{\eta_1\langle k \rangle + \epsilon}{t} \int_0^t \frac{F_1(U_a(s))H_1(I_a(s))}{I_a(s) + I_b(s)} ds + \frac{\eta_2\langle k \rangle + \epsilon}{t} \\ & \times \int_0^t \frac{F_2(U_b(s))H_2(I_b(s))}{I_a(s) + I_b(s)} ds - \frac{\sigma_1^2\langle k \rangle^2}{2t} \int_0^t \left(\frac{F_1(U_a(s))H_1(I_a(s))}{I_a(s) + I_b(s)} \right)^2 ds \\ & - \frac{\sigma_2^2\langle k \rangle^2}{2t} \int_0^t \left(\frac{F_2(U_b(s))H_2(I_b(s))}{I_a(s) + I_b(s)} \right)^2 ds - \min\{u + \iota_1 + \epsilon_1, u + \iota_2 \\ & + \epsilon_2\} + \frac{M_1(t)}{t} + \frac{M_2(t)}{t}, \end{aligned} \quad (A1)$$

where

$$\begin{aligned} M_1(t) &= \int_0^t \frac{\sigma_1\langle k \rangle F_1(U_a(s))H_1(I_a(s))}{I_a(s) + I_b(s)} dB_1(s), \\ M_2(t) &= \int_0^t \frac{\sigma_2\langle k \rangle F_2(U_b(s))H_2(I_b(s))}{I_a(s) + I_b(s)} dB_2(s) \end{aligned}$$

are local martingales with quadratic variations

$$\langle M_1, M_1 \rangle_t = \sigma_1^2\langle k \rangle^2 \int_0^t \left(\frac{F_1(U_a(s))H_1(I_a(s))}{I_a(s) + I_b(s)} \right)^2 ds \leq \sigma_1^2\langle k \rangle^2 (D_{\Gamma_0})^2 (H'(0))^2 (\Gamma_0)^2 t,$$

$$\langle M_2, M_2 \rangle_t = \sigma_2^2\langle k \rangle^2 \int_0^t \left(\frac{F_2(U_b(s))H_2(I_b(s))}{I_a(s) + I_b(s)} \right)^2 ds \leq \sigma_2^2\langle k \rangle^2 (D_{\Gamma_0})^2 (H'(0))^2 (\Gamma_0)^2 t.$$

Thus $\limsup_{t \rightarrow \infty} \frac{\langle M_i, M_i \rangle_t}{t} < \infty$, so $\lim_{t \rightarrow \infty} \frac{M_i(t)}{t} = 0$.

Compute maximum of $\Xi_1(x), \Xi_2(x)$:

$$\begin{aligned} \Xi_1(x)_{\max} &= -\frac{\sigma_1^2\langle k \rangle^2}{2} \left(\frac{\eta_1\langle k \rangle + \epsilon}{\sigma_1^2\langle k \rangle^2} \right)^2 + \frac{(\eta_1\langle k \rangle + \epsilon)^2}{\sigma_1^2\langle k \rangle^2} - \min\{u + \iota_1 + \epsilon_1, u + \iota_2 + \epsilon_2\} \\ &= \frac{1}{2} \left(\frac{\eta_1\langle k \rangle + \epsilon}{\sigma_1\langle k \rangle} \right)^2 - \min\{u + \iota_1 + \epsilon_1, u + \iota_2 + \epsilon_2\}, \\ \Xi_2(x)_{\max} &= -\frac{\sigma_2^2\langle k \rangle^2}{2} \left(\frac{\eta_2\langle k \rangle + \epsilon}{\sigma_2^2\langle k \rangle^2} \right)^2 + \frac{(\eta_2\langle k \rangle + \epsilon)^2}{\sigma_2^2\langle k \rangle^2} = \frac{1}{2} \left(\frac{\eta_2\langle k \rangle + \epsilon}{\sigma_2\langle k \rangle} \right)^2. \end{aligned}$$

Substitute the maximum upper bound:

$$\begin{aligned} \frac{\ln(I_a(t) + I_b(t))}{t} &\leq \frac{\ln(I_a(0) + I_b(0))}{t} + \frac{1}{2} \left(\frac{\eta_1 \langle k \rangle + \epsilon}{\sigma_1 \langle k \rangle} \right)^2 - \min\{u + \iota_1 + \epsilon_1, u + \iota_2 + \epsilon_2\} \\ &\quad + \frac{1}{2} \left(\frac{\eta_2 \langle k \rangle + \epsilon}{\sigma_2 \langle k \rangle} \right)^2 + \frac{M_1(t)}{t} + \frac{M_2(t)}{t}, \end{aligned}$$

Take $t \rightarrow \infty$ and then $\epsilon \rightarrow 0$ to get conclusion (i).

For condition (ii), suppose $\sigma_i^2 \langle k \rangle < \frac{\eta_i}{F_i(\Gamma_0)H'(0)}$, and there exists small $v > 0$ such that $F_i(\Gamma_0 + v)H'(0) \leq \frac{\eta_i \langle k \rangle + \epsilon}{\sigma_i^2 \langle k \rangle^2}$. For $t \geq T_0$:

$$\begin{aligned} \frac{\ln(I_a(t) + I_b(t))}{t} &\leq \frac{\ln(I_a(0) + I_b(0))}{t} + \frac{1}{t} \int_0^{T_0} \Xi_1 \left(\frac{F_1(U_a(s))H_1(I_a(s))}{I_a(s) + I_b(s)} \right) ds \\ &\quad + \Xi_1(F_1(\Gamma_0 + v)H'(0)) \frac{t - T_0}{t} + \frac{1}{t} \int_0^{T_0} \Xi_2 \left(\frac{F_2(U_b(s))H_2(I_b(s))}{I_a(s) + I_b(s)} \right) ds \\ &\quad + \Xi_2(F_2(\Gamma_0 + v)H'(0)) \frac{t - T_0}{t} + \frac{M_1(t)}{t} + \frac{M_2(t)}{t}. \end{aligned}$$

Let $\epsilon, v \rightarrow 0$, and we obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\ln(I_a(t) + I_b(t))}{t} &\leq \Xi_1(F_1(\Gamma_0)H'(0)) + \Xi_2(F_2(\Gamma_0)H'(0)) \\ &= \min\{u + \iota_1 + \epsilon_1, u + \iota_2 + \epsilon_2\} (R_0^e - 1) < 0, \end{aligned}$$

which completes the proof of statement (ii).

Appendix C

This appendix contains all tedious generator expansions, composite Lyapunov operator bounding and eight subdomain piecewise estimates removed from the proof of Theorem 3 in Section 5. It suffices to verify conditions (i) and (ii) in Lemma 3. The proof is carried out in two steps.

Step 1. The diffusion matrix of model (17) is expressed as

$$\tilde{A} = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{pmatrix},$$

where $A_1 = \sigma_1^2 \langle k \rangle^2 (F_1(U_a)H_1(I_a))^2$, $A_2 = \sigma_2^2 \langle k \rangle^2 (F_2(U_b)H_2(I_b))^2$. Choose

$$\tilde{M} = \min_{(U_a, I_a, U_b, I_b) \in \overline{D}_k} \{ \sigma_1^2 \langle k \rangle^2 (F_1(U_a)H_1(I_a))^2, \sigma_2^2 \langle k \rangle^2 (F_2(U_b)H_2(I_b))^2 \},$$

for any $(U_a, I_a, U_b, I_b) \in \overline{D}_k$, $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in R_4^+$.

$$\begin{aligned} \sum_{i,j=1}^4 a_{ij}(U_a, I_a, U_b, I_b) \zeta_i \zeta_j &= \sigma_1^2 \langle k \rangle^2 (F_1(U_a)H_1(I_a))^2 \zeta_1^2 + \sigma_1^2 \langle k \rangle^2 (F_1(U_a)H_1(I_a))^2 \zeta_2^2 \\ &\quad + \sigma_2^2 \langle k \rangle^2 (F_2(U_b)H_2(I_b))^2 \zeta_3^2 + \sigma_2^2 \langle k \rangle^2 (F_2(U_b)H_2(I_b))^2 \zeta_4^2 \\ &\geq \tilde{M} \|\zeta\|^2, \end{aligned}$$

where $\overline{D}_k = [\frac{1}{k}, k] \times [\frac{1}{k}, k] \times [\frac{1}{k}, k] \times [\frac{1}{k}, k]$ and $k > 1$ is a sufficiently large integer. So condition (i) of Lemma 3 is satisfied.

Step 2. We define a C^2 -function $V : R_4^+ \rightarrow R$ as follows:

$$\tilde{V}(U_a, I_a, U_b, I_b) = \Theta V_1 + V_2 + V_3,$$

where

$$V_1 = -c_1 \ln U_a(t) + c_2 U_a(t) - I_a(t) - c_3 \ln U_b(t) + U_b(t) - I_b(t),$$

$$V_2 = -\ln U_a(t) - \ln U_b(t),$$

$$V_3 = \frac{1}{\delta + 1} (U_a(t) + I_a(t) + U_b(t) + I_b(t))^{\delta+1}.$$

$\Theta > 0$ is a large enough constant and c_1, c_2, c_3 are constants determined later, which satisfies

$$\Theta T_2 + F_2 \leq -2, \quad (A2)$$

$$\Theta T_2 + F_3 \leq -2, \quad (A3)$$

F_2, F_3 are a determined later. Obviously,

$$\lim_{t \rightarrow \infty} \inf_{(U_a, I_a, U_b, I_b) \in D_k} \tilde{V}(U_a, I_a, U_b, I_b) = +\infty,$$

where $D_k = (\frac{1}{k}, k) \times (\frac{1}{k}, k) \times (\frac{1}{k}, k) \times (\frac{1}{k}, k)$.

It follows that $\tilde{V}(U_a, I_a, U_b, I_b)$ denotes a continuous function, and a minimum point $(\bar{U}_a, \bar{I}_a, \bar{U}_b, \bar{I}_b) \in R_4^+$ exists.

Define a non-negative C^2 function $\bar{V}(U_a, I_a, U_b, I_b) : R_4^+ \rightarrow R$ as follows:

$$\bar{V}(U_a, I_a, U_b, I_b) = \tilde{V}(U_a, I_a, U_b, I_b) - \tilde{V}(\bar{U}_a, \bar{I}_a, \bar{U}_b, \bar{I}_b).$$

Through the application of Itô's formula, we deduce that

$$\begin{aligned} \mathcal{L}V_1 &= \left(-\frac{c_1}{U_a} + c_2 \right) \left(T_1 - \eta_1 \langle k \rangle F_1(U_a) H_1(I_a) + \int_0^\infty I_a(t - \theta) \chi(\theta) d\theta - (u + \varsigma_1) U_a \right) \\ &\quad + \frac{1}{2} \frac{c_1}{U_a^2} (-\sigma_1 \langle k \rangle F_1(U_a) H_1(I_a))^2 - \left(\eta_1 \langle k \rangle F_1(U_a) H_1(I_a) - (u + \iota_1 + \varepsilon_1 + \pi) I_a \right) \\ &\quad + \left(-\frac{c_3}{U_b} + 1 \right) \left(T_2 - \eta_2 \langle k \rangle F_2(U_b) H_2(I_b) + \int_0^\infty I_b(t - \lambda) \psi(\lambda) d\lambda - (u + \varsigma_2) U_b \right) \\ &\quad + \frac{1}{2} \frac{c_3}{U_b^2} (-\sigma_2 \langle k \rangle F_2(U_b) H_2(I_b))^2 - \left(\eta_2 \langle k \rangle F_2(U_b) H_2(I_b) + \pi I_a - (u + \iota_2 + \varepsilon_2) I_b \right) \\ &\leq -\frac{c_1 T_1}{U_a} + c_1 \eta_1 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + c_1 (u + \varsigma_1) + c_2 T_1 + \frac{c_2 \iota_1 \Gamma_0}{u} - c_2 (u + \varsigma_1) U_a \\ &\quad + \frac{1}{2} c_1 \sigma_1^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 + (u + \iota_1 + \varepsilon_1) I_a - \frac{c_3 T_2}{U_b} + c_3 \eta_2 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + \\ &\quad c_3 (u + \varsigma_2) + T_2 + \frac{\iota_2 \Gamma_0}{u} - (u + \varsigma_2) U_b + \frac{1}{2} c_3 \sigma_2^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 + (u + \iota_2 + \varepsilon_2) I_b \\ &\leq -4 \sqrt[4]{c_1 c_2 c_3 T_1 T_2 (u + \varsigma_1) (u + \varsigma_2)} + c_1 \left(\eta_1 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + u + \varsigma_1 + \frac{1}{2} \sigma_1^2 \langle k \rangle^2 \right. \\ &\quad \times (D_{\Gamma_0} H'(0) \Gamma_0)^2 \Big) + c_2 \left(T_1 + \frac{\iota_1 \Gamma_0}{u} \right) + c_3 \left(\eta_2 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + u + \varsigma_2 + \frac{1}{2} \sigma_2^2 \langle k \rangle^2 \right. \\ &\quad \times (D_{\Gamma_0} H'(0) \Gamma_0)^2 \Big) + T_2 + \frac{\iota_2 \Gamma_0}{u} + (u + \iota_1 + \varepsilon_1) I_a + (u + \iota_2 + \varepsilon_2) I_b. \end{aligned}$$

Let

$$\begin{aligned} c_1 \left(\eta_1 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + u + \varsigma_1 + \frac{1}{2} \sigma_1^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 \right) &= c_2 \left(T_1 + \frac{\iota_1 \Gamma_0}{u} \right) \\ &= c_3 \left(\eta_2 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + u + \varsigma_2 + \frac{1}{2} \sigma_2^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 \right). \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{L}V_1 &\leq -\frac{\iota_2 \Gamma_0}{u} (R_0^s - 1) + (u + \iota_1 + \varepsilon_1) I_a + (u + \iota_2 + \varepsilon_2) I_b + T_2 \\ &= -\alpha + (u + \iota_1 + \varepsilon_1) I_a + (u + \iota_2 + \varepsilon_2) I_b + T_2, \end{aligned}$$

$$\text{where } \alpha = \frac{\iota_2 \Gamma_0}{u} (R_0^s - 1).$$

$$\begin{aligned} \mathcal{L}V_2 &= -\frac{1}{U_a} \left(T_1 - \eta_1 \langle k \rangle F_1(U_a) H_1(I_a) + \int_0^\infty I_a(t-\theta) \chi(\theta) d\theta - (u + \varsigma_1) U_a \right) \\ &\quad + \frac{1}{2} \frac{1}{(U_a)^2} (-\sigma_1 \langle k \rangle F_1(U_a) H_1(I_a))^2 - \frac{1}{U_b} \left(T_2 - \eta_2 \langle k \rangle F_2(U_b) H_2(I_b) \right. \\ &\quad \left. + \int_0^\infty I_b(t-\lambda) \psi(\lambda) d\lambda - (u + \varsigma_2) U_b \right) + \frac{1}{2} \frac{1}{(U_b)^2} (-\sigma_2 \langle k \rangle F_2(U_b) H_2(I_b))^2 \\ &\leq -\frac{T_1}{U_a} - \frac{T_2}{U_b} + \eta_1 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + 2u + \varsigma_1 + \varsigma_2 + \frac{1}{2} \sigma_1^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 \\ &\quad + \eta_2 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + \frac{1}{2} \sigma_2^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2, \\ \mathcal{L}V_3 &= (U_a + I_a + U_b + I_b)^\delta \left(T_1 + \int_0^\infty I_a(t-\theta) \chi(\theta) d\theta - (u + \varsigma_1) U_a - (u + \iota_1 + \varepsilon_1) I_a \right. \\ &\quad \left. + T_2 + \int_0^\infty I_b(t-\lambda) \psi(\lambda) d\lambda - (u + \varsigma_2) U_b - (u + \iota_2 + \varepsilon_2) I_b \right) + \delta (U_a + I_a + U_b \\ &\quad + I_b)^{\delta-1} \left(\sigma_1^2 \langle k \rangle^2 (F_1(U_a) H_1(I_a))^2 + \sigma_2^2 \langle k \rangle^2 (F_2(U_b) H_2(I_b))^2 \right) \\ &\leq (U_a + I_a + U_b + I_b)^\delta \left(T_1 + T_2 + \frac{(\iota_1 + \iota_2) \Gamma_0}{u} - u (U_a + I_a + U_b + I_b) \right) + \frac{1}{2} \delta (U_a \\ &\quad + I_a + U_b + I_b)^{\delta-1} \left(\sigma_1^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 (U_a)^2 + \sigma_1^2 \langle k \rangle^2 (F_1(\Gamma_0))^2 (I_a)^2 \right. \\ &\quad \left. + \sigma_2^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 (U_b)^2 + \sigma_2^2 \langle k \rangle^2 (F_2(\Gamma_0))^2 (I_b)^2 \right) \\ &\leq (U_a + I_a + U_b + I_b)^\delta \left(T_1 + T_2 + \frac{(\iota_1 + \iota_2) \Gamma_0}{u} \right) - \left(u - \frac{1}{2} \delta \langle k \rangle^2 \left(\sigma_1^2 D_{\Gamma_0}^2 (H'(0))^2 \Gamma_0^2 \right. \right. \\ &\quad \left. \left. \vee \sigma_1^2 (F_1(\Gamma_0) H'(0))^2 \vee \sigma_2^2 D_{\Gamma_0}^2 (H'(0))^2 \Gamma_0^2 \vee \sigma_2^2 (F_2(\Gamma_0) H'(0))^2 \right) \right) (U_a + I_a + U_b + I_b)^{\delta+1} \\ &= B - \frac{\phi}{2} (U_a + I_a + U_b + I_b)^{\delta+1} \\ &\leq B - \frac{\phi}{2} (U_a^{\delta+1} + I_a^{\delta+1} + U_b^{\delta+1} + I_b^{\delta+1}), \end{aligned}$$

where

$$\begin{aligned} B &= \sup_{D \in R_4^+} \left\{ (U_a + I_a + U_b + I_b)^\delta \left(T_1 + T_2 + \frac{(\iota_1 + \iota_2) \Gamma_0}{u} \right) - \frac{\phi}{2} (U_a + I_a + U_b + I_b)^{\delta+1} \right\}, \\ \phi &= u - \frac{1}{2} \delta \langle k \rangle^2 \left(\sigma_1^2 D_{\Gamma_0}^2 (H'(0))^2 \Gamma_0^2 \vee \sigma_1^2 (F_1(\Gamma_0) H'(0))^2 \vee \sigma_2^2 D_{\Gamma_0}^2 (H'(0))^2 \Gamma_0^2 \right. \\ &\quad \left. \vee \sigma_2^2 (F_2(\Gamma_0) H'(0))^2 \right), \quad D = (U_a, I_a, U_b, I_b). \end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}\bar{V} \leq & -\Theta\alpha + \Theta(u + \iota_1 + \varepsilon_1)I_a + \Theta(u + \iota_2 + \varepsilon_2)I_b + \Theta T_2 - \frac{T_1}{U_a} - \frac{T_2}{U_b} + \eta_1 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 \\ & + \frac{1}{2} \sigma_1^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 + \eta_2 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + \frac{1}{2} \sigma_2^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 + 2u + \varsigma_1 \\ & + \varsigma_2 + B - \frac{\phi}{2} (U_a^{\delta+1} + I_a^{\delta+1} + U_b^{\delta+1} + I_b^{\delta+1}).\end{aligned}$$

Define

$$\begin{aligned}F_1 &= \sup_{D \in R_4^+} \left\{ \Theta(u + \iota_1 + \varepsilon_1)I_a + \Theta(u + \iota_2 + \varepsilon_2)I_b + \Theta T_2 + \eta_1 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + 2u + \varsigma_1 \right. \\ &\quad \left. + \varsigma_2 + \frac{1}{2} \sigma_1^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 + \eta_2 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + \frac{1}{2} \sigma_2^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 + B \right. \\ &\quad \left. - \frac{\phi}{2} (U_a^{\delta+1} + I_a^{\delta+1} + U_b^{\delta+1} + I_b^{\delta+1}) \right\}, \\ F_2 &= \sup_{D \in R_4^+} \left\{ \Theta(u + \iota_2 + \varepsilon_2)I_b + \eta_1 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + 2u + \varsigma_1 + \varsigma_2 + \frac{1}{2} \sigma_1^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 \right. \\ &\quad \left. + \eta_2 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + \frac{1}{2} \sigma_2^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 + B - \frac{\phi}{2} (U_a^{\delta+1} + I_a^{\delta+1} + U_b^{\delta+1} + I_b^{\delta+1}) \right\}, \\ F_3 &= \sup_{D \in R_4^+} \left\{ \Theta(u + \iota_1 + \varepsilon_1)I_a + \eta_1 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + 2u + \varsigma_1 + \varsigma_2 + \frac{1}{2} \sigma_1^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 \right. \\ &\quad \left. + \eta_2 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + \frac{1}{2} \sigma_2^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 + B - \frac{\phi}{2} (U_a^{\delta+1} + I_a^{\delta+1} + U_b^{\delta+1} + I_b^{\delta+1}) \right\}, \\ F_4 &= \sup_{D \in R_4^+} \left\{ \Theta(u + \iota_1 + \varepsilon_1)I_a + \Theta(u + \iota_2 + \varepsilon_2)I_b + \Theta T_2 + \eta_1 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + 2u + \varsigma_1 \right. \\ &\quad \left. + \varsigma_2 + \frac{1}{2} \sigma_1^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 + \eta_2 \langle k \rangle D_{\Gamma_0} H'(0) \Gamma_0 + \frac{1}{2} \sigma_2^2 \langle k \rangle^2 (D_{\Gamma_0} H'(0) \Gamma_0)^2 + B \right\}.\end{aligned}\tag{A4}$$

And further define a bounded closed set

$$D_\epsilon = \left\{ (U_a, I_a, U_b, I_b) \in R_4^+ : \epsilon \leq U_a \leq \frac{1}{\epsilon}, \epsilon \leq I_a \leq \frac{1}{\epsilon}, \epsilon \leq U_b \leq \frac{1}{\epsilon}, \epsilon \leq I_b \leq \frac{1}{\epsilon} \right\},$$

where ϵ is a positive constant that is sufficiently small. To facilitate the following analysis, eight domains form the partition of $R_4^+ \setminus D_\epsilon$,

$$\begin{aligned}D_1 &= \{(U_a, I_a, U_b, I_b) \in R_4^+ : 0 \leq U_a < \epsilon\}, \\ D_2 &= \{(U_a, I_a, U_b, I_b) \in R_4^+ : 0 \leq I_a < \epsilon\}, \\ D_3 &= \{(U_a, I_a, U_b, I_b) \in R_4^+ : 0 \leq U_b < \epsilon\}, \\ D_4 &= \{(U_a, I_a, U_b, I_b) \in R_4^+ : 0 \leq I_b < \epsilon\}, \\ D_5 &= \left\{ (U_a, I_a, U_b, I_b) \in R_4^+ : U_a > \frac{1}{\epsilon} \right\}, \\ D_6 &= \left\{ (U_a, I_a, U_b, I_b) \in R_4^+ : I_a > \frac{1}{\epsilon} \right\}, \\ D_7 &= \left\{ (U_a, I_a, U_b, I_b) \in R_4^+ : U_b > \frac{1}{\epsilon} \right\}, \\ D_8 &= \left\{ (U_a, I_a, U_b, I_b) \in R_4^+ : I_b > \frac{1}{\epsilon} \right\}.\end{aligned}$$

It will next be shown that $\mathcal{L}\bar{V}(U_a, I_a, U_b, I_b) \leq -1$ on $R_4^+ \setminus D_\epsilon$.

Case 1. When $(U_a, I_a, U_b, I_b) \in D_1$, it is straightforward to verify that

$$\mathcal{L}\bar{V} \leq -\frac{T_1}{U_a} + F_1 \leq -\frac{T_1}{\epsilon} + F_1, \quad (\text{A5})$$

take a constant $\epsilon > 0$ small enough to ensure that $-T_1/\epsilon + F_1 \leq -1$, then

$$\mathcal{L}\bar{V} \leq -1 \quad \text{for any } (U_a, I_a, U_b, I_b) \in D_1.$$

Case 2. If $(U_a, I_a, U_b, I_b) \in D_2$, it can be easily shown that

$$\mathcal{L}\bar{V} \leq \Theta(u + \iota_1 + \epsilon_1)I_a + \Theta T_2 + F_2 \leq \Theta(u + \iota_1)\epsilon + \Theta T_2 + F_2, \quad (\text{A6})$$

where F_2 is given by (25). Take a sufficiently small $\epsilon > 0$ with $\Theta(u + \iota_1 + \epsilon_1)\epsilon \leq 1$. According to (23), one can get that

$$\mathcal{L}\bar{V} \leq -1 \quad \text{for any } (U_a, I_a, U_b, I_b) \in D_2.$$

Case 3. If $(U_a, I_a, U_b, I_b) \in D_3$, it follows that

$$\mathcal{L}\bar{V} \leq -\frac{T_2}{U_b} + F_1 \leq -\frac{T_2}{\epsilon} + F_1, \quad (\text{A7})$$

we choose a small enough constant $\epsilon > 0$, such that $-T_2/\epsilon + F_1 \leq -1$, and thus

$$\mathcal{L}\bar{V} \leq -1 \quad \text{for any } (U_a, I_a, U_b, I_b) \in D_3.$$

Case 4. If $(U_a, I_a, U_b, I_b) \in D_4$, we have

$$\mathcal{L}\bar{V} \leq \Theta(u + \iota_2 + \epsilon_2)I_b + \Theta T_2 + F_3 \leq \Theta(u + \iota_2 + \epsilon_2)\epsilon + \Theta T_2 + F_3, \quad (\text{A8})$$

where F_3 is given by (25). We choose a constant $\epsilon > 0$ that is sufficiently small to satisfy $\Theta(u + \iota_2 + \epsilon_2)\epsilon \leq 1$. From (24), we have

$$\mathcal{L}\bar{V} \leq -1 \quad \text{for any } (U_a, I_a, U_b, I_b) \in D_4.$$

Case 5. If $(U_a, I_a, U_b, I_b) \in D_5$, it can be concluded that

$$\mathcal{L}\bar{V} \leq -\frac{\phi}{4}U_a^{\delta+1} + F_4 \leq -\frac{\phi}{4}\frac{1}{\epsilon^{\delta+1}} + F_4. \quad (\text{A9})$$

We select a sufficiently small constant $\epsilon > 0$ such that the inequality $-\frac{\phi}{4}\frac{1}{\epsilon^{\delta+1}} + F_4 \leq -1$ holds. Therefore,

$$\mathcal{L}\bar{V} \leq -1 \quad \text{for any } (U_a, I_a, U_b, I_b) \in D_5.$$

Case 6. If $(U_a, I_a, U_b, I_b) \in D_6$, we can be inferred that

$$\mathcal{L}\bar{V} \leq -\frac{\phi}{4}I_a^{\delta+1} + F_4 \leq -\frac{\phi}{4}\frac{1}{\epsilon^{\delta+1}} + F_4. \quad (\text{A10})$$

Choose the same ϵ as Case 5, and thus

$$\mathcal{L}\bar{V} \leq -1 \quad \text{for any } (U_a, I_a, U_b, I_b) \in D_6.$$

Similarly, we can obtain results concerning Case 7 and Case 8. Hence, we finally reach the deduction that

$$\mathcal{L}\bar{V} \leq -1 \quad \text{for all } (U_a, I_a, U_b, I_b) \in R_4^+ \setminus D_\epsilon.$$

In summary, the second condition of Lemma 3 is satisfied, and thus Lemma 3 is proven. Therefore, the model has a unique ergodic stationary distribution.

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