

Article

When Does Global Rationality Exist? A Representation Theory for Evaluative Systems

Pascal Stiefenhofer 

Newcastle University Business School, Newcastle University, Newcastle upon Tyne NE1 4SE, UK;
pascal.stiefenhofer@newcastle.ac.uk

Highlights

Please indicate how your work links to systems science via your contributions to systems practice, theory, and/or methodology.

- This paper develops a systems framework for testing when locally defined evaluations across changing feasible configurations admit a globally coherent scalar representation.
- We establish zero circulation and path independence as exact conditions for global scalar representability in systems with architecture-dependent feasible changes.

What are the main findings and/or the implications of the main findings?

- Local rationality can exist without global rationality, so locally coherent decisions need not be representable by a single global utility, welfare, value, or potential function.
- The utility application shows that global welfare coherence exists exactly under cross-channel compatibility, providing a test for when scalar welfare comparisons are mathematically justified.

Abstract

Architecture-dependent systems pose a local-to-global problem in which locally admissible and locally evaluated changes need not admit a globally coherent scalar representation. We develop a systems-theoretic framework combining system state and architecture, admissible directions of change, local evaluation, and induced dynamics within a differential-geometric structure. We establish that, under connectedness and bracket generation, a global scalar representation exists if and only if the horizontal evaluation form has zero circulation along every closed horizontal curve, with an equivalent result if horizontal evaluation is path-independent. When it exists, the representation is unique up to an additive constant and reproduces the primitive canonical rational correspondence. Along canonical rational trajectories, its rate of change equals the maximal local evaluative rate; with additional regularity, feasible inaction, and precompactness, this yields strict ascent outside rational equilibrium, excludes non-equilibrium recurrence, and confines limiting behaviour to rational equilibria. An endogenous-opportunity application shows when locally rational economic adjustments admit a global utility or welfare representation. Representability is equivalent to the cross-channel compatibility condition ($r \in \text{Im}(C^T)$), while distance from this subspace quantifies failure of global welfare coherence. The framework therefore distinguishes local optimisation from the stronger systems property of global evaluative coherence.



Academic Editor: Gianfranco Minati

Received: 12 August 2026

Revised: 2 September 2026

Accepted: 10 September 2026

Published: 11 September 2026

Copyright: © 2026 by the author.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

Keywords: evaluative systems; global scalar representability; architecture-dependent systems; horizontal distributions; path dependence; non-integrability; differential inclusions; complex systems

MSC: 91B06; 93A30; 37N40; 49J53

1. Introduction

Dynamic economic models commonly impose a global objective, even when current choice changes the conditions governing subsequent choice. Becker and Murphy [1], for example, define rationality through a consistent intertemporal utility-maximising plan, while past consumption affects subsequent incentives. The same structural dependence arises in investment, financing, technological adoption, and relationship formation, where current decisions alter future opportunities. A global objective is a legitimate modelling primitive, but its existence does not follow from local rationality. If rational choice is instead defined by the evaluation of currently feasible adjustments, a representation problem arises. Under what conditions are these local evaluations generated by a single global objective?

The problem has established economic antecedents. Demand behaviour can fail to be integrable to a utility function [2], and sequentially myopic optimisation need not coincide with global multiperiod optimisation [3]. The present problem reverses the usual order of construction. Local evaluation defines rational adjustment independently of any global objective, whose existence must instead be established over the accessible state space. Because current actions determine subsequent feasible directions, locally rational choice may be well defined at every state, even when evaluation has nonzero circulation around a closed accessible adjustment cycle. Such circulation is incompatible with any global utility or value function. The central representation theorem gives necessary and sufficient conditions for this obstruction to vanish.

The problem matters precisely because endogenous opportunity structures are common. Path dependence and technological lock-in [4,5], organisational adaptation [6,7], technological recombination [8], and socio-technical and complex-system dynamics [9–19] all describe systems in which realised actions condition subsequent possibilities. In such systems, two adjustment paths can connect the same economic configurations while passing through different intermediate opportunity structures. A global utility or value representation requires the evaluations accumulated along those paths to be mutually compatible. That requirement is additional to local optimisation.

Economics has a substantial literature on related representation problems. Revealed-preference theory characterises when observed choice can be rationalised by preferences or utility [20–24], with extensions incorporating limited attention, information acquisition, reference dependence, and state-dependent utility [25–28]. Demand integrability asks when demand is consistent with utility maximisation [29,30]; potential-game theory asks when incentive differences admit a common potential [31]; inverse optimisation and reinforcement learning recover objectives that rationalise behaviour [32–34]; and notions of local or myopic rationality study optimisation under restricted information, horizons, or decision domains [3,35–37]. What remains to be characterised is when locally rational evaluations admit a global scalar representation when current actions endogenously determine the directions of subsequent feasible adjustment.

We take local evaluation, rather than a global objective, as primitive. At each state (z), a covector (α_z) evaluates the feasible directions ($A(z)$), and local rationality is defined by

$$R(z) = \arg \max_{v \in A(z)} \alpha_z(v).$$

Feasible motion through the state–architecture manifold is generated by a horizontal distribution (G), on which local evaluation induces the horizontal form (Θ). Global ratio-

nality is then a representation property. It holds precisely when there exists a scalar (V) such that

$$\exists V \in C^\infty(E) \quad \text{with} \quad dV|_G = \Theta.$$

Local rationality is therefore defined independently of V . The central representation theorem determines whether such a V exists, proving that global rationality holds if and only if horizontal evaluation has zero circulation around every closed accessible curve, equivalently if and only if it is path-independent.

Accessibility and integrability solve complementary parts of this problem. Differential inclusions and viability theory characterise state-dependent feasible motion [38–40], while geometric control theory characterises horizontal accessibility and nonholonomic motion [41–44]. They do not, by themselves, determine whether evaluation over accessible motion admits a global scalar. Conversely, classical utility integrability does not, by itself, address primitive evaluation over an endogenously generated nonholonomic accessibility distribution. This failure is independent of uncertainty. Even when all local evaluations are known exactly, they need not admit a globally coherent representation [45–49].

The mathematical ingredients used below include established exactness, accessibility, and differential-inclusion results. The contribution is not a new version of the Chow–Rashevskii theorem or of path independence in ordinary differential geometry. It is the representation problem obtained when a primitive directional evaluator is defined before any scalar objective, while the architecture of the system determines the directions over which that evaluator must be mutually compatible. The results identify the obstruction, show what global representation adds to independently defined rational motion, and make the obstruction operational in an endogenous-opportunity system.

Theorem 1 gives the first main result. On a connected state–architecture manifold with a smooth constant-rank bracket-generating distribution, a global scalar representation exists if and only if

$$\oint_{\eta} \Theta = 0$$

for every closed horizontal curve (η), equivalently if horizontal evaluation is path-independent. When the condition fails, the local rational correspondence remains well defined, although no scalar objective represents its primitive evaluation.

Proposition 2 and Corollary 1 provide the second main result. Whenever representation exists, the recovered scalar represents the already defined canonical rational correspondence and satisfies

$$\frac{d}{dt} V(\gamma(t)) = \rho(\gamma(t))$$

along every canonical rational trajectory. Theorems 3 and 4 then convert representability, under their additional hypotheses, into strict ascent, exclusion of non-equilibrium recurrence, localisation of limit sets, and acyclic asymptotic ordering. These conclusions are derived from representability and are not imposed through a primitive gradient or Lyapunov structure.

Proposition 5 makes the third main result systems-operational. In the endogenous-opportunity application, global representation holds exactly when

$$r \in \text{Im}(C^T),$$

so $m - q$ cross-channel restrictions distinguish global coherence from pointwise optimisation. Proposition 6 quantifies failure by

$$\delta_{\text{GR}} = \left\| \left[I - C^T (CC^T)^{-1} C \right] r \right\|,$$

the Euclidean distance from the representable subspace. Thus, the application shows that the geometric formalism identifies a restriction that local optimisation alone does not reveal—namely, the compatibility between local evaluation and the architecture endogenously generated by current actions.

The remainder of the paper is organised as follows. Section 2 develops the evaluative structure of local rationality and the associated canonical dynamics. Section 3 establishes the representation theorem and derives its dynamical consequences. Section 4 applies the theory to endogenous opportunity formation and derives the conditions for global welfare coherence. Section 5 discusses the implications and scope of the framework, and Section 6 concludes the paper. Appendix A provides the proofs of the main theoretical results and supporting technical arguments.

2. The Evaluative Structure of Local Rationality

The framework begins with local directional evaluation rather than a prescribed global scalar objective. No utility, value, cost, potential, or Lyapunov function is assumed. Instead, evaluation, feasibility, and motion are represented by distinct geometric objects. A directional bundle specifies the directions on the basis of which local evaluation is defined, a state-dependent correspondence identifies those that are feasible, and a horizontal distribution determines how evaluable directions lift to motion on the state–architecture manifold. This separation makes it possible to ask whether locally specified evaluations admit a global scalar representation without assuming one in advance.

Let E denote the state–architecture manifold and X be the underlying state manifold, related by a smooth fibre bundle $(\pi : E \rightarrow X)$. Local evaluation is defined on a smooth constant-rank vector sub-bundle $(D \subseteq TX)$, while the feasibility of configuration $z \in E$ is specified by

$$A(z) \subseteq D_{\pi(z)}.$$

Thus, $D_{\pi(z)}$ contains all directions on the basis of which evaluation is defined at z , whereas $A(z)$ contains only those directions that are feasible under that configuration. The distinction separates evaluability from feasibility and, consequently, the domain of scalar representation from the domain of feasible system motion.

Definition 1 (Evaluative Structure). *An evaluative structure is a triple $\mathfrak{E} = (E, \pi, \mathcal{A})$, where $\pi : E \rightarrow X$ is a smooth fibre bundle with connected finite-dimensional smooth manifolds E and X , and $\mathcal{A} = (D, \alpha, A, G)$ consists of*

1. *A smooth constant-rank vector sub-bundle $(D \subseteq TX)$;*
2. *A smooth section $(\alpha \in \Gamma(\pi^* D^*))$ so that, for every $z \in E$, the map $(\alpha_z : D_{\pi(z)} \rightarrow \mathbb{R})$ is linear;*
3. *A correspondence $(A : E \rightrightarrows D)$ satisfying $\emptyset \neq A(z) \subseteq D_{\pi(z)}$ for every $z \in E$, with each $A(z)$ compact and convex;*
4. *A smooth constant-rank distribution $(G \subseteq TE)$ such that $d\pi|_G : G \rightarrow \pi^* D$ is a smooth vector-bundle isomorphism over E .*

Since $d\pi|_G$ is a smooth vector-bundle isomorphism, its inverse $(L := (d\pi|_G)^{-1} : \pi^* D \rightarrow G)$ is smooth. For every $z \in E$, $L_z : D_{\pi(z)} \rightarrow G_z$ is a linear isomorphism satisfying $d\pi_z \circ L_z = \text{id}_{D_{\pi(z)}}$. Therefore, the lifted feasible set is $\hat{A}(z) := L_z(A(z)) \subseteq G_z$.

The evaluative section (α) induces a smooth horizontal covector field $(\Theta \in \Gamma(G^*))$ defined by

$$\Theta_z(\xi) := \alpha_z(d\pi_z(\xi)), \quad z \in E, \quad \xi \in G_z. \quad (1)$$

Equivalently, $\Theta_z(L_z v) = \alpha_z(v)$ for every $v \in D_{\pi(z)}$. Hence, Θ transports the primitive local evaluation from D to the horizontal distribution (G) without introducing a global scalar representation.

A piecewise C^1 curve $(\gamma : [a, b] \rightarrow E)$ is horizontal if $\dot{\gamma}(t) \in G_{\gamma(t)}$ wherever the derivative exists. Its accumulated evaluation is the horizontal line integral $(\int_{\gamma} \Theta)$. Since $\hat{A}(z) \subseteq G_z$, every feasible trajectory is horizontal, but the converse need not hold. A horizontal velocity may belong to $G_z \setminus \hat{A}(z)$. Horizontal path independence and horizontal circulation therefore concern evaluation on the full distribution (G) , not only the directions realised by feasible or canonical rational motion.

The primitives $(E, \pi, D, \alpha, A, G)$ determine local evaluation, feasibility, and horizontal motion without imposing any global scalar objective. In general, $\int_{\gamma} \Theta$ may depend on the horizontal path (γ) , and there need not exist $V : E \rightarrow \mathbb{R}$ satisfying $dV|_G = \Theta$. The representation problem developed below determines the conditions under which such a smooth global scalar representation exists.

2.1. Canonical Rational Dynamics

An evaluative structure induces a local rationality rule independently of global scalar representability. At each $z \in E$, a feasible direction is locally rational if it attains the maximal instantaneous evaluation under α_z over $A(z) \subseteq D_{\pi(z)}$. Rationality is therefore defined locally, relative to the feasible directions available under the current system configuration, without presupposing a global objective or scalar ordering.

For an evaluative structure $(\mathfrak{E} = (E, \pi, A))$, we define the instantaneous rational value (ρ) and canonical rational correspondence (R) as

$$\rho(z) := \max_{v \in A(z)} \alpha_z(v), \quad R(z) := \arg \max_{v \in A(z)} \alpha_z(v). \quad (2)$$

The corresponding horizontal rational directions are $\hat{R}(z) := L_z(R(z)) \subseteq \hat{A}(z) \subseteq G_z$.

Proposition 1 (Pointwise Structure of the Rational Correspondence). *For every $z \in E$, set $R(z)$ is nonempty, compact, and convex.*

Proposition 1 is pointwise only. Definition 1 does not, by itself, imply the continuity of ρ , the upper hemicontinuity of R or $\hat{R}$, or local boundedness of the lifted correspondence. These properties require additional regularity of A and are established below.

The inclusions $(\hat{R}(z) \subseteq \hat{A}(z) \subseteq G_z)$ distinguish the full horizontal domain, the feasible horizontal directions, and the directions attaining maximal instantaneous evaluation. Representation concerns evaluation on G , whereas canonical rational motion is generated only by $\hat{R}$.

Let $I \subseteq \mathbb{R}$ be an interval. An absolutely continuous curve $(\gamma : I \rightarrow E)$ is called a canonical rational trajectory when

$$\dot{\gamma}(t) \in \hat{R}(\gamma(t)) \quad \text{for almost every } t \in I. \quad (3)$$

Equation (3) defines the canonical rational dynamics associated with $\mathfrak{E}$.

Absolute continuity is the natural trajectory class for (3): the derivative exists almost everywhere, while the set-valued correspondence $(\hat{R})$ may admit multiple rational velocities under the same configuration. Equation (3) asserts neither existence nor uniqueness; these require additional regularity of $\hat{R}$ and are established separately. Thus, canonical rational dynamics are determined entirely by the local objects $(A, \alpha, \text{ and } L)$. Neither R nor $\hat{R}$ presupposes a global scalar objective. Global representability is a logically subsequent

property: it asks whether the already-defined local evaluation admits a smooth scalar representation, i.e., $V \in C^\infty(E)$, satisfying $dV|_G = \Theta$.

2.2. Global Scalar Representability

The horizontal evaluation form ($\Theta \in \Gamma(G^*)$) represents the primitive local evaluation on the full horizontal distribution G . By construction, $\Theta_z(L_z v) = \alpha_z(v)$ for every $z \in E$ and $v \in D_{\pi(z)}$, so the horizontal lift preserves directional evaluation exactly. Evaluation and feasibility remain distinct. The covector (Θ_z) is defined on the entire fibre (G_z), whereas $\hat{A}(z) = L_z(A(z)) \subseteq G_z$ contains only the directions feasible at z . Since $L_z : A(z) \rightarrow \hat{A}(z)$ is a linear bijection preserving evaluation,

$$\hat{R}(z) = \arg \max_{\xi \in \hat{A}(z)} \Theta_z(\xi).$$

Hence, the canonical rational correspondence is determined entirely by the local evaluative structure and does not presuppose a global scalar objective. The global representation problem is whether there exists a single smooth function ($V \in C^\infty(E)$) whose differential reproduces the primitive evaluation on every horizontal direction, that is,

$$dV|_G = \Theta.$$

An evaluative structure ($\mathfrak{E}$) is globally scalar-representable when there exists $V \in C^\infty(E)$ such that

$$dV_z|_{G_z} = \Theta_z \quad \text{for every } z \in E. \quad (4)$$

Equivalently, $dV_z(\xi) = \Theta_z(\xi)$ for every $z \in E$ and $\xi \in G_z$. Any such V is a global scalar representation.

Global scalar representability is therefore a compatibility property of the horizontal evaluative geometry. It does not require Θ to be an exact one-form on E , since Θ is defined only on G . Rather, it requires Θ to be the restriction to G of the differential of a globally defined smooth function. This requirement holds on the full horizontal distribution and is therefore stronger than agreement only on the feasible sets ($\hat{A}(z)$) or along realised canonical rational trajectories.

If V is a global scalar representation, then for every $z \in E$ and $v \in D_{\pi(z)}$,

$$dV_z(L_z v) = \Theta_z(L_z v) = \alpha_z(v). \quad (5)$$

Thus, V reproduces the primitive local evaluation throughout its directional domain. Representability does not replace the local evaluative structure by a scalar objective; it establishes that this structure admits a globally coherent scalar representation. Existence is nevertheless a global property. Local evaluation may be smooth and canonical rational dynamics may be well defined while horizontal integration remains path-dependent. The Representation Theorem identifies this obstruction and characterises representability by horizontal path independence or, equivalently, by vanishing horizontal periods. Under bracket generation, these conditions extend across the connected state–architecture manifold.

3. Representation Theory

Global scalar representability requires the local evaluation encoded by $\Theta \in \Gamma(G^*)$ to be compatible with the horizontal geometry of E . The fundamental obstruction is horizontal circulation: nonzero accumulated evaluation around a closed horizontal curve is incompatible with the restriction to G of the differential of any globally defined smooth function.

For a piecewise C^1 horizontal curve $\gamma : [a, b] \rightarrow E$, we define the *horizontal evaluative integral* as

$$\int_{\gamma} \Theta := \int_a^b (\gamma^* \Theta)_t (\partial_t) dt = \int_a^b \Theta_{\gamma(t)}(\dot{\gamma}(t)) dt, \quad (6)$$

where $t \in [a, b]$ is the integration variable and $\gamma^* \Theta$ is the pullback of the horizontal covector along γ . Integration is taken over the finitely many C^1 segments. Since $\dot{\gamma}(t) \in G_{\gamma(t)}$ wherever the derivative exists and $\Theta_{\gamma(t)} \in G_{\gamma(t)}^*$, the integral is intrinsically defined on the horizontal distribution. No extension of Θ to a differential 1-form on E is required.

We call $\mathfrak{E}$ *evaluatively consistent* when

$$\oint_{\gamma} \Theta = 0 \quad (7)$$

for every piecewise C^1 closed horizontal curve (γ) in E .

Evaluative consistency is equivalent to horizontal path independence within each horizontal accessibility class. If γ_1 and γ_2 are piecewise C^1 horizontal curves with the same endpoints, then $\gamma_1 * \bar{\gamma}_2$ is a closed piecewise C^1 horizontal curve, where $\bar{\gamma}_2$ denotes the reverse of γ_2 . Evaluative consistency therefore gives

$$\int_{\gamma_1} \Theta = \int_{\gamma_2} \Theta.$$

Conversely, horizontal path independence implies a vanishing evaluative integral around every closed horizontal curve. Importantly, this condition concerns the full distribution (G) , not merely the feasible correspondence $(\hat{A})$ or the canonical rational correspondence $(\hat{R})$. It is therefore a compatibility condition on the evaluative geometry itself rather than on trajectories selected by the canonical rational dynamics.

Global scalar representability necessarily implies evaluative consistency. If $V \in C^\infty(E)$ satisfies $dV|_G = \Theta$, then every piecewise C^1 horizontal curve $(\gamma : [a, b] \rightarrow E)$ satisfies

$$\int_{\gamma} \Theta = V(\gamma(b)) - V(\gamma(a)). \quad (8)$$

Indeed, on each C^1 segment,

$$\Theta_{\gamma(t)}(\dot{\gamma}(t)) = dV_{\gamma(t)}(\dot{\gamma}(t)) = \frac{d}{dt}(V \circ \gamma)(t),$$

and integration over the segments yields (8). Hence, every closed horizontal curve has zero evaluative integral.

The converse is the substantive local-to-global step. Suppose that G is bracket-generating on the connected finite-dimensional manifold (E) . According to the Chow–Rashevskii theorem, any two points of E can be joined by a piecewise C^1 horizontal curve. If $\mathfrak{E}$ is evaluatively consistent, we fix $z_0 \in E$ and $c \in \mathbb{R}$ and define

$$V(z) := c + \int_{\gamma_{z_0,z}} \Theta,$$

where $\gamma_{z_0,z}$ is any piecewise C^1 horizontal curve from z_0 to z . Horizontal path independence makes V well defined. The remaining issue is regularity: path independence alone does not establish that V is smooth or that $dV|_G = \Theta$, as required by global scalar representability. Because Θ is defined only on the possibly non-integrable distribution (G) , this cannot be reduced to the ordinary exactness criterion for a differential 1-form on E . The Representation Theorem resolves this issue using finite compositions of flows of local

smooth sections of G , together with bracket generation, to establish smoothness and the differential identity. It thereby establishes the equivalence between evaluative consistency and global scalar representability, with uniqueness up to an additive constant.

Lemma 1 (Finite-Flow Local Parametrisation). *Let E be a finite-dimensional smooth manifold, and let $G \subseteq TE$ be a smooth constant-rank distribution. We define*

$$\mathcal{X}_G := \{X \in \Gamma_{\text{loc}}(TE) \mid X_z \in G_z \text{ whenever } X \text{ is defined at } z\},$$

and suppose that G is bracket-generating, i.e.,

$$\text{Lie}_z(G) = T_z E \quad \text{for every } z \in E. \quad (9)$$

Then, for every $q \in E$, there exist a point $(p \in \mathcal{O}(q))$, where $\mathcal{O}(q)$ denotes the $\mathcal{X}_G$ -orbit through q), vector fields $(X_1, \dots, X_m \in \mathcal{X}_G)$, an open set $(U \subseteq \mathbb{R}^m)$, and $u^ \in U$ such that the endpoint map, i.e.,*

$$\Phi(u_1, \dots, u_m) := \varphi_{u_m}^{X_m} \circ \dots \circ \varphi_{u_1}^{X_1}(p), \quad (10)$$

is well defined and smooth on U , satisfies

$$\Phi(u^*) = q, \quad (11)$$

and has a surjective differential, i.e.,

$$D\Phi_{u^*} : \mathbb{R}^m \longrightarrow T_q E. \quad (12)$$

Consequently, after restricting U to a sufficiently small neighbourhood of u^ , there exist an open neighbourhood $(W \subseteq E)$ of q and a smooth map, i.e.,*

$$\sigma : W \longrightarrow U, \quad (13)$$

such that

$$\Phi \circ \sigma = \text{id}_W. \quad (14)$$

If E is connected, then E consists of a single $\mathcal{X}_G$ -orbit. Consequently, any two points of E can be joined by a finite concatenation of integral curves of local smooth vector fields tangent to G .

Remark 1 (Role of the Finite-Flow Parametrisation). *Lemma 1 provides more than horizontal accessibility. Bracket generation ensures that points can be connected by finite concatenations of horizontal flow segments, whereas the submersion property yields local families of such concatenations whose parameters depend smoothly on the terminal point. This smooth local parametrisation is the mechanism used in the Representation Theorem to establish regularity of the scalar function defined by horizontal evaluative integrals.*

Theorem 1 (Representation Theorem). *Let E be a connected finite-dimensional smooth manifold, $G \subseteq TE$ be a smooth constant-rank bracket-generating distribution, and $\Theta \in \Gamma(G^*)$ be a smooth horizontal evaluation form. Then, the following statements are equivalent.*

(i) *Evaluative consistency: For every piecewise C^1 closed horizontal curve (γ) ,*

$$\oint_{\gamma} \Theta = 0. \quad (15)$$

(ii) *Global scalar representability: There exists $V \in C^\infty(E)$ such that*

$$dV|_G = \Theta. \quad (16)$$

Whenever these equivalent conditions hold, horizontal evaluative integrals are path-independent. Moreover, for any $z_0 \in E$ and $c \in \mathbb{R}$,

$$V(z) = c + \int_{\gamma_{z_0,z}} \Theta, \quad (17)$$

where $\gamma_{z_0,z}$ is any piecewise C^1 horizontal curve from z_0 to z defining a smooth global scalar representation. Every global scalar representation is obtained in this way, and any two global scalar representations differ by an additive constant.

Remark 2 (Horizontal Exactness). *Theorem 1 does not require E to be simply connected. Evaluative consistency directly imposes the global vanishing-period condition required for horizontal path independence. Nor must Θ be extended to a differential 1-form on E : since $\Theta \in \Gamma(G^*)$ is intrinsically defined on G , the relevant notion of exactness is the existence of $V \in C^\infty(E)$ satisfying*

$$dV|_G = \Theta.$$

Bracket generation then serves two roles: it yields global horizontal accessibility on connected E and, through Lemma 1, supplies the local finite-flow parametrisation used to establish the smoothness of the path-defined representation.

Global scalar representability links the primitive local rationality rule to the global representation without altering the underlying dynamics. The canonical rational correspondence remains determined by A , α , and the horizontal lift (L). Representability establishes that the same correspondence can be recovered exactly by maximising the differential of the global representation over feasible horizontal directions.

Proposition 2 (Scalar Characterisation of Canonical Rational Directions). *Let $\mathfrak{E}$ be globally scalar representable, and let $V \in C^\infty(E)$ satisfy*

$$dV|_G = \Theta.$$

Then, for every $z \in E$ and $v \in A(z)$,

$$dV_z(L_z v) = \alpha_z(v). \quad (18)$$

Consequently,

$$\hat{R}(z) = \arg \max_{\xi \in \hat{A}(z)} dV_z(\xi) \quad \text{for every } z \in E. \quad (19)$$

Hence, the primitive and scalar formulations select exactly the same canonical rational directions.

Corollary 1 (Exact Evolution Identity). *Under the hypotheses of Proposition 2, let $\gamma : I \rightarrow E$ be a canonical rational trajectory. Then,*

$$\frac{d}{dt} V(\gamma(t)) = \max_{\xi \in \hat{A}(\gamma(t))} dV_{\gamma(t)}(\xi) = \rho(\gamma(t)) \quad \text{for almost every } t \in I. \quad (20)$$

Equivalently,

$$\frac{d}{dt}V(\gamma(t)) = \max_{v \in A(\gamma(t))} \alpha_{\gamma(t)}(v) \quad \text{for almost every } t \in I. \quad (21)$$

Proposition 2 establishes pointwise equivalence between the primitive and represented formulations of canonical rationality, while Corollary 1 extends this equivalence to trajectories. Global scalar representability therefore does not replace the locally defined dynamics: it yields a scalar representation whose rate of change along every canonical rational trajectory equals exactly the maximal local evaluative rate. This exact evolution identity provides the basis for the ascent, recurrence, and limit-set results below.

3.1. Rational Ascent and Asymptotic Structure

The exact evolution identity in Corollary 1 relates global scalar representability directly to canonical rational trajectories. Representability alone, however, does not imply ascent because the maximal local evaluative rate may be negative. Non-negative evolution requires the additional condition of feasible inaction.

Throughout this subsection, let $\mathfrak{E} = (E, \pi, A)$ be globally scalar representable, with $V \in C^\infty(E)$ satisfying $dV|_G = \Theta$. Assume that A has nonempty compact convex values and that inaction is feasible:

$$0 \in A(z) \quad \text{for every } z \in E. \quad (22)$$

The continuity assumptions required for correspondence regularity and existence of canonical rational trajectories are introduced below.

Recall that $\rho(z) = \max_{v \in A(z)} \alpha_z(v)$. Since $\alpha_z(0) = 0$, feasible inaction implies $\rho(z) \geq 0$ for every $z \in E$. According to Proposition 2,

$$\rho(z) = \max_{\xi \in \hat{A}(z)} dV_z(\xi), \quad \hat{R}(z) = \arg \max_{\xi \in \hat{A}(z)} dV_z(\xi).$$

Consequently, every canonical rational trajectory (γ) , whenever defined, satisfies

$$\frac{d}{dt}V(\gamma(t)) = \rho(\gamma(t)) \geq 0 \quad \text{for almost every } t. \quad (23)$$

Hence, $V \circ \gamma$ is nondecreasing along canonical rational motion.

A state $(z \in E)$ is a rational equilibrium when $0 \in \hat{R}(z)$ and a strong rational equilibrium when $\hat{R}(z) = \{0\}$. We write

$$\mathcal{E} := \{z \in E : 0 \in \hat{R}(z)\}, \quad \mathcal{E}^s := \{z \in E : \hat{R}(z) = \{0\}\}.$$

The distinction is dynamical. At a rational equilibrium, zero velocity is canonical rational, but nonzero canonical rational velocities may also be available. A strong rational equilibrium excludes all such nonzero velocities, so every canonical rational motion from that configuration is stationary.

Proposition 3 (Zero-Rate Characterisation of Rational Equilibrium). *Under condition (22),*

$$\mathcal{E} = \rho^{-1}(0). \quad (24)$$

Consequently, $\rho(z) > 0$ for every $z \notin \mathcal{E}$.

Equations (23) and (24) establish the pointwise ascent structure: the canonical rate is zero precisely on $\mathcal{E}$ and strictly positive outside it. These properties alone, however, do not establish the existence or continuation of canonical rational trajectories, nor do they determine recurrence or asymptotic behaviour. The analysis therefore proceeds in three steps. First, continuity of the fibre-valued feasible correspondence is formulated intrinsically and used to establish the regularity of ρ , R , and $\hat{R}$. Second, these properties yield existence and continuation results for the canonical differential inclusion. Third, the exact ascent identity, together with precompactness, determines recurrence and limit-set structure. Since $A(z) \subseteq D_{\pi(z)}$ takes values in fibres that vary with z , its continuity must be defined locally through vector-bundle trivialisations.

Definition 2 (Continuity of a Fibre-Valued Correspondence). *Let $A : E \rightrightarrows D$ satisfy $A(z) \subseteq D_{\pi(z)}$ for every $z \in E$. The correspondence (A) is continuous if, for every vector-bundle trivialisation $(\tau : D|_U \rightarrow U \times \mathbb{R}^d)$, the induced correspondence, i.e.,*

$$A_\tau(z) := \text{pr}_2(\tau(A(z))), \quad z \in \pi^{-1}(U), \quad (25)$$

is both upper and lower hemicontinuous as a correspondence $(A_\tau : \pi^{-1}(U) \rightrightarrows \mathbb{R}^d)$.

The definition is independent of the chosen bundle coordinates. Let τ_1 and τ_2 be trivialisations over U_1 and U_2 . On $U_1 \cap U_2$, their fibre coordinates are related by a smooth transition map $(T : U_1 \cap U_2 \rightarrow \text{GL}(d, \mathbb{R}))$, and therefore,

$$A_{\tau_2}(z) = T(\pi(z))A_{\tau_1}(z). \quad (26)$$

The map expressed as $(z, w) \mapsto (z, T(\pi(z))w)$ is a homeomorphism with continuous inverse $(z, w) \mapsto (z, T(\pi(z))^{-1}w)$. Upper and lower hemicontinuity are preserved under this fibrewise change in coordinates. Hence, Definition 2 is intrinsic.

Proposition 4 (Regularity of the Canonical Rational Correspondence). *Let $\mathfrak{E} = (E, \pi, A)$ be an evaluative structure and suppose that $A : E \rightrightarrows D$ is continuous in the sense of Definition 2, with nonempty compact convex values. Then, the following properties hold.*

- (i) *The maximal evaluative rate $(\rho : E \rightarrow \mathbb{R})$ is continuous.*
- (ii) *The canonical rational correspondence (R) is upper hemicontinuous in the fibre-valued sense and has nonempty compact convex values.*
- (iii) *The lifted correspondence $(\hat{R} : E \rightrightarrows TE)$ is upper hemicontinuous and locally bounded, with nonempty compact convex values satisfying $\hat{R}(z) \subseteq G_z$ for every $z \in E$.*

Theorem 2 (Existence and Continuation of Canonical Rational Trajectories). *Assume the hypotheses of Proposition 4. Then, for every $z_0 \in E$, there exist $\varepsilon > 0$ and an absolutely continuous canonical rational trajectory $(\gamma : [0, \varepsilon] \rightarrow E)$ satisfying*

$$\gamma(0) = z_0, \quad \dot{\gamma}(t) \in \hat{R}(\gamma(t)) \quad \text{for almost every } t \in [0, \varepsilon]. \quad (27)$$

Every local solution admits a maximal forward extension, i.e., $\gamma : [0, T_{\max}) \rightarrow E$, where $0 < T_{\max} \leq \infty$. If $T_{\max} < \infty$, then $\gamma([0, T_{\max}))$ is not relatively compact in E . Equivalently,

$$\overline{\gamma([0, T_{\max}))} \text{ compact} \implies T_{\max} = \infty. \quad (28)$$

Hence, every precompact maximal canonical rational trajectory is forward complete.

Theorem 3 (Rational Ascent and Limit-Set Theorem). *Let $\mathfrak{E} = (E, \pi, \mathcal{A})$ be globally scalar-representable, and let $V \in C^\infty(E)$ satisfy $dV|_G = \Theta$. Assume that A is continuous in the sense of Definition 2, has nonempty compact convex values, and satisfies*

$$0 \in A(z) \quad \text{for every } z \in E. \quad (29)$$

Then, every canonical rational trajectory (γ) satisfies the following properties.

- (i) *For all $t_1 \leq t_2$ in the domain of γ ,*

$$V(\gamma(t_2)) - V(\gamma(t_1)) = \int_{t_1}^{t_2} \rho(\gamma(s)) \, ds \geq 0. \quad (30)$$

Hence, $V \circ \gamma$ is nondecreasing.

- (ii) *If*

$$\gamma([t_1, t_2]) \not\subseteq \mathcal{E}$$

for some $t_1 < t_2$, then

$$V(\gamma(t_2)) > V(\gamma(t_1)). \quad (31)$$

Thus, equality in (30) on a nondegenerate interval occurs if and only if the entire trajectory segment lies in $\mathcal{E}$.

- (iii) *Every periodic canonical rational trajectory is contained in $\mathcal{E}$. More generally, if $z = \gamma(t_0)$ is recurrent in the sense that there exists a sequence $(t_n \rightarrow \infty)$ with $\gamma(t_n) \rightarrow z$, then $z \in \mathcal{E}$.*
 (iv) *If $\gamma : [0, \infty) \rightarrow E$ is precompact, then $\omega(\gamma)$ is nonempty, compact, and connected, and*

$$\omega(\gamma) \subseteq \mathcal{E}. \quad (32)$$

Consequently, for any fixed Riemannian metric on E ,

$$\text{dist}(\gamma(t), \mathcal{E}) \longrightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (33)$$

- (v) *If the trajectory in (iv) is precompact and, in addition, $\mathcal{E}$ is discrete, then there exists $z^* \in \mathcal{E}$ such that*

$$\gamma(t) \longrightarrow z^* \quad \text{as } t \rightarrow \infty. \quad (34)$$

Remark 3 (Strong Equilibria Are Not Required). *The convergence conclusion requires discreteness of the rational-equilibrium set $(\mathcal{E})$ but not the stronger condition $(\widehat{R}(z) = \{0\})$. At a point $(z \in \mathcal{E} \setminus \mathcal{E}^s)$, the zero direction is canonically rational, although nonzero canonical rational directions may also be available. Theorem 3 nevertheless implies that every precompact forward trajectory has a nonempty connected omega-limit set contained in $\mathcal{E}$. When $\mathcal{E}$ is discrete, this limit set must be a singleton, which is sufficient for convergence.*

3.2. Global Rationality

Global scalar representability imposes structure beyond the pointwise characterisation of canonical rationality. Together with feasible inaction and the exact evolution identity, it globally orients canonical rational motion through the represented value (V) . Along every canonical rational trajectory, V is nondecreasing; for complete precompact trajectories not contained in the rational-equilibrium set, the backward and forward asymptotic levels are strictly ordered.

This global ordering is not imposed through a primitive Lyapunov function. It is derived from the local evaluative geometry through scalar representability and the exact evolution identity.

To distinguish asymptotic limit sets from the primitive evaluative section (α), we define the backward and forward limit sets of a complete trajectory ($\gamma : \mathbb{R} \rightarrow E$) as

$$\omega^-(\gamma) := \bigcap_{T>0} \overline{\gamma((-\infty, -T])}, \quad \omega^+(\gamma) := \bigcap_{T>0} \overline{\gamma([T, \infty)).}$$

Theorem 4 (Global Rationality and Acyclic Asymptotic Ordering). *Let $\mathfrak{E} = (E, \pi, \mathcal{A})$ be globally scalar-representable, and let $V \in C^\infty(E)$ satisfy $dV|_G = \Theta$. Assume that A is continuous in the sense of Definition 2, has nonempty compact convex values, and satisfies*

$$0 \in A(z) \quad \text{for every } z \in E.$$

Let $\gamma : \mathbb{R} \rightarrow E$ be a complete precompact canonical rational trajectory. Then, the following properties hold.

(i) *The finite limits of*

$$V_- := \lim_{t \rightarrow -\infty} V(\gamma(t)), \quad V_+ := \lim_{t \rightarrow \infty} V(\gamma(t)) \quad (35)$$

exist and satisfy $V_- \leq V_+$.

(ii) *The sets expressed as $\omega^-(\gamma)$ and $\omega^+(\gamma)$ are nonempty, compact, and connected, and*

$$\omega^-(\gamma) \subseteq \mathcal{E}, \quad \omega^+(\gamma) \subseteq \mathcal{E}. \quad (36)$$

Moreover,

$$V|_{\omega^-(\gamma)} \equiv V_-, \quad V|_{\omega^+(\gamma)} \equiv V_+. \quad (37)$$

(iii) *If $\gamma(\mathbb{R}) \not\subseteq \mathcal{E}$, then*

$$V_- < V_+. \quad (38)$$

Hence, every complete, precompact canonical rational trajectory not contained in $\mathcal{E}$ is asymptotically oriented from a strictly lower backward V -level to a strictly higher forward V -level.

(iv) *If $z^-, z^+ \in \mathcal{E}$ satisfy*

$$\omega^-(\gamma) = \{z^-\}, \quad \omega^+(\gamma) = \{z^+\},$$

and $\gamma(\mathbb{R}) \not\subseteq \mathcal{E}$, then

$$V(z^-) < V(z^+). \quad (39)$$

Consequently, no finite directed cycle of complete, precompact non-equilibrium canonical rational connections between rational equilibria exists.

Theorem 4 strengthens the one-sided asymptotic conclusions of Theorem 3. Complete, precompact canonical rational trajectories have both backward and forward limit sets in $\mathcal{E}$, and every trajectory not contained in $\mathcal{E}$ strictly separates its backward and forward represented-value levels. This induces an orientation on equilibrium-to-equilibrium canonical connections.

For $z^-, z^+ \in \mathcal{E}$, we write $z^- \prec z^+$ when there exists a complete, precompact canonical rational trajectory ($\gamma : \mathbb{R} \rightarrow E$) such that

$$\omega^-(\gamma) = \{z^-\}, \quad \omega^+(\gamma) = \{z^+\}, \quad \gamma(\mathbb{R}) \not\subseteq \mathcal{E}.$$

The $\prec$ relation records direct canonical asymptotic connections. It need not be transitive. In particular, $z_1 \prec z_2$ and $z_2 \prec z_3$ do not imply the existence of a single canonical trajectory directly connecting z_1 to z_3 . Let $\prec^+$ denote the transitive closure of $\prec$.

Corollary 2 (Asymptotic Rational Order). For $z^-, z^+ \in \mathcal{E}$,

$$z^- \prec^+ z^+ \implies V(z^-) < V(z^+). \quad (40)$$

Consequently, $\prec^+$ is a strict partial order on the subset of $\mathcal{E}$ participating in the canonical asymptotic connection structure. In particular, the connection structure is acyclic.

Global rationality therefore entails more than monotonicity along individual trajectories. The scalar representation derived from the local evaluative geometry strictly orders complete precompact, non-equilibrium transitions between rational equilibria and excludes directed cycles among them. Hence, the asymptotic order of rational motion is derived from the globally representable local evaluative structure rather than imposed as a primitive ordering of the dynamics.

4. Global Welfare Coherence Under Endogenous Opportunity Formation

Economic representation problems usually ask whether observed choice or demand can be rationalised by preferences or utility. Revealed-preference theory provides behavioural conditions for such rationalisation [20–24], with extensions allowing limited attention, information acquisition, reference dependence, and state-dependent utility [25–28]. Classical integrability theory similarly asks when local demand restrictions are compatible with utility maximisation [29,30].

The representation problem developed here has a different primitive object. A feasible-direction correspondence ($A(z)$) and a local evaluative covector (α_z) determine rational adjustment before any global objective is assumed. Theorem 1 then determines whether evaluation over the resulting accessible state space is generated by a single scalar. The distinction becomes economically substantive when current actions alter subsequent opportunities. Collateral, lending relationships, earnings, technological adoption, and relationship-specific investment can all change the future domain of feasible adjustment. Evidence from corporate finance documents precisely such endogenous opportunity formation [50–53].

The application below demonstrates the role of the full representation machinery. Local maximisation will remain well defined independently of global welfare coherence. Endogenous opportunity formation generates a nonholonomic accessibility structure. Theorem 1 then reduces global welfare coherence to an explicit cross-channel restriction, Proposition 6 quantifies its failure, and the dynamical results of Section 3 determine what follows when representation exists.

Let

$$X \subseteq \mathbb{R}^{2m}$$

be connected and open and define

$$E = X \times \mathbb{R}^q,$$

with coordinates of

$$z = (x_1, y_1, \dots, x_m, y_m, a_1, \dots, a_q).$$

The x_j variables represent directly economic positions such as productive investment, technological adoption, relationship capital, or market participation. The y_j variables determine how strongly adjustment through channel j changes subsequent opportunities. The vector expressed as

$$a = (a_1, \dots, a_q)$$

records the resulting opportunity architecture. In a financing interpretation, its components may represent distinct dimensions of financing accessibility. Let

$$\pi : E \rightarrow X$$

be the canonical projection and take $D = TX$.

Let

$$C = (c_{\ell j}) \in \mathbb{R}^{q \times m}, \quad \text{rank } C = q < m,$$

describe how the m economic adjustment channels generate the q architectural dimensions. We define

$$H_{2j-1} = \frac{\partial}{\partial x_j} + y_j \sum_{\ell=1}^q c_{\ell j} \frac{\partial}{\partial a_\ell}, \quad (41)$$

$$H_{2j} = \frac{\partial}{\partial y_j}, \quad j = 1, \dots, m, \quad (42)$$

and

$$G = \text{span}\{H_1, \dots, H_{2m}\}.$$

Since

$$d\pi(H_{2j-1}) = \frac{\partial}{\partial x_j}, \quad d\pi(H_{2j}) = \frac{\partial}{\partial y_j},$$

fields $H_1, \dots, H_{2m}$ are pointwise linearly independent. Therefore, G is a smooth constant-rank distribution, and

$$d\pi|_{G_z} : G_z \longrightarrow D_{\pi(z)}$$

is an isomorphism for every $z \in E$. The horizontal lift, i.e.,

$$L_z = (d\pi|_{G_z})^{-1},$$

is consequently well defined.

A horizontal trajectory satisfies

$$\dot{a} = C \begin{pmatrix} y_1 \dot{x}_1 \\ \vdots \\ y_m \dot{x}_m \end{pmatrix}. \quad (43)$$

Equation (43) makes the opportunity architecture endogenous. The state (a) is not an independent instantaneous control but is generated by sequences of directly economic adjustments.

The same construction verifies the accessibility hypothesis used by Theorem 1. For every j ,

$$[H_{2j-1}, H_{2j}] = - \sum_{\ell=1}^q c_{\ell j} \frac{\partial}{\partial a_\ell}.$$

Because $\text{rank } C = q$, these brackets span all architectural directions. Hence,

$$\text{Lie}_z(G) = T_z E \quad \text{for every } z \in E,$$

and G is bracket-generating of step two. Since X is connected, $E = X \times \mathbb{R}^q$ is connected. The present economy therefore satisfies the connectedness, smooth constant-rank, and bracket-generation conditions under which Theorem 1 gives a global representation criterion.

Let

$$U_0 : X \rightarrow \mathbb{R}$$

be smooth, and let

$$r = (r_1, \dots, r_m)^\top \in \mathbb{R}^m$$

collect the primitive marginal evaluations of architecture-producing adjustment through the m channels. For

$$u = (u_{x_1}, u_{y_1}, \dots, u_{x_m}, u_{y_m}) \in D_{\pi(z)},$$

define

$$\alpha_z(u) = (dU_0)_{\pi(z)}(u) + \sum_{j=1}^m r_j y_j u_{x_j}. \quad (44)$$

No welfare or value function on E is assumed. The horizontal evaluation form of the general theory is

$$\Theta_z(L_z u) = \alpha_z(u).$$

State-dependent feasibility is given by

$$A(z) = \left\{ u \in D_{\pi(z)} \mid u^\top M(z) u \leq 1 \right\}, \quad (45)$$

where $M(z)$ is smooth and positive definite. Thus, $A(z)$ is nonempty, compact, and strictly convex and contains 0, while

$$\hat{A}(z) = L_z(A(z)) \subseteq G_z.$$

Let $g(z)$ be the coordinate vector of the α_z covector in (44). The maximal primitive evaluation rate is

$$\rho(z) = \max_{u \in A(z)} \alpha_z(u) = \sqrt{g(z)^\top M(z)^{-1} g(z)}.$$

If $g(z) \neq 0$, strict convexity of (45) gives the unique locally rational adjustment, i.e.,

$$u^*(z) = \frac{M(z)^{-1} g(z)}{\sqrt{g(z)^\top M(z)^{-1} g(z)}}. \quad (46)$$

If $g(z) = 0$, every feasible direction is locally maximal, and $\rho(z) = 0$. Hence,

$$R(z) = \arg \max_{u \in A(z)} \alpha_z(u), \quad \hat{R}(z) = L_z(R(z)),$$

with

$$R(z) = \{u^*(z)\} \quad \text{whenever } g(z) \neq 0.$$

Equations (44)–(46) therefore determine local rationality without a global scalar objective.

Theorem 1 now asks whether the independently specified local evaluator and endogenous accessibility structure are globally compatible. In this economy, the abstract zero-circulation condition admits an exact algebraic characterisation.

Proposition 5 (Architecture compatibility and global welfare coherence). *For the economic system defined above, the following statements are equivalent.*

1. There exists $V \in C^\infty(E)$ such that

$$dV|_G = \Theta.$$

2. Every piecewise C^1 closed horizontal economic adjustment cycle (Γ) satisfies

$$\oint_{\Gamma} \Theta = 0.$$

3.

$$r \in \text{Im}(C^T).$$

4. There exists a unique $\beta \in \mathbb{R}^q$ such that

$$r = C^T \beta.$$

Whenever these conditions hold,

$$V(x, a) = U_0(x) + \beta^T a$$

is a global scalar representation.

Proposition 5 converts Theorem 1 into an overidentifying economic restriction. The β vector gives common shadow values to the q architectural dimensions and requires

$$r_j = C_{\cdot j}^T \beta, \quad j = 1, \dots, m.$$

Since

$$\text{rank } C = q < m,$$

the representable subspace ($\text{Im}(C^T)$) has codimension $m - q$. Global welfare coherence therefore imposes exactly $m - q$ independent cross-channel restrictions that are absent from the pointwise maximisation problem (46).

The smallest overidentified economy makes the distinction explicit. Let

$$m = 3, \quad q = 2, \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The first two channels generate the architectural dimensions separately, while the third generates both. Since

$$\ker C = \text{span} \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\},$$

Proposition 5 gives the necessary and sufficient condition, i.e.,

$$r_3 = r_1 + r_2.$$

When it holds,

$$\beta = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}, \quad V(x, a) = U_0(x) + r_1 a_1 + r_2 a_2.$$

The failure case shows why the horizontal machinery is required. Let

$$r = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \quad A = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \in \ker C.$$

Then,

$$CA = 0, \quad r^T A = 1.$$

To connect this algebraic incompatibility to the geometry of Theorem 1, let Γ be a closed horizontal cycle and define

$$A_j = \oint_{\Gamma} y_j dx_j, \quad A = (A_1, \dots, A_m)^T.$$

Integrating (43) gives

$$\Delta a = CA.$$

Thus, $A \in \ker C$ corresponds to zero net architectural displacement. Conversely, because X is open, sufficiently small rectangular horizontal loops in the (x_j, y_j) -planes can realise any sufficiently small scalar multiple of such an A . Concatenating these loops produces a closed horizontal adjustment cycle with $CA = 0$.

Using (44),

$$\oint_{\Gamma} \Theta = \oint_{\Gamma} dU_0 + \sum_{j=1}^m r_j \oint_{\Gamma} y_j dx_j = r^T A,$$

because the exact baseline term has zero circulation. For the vector above,

$$\oint_{\Gamma} \Theta = 1 \neq 0.$$

The zero-circulation condition in Theorem 1 therefore fails, and no $V \in C^\infty(E)$ satisfying $dV|_G = \Theta$ exists. Nevertheless, (45) and (46) continue to determine locally rational adjustment. Pointwise rationality is therefore preserved while global welfare coherence fails. The obstructing cycle belongs to the horizontal accessibility geometry generated by G and need not, itself, be a canonical rational trajectory generated by $\hat{R}$.

The same geometry yields a quantitative measure of failure. We define

$$P_C = C^T(CC^T)^{-1}C.$$

Since C has full row rank, P_C is the orthogonal projection onto $\text{Im}(C^T)$. Let

$$\delta_{GR} = \|(I - P_C)r\|. \quad (47)$$

Proposition 6 (Distance to global representability). *For the economic system above,*

$$\delta_{GR} = \text{dist}(r, \text{Im}(C^T)) = \sup_{\substack{A \in \ker C \\ \|A\|=1}} |r^T A|. \quad (48)$$

Consequently,

$$\delta_{GR} = 0$$

if and only if the equivalent conditions of Proposition 5 hold.

Equation (47) is the minimum Euclidean perturbation of the primitive channel evaluations required for global welfare coherence. Equation (48) gives the equivalent maximal primitive circulation coefficient over unit architecture-neutral signed-area vectors. In the three-channel economy,

$$\delta_{GR} = \frac{|r_3 - r_1 - r_2|}{\sqrt{3}}.$$

Hence, for $r = (1, 1, 3)^\top$,

$$\delta_{GR} = \frac{1}{\sqrt{3}}.$$

The abstract obstruction identified by Theorem 1 is therefore both economically interpretable and quantitatively measurable.

When

$$\delta_{GR} = 0,$$

the representation machinery delivers more than the existence of V . Proposition 2 gives

$$dV_z(L_z v) = \alpha_z(v) \quad \text{for every } v \in A(z),$$

and therefore,

$$\hat{R}(z) = \arg \max_{\xi \in \hat{A}(z)} dV_z(\xi).$$

The recovered scalar consequently represents the rational correspondence defined independently by (44) and (45).

Corollary 1 then yields, along every canonical rational trajectory, i.e.,

$$\dot{\gamma}(t) \in \hat{R}(\gamma(t)),$$

the exact evolution identity, i.e.,

$$\frac{d}{dt} V(\gamma(t)) = \rho(\gamma(t)) \quad \text{for almost every } t. \quad (49)$$

Since $0 \in A(z)$ according to (45),

$$\rho(z) \geq 0,$$

while the ellipsoidal structure gives

$$\rho(z) = 0 \iff g(z) = 0.$$

Thus, the static compatibility condition in Proposition 5 becomes an exact law of welfare evolution along the rational dynamics.

Under the hypotheses of Theorem 3, (49) implies strict ascent away from rational equilibrium, excludes non-equilibrium recurrence, and places the omega-limit set of every precompact canonical rational trajectory in the rational-equilibrium set. Under the additional hypotheses of Theorem 4, complete precompact rational trajectories are ordered by their asymptotic V -levels, and finite cycles of non-equilibrium connections are excluded. The representation condition therefore determines not only whether a global welfare scalar exists but also whether independently specified locally rational motion admits a global acyclic ordering.

The empirical literature motivating (43) does not separately identify theoretical objects C and r . Reduced-form estimates of collateral, relationship, or earnings effects cannot, without additional identifying assumptions, be interpreted simultaneously as estimates of the architecture-production matrix and the primitive evaluation vector. The theory nevertheless yields a testable structural restriction whenever these objects can be separately identified. Proposition 5 requires

$$(I - P_C)r = 0,$$

while Proposition 6 measures its violation by

$$\delta_{GR} = \|(I - P_C)r\|.$$

Observed behaviour satisfying the local optimisation problem (46) therefore does not establish global welfare coherence. That stronger conclusion follows if and only if the zero-circulation condition of Theorem 1 holds.

The application consequently uses the complete local-to-global theorem chain. Equation (44) specifies primitive evaluation; (45) and (46) determine local rationality; (43) generates the endogenous accessibility structure; Proposition 5 determines global representability; Proposition 6 measures its failure; and Corollary 1, together with Theorems 3 and 4, determines the global dynamics whenever representation exists.

5. Discussion

5.1. Rationality as a Systems Property

Theorem 1 establishes a systems distinction between local rationality and global evaluative coherence. Local rationality requires well-defined evaluation and selection under each configuration. Global scalar rationality additionally requires

$$dV|_G = \Theta$$

or, equivalently, zero circulation on every closed accessible curve. Pointwise optimisation therefore does not imply system-wide rationality. The latter is an integrability property generated jointly by local evaluation and the architecture of accessibility.

This places rationality alongside other system-level properties that depend on relations among components rather than on components considered separately. Higher-order interactions, information pathways, and network architecture can determine collective dynamics, resilience, and critical behaviour [15–19]. The corresponding property here is evaluative coherence. A system may possess locally rational responses everywhere while their composition through accessible paths prevents any global scalar ordering.

The circulation criterion makes this failure operational. If an accessible closed curve (Γ) satisfies

$$\oint_{\Gamma} \Theta \neq 0,$$

evaluation is path-dependent, even though the system returns to the same configuration. This strengthens conventional accounts of path dependence, where history changes the state ultimately reached [4–8]. Here, history can instead change accumulated evaluation between identical endpoints. Nonzero circulation therefore identifies structural path dependence in the evaluative system itself.

This distinction also limits what state augmentation can achieve. Adding history, institutions, technology, or opportunity variables may render system evolution Markovian, but it does not imply integrability of the resulting evaluative field. For systems in which actions reshape future opportunities, state completeness and scalar representability are separate properties. Practically, expanding a model until it predicts system evolution does not establish that the same model admits a coherent welfare, reward, or performance function. Theorem 1 provides the additional test required to establish that stronger property.

5.2. Relation to Rationality, Integrability, and Systems Theory

Theorem 1 connects two literatures that usually address different questions. Revealed-preference and integrability theory study when local or observed choice relations admit a scalar utility representation [22,23], while potential games ask when decentralised incen-

tives admit a common potential [31]. Viability theory and geometric control instead determine which motions and states are accessible under system constraints [38,40,43,44]. Here, these questions are coupled. The distribution (G) determines the accessible geometry, and

$$dV|_G = \Theta$$

holds precisely when the primitive directional evaluation is integrable over that geometry. Global rationality is therefore a compatibility property of evaluation and system accessibility, not a consequence of pointwise optimisation.

This distinction also separates representation from inverse optimisation. Inverse optimal control and inverse reinforcement learning seek an objective capable of rationalising observed behaviour [32–34]. Theorem 1 addresses the logically prior existence question. If

$$\oint_{\Gamma} \Theta \neq 0$$

for an accessible closed curve, no smooth scalar satisfying $dV|_G = \Theta$ exists. Additional data may reduce uncertainty about an unknown objective but cannot identify an objective that the evaluative architecture does not admit. This provides a practical diagnostic distinction between parameter identification and structural non-representability.

The economic results make that distinction testable. Proposition 5 gives the exact compatibility condition, i.e.,

$$r \in \text{Im}(C^T),$$

while Proposition 6 measures its failure by

$$\delta_{GR} = \text{dist}(r, \text{Im}(C^T)).$$

Thus, local optimisation may remain well defined, even when the architecture precludes a common scalar representation. This complements systems research in which network architecture, interdependence, and stakeholder interaction generate system-level properties not visible when components are considered separately [54–56].

Representability also changes what can be concluded dynamically. Corollary 1 gives

$$\frac{d}{dt} V(\gamma(t)) = \rho(\gamma(t)),$$

and Theorem 3 converts this identity, under its stated regularity conditions, into strict ascent away from equilibrium and exclusion of non-equilibrium recurrence. Theorem 4 then orders complete non-equilibrium connections by asymptotic V -levels. Acyclic global organisation is therefore derived from evaluative integrability rather than imposed through a gradient system or an exogenous Lyapunov function. This is the principal systems consequence of scalar representation.

Finally, structural non-representability should not be confused with uncertainty. Knightian uncertainty, ambiguity, and deep uncertainty concern incomplete knowledge of outcomes, probabilities, or models [45–47,49]. Nonzero circulation can persist even when

$$(E, \pi, D, \alpha, A, G)$$

is completely known. Learning can therefore reduce epistemic uncertainty without restoring global rationality. Restoration requires a change in the evaluative structure, the accessibility architecture, or their compatibility.

5.3. Policy and Governance Implications

Theorem 1 has a direct implication for policy systems in which current interventions alter future opportunities. Policy analysis often begins with a scalar welfare or performance criterion. The theorem shows that, when evaluation is specified locally and interventions reshape the accessible system, the existence of such a scalar must first be established. Local optimality of individual interventions does not imply global policy coherence.

Proposition 5 makes this requirement operational. Global coherence in the economic application requires

$$r \in \text{Im}(C^T).$$

Channel-specific evaluations must therefore be compatible with the architecture through which interventions modify future opportunities. This is relevant to multi-instrument systems in which financing, technology, infrastructure, institutional relationships, or market access are both affected by current policy and constrain subsequent action. Evaluating each instrument separately can miss incompatibilities generated by their joint effect on the opportunity architecture.

The practical implication is a change in the sequence of policy analysis. Where interventions reshape future feasible sets, representability should be tested before a scalar objective is used for system-wide optimisation. If C and r are separately identified on substantively meaningful scales, Proposition 6 provides the diagnostic, i.e.,

$$\delta_{GR} = \|(I - P_C)r\|.$$

The condition of $\delta_{GR} = 0$ supports scalar representation, whereas $\delta_{GR} > 0$ identifies structural incompatibility between local evaluation and opportunity formation. The distinction matters for intervention design because failure may require revision of evaluative weights, redesign of the opportunity architecture, or abandonment of a single scalar representation rather than further optimisation of an incompatible objective.

The same diagnostic logic applies to socio-technical and multi-agent systems, where algorithmic decisions, human judgement, regulatory constraints, and heterogeneous objectives interact through a common architecture [57–61]. Related systems research emphasises that network architecture and stakeholder interaction can determine system-level outcomes [55,56]. The present results add a representation question to this systems perspective. Apparent misalignment need not result from incorrectly calibrated parameters within a common objective. The local evaluators, themselves, may be incompatible with any common scalar representation over the accessible architecture.

The policy contribution is therefore diagnostic rather than prescriptive. The theory does not determine welfare weights or preferred institutional designs. It determines whether scalar policy evaluation is compatible with the system architecture and, where it is not, whether attention should shift from optimisation within the existing model towards the evaluative structure, the opportunity architecture, or their interaction.

5.4. Limitations and Research Extensions

The assumptions underlying the results delimit their scope and suggest several extensions. First, Theorem 1 uses bracket generation to obtain horizontal accessibility on the connected state space. Real systems may instead be modular or institutionally segmented, as in systems research, where resilience depends on connections among subsystems [54,55]. Without bracket generation, representation becomes orbitwise, i.e.,

$$dV_O|_G = \Theta,$$

on each accessibility orbit ($\mathcal{O}$). This extension could distinguish incoherence within subsystems from incoherence generated by their coupling.

Second, the condition of

$$dV|_G = \Theta$$

is stronger than representation on feasible directions or realised trajectories. Hence,

horizontal representation $\implies$ feasible representation $\implies$ trajectory representation,

where the reverse implications require additional conditions. Characterising these conditions would clarify when an objective estimated from observed trajectories remains valid for counterfactual actions or system-wide evaluation.

Third, the framework is deterministic and single-evaluative. Networked and socio-technical systems may instead contain heterogeneous decision centres, adaptive behaviour, and distributed objectives [55,56]. Replacing α with

$$\{\alpha^i\}_{i \in N}$$

would allow interacting evaluators to admit a common potential, require a vector-valued representation, or be structurally incompatible. Dynamic or stochastic extensions of α_t , A_t , and G_t could similarly determine whether evaluative coherence survives learning, shocks, and architectural adaptation, including irreversible changes in flow architecture [62].

Fourth, the asymptotic results require regularity, completeness, and precompactness beyond the conditions needed for representation. Systems with switching regimes, discontinuous adjustment, cascading failures, or unbounded trajectories may therefore admit a scalar representation without satisfying the global dynamical conclusions. Hybrid and piecewise-smooth extensions could identify whether coherence survives structural change and whether reconfiguration can restore it, building on related stability analysis for nonsmooth dynamics [54,55,63].

Finally, Propositions 5 and 6 provide structural restrictions rather than an empirical identification strategy. Given separately identified C and r , the theory yields the testable restriction of

$$H_0: (I - P_C)r = 0$$

and the departure measure expressed as

$$\delta_{GR} = \|(I - P_C)r\|.$$

Statistical inference for these objects would make global rational coherence empirically testable and help distinguish failures arising from local evaluation, opportunity architecture, or their incompatibility.

These extensions preserve the paper's central systems question of whether the architecture of feasible system evolution permits a globally coherent scalar representation.

6. Conclusions

This paper develops a systems account of rationality that separates locally rational adjustment from global scalar coherence. Rather than assuming a utility, welfare, value, potential, or Lyapunov function, the framework asks when local evaluations generated by feasible system change can be integrated into a common global representation. Global coherence is therefore a property to be established rather than imposed.

The endogenous-opportunity application gives this distinction a direct economic interpretation. When current actions reshape future opportunities, local optimisation

may remain well defined even though no common utility or welfare representation exists. The failure of global rationality may therefore arise from incompatibility between local evaluation and the architecture through which feasible choices are connected rather than from locally irrational behaviour.

For systems analysis, this provides a practical diagnostic principle. Before using a scalar objective for optimisation, control, policy evaluation, or system design, one can test whether local evaluations are globally compatible. Failure of this test indicates that architectural reconfiguration may be required rather than further optimisation of an incoherent objective. The framework therefore shifts the systems question from how best to optimise a given objective to whether the architecture of feasible system evolution permits a coherent objective to exist.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The author declares no conflicts of interest.

Appendix A. Proofs of the Main Results

For completeness, the proofs of all lemmas, propositions, theorems, and corollaries are collected here in the order in which the results appear in the main text.

Proof of Proposition 1. Fix $z \in E$. Since $A(z)$ is nonempty and compact and $\alpha_z : D_{\pi(z)} \rightarrow \mathbb{R}$ is continuous, the maximum in (2) is attained. Hence, $R(z)$ is nonempty. Because $R(z)$ is closed in the compact set $(A(z))$, it is compact.

Let $v_1, v_2 \in R(z)$ and $\lambda \in [0, 1]$. Convexity of $A(z)$ gives $\lambda v_1 + (1 - \lambda)v_2 \in A(z)$, while linearity of α_z implies

$$\alpha_z(\lambda v_1 + (1 - \lambda)v_2) = \lambda \alpha_z(v_1) + (1 - \lambda) \alpha_z(v_2) = \rho(z).$$

Thus, $\lambda v_1 + (1 - \lambda)v_2 \in R(z)$, proving convexity. $\square$

Proof of Lemma 1. Fix $q \in E$ and let $\mathcal{O}(q)$ denote its $\mathcal{X}_G$ -orbit. According to the Stefan–Sussmann orbit theorem, $\mathcal{O}(q)$ is a connected, immersed submanifold of E , and its tangent distribution is the smallest distribution containing the values of the vector fields in $\mathcal{X}_G$ and invariant under their local flows [64]. In particular, for every $z \in \mathcal{O}(q)$,

$$\text{Lie}_z(G) \subseteq T_z \mathcal{O}(q) \subseteq T_z E. \quad (\text{A1})$$

According to the bracket-generating hypothesis (9),

$$T_z \mathcal{O}(q) = T_z E \quad \text{for every } z \in \mathcal{O}(q). \quad (\text{A2})$$

Hence, $\dim \mathcal{O}(q) = \dim E$; therefore, $\mathcal{O}(q)$ is open in E .

The finite-composition description of the orbit supplied by the Stefan–Sussmann theorem implies that $T_q \mathcal{O}(q)$ is spanned by endpoint variations generated by finite compositions of local flows of vector fields in $\mathcal{X}_G$ [64]. Since $T_q \mathcal{O}(q)$ is finite-dimensional, finitely many such endpoint variations may be selected to span $T_q \mathcal{O}(q)$. Collecting the corresponding flow parameters into a single finite composition, one may choose vector fields $(X_1, \dots, X_m \in \mathcal{X}_G)$, a point $(p \in \mathcal{O}(q))$, and a parameter $(u^* \in \mathbb{R}^m)$

such that the endpoint map (Φ) defined in (10) is smooth on an open neighbourhood (U of u^*), satisfies $\Phi(u^*) = q$, and

$$\text{Im } D\Phi_{u^*} = T_q\mathcal{O}(q). \quad (\text{A3})$$

Collecting the corresponding flow parameters into the endpoint map (Φ) therefore gives

$$\text{Im } D\Phi_{u^*} = T_q\mathcal{O}(q).$$

Combining this identity with (A2) yields

$$\text{Im } D\Phi_{u^*} = T_qE,$$

so $D\Phi_{u^*}$ is surjective.

According to the submersion theorem, after restricting U to a sufficiently small neighbourhood of u^* , there exist an open neighbourhood ($W \subseteq E$) of q and a smooth local right inverse, i.e.,

$$\sigma : W \longrightarrow U,$$

such that

$$\Phi \circ \sigma = \text{id}_W.$$

Hence, every point sufficiently close to q can be reached from the fixed point (p) by a finite concatenation of horizontal flow segments whose flow parameters depend smoothly on the terminal point.

The global accessibility statement remains to be established. The $\mathcal{X}_G$ -orbits form a partition of E . The argument above applies to every orbit and shows that each is open. If E is connected, it cannot be partitioned into more than one nonempty, disjoint, open orbit. Hence,

$$\mathcal{O}(q) = E \quad \text{for every } q \in E.$$

According to the definition of the orbit relation, any two points of E can therefore be joined by a finite concatenation of integral curves of local smooth vector fields belonging to $\mathcal{X}_G$. $\square$

Proof of Theorem 1. Since G is bracket-generating and E is connected, Lemma 1 implies that E consists of a single $\mathcal{X}_G$ -orbit. Hence, any two points of E can be joined by a finite concatenation of integral curves of local smooth vector fields tangent to G and, therefore, by a piecewise C^1 horizontal curve.

Suppose first that (ii) holds. Let $\gamma : [a, b] \rightarrow E$ be a piecewise C^1 horizontal curve. On each C^1 segment,

$$\dot{\gamma}(t) \in G_{\gamma(t)},$$

and consequently,

$$\Theta_{\gamma(t)}(\dot{\gamma}(t)) = dV_{\gamma(t)}(\dot{\gamma}(t)) = \frac{d}{dt}(V \circ \gamma)(t).$$

Integration over the finitely many C^1 segments gives

$$\int_{\gamma} \Theta = V(\gamma(b)) - V(\gamma(a)). \quad (\text{A4})$$

If γ is closed, the right-hand side vanishes. Thus, (i) follows.

Conversely, suppose that (i) holds. We first establish horizontal path independence. Let γ_1 and γ_2 be piecewise C^1 horizontal curves with the same initial and terminal points. The concatenation is expressed as

$$\gamma_1 * \overline{\gamma_2},$$

where $\overline{\gamma_2}$ denotes that γ_2 with reversed orientation is a closed, piecewise C^1 horizontal curve. Hence,

$$0 = \oint_{\gamma_1 * \overline{\gamma_2}} \Theta = \int_{\gamma_1} \Theta - \int_{\gamma_2} \Theta.$$

Therefore,

$$\int_{\gamma_1} \Theta = \int_{\gamma_2} \Theta. \quad (\text{A5})$$

Fix $z_0 \in E$ and $c \in \mathbb{R}$. For each $z \in E$, choose a piecewise C^1 horizontal curve $(\gamma_{z_0,z})$ from z_0 to z and define

$$V(z) := c + \int_{\gamma_{z_0,z}} \Theta. \quad (\text{A6})$$

Horizontal accessibility guarantees the existence of $\gamma_{z_0,z}$, while (A5) makes V independent of the chosen horizontal curve. Thus, $V : E \rightarrow \mathbb{R}$ is well defined.

The nontrivial point is to establish the smoothness of this path-defined function. Fix $q \in E$. According to Lemma 1, there exist $p \in E$; local smooth vector fields $(X_1, \dots, X_m \in \mathcal{X}_G)$; an open set $(U \subseteq \mathbb{R}^m)$; a point $(u^* \in U)$; an open neighbourhood $(W \subseteq E \text{ of } q)$; and a smooth map, i.e.,

$$\sigma = (\sigma_1, \dots, \sigma_m) : W \longrightarrow U.$$

such that

$$\Phi \circ \sigma = \text{id}_W, \quad (\text{A7})$$

where

$$\Phi(u_1, \dots, u_m) = \varphi_{u_m}^{X_m} \circ \dots \circ \varphi_{u_1}^{X_1}(p). \quad (\text{A8})$$

After shrinking U and W if necessary, all flow compositions appearing below are defined on the required parameter domains.

For $y \in W$, define

$$p_0(y) := p,$$

and recursively,

$$p_j(y) := \varphi_{\sigma_j(y)}^{X_j}(p_{j-1}(y)), \quad j = 1, \dots, m. \quad (\text{A9})$$

Smooth dependence of local flows on time and initial conditions inductively implies that every map $(p_j : W \rightarrow E)$ is smooth. Moreover,

$$p_m(y) = \Phi(\sigma(y)) = y.$$

For each j , let $\eta_{j,y}$ be the oriented flow segment, i.e.,

$$s \longmapsto \varphi_s^{X_j}(p_{j-1}(y)),$$

with s running from 0 to $\sigma_j(y)$. Negative values of $\sigma_j(y)$ are interpreted with the corresponding reversed orientation. Since X_j is tangent to G , each $\eta_{j,y}$ is horizontal. Their concatenation, i.e.,

$$\eta_y := \eta_{1,y} * \dots * \eta_{m,y},$$

is therefore a piecewise C^1 horizontal curve from p to y .

Choose, once and for all, a piecewise C^1 horizontal curve from z_0 to p . According to horizontal path independence,

$$V(y) = V(p) + \sum_{j=1}^m I_j(y), \quad I_j(y) := \int_0^{\sigma_j(y)} F_j(s, y) ds, \quad (\text{A10})$$

where

$$F_j(s, y) := \Theta_{\varphi_s^{X_j}(p_{j-1}(y))} \left(X_j(\varphi_s^{X_j}(p_{j-1}(y))) \right). \quad (\text{A11})$$

For each j , the F_j function is smooth jointly in (s, y) on its domain. Indeed, the local flow of X_j depends smoothly on time and initial conditions, p_{j-1} is smooth, and both X_j and Θ are smooth. Since σ_j is smooth, smooth dependence of parameterised integrals on their parameters gives

$$I_j \in C^\infty(W).$$

Equation (A10) therefore implies

$$V|_W \in C^\infty(W).$$

Since $q \in E$ was arbitrary,

$$V \in C^\infty(E). \quad (\text{A12})$$

The horizontal differential of V remains to be identified. Fix $z \in E$ and $\xi \in G_z$. Since G is a smooth vector sub-bundle of TE , there exists a local smooth section ($X \in \mathcal{X}_G$) satisfying

$$X_z = \xi.$$

Let φ_t^X denote its local flow. For a sufficiently small t , the curve expressed as

$$s \mapsto \varphi_s^X(z)$$

is horizontal. Horizontal path independence and (A6) therefore give

$$V(\varphi_t^X(z)) - V(z) = \int_0^t \Theta_{\varphi_s^X(z)} \left(X_{\varphi_s^X(z)} \right) ds. \quad (\text{A13})$$

Differentiating at $t = 0$ yields

$$dV_z(\xi) = \Theta_z(\xi).$$

Since $z \in E$ and $\xi \in G_z$ were arbitrary,

$$dV|_G = \Theta.$$

Thus, (ii) holds.

Equation (A4) further shows that every global scalar representation satisfies

$$V(z) = V(z_0) + \int_{\gamma_{z_0, z}} \Theta,$$

so every representation has the form of (17).

Finally, let $V_1, V_2 \in C^\infty(E)$ satisfy

$$dV_1|_G = dV_2|_G = \Theta,$$

and define $H := V_1 - V_2$. Then,

$$dH|_G = 0.$$

Given an arbitrary $p, q \in E$, horizontal accessibility provides a piecewise C^1 horizontal curve (γ) joining p to q . On every C^1 segment,

$$\frac{d}{dt}(H \circ \gamma)(t) = dH_{\gamma(t)}(\dot{\gamma}(t)) = 0.$$

Hence,

$$H(q) = H(p).$$

Since, p and q were arbitrary, H is constant on E . Therefore, the global scalar representation is unique up to an additive constant. $\square$

Proof of Proposition 2. Fix $z \in E$ and $v \in A(z)$. Since

$$L_z v \in \hat{A}(z) \subseteq G_z$$

and

$$d\pi_z(L_z v) = v,$$

the representation identity ($dV|_G = \Theta$) gives

$$dV_z(L_z v) = \Theta_z(L_z v) = \alpha_z(d\pi_z(L_z v)) = \alpha_z(v).$$

This proves (18).

Since

$$L_z : A(z) \longrightarrow \hat{A}(z)$$

is a bijection, maximisation over $\hat{A}(z)$ may be transferred through L_z . Using (18),

$$\arg \max_{\xi \in \hat{A}(z)} dV_z(\xi) = L_z \left(\arg \max_{v \in A(z)} dV_z(L_z v) \right) \quad (\text{A14})$$

$$= L_z \left(\arg \max_{v \in A(z)} \alpha_z(v) \right) \quad (\text{A15})$$

$$= L_z(R(z)) = \hat{R}(z). \quad (\text{A16})$$

This proves (19). $\square$

Proof of Corollary 1. Since $V \in C^\infty(E)$ and γ is absolutely continuous, $V \circ \gamma$ is absolutely continuous on every compact subinterval of I . The chain rule therefore gives

$$\frac{d}{dt} V(\gamma(t)) = dV_{\gamma(t)}(\dot{\gamma}(t)) \quad (\text{A17})$$

for almost every $t \in I$.

For almost every such t , the canonical rational trajectory condition gives

$$\dot{\gamma}(t) \in \hat{R}(\gamma(t)).$$

According to Proposition 2,

$$\hat{R}(\gamma(t)) = \arg \max_{\xi \in \hat{A}(\gamma(t))} dV_{\gamma(t)}(\xi).$$

Hence,

$$dV_{\gamma(t)}(\dot{\gamma}(t)) = \max_{\xi \in \hat{A}(\gamma(t))} dV_{\gamma(t)}(\xi).$$

The bijection expressed as

$$L_{\gamma(t)} : A(\gamma(t)) \longrightarrow \hat{A}(\gamma(t))$$

and (18) further imply

$$\max_{\xi \in \hat{A}(\gamma(t))} dV_{\gamma(t)}(\xi) = \max_{v \in A(\gamma(t))} \alpha_{\gamma(t)}(v) = \rho(\gamma(t)).$$

Combining these identities with (A17) proves (20) and (21). $\square$

Proof of Proposition 3. Fix $z \in E$. Since $0 \in A(z)$ and $L_z(0) = 0$, one has $0 \in \hat{A}(z)$. If $z \in \mathcal{E}$, then $0 \in \hat{R}(z)$, so the zero direction attains the maximal represented rate. Hence,

$$\rho(z) = \max_{\xi \in \hat{A}(z)} dV_z(\xi) = dV_z(0) = 0.$$

Conversely, if $\rho(z) = 0$, then

$$dV_z(0) = 0 = \max_{\xi \in \hat{A}(z)} dV_z(\xi).$$

Thus, $0 \in \hat{R}(z)$; therefore, $z \in \mathcal{E}$. This proves (24). Since $\rho \geq 0$, it follows that $\rho(z) > 0$ whenever $z \notin \mathcal{E}$. $\square$

Proof of Proposition 4. Fix $z_0 \in E$. Choose an open neighbourhood ($U \subseteq X$) of $\pi(z_0)$ and a vector-bundle trivialisation ($\tau : D|_U \rightarrow U \times \mathbb{R}^d$). Let $\tau_x^{\text{fib}} : D_x \rightarrow \mathbb{R}^d$ denote the associated fibre isomorphism and write

$$A_\tau(z) := \tau_{\pi(z)}^{\text{fib}}(A(z)), \quad z \in \pi^{-1}(U).$$

By assumption, A_τ is continuous with nonempty compact convex values.

The smooth section ($\alpha \in \Gamma(\pi^* D^*)$) has the local coordinate representation ($a_\tau : \pi^{-1}(U) \rightarrow (\mathbb{R}^d)^*$) defined by

$$a_\tau(z)(w) := \alpha_z((\tau_{\pi(z)}^{\text{fib}})^{-1}w).$$

Hence, a_τ is smooth, and $f_\tau(z, w) := a_\tau(z)(w)$ is continuous on $\pi^{-1}(U) \times \mathbb{R}^d$. The local maximisation problem is therefore

$$\rho(z) = \max_{w \in A_\tau(z)} f_\tau(z, w), \quad R_\tau(z) = \arg \max_{w \in A_\tau(z)} f_\tau(z, w). \quad (\text{A18})$$

Berge's maximum theorem applied to (A18) gives continuity of $\rho|_{\pi^{-1}(U)}$ and upper hemicontinuity of R_τ , with nonempty compact values. Since z_0 was arbitrary, ρ is continuous on E . The coordinate invariance established in (26) shows that R is intrinsically upper hemicontinuous and compact-valued.

Convexity remains to be verified. Fix $z \in E$, let $v_1, v_2 \in R(z)$, and take $\lambda \in [0, 1]$. Since $A(z)$ is convex,

$$v_\lambda := \lambda v_1 + (1 - \lambda)v_2 \in A(z).$$

The linearity of α_z gives

$$\alpha_z(v_\lambda) = \lambda \alpha_z(v_1) + (1 - \lambda) \alpha_z(v_2) = \rho(z),$$

so $v_\lambda \in R(z)$. Thus, $R(z)$ is convex, proving (i) and (ii).

For (iii), recall that $\hat{R}(z) = L_z(R(z))$, where $L : \pi^*D \rightarrow G$ is a smooth vector-bundle isomorphism. Each fibre map $(L_z : D_{\pi(z)} \rightarrow G_z)$ is a linear homeomorphism. Hence, $\hat{R}(z)$ is nonempty, compact, and convex, and $\hat{R}(z) \subseteq G_z$.

Choose compatible local trivialisations of π^*D and G over an open neighbourhood $W \subseteq E$. In these coordinates,

$$L(z, w) = (z, \Lambda(z)w),$$

where $\Lambda(z) \in \text{GL}(d, \mathbb{R})$ depends smoothly on z . Consequently,

$$\hat{R}_{\text{loc}}(z) = \Lambda(z)R_\tau(z). \quad (\text{A19})$$

We first establish local boundedness. Fix $z_0 \in W$. Since $R_\tau(z_0)$ is compact, choose a bounded open set $(O \subseteq \mathbb{R}^d)$ containing $R_\tau(z_0)$. Upper hemicontinuity of R_τ gives a neighbourhood $(W_0 \subseteq W)$ of z_0 such that $R_\tau(z) \subseteq O$ for every $z \in W_0$. After shrinking W_0 so that $\bar{W}_0$ is contained in the chosen coordinate neighbourhood, continuity of Λ gives

$$\sup_{z \in W_0} \|\Lambda(z)\| < \infty.$$

Since O is bounded, (A19) implies that $\hat{R}_{\text{loc}}$ is uniformly bounded on W_0 . Thus, $\hat{R}$ is locally bounded.

To prove upper hemicontinuity, let $z_n \rightarrow z$ in W_0 and suppose $\xi_n \in \hat{R}_{\text{loc}}(z_n)$ with $\xi_n \rightarrow \xi$. Set

$$w_n := \Lambda(z_n)^{-1}\xi_n.$$

Then, $w_n \in R_\tau(z_n)$, and continuity of Λ^{-1} gives $w_n \rightarrow w := \Lambda(z)^{-1}\xi$. Since R_τ is upper hemicontinuous with compact values, its graph is closed. Hence, $w \in R_\tau(z)$; therefore, $\xi = \Lambda(z)w \in \hat{R}_{\text{loc}}(z)$.

Thus, $\hat{R}_{\text{loc}}$ has a closed graph. Together with local boundedness and compact values, the finite-dimensional closed-graph characterisation of upper hemicontinuity yields upper hemicontinuity of $\hat{R}_{\text{loc}}$ and, hence, of $\hat{R}$.

Equivalently, after fixing any Riemannian metric on E , local boundedness means that for every $z_0 \in E$, there exist a neighbourhood (W) of z_0 and $C < \infty$ such that

$$\|\xi\| \leq C \quad \text{for every } z \in W, \quad \xi \in \hat{R}(z). \quad (\text{A20})$$

This completes the proof. $\square$

Proof of Theorem 2. Fix $z_0 \in E$, and choose a coordinate chart $(\psi : U \rightarrow \Omega \subseteq \mathbb{R}^n)$ with $z_0 \in U$. In local coordinates, the canonical differential inclusion becomes

$$\dot{x}(t) \in F(x(t)), \quad F(x) := d\psi_{\psi^{-1}(x)}(\hat{R}(\psi^{-1}(x))). \quad (\text{A21})$$

According to Proposition 4, $\hat{R}$ is upper hemicontinuous and locally bounded, with nonempty compact convex values. The bundle map $((z, \xi) \mapsto (\psi(z), d\psi_z\xi))$ is a local vector-bundle isomorphism. These properties are therefore preserved in coordinates, so $F : \Omega \rightrightarrows \mathbb{R}^n$ is upper hemicontinuous and locally bounded with nonempty compact convex values.

The standard finite-dimensional existence theorem for differential inclusions with these properties yields $\varepsilon > 0$ and an absolutely continuous curve $(x : [0, \varepsilon] \rightarrow \Omega)$ satisfying $x(0) = \psi(z_0)$ and $\dot{x}(t) \in F(x(t))$ almost everywhere. Consequently, $\gamma := \psi^{-1} \circ x$ is absolutely continuous, i.e., $\gamma(0) = z_0$, and satisfies (27).

A maximal forward extension is obtained by ordering all forward extensions of the local solution by domain inclusion and agreement on common domains. The union of every totally ordered family is, again, a canonical rational trajectory on each compact subinterval of its domain. A maximal element therefore exists. Its domain may be written as $[0, T_{\max})$ for some $0 < T_{\max} \leq \infty$, since any finite terminal point belonging to the domain would admit a further local extension according to the existence result just proven.

Suppose, toward a contradiction, that $T_{\max} < \infty$ and that $\gamma([0, T_{\max}))$ is relatively compact. Set

$$K := \overline{\gamma([0, T_{\max}))}.$$

Then, K is compact. Fix a Riemannian metric on E , with an associated norm of $\|\cdot\|$ and distance of d .

Local boundedness of $\hat{R}$ and compactness of K give a constant $C < \infty$ such that

$$\|\xi\| \leq C \quad \text{for every } z \in K, \quad \xi \in \hat{R}(z). \quad (\text{A22})$$

Since $\dot{\gamma}(t) \in \hat{R}(\gamma(t))$ almost everywhere, $\|\dot{\gamma}(t)\| \leq C$ for almost every $t < T_{\max}$. Hence, for $0 \leq s < t < T_{\max}$,

$$d(\gamma(s), \gamma(t)) \leq \int_s^t \|\dot{\gamma}(u)\| du \leq C(t - s).$$

Thus, γ is C -Lipschitz with respect to the time parameter on $[0, T_{\max})$. In particular, $\gamma(t)$ is Cauchy as $t \uparrow T_{\max}$. Since its image has compact closure, there exists a unique $z^* \in K$ such that

$$\gamma(t) \longrightarrow z^* \quad \text{as } t \uparrow T_{\max}. \quad (\text{A23})$$

Define $\gamma(T_{\max}) := z^*$. The preceding Lipschitz estimate extends to the closed interval $[0, T_{\max}]$, so this extension of γ is absolutely continuous.

Apply the local existence result with an initial state of z^* . After translating the time parameter, there exist $\delta > 0$ and an absolutely continuous canonical rational trajectory $(\eta : [T_{\max}, T_{\max} + \delta] \rightarrow E)$ satisfying $\eta(T_{\max}) = z^*$. Define

$$\tilde{\gamma}(t) := \begin{cases} \gamma(t), & 0 \leq t \leq T_{\max}, \\ \eta(t), & T_{\max} \leq t \leq T_{\max} + \delta. \end{cases}$$

The two absolutely continuous pieces agree at $T_{\max}$, so $\tilde{\gamma}$ is absolutely continuous on $[0, T_{\max} + \delta]$. Moreover,

$$\dot{\tilde{\gamma}}(t) \in \hat{R}(\tilde{\gamma}(t))$$

for almost every t because the inclusion holds almost everywhere on each piece and the joining time has a measure of zero. Hence, $\tilde{\gamma}$ is a canonical rational trajectory extending γ beyond $T_{\max}$, contradicting maximality.

Therefore, a maximal forward solution with a finite $T_{\max}$ cannot have a relatively compact image. This proves (28). $\square$

Proof of Theorem 3. According to Proposition 4, the maximal evaluative rate (ρ) is continuous. Condition (29) and the linearity of α_z give

$$\rho(z) = \max_{v \in A(z)} \alpha_z(v) \geq \alpha_z(0) = 0$$

for every $z \in E$. Moreover, Proposition 3 gives

$$\mathcal{E} = \rho^{-1}(0). \quad (\text{A24})$$

Let $\gamma : I \rightarrow E$ be a canonical rational trajectory. Accordign to Corollary 1,

$$\frac{d}{dt}V(\gamma(t)) = \rho(\gamma(t))$$

for almost every $t \in I$. Since $V \circ \gamma$ is absolutely continuous on compact subintervals of I , integration yields (30). This proves (i).

For (ii), suppose that $\gamma([t_1, t_2]) \not\subseteq \mathcal{E}$. Then, there exists $t_0 \in [t_1, t_2]$ such that $\gamma(t_0) \notin \mathcal{E}$. According to (A24),

$$\rho(\gamma(t_0)) > 0.$$

Continuity of $\rho \circ \gamma$ therefore gives a nondegenerate interval $(J \subseteq [t_1, t_2])$ and a constant ($c > 0$) such that

$$\rho(\gamma(t)) \geq c \quad \text{for every } t \in J.$$

Consequently,

$$V(\gamma(t_2)) - V(\gamma(t_1)) = \int_{t_1}^{t_2} \rho(\gamma(s)) ds \geq \int_J \rho(\gamma(s)) ds \geq c|J| > 0.$$

This proves (31). Conversely, if $\gamma([t_1, t_2]) \subseteq \mathcal{E}$, then $\rho(\gamma(t)) = 0$ throughout the interval, so (30) gives equality.

For (iii), suppose first that γ is periodic with period $T > 0$. Then,

$$0 = V(\gamma(T)) - V(\gamma(0)) = \int_0^T \rho(\gamma(s)) ds.$$

Since $\rho \circ \gamma$ is continuous and non-negative, it follows that $\rho(\gamma(s)) = 0$ for every $s \in [0, T]$. Hence,

$$\gamma([0, T]) \subseteq \mathcal{E}.$$

Now, let $z = \gamma(t_0)$ be recurrent and suppose, toward a contradiction, that $z \notin \mathcal{E}$. According to part (ii), there exists $t_1 > t_0$ such that

$$V(\gamma(t_1)) > V(\gamma(t_0)) = V(z).$$

Monotonicity then gives

$$V(\gamma(t)) \geq V(\gamma(t_1)) > V(z) \quad \text{for every } t \geq t_1.$$

Recurrence provides $t_n \rightarrow \infty$ such that $\gamma(t_n) \rightarrow z$. For a sufficiently large n , $t_n \geq t_1$, whereas continuity of V gives

$$V(\gamma(t_n)) \rightarrow V(z),$$

contradicting the preceding strict inequality. Therefore, $z \in \mathcal{E}$.

For (iv), suppose that $\gamma : [0, \infty) \rightarrow E$ is precompact and set

$$K := \overline{\gamma([0, \infty))}.$$

Then, K is compact. Since V is continuous on K and $V \circ \gamma$ is nondecreasing, the finite limit of

$$V_\infty := \lim_{t \rightarrow \infty} V(\gamma(t))$$

exists. Part (i) therefore gives

$$\int_0^\infty \rho(\gamma(s)) ds = V_\infty - V(\gamma(0)) < \infty. \quad (\text{A25})$$

For $T \geq 0$, define

$$K_T := \overline{\gamma([T, \infty))}.$$

Each K_T is nonempty and compact. It is also connected because $\gamma([T, \infty))$ is the continuous image of the connected interval $([T, \infty))$, and the closure of a connected set is connected. The family $((K_T)_{T \geq 0})$ is decreasing, and therefore,

$$\omega(\gamma) = \bigcap_{T \geq 0} K_T$$

is nonempty, compact, and connected.

Equation (32) remains to be proven. Fix a Riemannian metric on E and suppose, toward a contradiction, that

$$z^* \in \omega(\gamma) \setminus \mathcal{E}.$$

Then, $\rho(z^*) > 0$. According to the continuity of ρ , there exist $r > 0$ and $c > 0$ such that

$$\rho(z) \geq c \quad \text{whenever } d(z, z^*) < r. \quad (\text{A26})$$

According to Proposition 4, $\hat{R}$ is locally bounded. Compactness of K therefore yields $C < \infty$ such that

$$\|\xi\| \leq C \quad \text{for every } z \in K, \quad \xi \in \hat{R}(z). \quad (\text{A27})$$

If $C = 0$, then $\dot{\gamma}(t) = 0$ for almost every t , so absolute continuity implies that γ is constant. The exact evolution identity then gives $\rho(\gamma(t)) = 0$ almost everywhere, and continuity of $\rho \circ \gamma$ gives $\rho(\gamma(t)) = 0$ everywhere. Hence, $\omega(\gamma) \subseteq \mathcal{E}$, contrary to the choice of z^* . Thus, $C > 0$.

Set

$$\tau := \frac{r}{2C}.$$

Since $z^* \in \omega(\gamma)$, there exists a sequence $(t_n \rightarrow \infty)$ such that $\gamma(t_n) \rightarrow z^*$. Passing to a subsequence, we may assume

$$d(\gamma(t_n), z^*) < \frac{r}{2}, \quad t_{n+1} > t_n + \tau.$$

For every $s \in [t_n, t_n + \tau]$, absolute continuity and (A27) give

$$d(\gamma(s), \gamma(t_n)) \leq \int_{t_n}^s \|\dot{\gamma}(u)\| du \leq C(s - t_n) \leq \frac{r}{2}.$$

Hence,

$$d(\gamma(s), z^*) < r$$

throughout $[t_n, t_n + \tau]$. According to (A26),

$$\int_{t_n}^{t_n + \tau} \rho(\gamma(s)) ds \geq c\tau.$$

The intervals $[t_n, t_n + \tau]$ are pairwise disjoint. Summing over n therefore gives

$$\int_0^\infty \rho(\gamma(s)) ds = \infty,$$

contradicting (A25). Thus,

$$\omega(\gamma) \subseteq \mathcal{E}.$$

To establish (33), suppose, instead, that the asserted convergence fails. Then, there exist $\varepsilon > 0$ and $t_n \rightarrow \infty$ such that

$$\text{dist}(\gamma(t_n), \mathcal{E}) \geq \varepsilon.$$

Precompactness yields a convergent subsequence, again denoted by $\gamma(t_n)$, with

$$\gamma(t_n) \longrightarrow \bar{z}.$$

Since $t_n \rightarrow \infty$, $\bar{z} \in \omega(\gamma) \subseteq \mathcal{E}$ necessarily holds. Because $\mathcal{E} = \rho^{-1}(0)$ is closed, the distance function to $\mathcal{E}$ is continuous. Hence,

$$0 = \text{dist}(\bar{z}, \mathcal{E}) = \lim_{n \rightarrow \infty} \text{dist}(\gamma(t_n), \mathcal{E}) \geq \varepsilon,$$

which is a contradiction. This proves (iv).

Finally, suppose that $\mathcal{E}$ is discrete. According to (iv), $\omega(\gamma)$ is a nonempty connected subset of the discrete set $(\mathcal{E})$. Therefore,

$$\omega(\gamma) = \{z^*\}$$

for some $z^* \in \mathcal{E}$.

If $\gamma(t) \not\rightarrow z^*$, there exist $\varepsilon > 0$ and $t_n \rightarrow \infty$ such that

$$d(\gamma(t_n), z^*) \geq \varepsilon.$$

Precompactness gives a convergent subsequence. Its limit belongs to $\omega(\gamma) = \{z^*\}$, contradicting the preceding inequality. Therefore,

$$\gamma(t) \longrightarrow z^* \quad \text{as } t \rightarrow \infty.$$

This proves (v). $\square$

Proof of Theorem 4. According to Corollary 1 and feasible inaction,

$$\frac{d}{dt}V(\gamma(t)) = \rho(\gamma(t)) \geq 0$$

for almost every $t \in \mathbb{R}$. Hence, $V \circ \gamma$ is nondecreasing.

Since $\gamma(\mathbb{R})$ is precompact, $K := \overline{\gamma(\mathbb{R})}$ is compact. Continuity of V implies that $V(K)$ is bounded. Every bounded, nondecreasing, real-valued function on $\mathbb{R}$ has finite limits at both temporal ends. This proves (35).

For $T > 0$, define

$$K_T^- := \overline{\gamma((-\infty, -T])}, \quad K_T^+ := \overline{\gamma([T, \infty))}.$$

Each $K_T^\pm$ is nonempty, compact, and connected. The corresponding time interval is connected, its image under γ is connected, and closure preserves connectedness. The $(K_T^-)_{T>0}$ and $(K_T^+)_{T>0}$ families are nested. Compactness of K therefore implies that $\omega^-(\gamma)$ and $\omega^+(\gamma)$ are nonempty, compact, and connected.

The forward inclusion $(\omega^+(\gamma) \subseteq \mathcal{E})$ follows from Theorem 3. We prove the backward inclusion.

For every $T < 0$, the exact evolution identity gives

$$V(\gamma(0)) - V(\gamma(T)) = \int_T^0 \rho(\gamma(s)) ds.$$

Letting $T \rightarrow -\infty$ and using $V(\gamma(T)) \rightarrow V_-$ yields

$$\int_{-\infty}^0 \rho(\gamma(s)) ds = V(\gamma(0)) - V_- < \infty. \quad (\text{A28})$$

Suppose, toward a contradiction, that $z^* \in \omega^-(\gamma) \setminus \mathcal{E}$. According to Proposition 3, $\rho(z^*) > 0$. Continuity of ρ gives $r > 0$ and $c > 0$ such that

$$\rho(z) \geq c \quad \text{whenever } d(z, z^*) < r. \quad (\text{A29})$$

Fix a Riemannian metric on E . According to Proposition 4, $\hat{R}$ is locally bounded. Compactness of K therefore gives $C < \infty$ such that

$$\|\xi\| \leq C \quad \text{for every } z \in K, \quad \xi \in \hat{R}(z). \quad (\text{A30})$$

If $C = 0$, then $\dot{\gamma} = 0$ almost everywhere, so γ is constant. The exact evolution identity then gives $\rho(\gamma(t)) = 0$, and the desired inclusion follows. Assume $C > 0$ and set

$$\tau := \frac{r}{2C}.$$

Since $z^* \in \omega^-(\gamma)$, there exists $t_n \rightarrow -\infty$ such that $\gamma(t_n) \rightarrow z^*$. Passing to a subsequence, we may assume

$$d(\gamma(t_n), z^*) < \frac{r}{2},$$

that $t_n + \tau \leq 0$, and that the intervals $[t_n, t_n + \tau]$ are pairwise disjoint. For every $s \in [t_n, t_n + \tau]$, absolute continuity and (A30) give

$$d(\gamma(s), \gamma(t_n)) \leq \int_{t_n}^s \|\dot{\gamma}(u)\| du \leq C(s - t_n) \leq \frac{r}{2}.$$

Hence, $d(\gamma(s), z^*) < r$ throughout the interval, and (A29) gives

$$\int_{t_n}^{t_n + \tau} \rho(\gamma(s)) ds \geq c\tau.$$

Summing over infinitely many pairwise disjoint intervals contradicts (A28). Therefore,

$$\omega^-(\gamma) \subseteq \mathcal{E},$$

which, together with the forward inclusion, proves (36).

Let $z \in \omega^+(\gamma)$. There exists $t_n \rightarrow \infty$ such that $\gamma(t_n) \rightarrow z$. Continuity of V gives

$$V(z) = \lim_{n \rightarrow \infty} V(\gamma(t_n)) = V_+.$$

Hence, $V|_{\omega^+(\gamma)} \equiv V_+$. The same argument with $t_n \rightarrow -\infty$ yields $V|_{\omega^-(\gamma)} \equiv V_-$, proving (37).

For (iii), suppose $\gamma(\mathbb{R}) \not\subseteq \mathcal{E}$. Choose $t_0 \in \mathbb{R}$ such that $\gamma(t_0) \notin \mathcal{E}$. According to $\mathcal{E} = \rho^{-1}(0)$,

$$\rho(\gamma(t_0)) > 0.$$

Continuity of $\rho \circ \gamma$ gives $t_1 < t_0 < t_2$ and $c > 0$ such that $\rho(\gamma(t)) \geq c$ on $[t_1, t_2]$. Therefore,

$$V(\gamma(t_2)) - V(\gamma(t_1)) = \int_{t_1}^{t_2} \rho(\gamma(s)) ds \geq c(t_2 - t_1) > 0.$$

Monotonicity gives

$$V_- \leq V(\gamma(t_1)) < V(\gamma(t_2)) \leq V_+,$$

and, hence,

$$V_- < V_+.$$

For (iv), if $\omega^-(\gamma) = \{z^-\}$ and $\omega^+(\gamma) = \{z^+\}$, then (37) gives

$$V(z^-) = V_-, \quad V(z^+) = V_+.$$

Part (iii) therefore yields

$$V(z^-) < V(z^+).$$

Suppose, finally, that a finite directed cycle exists, i.e.,

$$z_1 \longrightarrow z_2 \longrightarrow \cdots \longrightarrow z_m \longrightarrow z_1,$$

with every arrow realised by a complete, precompact, canonical, rational trajectory not contained in $\mathcal{E}$, whose singleton backward and forward limit sets are the source and target equilibria. Applying (39) to each connection gives

$$V(z_1) < V(z_2) < \cdots < V(z_m) < V(z_1),$$

which is impossible. Hence, no such directed cycle exists. $\square$

Proof of Corollary 2. If $z^- \prec^+ z^+$, then there exist $m \geq 1$ and equilibria, i.e.,

$$z_0 = z^-, \quad z_1, \dots, z_m = z^+,$$

such that

$$z_{j-1} \prec z_j \quad \text{for } j = 1, \dots, m.$$

Theorem 4 gives

$$V(z_0) < V(z_1) < \cdots < V(z_m),$$

and, hence, $V(z^-) < V(z^+)$. Therefore, $z \not\prec^+ z$ for every equilibrium participating in the relation. Since $\prec^+$ is transitive by construction, it is a strict partial order. A directed cycle would imply $z \prec^+ z$ for some z , which is impossible. $\square$

Proof of Proposition 5. The equivalence of the first two statements follows from Theorem 1. Set E is connected, and the construction above establishes that G is smooth, constant-rank, and bracket-generating.

Identify U_0 with its pullback $U_0 \circ \pi$. Suppose first that V exists and write

$$W = V - U_0.$$

Since

$$dW(H_{2j}) = 0,$$

we obtain

$$\frac{\partial W}{\partial y_j} = 0 \quad \text{for every } j.$$

The remaining horizontal identities give

$$\frac{\partial W}{\partial x_j} + y_j C_{\cdot j}^T \nabla_a W = r_j y_j.$$

Because W and $\nabla_a W$ are independent of y_j , this identity for all y_j values implies

$$\frac{\partial W}{\partial x_j} = 0, \quad C_{\cdot j}^T \nabla_a W = r_j.$$

Consequently,

$$C^T \nabla_a W = r.$$

A necessary condition is therefore

$$r \in \text{Im}(C^T).$$

Conversely, if

$$r = C^T \beta,$$

then

$$V(x, a) = U_0(x) + \beta^T a$$

satisfies $dV|_G = \Theta$. Since C has full row rank, $C^T : \mathbb{R}^q \rightarrow \mathbb{R}^m$ is injective, so β is unique.

For the cycle characterisation, consider a concatenated horizontal adjustment whose projection onto each (x_j, y_j) -plane is closed and define

$$A_j = \oint y_j dx_j, \quad A = (A_1, \dots, A_m)^T.$$

The directly economic coordinates return to their initial values, while (43) gives the net architectural displacement, i.e.,

$$\Delta a = CA.$$

Hence, the lifted horizontal adjustment is closed whenever

$$A \in \ker C.$$

Conversely, fix $A \in \ker C$. Because X is open, for every base point, there exists $\varepsilon > 0$ such that, for a sufficiently small $|t| < \varepsilon$, oriented rectangular loops can be constructed in the (x_j, y_j) -planes with signed areas tA_j while remaining inside X . Concatenating these loops gives a closed horizontal curve because

$$\Delta a = tCA = 0.$$

The baseline term is exact and therefore integrates to zero around the closed projection. The remaining primitive evaluation satisfies

$$\oint_{\Gamma_t} \Theta = t r^T A.$$

Zero circulation around every closed horizontal curve therefore implies

$$r^T A = 0 \quad \text{for every } A \in \ker C.$$

Conversely, if this orthogonality condition holds, every closed horizontal curve has zero primitive circulation. Since

$$(\ker C)^\perp = \operatorname{Im}(C^\top),$$

the condition is equivalent to

$$r \in \operatorname{Im}(C^\top).$$

□

Proof of Proposition 6. Write

$$r^\parallel = P_C r, \quad r^\perp = (I - P_C)r.$$

Orthogonal projection gives

$$r^\parallel \in \operatorname{Im}(C^\top), \quad r^\perp \in \operatorname{Im}(C^\top)^\perp = \ker C.$$

Hence,

$$\delta_{GR} = \|r^\perp\| = \operatorname{dist}(r, \operatorname{Im}(C^\top)).$$

For every $A \in \ker C$ with $\|A\| = 1$,

$$r^\top A = (r^\perp)^\top A,$$

so Cauchy–Schwarz gives

$$|r^\top A| \leq \|r^\perp\|.$$

If $r^\perp \neq 0$, equality is attained at

$$A^* = \frac{r^\perp}{\|r^\perp\|}.$$

If $r^\perp = 0$, both sides of (48) vanish. □

References

1. Becker, G.S.; Murphy, K.M. A theory of rational addiction. *J. Political Econ.* **1988**, *96*, 675–700. <https://doi.org/10.1086/261558>.
2. Russell, T. How quasi-rational are you? A behavioral interpretation of a two form which measures non-integrability of a system of demand equations. *Econ. Lett.* **1997**, *56*, 181–186. [https://doi.org/10.1016/S0165-1765\(97\)81898-8](https://doi.org/10.1016/S0165-1765(97)81898-8).
3. Tesfatsion, L. Global and approximate global optimality of myopic economic decisions. *J. Econ. Dyn. Control* **1980**, *2*, 135–160. [https://doi.org/10.1016/0165-1889\(80\)90056-1](https://doi.org/10.1016/0165-1889(80)90056-1).
4. Arthur, W.B. Competing Technologies, Increasing Returns, and Lock-In by Historical Events. *Econ. J.* **1989**, *99*, 116–131. <https://doi.org/10.2307/2234208>.
5. David, P.A. Clio and the Economics of QWERTY. *Am. Econ. Rev.* **1985**, *75*, 332–337.
6. Levinthal, D.A. Adaptation on Rugged Landscapes. *Manag. Sci.* **1997**, *43*, 934–950. <https://doi.org/10.1287/mnsc.43.7.934>.
7. Rivkin, J.W. Imitation of Complex Strategies. *Manag. Sci.* **2000**, *46*, 824–844. <https://doi.org/10.1287/mnsc.46.6.824.11940>.
8. Fleming, L.; Sorenson, O. Technology as a Complex Adaptive System. *Res. Policy* **2001**, *30*, 1019–1039.
9. Forrester, J.W. *Industrial Dynamics*; MIT Press: Cambridge, MA, USA, 1961.
10. Meadows, D.H. *Thinking in Systems: A Primer*; Chelsea Green Publishing: White River Junction, VT, USA, 2008.
11. Mitchell, M. *Complexity: A Guided Tour*; Oxford University Press: New York, NY, USA, 2009.
12. Batty, M. Building a Science of Cities. *Cities* **2012**, *29*, S9–S16. <https://doi.org/10.1016/j.cities.2011.11.008>.
13. Cabrera, D.; Cabrera, L.; Powers, E. A Unifying Theory of Systems Thinking with Psychosocial Applications. *Syst. Res. Behav. Sci.* **2015**, *32*, 534–545. <https://doi.org/10.1002/sres.2351>.
14. Geels, F.W. From Sectoral Systems of Innovation to Socio-Technical Systems: Insights about Dynamics and Change from Sociology and Institutional Theory. *Res. Policy* **2004**, *33*, 897–920. <https://doi.org/10.1016/j.respol.2004.01.015>.

15. Zhang, Y.; Lucas, M.; Battiston, F. Higher-Order Interactions Shape Collective Dynamics Differently in Hypergraphs and Simplicial Complexes. *Nat. Commun.* **2023**, *14*, 1605. <https://doi.org/10.1038/s41467-023-37190-9>.
16. Ghavasieh, A.; De Domenico, M. Diversity of Information Pathways Drives Sparsity in Real-World Networks. *Nat. Phys.* **2024**, *20*, 512–519. <https://doi.org/10.1038/s41567-023-02330-x>.
17. Meena, C.; Hens, C.; Acharyya, S.; Haber, S.; Boccaletti, S.; Barzel, B. Emergent Stability in Complex Network Dynamics. *Nat. Phys.* **2023**, *19*, 1033–1042. <https://doi.org/10.1038/s41567-023-02020-8>.
18. Artime, O.; Grassia, M.; De Domenico, M.; Gleeson, J.P.; Makse, H.A.; Mangioni, G.; Perc, M.; Radicchi, F. Robustness and Resilience of Complex Networks. *Nat. Rev. Phys.* **2024**, *6*, 114–131. <https://doi.org/10.1038/s42254-023-00676-y>.
19. Moran, J.; Romeijnders, M.; Le Doussal, P.; Pijpers, F.P.; Weitzel, U.; Panja, D.; Bouchaud, J.-P. Timeliness Criticality in Complex Systems. *Nat. Phys.* **2024**, *20*, 1352–1358. <https://doi.org/10.1038/s41567-024-02525-w>.
20. Samuelson, P.A. A note on the pure theory of consumer's behaviour. *Economica* **1938**, *5*, 61–71. <https://doi.org/10.2307/2548836>.
21. Houthakker, H.S. Revealed preference and the utility function. *Economica* **1950**, *17*, 159–174. <https://doi.org/10.2307/2549382>.
22. Afriat, S.N. The Construction of Utility Functions from Expenditure Data. *Int. Econ. Rev.* **1967**, *8*, 67–77. <https://doi.org/10.2307/2525382>.
23. Varian, H.R. The Nonparametric Approach to Demand Analysis. *Econometrica* **1982**, *50*, 945–973. <https://doi.org/10.2307/1912771>.
24. Nishimura, H.; Ok, E.A.; Quah, J.K.-H. A comprehensive approach to revealed preference theory. *Am. Econ. Rev.* **2017**, *107*, 1239–1263. <https://doi.org/10.1257/aer.20150947>.
25. Masatlioglu, Y.; Nakajima, D.; Ozbay, E.Y. Revealed attention. *Am. Econ. Rev.* **2012**, *102*, 2183–2205. <https://doi.org/10.1257/aer.102.5.2183>.
26. Caplin, A.; Dean, M. Revealed preference, rational inattention, and costly information acquisition. *Am. Econ. Rev.* **2015**, *105*, 2183–2203. <https://doi.org/10.1257/aer.20140117>.
27. Ok, E.A.; Ortoleva, P.; Riella, G. Revealed (P)Reference theory. *Am. Econ. Rev.* **2015**, *105*, 299–321. <https://doi.org/10.1257/aer.2011046>.
28. Lu, J. Bayesian identification: A theory for state-dependent utilities. *Am. Econ. Rev.* **2019**, *109*, 3192–3228. <https://doi.org/10.1257/aer.20171534>.
29. Samuelson, P.A. The problem of integrability in utility theory. *Economica* **1950**, *17*, 355–385.
30. Hurwicz, L.; Uzawa, H. On the integrability of demand functions. In *Preferences, Utility, and Demand*; Chipman, J.S., Hurwicz, L., Richter, M.K., Sonnenschein, H.F., Eds.; Harcourt Brace Jovanovich: New York, NY, USA, 1971.
31. Monderer, D.; Shapley, L.S. Potential Games. *Games Econ. Behav.* **1996**, *14*, 124–143. <https://doi.org/10.1006/game.1996.0044>.
32. Kalman, R.E. When Is a Linear Control System Optimal? *J. Basic Eng.* **1964**, *86*, 51–60. <https://doi.org/10.1115/1.3653115>.
33. Ng, A.Y.; Russell, S.J. Algorithms for Inverse Reinforcement Learning. In Proceedings of the Seventeenth International Conference on Machine Learning (ICML), Stanford, CA, USA, 29 June–2 July 2000; pp. 663–670.
34. Skalse, J.; Abate, A. Partial Identifiability and Misspecification in Inverse Reinforcement Learning. *Artif. Intell.* **2026**, *356*, 104525. <https://doi.org/10.1016/j.artint.2026.104525>.
35. Evans, D.; Li, J.; McGough, B. Local rationality. *J. Econ. Behav. Organ.* **2023**, *205*, 216–236. <https://doi.org/10.1016/j.jebo.2022.11.005>.
36. Sobel, M.J. Myopic solutions of Markov decision processes and stochastic games. *Oper. Res.* **1981**, *29*, 995–1009. <https://doi.org/10.1287/opre.29.5.995>.
37. Glazer, R.; Steckel, J.H.; Winer, R.S. Locally rational decision making: The distracting effect of information on managerial performance. *Manag. Sci.* **1992**, *38*, 212–226. <https://doi.org/10.1287/mnsc.38.2.212>.
38. Aubin, J.-P. *Viability Theory*; Birkhäuser: Boston, MA, USA, 1991.
39. Aubin, J.-P.; Cellina, A. *Differential Inclusions: Set-Valued Maps and Viability Theory*; Springer: Berlin, Germany, 1984.
40. Clarke, F.H.; Ledyaev, Y.S.; Stern, R.J.; Wolenski, P.R. *Nonsmooth Analysis and Control Theory*; Springer: New York, NY, USA, 1998.
41. Chow, W.-L. Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung. *Math. Ann.* **1939**, *117*, 98–105. <https://doi.org/10.1007/BF01450011>.
42. Rashevskii, P.K. Any Two Points of a Completely Nonholonomic Space May Be Connected by an Admissible Line. *Uchenye Zap. Mosk. Gos. Univ.* **1938**, *2*, 83–94. (In Russian)
43. Montgomery, R. *A Tour of Subriemannian Geometries, Their Geodesics and Applications*; American Mathematical Society: Providence, RI, USA, 2002.
44. Bullo, F.; Lewis, A.D. *Geometric Control of Mechanical Systems: Modeling, Analysis, and Design for Simple Mechanical Control Systems*; Texts in Applied Mathematics; Springer: New York, NY, USA, 2004; Volume 49.
45. Knight, F.H. *Risk, Uncertainty and Profit*; Houghton Mifflin: Boston, MA, USA, 1921.
46. Ellsberg, D. Risk, Ambiguity, and the Savage Axioms. *Q. J. Econ.* **1961**, *75*, 643–669.
47. Gilboa, I.; Schmeidler, D. Maxmin Expected Utility. *J. Math. Econ.* **1989**, *18*, 141–153. [https://doi.org/10.1016/0304-4068\(89\)90018-9](https://doi.org/10.1016/0304-4068(89)90018-9).

48. Walker, W.E.; Harremoës, P.; Rotmans, J.; van der Sluijs, J.P.; van Asselt, M.B.A.; Janssen, P.; Kreyer von Krauss, M.P. Defining Uncertainty: A Conceptual Basis for Uncertainty Management in Model-Based Decision Support. *Integr. Assess.* **2003**, *4*, 5–17. <https://doi.org/10.1076/iaij.4.1.5.16466>.
49. Lempert, R.J. Robust Decision Making (RDM). In *Decision Making Under Deep Uncertainty*; Marchau, V.A.W.J., Walker, W.E., Bloemen, P.J.T.M., Popper, S.W.E., Eds.; Springer: Cham, Switzerland, 2019; pp. 23–51. https://doi.org/10.1007/978-3-030-05252-2_2.
50. Gan, J. Collateral, debt capacity, and corporate investment: Evidence from a natural experiment. *J. Financ. Econ.* **2007**, *85*, 709–734. <https://doi.org/10.1016/j.jfineco.2006.06.007>.
51. Chaney, T.; Sraer, D.; Thesmar, D. The collateral channel: How real estate shocks affect corporate investment. *Am. Econ. Rev.* **2012**, *102*, 2381–2409. <https://doi.org/10.1257/aer.102.6.2381>.
52. Anderson, G.; Bahaj, S.; Chavaz, M.; Foulis, A.; Pinter, G. Lending relationships and the collateral channel. In *Bank of England Staff Working Paper*; Bank of England: London, UK, 2018.
53. Drechsel, T. Earnings-based borrowing constraints and macroeconomic fluctuations. *Am. Econ. J. Macroecon.* **2023**, *15*, 1–34. <https://doi.org/10.1257/mac.20210099>.
54. Chen, W.; Li, W.; Zhang, T. Complex Network-Based Resilience Capability Assessment for a Combat System of Systems. *Systems* **2024**, *12*, 31. <https://doi.org/10.3390/systems12010031>.
55. Zhang, H.; Jiang, S.; Lin, X.; Yu, X.; Zheng, W. A Networked Game Theoretic Model for Evaluating Resilience in Megaprojects: Integrating Stakeholder Interactions and Lifecycle Adaptability. *Systems* **2025**, *13*, 122. <https://doi.org/10.3390/systems13020122>.
56. Cheng, S.; Fan, Q. Research on the Selection of Multi-Agent Interaction Modes in Complex Product R&D Networks Under Disruption Events. *Systems* **2025**, *13*, 836. <https://doi.org/10.3390/systems13100836>.
57. Taeihagh, A. Governance of Artificial Intelligence. *Policy Soc.* **2021**, *40*, 137–157. <https://doi.org/10.1080/14494035.2021.1928377>.
58. Papagiannidis, E.; Mikalef, P.; Conboy, K. Responsible Artificial Intelligence Governance: A Review and Research Framework. *J. Strateg. Inf. Syst.* **2025**, *34*, 101885. <https://doi.org/10.1016/j.jsis.2024.101885>.
59. Selten, F.; Klievink, B. Organizing Public Sector AI Adoption: Navigating between Separation and Integration. *Gov. Inf. Q.* **2024**, *41*, 101885. <https://doi.org/10.1016/j.giq.2023.101885>.
60. Kolt, N. Lessons from Complex Systems Science for AI Governance. *Patterns* **2025**, *6*, 101341. <https://doi.org/10.1016/j.patter.2025.101341>.
61. Stiefenhofer, P.; Deniz, C. Artificial General Intelligence and Planetary Justice: A Framework for Safe and Just Transitions. *Challenges* **2025**, *16*, 59. <https://doi.org/10.3390/challe16040059>.
62. Stiefenhofer, P.; Deniz, C. Constructal Design under Discontinuities: Emergence and Stability of Flow Architectures under Irreversible Constraints. *Int. Commun. Heat Mass Transf.* **2026**, *179*, 112064. <https://doi.org/10.1016/j.icheatmasstransfer.2026.112064>.
63. Stiefenhofer, P. Exponential Stability of Nonsmooth Periodic Orbits in the Complex Space. *Phys. Lett. A* **2025**, *560*, 130935. <https://doi.org/10.1016/j.physleta.2025.130935>.
64. Susmann, H.J. Orbits of Families of Vector Fields and Integrability of Distributions. *Trans. Am. Math. Soc.* **1973**, *180*, 171–188. <https://doi.org/10.1090/S0002-9947-1973-0321133-2>.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.