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AI-Enabled Edge-Based Intraoral Wearable System for Early Detection and Management of Dental Caries

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Abstract

Dental caries remains one of the most prevalent yet preventable non-communicable diseases worldwide, disproportionately affecting populations with limited access to dental care and persistent socioeconomic inequalities. Early-stage lesions frequently remain undetected because of their asymptomatic nature, inadequate screening infrastructure, and the absence of continuous monitoring technologies, resulting in preventable complications and increased healthcare costs. To address these challenges, this study proposes an Internet of Things (IoT)-enabled intraoral wearable sensing device (I-OWSD) for continuous, quantitative, real-time monitoring of biomarkers associated with caries progression. The proposed framework integrates intraoral wearable sensing, cloud-based telemedicine services, and artificial intelligence (AI)-assisted analytics to support preventive oral healthcare and remote clinical decision-making. Two primary contributions are presented. First, a fractional-order delay-type model (FODM) based on the Caputo–Fabrizio derivative is proposed to capture the memory-dependent and nonlocal dynamics of caries progression. Mathematical analysis establishes the model’s non-negativity, boundedness, existence, uniqueness, and stability properties. Second, a biocompatible intraoral sensor interface is designed to enable continuous data acquisition and secure wireless communication with digital health platforms. Simulation results based on the proposed FODM suggest that, under an estimated adoption rate of 67.49%, the I-OWSD framework could reduce caries prevalence by approximately 15% while improving opportunities for early intervention and preventive care. The findings demonstrate the potential of combining fractional-order modelling, wearable sensing, and AI-driven teledentistry to advance continuous oral health monitoring and preventive dental care.

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1. Introduction

Recent advancements in AI-enabled IoT wearable health technologies demonstrate a rapid transition from research prototypes to practical multimodal monitoring systems. This trend highlights the relevance of the proposed I-OWSD framework. For instance, an all-in-one smartphone-based intelligent photonic mHealth system integrates flexible photonic sensors with deep learning [1]. It enables the monitoring of plantar biomechanics and cardiorespiratory signals with clinical-grade accuracy. The system also incorporates signal processing and temporal graph neural networks for robust physiological interpretation.

In digital health systems, AI-integrated wearables are increasingly used to continuously acquire and process physiological and biochemical data. These systems leverage machine learning (ML) for early detection, pattern recognition, and personalized interventions across cardiovascular, metabolic, and activity domains [2]. Such developments demonstrate the potential of combining real-time sensing, edge computation, and AI analytics in wearable platforms. This provides strong motivation for applying similar interdisciplinary approaches to intraoral monitoring for early dental disease detection [3].

Many countries are adopting wearable expert systems to enhance healthcare delivery and reduce pressure on overstretched systems. In the United Kingdom, the National Health Service (NHS) is exploring the deployment of smartwatches and wearable devices at scale, signalling a shift towards preventative and data-driven care [4]. This approach is particularly relevant for addressing global health challenges such as dental caries. Dental caries is a multifactorial disease affecting approximately 2.3 billion people worldwide. It disproportionately impacts underserved populations [5]. According to the Global Burden of Disease Study 2019, untreated caries remains the most prevalent health condition, contributing to oral diseases affecting over 3.5 billion individuals globally [5].

Dental caries is initiated by a complex interplay of dietary sugars, oral microbiota, and host factors, leading to progressive enamel and dentin demineralization. The process is primarily driven by acidogenic and aciduric bacteria such as *Streptococcus mutans* and *Lactobacillus* spp. [6]. Despite improvements in oral health strategies, early-stage lesions often remain asymptomatic and undetected, resulting in irreversible damage and elevated treatment costs disproportionately affecting the low-income population [7].

Against this backdrop, the emergence of IoT-enabled wearable technologies offers a promising solution for real-time monitoring and early detection of caries. Caries pathogenesis begins with biofilm formation, where bacteria metabolize carbohydrates into acids that reduce oral pH and compromise enamel integrity [8]. This acidic environment is further sustained by acid-tolerant bacteria such as *Lactobacilli* and *Actinomyces* [9]. While saliva typically buffers these acids and aids in remineralization, factors like xerostomia, suboptimal oral hygiene, and a poor diet increase vulnerability [10]. Caries can be anatomically and clinically classified: for example, as pit and fissure, smooth surface, or root lesions; as acute or chronic; and by activity levels [6].

IoT expert systems are transforming healthcare by integrating sensors, connectivity, and advanced analytics for continuous, personalised monitoring [11,12]. In dentistry, these systems enhance both diagnostic and preventative capabilities by tracking salivary biomarkers, plaque accumulation, and bacterial activity in real time [13]. Wearable biosensors and smartwatches, long applied in chronic disease management [14], are now

being tailored for oral health applications. For example, dental IoT sensors can monitor oral pH, detect specific bacterial strains, and assess tooth surface integrity. This enables timely intervention before irreversible damage occurs [13]. Salivary pH sensors can detect acidification events, while bacterial biosensors identify the presence of *S. mutans* with high sensitivity [14].

Additional sensing technologies further extend these capabilities. Pressure sensors integrated into oral appliances can detect bruxism by tracking occlusal forces [15]. Smart toothbrushes promote optimal hygiene through behaviour monitoring and feedback [16]. These innovations rely on accurate sensor calibration, clinical validation, and robust signal processing techniques [17]. Furthermore, ensuring user-friendly and biocompatible designs is essential for intraoral usability and long-term adoption [18].

Machine learning techniques are increasingly applied to interpret intraoral data. Algorithms such as support vector machines (SVMs), random forests, and convolutional neural networks (CNNs) are used to analyse biomarker trends and intraoral images for caries detection [19]. Cloud infrastructure enables real-time feedback and mobile alerts [20], while teledentistry platforms support remote consultations and personalised treatment planning [21].

Despite these advances, several challenges remain. Sensor miniaturisation for intraoral deployment is still technically demanding [22]. Data security during transmission and storage remains a critical concern [23]. In addition, large-scale clinical validation is required to ensure reliability and generalisability [13,24]. The integration of artificial intelligence for autonomous diagnosis presents further opportunities, but also introduces complexity in model validation and deployment [25]. Addressing these challenges is essential for effective prevention and early detection of dental caries, particularly in resource-limited settings.

The proposed I-OWSD system supports remote screening, continuous monitoring, and improved access to care, particularly valuable in underserved regions. Early wearable systems focused primarily on vital signs such as temperature [24,26], blood pressure, and heart rate [27,28]. Modern systems, however, have evolved to detect biochemical and physiological markers. The oral cavity provides an ideal environment for such devices, due to its accessibility and minimal disruption to daily activities [29]. For example, Onuora et al. [27] developed a wearable healthcare monitoring device using Caputo fractional calculus, demonstrating efficient and low-latency health monitoring.

Given the essential role of oral health in speech, nutrition, and overall quality of life, the burden of dental diseases is significant. The World Health Organization reports that dental caries in permanent teeth is the most common global disease [30]. Without timely intervention, it can lead to pain, infection, and disfigurement [31]. The proposed I-OWSD system offers the potential to improve early diagnosis and treatment monitoring. It can reduce patient discomfort and healthcare costs, particularly in regions with limited access to dental professionals [32,33].

Recent studies [34–37] have advanced the theoretical, clinical, economic, and ethical foundations of AI-assisted healthcare. These works demonstrate how autonomous AI agents can address complex system integration challenges [38]. By leveraging deep learning and large language model APIs, such systems enable early disease prediction and improved clinical decision-making.

Complementing these AI approaches, computational modelling provides deeper insights into disease dynamics. Fractional calculus has gained increasing attention for modelling biological systems with memory-dependent behaviour. It has been successfully applied to dental caries progression and other diseases [39–44]. For example, Baleanu et al. [45] modelled liver disease using Caputo–Fabrizio derivatives, while subsequent

studies explored non-singular kernel formulations [46,47]. Efficient numerical methods such as the FMSDTM [48] further enhance the solvability and precision of these models.

Against this backdrop, the integration of wearable IoT systems, AI-driven analytics, and fractional modelling offers a powerful interdisciplinary approach to dental caries management. The novelty of this study lies in the development of an FODM based on the Caputo–Fabrizio derivative. This model captures memory effects in dental caries progression and is integrated with an AI-powered I-OWSD framework for real-time monitoring.

Unlike conventional approaches that rely on periodic measurements, the proposed system enables continuous intraoral monitoring, secure data transmission, and cloud-based analytics. The model is rigorously validated for stability and predictive performance. Simulation results demonstrate improved population-level outcomes compared with existing methods. This integrated framework provides a scalable, accurate, and clinically relevant solution for preventive oral healthcare, particularly for high-risk and underserved populations.

Summary of Main Contributions

The key contributions of this work are as follows:

- Developing a novel fractional-order delay model (FODM) based on the constant proportional Caputo–Fabrizio (CPCF) operator to capture the memory-dependent progression of dental cavities over time.
- Developing a delay-type, non-integer derivative model (non-classical order) targeted to reflect the slow and persistent nature of cavity formation, supporting long-term oral health monitoring.
- Formulating an intelligent expert system (I-OWSDs) that simplifies tracking and control of dental cavities, making it easier for researchers and clinicians to analyze cavity behaviour and treatment effectiveness.
- Developing a numerical validation scheme to demonstrate the model’s effectiveness in supporting early diagnosis, reducing cavity spread, and improving treatment outcomes.

The article is organized as follows: Section 2 introduces the basic concepts and model formulation using the constant proportional Caputo–Fabrizio (CPCF) method. Section 3 provides an analysis of the proposed model, including non-negative bounded solutions, existence and uniqueness, and stability analysis, with a focus on wearables. Section 4 discusses the analysis of the model’s generalized proportional operators and the eigenfunctions of CPCF. Section 5 presents the key graphical analyses and their discussions. Finally, Section 6 offers the conclusions.

2. Telemedicine Edge Preliminaries and System Model Formulation

In a deployed setting (see Subsection 5.2), patients wear the I-OWSD intraorally, continuously sensing biomarkers relevant to dental caries, including pH, temperature, and enzymatic activity. The device integrates ion-selective microelectrodes for pH monitoring, microfluidic sensors for salivary flow and enzymatic composition, fluorescence-based optical sensors for *Streptococcus mutans* detection, and high-resolution thermistors for localized temperature mapping. Sensor signals are digitized and pre-processed on-device to reduce noise and compress data, then transmitted via low-power IoT protocols (BLE, LoRa, or Zigbee) to paired smartphones or edge gateways [38]. Edge processing performs real-time anomaly detection and preliminary AI-based classification, while secure cloud analytics combine sensor- and image-derived features using SVM, neural networks, and hybrid decision fusion to classify oral conditions, assess risk, and

predict disease progression. Clinicians access dashboards remotely for personalised interventions, while patients receive automated alerts to support preventive care and treatment adherence. The system is designed for scalability, allowing multiple devices to operate concurrently in community or clinical environments. Data confidentiality, integrity, low energy consumption, and interoperability are maintained throughout. To mathematically characterise temporal dynamics of dental cavity progression, fractional-order derivatives are used as a telemedicine edge modelling tool. These derivatives capture memory-dependent, non-local behaviours inherent in oral disease progression. Section 2.1 presents the necessary mathematical preliminaries and definitions underpinning the proposed model.

2.1. Preliminaries

Definition 1. The Caputo derivate of a differentiable function $Q(t)$ of order $q \in (0,1)$ with the starting point $t=0$ is defined as follows:

$${}^C D_t^q Q(t) = \frac{1}{\Gamma(1-q)} \int_0^t Q'(\sigma)(t-\sigma)^{-q} d\sigma, \quad (1)$$

Definition 2. The Riemann–Liouville (RL) integral is defined using the following formula, assuming that $Q(t)$ is an integrable function with $0 < q < 1$:

$${}^{RL} D_t^q Q(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\sigma)^{q-1} d\sigma, \quad (2)$$

Definition 3. The differential operator derived from [49,50] that may be expressed as a general proportional or conformable operator is defined as follows:

$${}^P D_t^q Q(t) = Z_1(q, t)Q(t) + Z_0(q, t)Q'(t), \quad (3)$$

$$\left. \begin{aligned} \lim_{q \rightarrow 0^+} Z_0(q, t) &= 0; & \lim_{q \rightarrow 1^-} Z_0(q, t) &= 1; & Z_0(q, t) &\neq 0, q \in (0, 1]; \\ \lim_{q \rightarrow 0^+} Z_1(q, t) &= 1; & \lim_{q \rightarrow 1^-} Z_1(q, t) &= 0; & Z_1(q, t) &\neq 0, q \in [0, 1]; \end{aligned} \right\} \quad (4)$$

where $q \in [0, 1]$. Z_1 and Z_0 are functions of t that ensure the subsequent criterion $\forall t \in \mathbb{R}$:

Consider this operator a generalization of the common differentiation operator ($DQ(t) = Q'(t)$), which depends on q . We are also interested in the constant proportionate, or CP, of the specific scenario:

$${}^{CP} D_t^q Q(t) = Z_1(q)Q(t) + Z_0(q)Q'(t), \quad (5)$$

Definition 4. Let $0 < q < 1$ and $Q \in \mathbb{G}'(0, t_f)$, $t_f \in \mathbb{R}^+$. The Caputo–Fabrizio derivative of order q of an integrable function $Q(t)$ is indicated by CF and is defined [51] as

$${}^{CF} D_{0,t}^q Q(t) = \frac{\mathcal{U}(q)}{1-q} \int_0^t Q'(\varrho) \exp\left[-\frac{q}{1-q}(t-\varrho)\right] d\varrho, \quad (6)$$

where $\mathbb{G}'(0, t_f)$ denote the Sobolev space and $\mathcal{U}(q)$, $\mathcal{U}(0) = \mathcal{U}(1) = 1$ is a normalization function.

Definition 5. CPCF, a new hybrid fractional operator formed by combining the CP (5) and CF (6) operators, is defined in [52] as follows:

$${}^{CPCF} D_t^q Q(t) = \frac{\mathcal{U}(q)}{1-q} \int_0^t [Z_1(q)Q(\varrho) + Z_0(q)Q'(\varrho)] \exp\left(-\frac{q}{1-q}(t-\varrho)\right) d\varrho \quad (7)$$

$$\begin{aligned}
&= \frac{U(q)Z_1(q)}{1-q} \int_0^t Q(q) \exp\left(-\frac{q}{1-q}(t-q)\right) dq \\
&\quad + \frac{U(q)Z_0(q)}{1-q} \int_0^t Q'(q) \exp\left(-\frac{q}{1-q}(t-q)\right) dq \\
&= Q(q) \cdot \frac{U(q)Z_1(q)}{1-q} \left(-\frac{q}{1-q}t\right) + Q'(q) \cdot \frac{U(q)Z_0(q)}{1-q} \left(-\frac{q}{1-q}t\right)
\end{aligned}$$

2.2. Fractional-Order Model Formulation

This section presents the edge-layer model of the telemedicine infrastructure. It enables real-time monitoring of high-risk patients using I-OWSD devices. The model uses on-device edge processing with a hardware accelerator for efficient data offloading [53]. This approach reduces latency, improves detection accuracy, optimizes resource utilization, and allows context-aware, patient-specific processing before sending relevant data to the cloud for advanced analytics.

To capture oral health progression, we define compartments to represent specific oral cavity conditions. This provides a structured framework for individualized health management at the edge. The fractional-order formulation models memory effects and biological non-linearities. It improves prediction fidelity for early dental caries detection. The model is developed under the following assumptions:

- The total population $N(t)$ is constant.
- The coefficient of transmission of dental cavity infection is constant.
- Individuals in the population put on oral wearable sensory devices.
- Recovered dental cavity-infected individuals after medical consultation and treatment return to the susceptible class due to weak immunity at a rate φ .
- Susceptible pregnancies after normal medical consultation due to data transmitted from the wearables return to susceptible class at a rate ξ .
- The I-OWSDs collect dental health data from the target population at the edge (on-device inferencing) and securely transmit it to both outpost and cloud-native healthcare providers with qualified clinicians. This setup enables real-time edge-to-cloud integration of the proposed FODM, leveraging 5G/6G telehealth infrastructure to support remote diagnostics in underserved populations.
- Upon receipt, healthcare providers can immediately analyse the data and initiate appropriate clinical interventions to improve patient outcomes.

2.3. Parameters and Variables

This subsection introduces the variables and parameters employed in the development of the proposed I-OWSD-based dental cavity monitoring framework. To represent the temporal evolution of dental caries, an integer-order nonlinear delay differential model is established. The framework partitions the population into six mutually exclusive compartments, each corresponding to a specific health condition or stage in the cavity progression pathway. For population-level analysis, the state variables are expressed as proportions of the total population, by $N(t)$, rather than absolute counts. This normalised representation provides a more realistic description of the distribution of individuals across the various health states over time. The force of data transmission generated by wearable-device users is defined as

$$\chi = \pi(S + \eta(E + I)), \quad (8)$$

where χ denotes the rate at which health-related information collected by the wearable devices is communicated to healthcare providers. The parameter π represents the

effective transmission coefficient, describing the frequency with which oral-health abnormalities are relayed through the monitoring system. The modification factor η captures the relative contribution of exposed and infected individuals to the overall data-sharing process.

Following the wearable-assessment framework adopted in [54], the device utility function is expressed as

$$\theta = \frac{A + P + t + h}{\rho}, \quad (9)$$

where A denotes the capability of the device to identify infection, P represents the population affected by dental infection, t is the monitoring duration, h corresponds to healthcare facilities involved in detection and intervention, and ρ characterises the level of sustained wearable-device utilisation within the target population. The variables are denoted as $S(t)$ = Cavity-susceptible individuals with I-OWSDs, $E(t)$ = Cavity-exposed individuals with I-OWSDs, $I(t)$ = Cavity-Infected individuals with I-OWSDs, $H(t)$ = Consultation and Treated individuals with I-OWSDs, $R(t)$ = Cavity-recovered individuals with I-OWSDs and $W(t)$ = I-OWSDs.

Using the information presented in Table 1 and Figure 1, a fractional-order mathematical model is formulated to describe the dynamics of the proposed system.

Table 1. Description of Parameters for Wearable Telemedicine Device.

Parameters	Description	Values with I-OWSDs	Values Without I-OWSDs	References
ψ	Recruitment rate of susceptible population with oral wearable sensory device.	231	231	Optimal
μ	Natural death rate.	0.02	0.02	Optimal
η	Modified parameter that accounts for the rate at which data collected from the exposed and infected individuals are shared with the healthcare provider.	8×10^{-6}	0	Optimal
Λ	The rate of individuals who do not use I-OWSDs and who are not aware of which dental disease they are infected with. Most of such population lives in rural areas and slums where people are unaware of dental infections.	0.0034	0.0034	[55]
β	Infection transmission coefficient (i.e., effective contact rate).	0.0006	0.0006	[55]
ζ	The transmission rate from exposed to infectious.	1.5×10^{-5}	1.5×10^{-5}	[55]
ϖ	Death rate due to infection.	0.00428	0.00428	Optimal
$v, \gamma, \emptyset$	The rate at which the biosensor of the wearable devices collects data from susceptible, exposed, and infected individuals with I-OWSDs, respectively.	0.00060, 0.000028, 0.0000028	0, 0, 0	Optimal
$\Delta, \Delta_1, \Delta_2$	The rate of movement of susceptible, exposed, and infected individuals with I-OWSDs, respectively, to consultation and treatment class.	0.00007, 0.0007, 0.045	0.00007, 0.0007, 0.045	Optimal
φ	The rate movement of individuals with I-OWSDs who, after normal consultation with the healthcare providers, return to the susceptible class.	0.8	0.8	Optimal
ξ	The rate at which the recovered individuals with I-OWSDs move to the susceptible class because of impaired immune function.	0.002	0.002	Optimal
σ	Recovery rate of the exposed, through the use of homemade treatment.	6.6×10^{-6}	6.6×10^{-6}	[55]

σ_1	Recovery rate of the infected, through the use of homemade treatment.	6.6×10^{-6}	6.6×10^{-6}	[55]
ε	The rate of infected individuals without I-OWSDs who attend hospital for consultation and treatment.	0.0033	0.0033	Optimal
τ	The delay in time t corresponds to the latent period of the cavity recognition.	0, 10, 30, 50, 70	0, 10, 30, 50, 70	[55]
ρ	Persistent use of wearable devices.	32	32	[54]
π	The effective data transmission rate.	$8E - 6$	0	[54]
ω	The rate constant at which the Monitored and Treated individuals with I-OWSDs recover.	0.0001	0.0001	Optimal
A	The identification of infection.	420	420	[54]
h	The infirmaries of identification.	0.95	0.95	[54]
P	Individuals with different forms of cavity.	1.3	1.3	Optimal
δ	The work functionality of the device.	0.95	0.95	[54]
t	Time.	360	360	Optimal

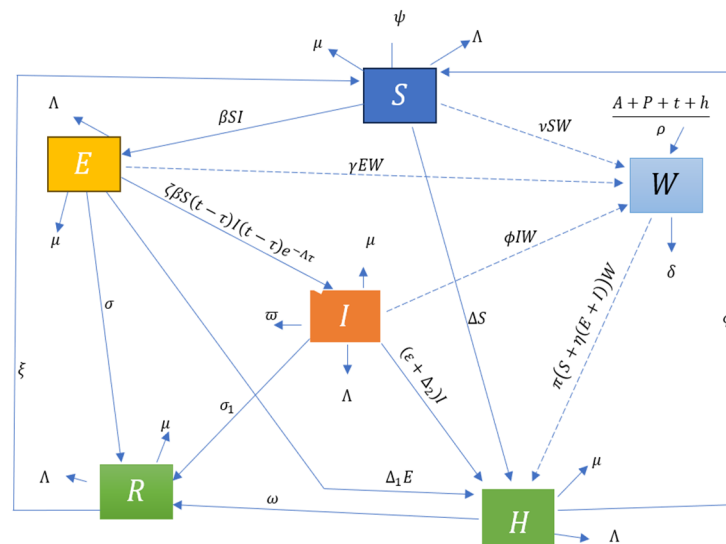


Figure 1. Flow diagram for optimising dental cavity care with computational and analytical insights for target I-OWSDs.

For a fractional order q , $0 < q \leq 1$, the following chemical reactions can be expressed as a series of fractional differential equations:

$$\left. \begin{aligned} {}^{CPCF}_0 D_t^q S(t) &= \psi + \xi R + \phi H - (\beta I + \Lambda + vW + \mu + \Delta)S \\ {}^{CPCF}_0 D_t^q E(t) &= \beta SI - \zeta \beta S(t - \tau) I(t - \tau) e^{-\Lambda \tau} - (\mu + \Lambda + \sigma + \Delta_1 + \gamma W)E \\ {}^{CPCF}_0 D_t^q I(t) &= \zeta \beta S(t - \tau) I(t - \tau) e^{-\Lambda \tau} - (\mu + \sigma_1 + \Lambda + \phi W + \omega + \varepsilon + \Delta_2)I \\ {}^{CPCF}_0 D_t^q H(t) &= \Delta S + \Delta_1 E + (\varepsilon + \Delta_2)I + \chi W - (\Lambda + \mu + \phi + \omega)H \\ {}^{CPCF}_0 D_t^q R(t) &= \omega H + \sigma E + \sigma_1 I - (\Lambda + \mu + \xi)R \\ {}^{CPCF}_0 D_t^q W(t) &= \theta + (vS + \gamma E + \phi I)W - (\chi + \delta)W \end{aligned} \right\} \quad (10)$$

along with the initial value conditions $S(t) \geq 0, E(t) \geq 0, I(t) \geq 0, H(t) \geq 0, R(t) \geq 0, W(t) \geq 0$.

The total population participating with the I-OWSDs monitoring framework at time t is represented by $N(t)$, which is obtained as the aggregate of all epidemiological compartments within the system, thus:

$${}^{CPCF}_0 D_t^q N_1(t) = {}^{CPCF}_0 D_t^q S(t) + {}^{CPCF}_0 D_t^q E(t) + {}^{CPCF}_0 D_t^q I(t) + {}^{CPCF}_0 D_t^q H(t) + {}^{CPCF}_0 D_t^q W(t) \quad (11a)$$

Furthermore, the total population linked with the device at time t is denoted by (11b).

$${}^{CPCF}_0D_t^q N_2(t) = {}^{CPCF}_0D_t^q W(t) \quad (11b)$$

Therefore, we have the entire population of the proposed model at time t as

$${}^{CPCF}_0D_t^q N(t) = {}^{CPCF}_0D_t^q N_1(t) + {}^{CPCF}_0D_t^q N_2(t) \quad (12a)$$

$${}^{CPCF}_0D_t^q N(t) = {}^{CPCF}_0D_t^q S(t) + {}^{CPCF}_0D_t^q E(t) + {}^{CPCF}_0D_t^q I(t) + {}^{CPCF}_0D_t^q H(t) + {}^{CPCF}_0D_t^q W(t) \quad (12b)$$

where $W(t)$ represents individuals actively utilising the proposed intraoral wearable sensing devices for continuous oral-health monitoring and data transmission. The parameter τ represents a clinically meaningful time delay corresponding to the latent period between initial tooth demineralization and its eventual detection or diagnosis. This delay arises due to the asymptomatic nature of early dental caries, limited access to routine screening, and variability in diagnostic efficiency. In the model, the delayed term $S(t - \tau)I(t - \tau)e^{-\Lambda\tau}$ captures how infection dynamics at time t depend on prior states, while accounting for attrition during the delay. Smaller values of τ indicate rapid detection via the I-OWSD framework, whereas larger values reflect delayed diagnosis, thereby establishing a direct link between model parameters and real-world clinical conditions.

3. Proposed Model Analysis

The non-negative bounded solution is derived in Appendix A (Equations (A1)–(A3)), while existence and uniqueness are addressed in Appendix B (Equations (B1)–(B15)). Stability analysis of the proposed mode is presented in Appendix C (Equations (C1)–(C12)). The model employs the Laplace transform of the CPCF operator to solve complex fractional differential equations, capturing system dynamics involving $\mathbf{S}(t), \mathbf{E}(t), \mathbf{I}(t), \mathbf{H}(t), \mathbf{R}(t)$ and $\mathbf{W}(t)$. This method handles nonlinearities and memory effects, offering valuable insights into its dynamic behavior.

3.1. Non-Negative Bounded Solutions

First, we check the non-negative and boundedness of the solution, with the help of a determination of a feasible region.

Theorem 1. The closed set $\Pi = \Pi_1 \times \Pi_2$ where $\Pi_1 = \{(S(t), E(t), I(t), H(t), R(t)) \in \mathbb{R}_{\geq 0}^5: 0 \leq N_1 \leq \frac{\psi}{\Lambda + \mu}\}$ and $\Pi_2 = \{W(t) \in \mathbb{R}_{\geq 0}^1: 0 \leq N_2 \leq \frac{\theta}{\delta}\}$ is non-negative.

Proof. The boundedness of the solution for the model system (10) can be explained by the following Equations (11a) and (11b), which are completed in Appendix A.

3.2. Existence and Uniqueness Analysis

The validity of the equational framework that ensures the survival of fractional calculus is examined in this:

Theorem 2. The solution of the model system (10) is positive, provided the initial conditions are positive for all $t > 0$ and assuming there exist non-negative constants, $\mathfrak{I}_h$ and $\widehat{\mathfrak{I}}_h$, such that

1. $|\mathcal{M}_h(x_h, t) - \mathcal{M}'_h(x_h, t)| \leq \mathfrak{I}_h |x_h - x'_h|, h = 1, \dots, 6.$
2. $|\mathcal{M}_h(x_h, t)|^2 \leq \widehat{\mathfrak{I}}_h (1 + |x_h|), \forall (x, t) \in \mathbb{R}^4 \times [0, t_f].$

Proof. Model system (10) is simplified and presented as the system (B1), as shown in Appendix B, with the completion of the Proof.

3.3. Stability Analysis of the Proposed Model

We conduct a qualitative study of the model system (10) to enable us to understand its dynamic properties. For a fractional-order model of oral wearable sensory devices, the stability analysis would involve studying the behaviour of the system over time.

Theorem 3. Let $\mathbb{G}$ represent a self-map on Banach space $\mathfrak{B}$. This implies that the following inequality is true for all instances of $\mathcal{G}$.

$$\|\mathbb{G}_a^* - \mathbb{G}_b^*\| \leq \mathcal{C}\|a - \mathbb{G}^*\| + c\|a - b\|, \quad (13)$$

with $e \in [0, 1)$, $\mathcal{H} \geq 0$, if we assume that $\mathbb{G}^*$ is Picard $\mathbb{G}^*$ stable.

Proof. Let's examine the equations associated with the model system (10) by having $\mathbb{G}^*$ as Picard $\mathbb{G}^*$ stable to obtain the following equations, as shown in Appendix C. (See Equations (C1)–(C12) in Appendix C.). $\square$

4. Generalized Proportional Operators

Theorem 5. The Laplace transform of the CPCF operator is given as follows:

$$\left. \begin{aligned} \mathcal{L}[{}^{CPCF}_0 D_t^\varrho S(t)] &= \Omega \mathcal{L}[S(t)] - \mathbb{X}S(0), \\ \mathcal{L}[{}^{CPCF}_0 D_t^\varrho E(t)] &= \Omega \mathcal{L}[E(t)] - \mathbb{X}E(0), \\ \mathcal{L}[{}^{CPCF}_0 D_t^\varrho I(t)] &= \Omega \mathcal{L}[I(t)] - \mathbb{X}I(0), \\ \mathcal{L}[{}^{CPCF}_0 D_t^\varrho H(t)] &= \Omega \mathcal{L}[H(t)] - \mathbb{X}H(0), \\ \mathcal{L}[{}^{CPCF}_0 D_t^\varrho R(t)] &= \Omega \mathcal{L}[R(t)] - \mathbb{X}R(0), \\ \mathcal{L}[{}^{CPCF}_0 D_t^\varrho W(t)] &= \Omega \mathcal{L}[W(t)] - \mathbb{X}W(0). \end{aligned} \right\}, \quad (14)$$

where $\Omega = \frac{u(\varrho)Z_1(\varrho)}{\varrho + \varsigma(1-\varrho)} + \frac{\varsigma u(\varrho)Z_0(\varrho)}{\varrho + \varsigma(1-\varrho)}$ and $\mathbb{X} = \frac{u(\varrho)Z_0(\varrho)}{\varrho + \varsigma(1-\varrho)}$.

Proof. From Equation (10), we obtain the Laplace transform of the CPCF operator for the variables S, E, I, H, R and W , and this is given in Appendix D. $\square$

Eigenfunctions of CPCF

Theorem 5. Consider the following system differential equations with the CPCF operator

$$\left. \begin{aligned} {}^{CPCF}_0 D_t^\varrho S(t) &= P_1(t, S, E, I, H, R, W), \\ {}^{CPCF}_0 D_t^\varrho E(t) &= P_2(t, S, E, I, H, R, W), \\ {}^{CPCF}_0 D_t^\varrho I(t) &= P_3(t, S, E, I, H, R, W), \\ {}^{CPCF}_0 D_t^\varrho H(t) &= P_4(t, S, E, I, H, R, W), \\ {}^{CPCF}_0 D_t^\varrho R(t) &= P_5(t, S, E, I, H, R, W), \\ {}^{CPCF}_0 D_t^\varrho W(t) &= P_6(t, S, E, I, H, R, W). \end{aligned} \right\} \quad (15)$$

Assuming that $S(0) = E(0) = I(0) = H(0) = R(0) = W(0) = 0$, and by utilising the Laplace transform, we have

$$\left. \begin{aligned} S(t) &= \mathbb{Q}[\varrho \mathbb{R} \mathcal{K}_t^{\varrho+i} P_1(t, \vartheta) + (1-\varrho) \mathbb{R} \mathcal{K}_t^{\varrho+i} P_1(t, \vartheta)], \\ E(t) &= \mathbb{Q}[\varrho \mathbb{R} \mathcal{K}_t^{\varrho+i} P_2(t, \vartheta) + (1-\varrho) \mathbb{R} \mathcal{K}_t^{\varrho+i} P_2(t, \vartheta)], \\ I(t) &= \mathbb{Q}[\varrho \mathbb{R} \mathcal{K}_t^{\varrho+i} P_3(t, \vartheta) + (1-\varrho) \mathbb{R} \mathcal{K}_t^{\varrho+i} P_3(t, \vartheta)], \\ H(t) &= \mathbb{Q}[\varrho \mathbb{R} \mathcal{K}_t^{\varrho+i} P_4(t, \vartheta) + (1-\varrho) \mathbb{R} \mathcal{K}_t^{\varrho+i} P_4(t, \vartheta)], \\ R(t) &= \mathbb{Q}[\varrho \mathbb{R} \mathcal{K}_t^{\varrho+i} P_5(t, \vartheta) + (1-\varrho) \mathbb{R} \mathcal{K}_t^{\varrho+i} P_5(t, \vartheta)], \\ W(t) &= \mathbb{Q}[\varrho \mathbb{R} \mathcal{K}_t^{\varrho+i} P_6(t, \vartheta) + (1-\varrho) \mathbb{R} \mathcal{K}_t^{\varrho+i} P_6(t, \vartheta)]. \end{aligned} \right\} \quad (16)$$

where $\mathbb{Q} = \frac{1}{u(\varrho)Z_0(\varrho)}$, and $\mathbb{R} = \sum_{i=0}^{\infty} \left(-\frac{Z_1(\varrho)}{Z_0(\varrho)}\right)^i$.

Proof. Using theorem 6, we have $\square$

$$\left. \begin{aligned} \mathcal{B}\{S(t)\}\mathbb{S} &= \mathcal{B}\{P_1(t, \vartheta)\}, \\ \mathcal{B}\{E(t)\}\mathbb{S} &= \mathcal{B}\{P_2(t, \vartheta)\}, \\ \mathcal{B}\{I(t)\}\mathbb{S} &= \mathcal{B}\{P_3(t, \vartheta)\}, \\ \mathcal{B}\{H(t)\}\mathbb{S} &= \mathcal{B}\{P_4(t, \vartheta)\}, \\ \mathcal{B}\{R(t)\}\mathbb{S} &= \mathcal{B}\{P_5(t, \vartheta)\}, \\ \mathcal{B}\{W(t)\}\mathbb{S} &= \mathcal{B}\{P_6(t, \vartheta)\} \end{aligned} \right\} \quad (17)$$

$$\text{with } \mathbb{S} = \left[\frac{u(q)}{q + \varsigma(1-q)} (Z_1(q) + \varsigma Z_0(q)) \right].$$

This gives

$$\left. \begin{aligned} \mathcal{B}\{S(t)\} &= \mathbb{Q}[q\mathbb{W}\varsigma^{-i-1} + (1-q)\mathbb{R}\varsigma^{-i}] \mathcal{B}\{P_1(t, \vartheta)\}, \\ \mathcal{B}\{E(t)\} &= \mathbb{Q}[q\mathbb{W}\varsigma^{-i-1} + (1-q)\mathbb{R}\varsigma^{-i}] \mathcal{B}\{P_2(t, \vartheta)\}, \\ \mathcal{B}\{I(t)\} &= \mathbb{Q}[q\mathbb{W}\varsigma^{-i-1} + (1-q)\mathbb{R}\varsigma^{-i}] \mathcal{B}\{P_3(t, \vartheta)\}, \\ \mathcal{B}\{H(t)\} &= \mathbb{Q}[q\mathbb{W}\varsigma^{-i-1} + (1-q)\mathbb{R}\varsigma^{-i}] \mathcal{B}\{P_4(t, \vartheta)\}, \\ \mathcal{B}\{R(t)\} &= \mathbb{Q}[q\mathbb{W}\varsigma^{-i-1} + (1-q)\mathbb{R}\varsigma^{-i}] \mathcal{B}\{P_5(t, \vartheta)\}, \\ \mathcal{B}\{W(t)\} &= \mathbb{Q}[q\mathbb{W}\varsigma^{-i-1} + (1-q)\mathbb{R}\varsigma^{-i}] \mathcal{B}\{P_6(t, \vartheta)\}. \end{aligned} \right\} \quad (18)$$

$$\mathbb{W} = \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)} \right)^n.$$

Therefore, we obtain

$$\varsigma^{-i}\{P(t)\} = \mathcal{B}\left\{ \frac{t^i}{\mathbb{Q}(i+1)} \right\} \mathcal{B}\{P(t)\} = \mathcal{B}\left\{ P(t) \cdot \frac{t^i}{\mathbb{Q}(i+1)} \right\} = \mathcal{B}\{0\mathcal{K}_t^{q+i}P(t)\} \quad (19)$$

$$\varsigma^{-i-1}\{P(t)\} = \mathcal{B}\left\{ \frac{t^i}{\mathbb{Q}(i)} \right\} \mathcal{B}\{P(t)\} = \mathcal{B}\left\{ P(t) \cdot \frac{t^i}{\mathbb{Q}(i)} \right\} = \mathcal{B}\{0\mathcal{K}_t^iP(t)\} \quad (20)$$

Hence utilizing the inverse Laplace transform, we have

$$\left. \begin{aligned} \mathcal{B}\{S(t)\} &= \mathbb{Q}[q\mathbb{W}\mathcal{B}\{0\mathcal{K}_t^{q+i}P_1(t, \vartheta)\} + (1-q)\mathbb{R}\varsigma^{-i}] \mathcal{B}\{0\mathcal{K}_t^iP_1(t, \vartheta)\}, \\ \mathcal{B}\{E(t)\} &= \mathbb{Q}[q\mathbb{W}\mathcal{B}\{0\mathcal{K}_t^{q+i}P_2(t, \vartheta)\} + (1-q)\mathbb{R}\varsigma^{-i}] \mathcal{B}\{0\mathcal{K}_t^iP_2(t, \vartheta)\}, \\ \mathcal{B}\{I(t)\} &= \mathbb{Q}[q\mathbb{W}\mathcal{B}\{0\mathcal{K}_t^{q+i}P_3(t, \vartheta)\} + (1-q)\mathbb{R}\varsigma^{-i}] \mathcal{B}\{0\mathcal{K}_t^iP_3(t, \vartheta)\}v \\ \mathcal{B}\{H(t)\} &= \mathbb{Q}[q\mathbb{W}\mathcal{B}\{0\mathcal{K}_t^{q+i}P_4(t, \vartheta)\} + (1-q)\mathbb{R}\varsigma^{-i}] \mathcal{B}\{0\mathcal{K}_t^iP_4(t, \vartheta)\}, \\ \mathcal{B}\{R(t)\} &= \mathbb{Q}[q\mathbb{W}\mathcal{B}\{0\mathcal{K}_t^{q+i}P_5(t, \vartheta)\} + (1-q)\mathbb{R}\varsigma^{-i}] \mathcal{B}\{0\mathcal{K}_t^iP_5(t, \vartheta)\}, \\ \mathcal{B}\{W(t)\} &= \mathbb{Q}[q\mathbb{W}\mathcal{B}\{0\mathcal{K}_t^{q+i}P_6(t, \vartheta)\} + (1-q)\mathbb{R}\varsigma^{-i}] \mathcal{B}\{0\mathcal{K}_t^iP_6(t, \vartheta)\}. \end{aligned} \right\} \quad (21)$$

5. Model Numerical Scheme

The model numerical scheme was obtained by applying the Laplace transform on both sides of Equation (10) and using Theorem 4, and it is completed in Appendix E:

$$\begin{aligned} \mathcal{B}\{S(t)\} &= \mathbb{K}S_0 + \mathbb{H}(\psi + \xi\mathcal{B}\{R\} + \varphi\mathcal{B}\{H\} - \beta\mathcal{B}\{SI\} - \nu\mathcal{B}\{SW\} - (\Lambda + \mu + \Delta_1)\mathcal{B}\{S\}), \\ \mathcal{B}\{E(t)\} &= \mathbb{K}E_0 + \mathbb{H}(\beta\mathcal{B}\{SI\} - \zeta\beta\mathcal{B}\{SI\}(t-\tau)(t-\tau)e^{-\Lambda\tau} - \Delta_1\mathcal{B}\{E\} - \gamma\mathcal{B}\{EW\} - (\mu + \Lambda + \sigma)\mathcal{B}\{E\}), \\ \mathcal{B}\{I(t)\} &= \mathbb{K}I_0 + \mathbb{H}(\zeta\beta\mathcal{B}\{SI\}(t-\tau)(t-\tau)e^{-\Lambda\tau} - \phi\mathcal{B}\{IW\} - (\mu + \sigma_1 + \Lambda + \varpi + \varepsilon + \Delta_2)\mathcal{B}\{I\}), \\ \mathcal{B}\{H(t)\} &= \mathbb{K}H_0 + \mathbb{H}(\Delta\mathcal{B}\{S\} + \Delta_1\mathcal{B}\{E\} + (\varepsilon + \Delta_2)\mathcal{B}\{I\} + \chi\mathcal{B}\{W\} - (\Lambda + \mu + \varphi + \omega)\mathcal{B}\{H\}), \\ \mathcal{B}\{R(t)\} &= \mathbb{K}R_0 + \mathbb{H}(\omega\mathcal{B}\{H\} + \sigma\mathcal{B}\{E\} + \sigma_1\mathcal{B}\{I\} - (\Lambda + \mu + \xi)\mathcal{B}\{R\}), \\ \mathcal{B}\{W(t)\} &= \mathbb{K}W_0 + \mathbb{H}(\theta + (\nu\mathcal{B}\{SW\} + \gamma\mathcal{B}\{EW\} + \phi\mathcal{B}\{IW\}) - (\chi + \delta)\mathcal{B}\{W\}), \end{aligned}$$

$$\text{where } \mathbb{K} = \frac{1}{\varsigma + \frac{Z_1(q)}{Z_0(q)}} \text{ and } \mathbb{H} = \frac{q + \varsigma(1-q)}{u(q)Z_1(q) + \varsigma u(q)Z_0(q)}.$$

The iterative solution in Equation (E5) models nonlinear, memory-dependent, and delay-coupled systems using fractal–fractional calculus as derived in Appendix E, for the state variables $S_{g+1}(t), E, I_{g+1}(t), H_{g+1}(t), R_{g+1}(t), W_{g+1}(t)$. Using operators $Z_0(q), Z_1(q)$, and $\mathcal{U}(q)$, the method captures hereditary and non-local dynamics. Its recursive form supports successive approximations and handles time delays effectively. The series incorporates delays via convolution with such terms as $SI(t-\tau), EW(t), IW(t)$, enabling the inclusion of real-world effects. The formulation is numerically tractable, enabling sensitivity analysis, policy simulations, and integration with numerical schemes. Overall, it bridges rigorous mathematics with decision-making by incorporating external

inputs, parameter uncertainty, and control functions, supporting optimal control, robustness analysis, and AI-driven applications.

5.1. Numerical Simulation and Validation

A. Setup Description

This section discusses experiments to derive the solution dynamics of our model, using the FODM Equation (10) with the Caputo-Fabrizio scheme. The focus is on detecting data for susceptible, exposed, and infected populations with dental cavities using I-OWSDs. We captured the effective data transmission rate and the reduction of dental cavity patients via real-time decision expert agents. The model for dental cavity dynamics with I-OWSD was implemented using AWS EC2 instances, each provisioned with configurations to host specific simulation elements (e.g., S, E, I, H, R , and W compartments). Any cloud platform can serve, although the computational demands are scaled using AWS Auto Scaling. AWS S3 was utilised for storing I-OWSDs data streams, while Lambda supported serverless execution for complex data processing. To manage parallel VMs handling FODs for a time-bound use case, AWS SimSpace Weaver was employed. Real-time data monitoring and updates, including real-time wearable inputs, were handled by AWS Kinesis, and CloudWatch was configured for resource utilisation monitoring. While AWS was used in this study for its simplicity and availability at the time, the simulation framework is cloud-agnostic and can be replicated using any multicloud or individual cloud system. Furthermore, open-source tools and platforms can also be adopted to implement and validate the simulation across diverse environments.

The system model (8) was simulated using initial conditions $S(0) = 2000$, $E(0) = 200$, $I(0) = 20$, $H(0) = 15$, $R(0) = 5$, $W(0) = 343$ [54,55]. The parameter values used in the simulation, provided in Table 1, were fixed for simulation and are not based on real data, which remains a limitation of this study.

B. Result Evaluation

Figures 2a–c and 3a–f illustrate the dynamic behavior of dental caries progression under three scenarios: absence of I-OWSDs, low-intensity deployment, and high-intensity deployment. In the absence of I-OWSDs, the system exhibits a classical uncontrolled epidemic-like progression of dental caries, characterized by a rapid increase in the exposed and infected compartments over time. Mathematically, this reflects dominant transition rates from susceptible to exposed and infected states, with limited feedback or control influence. Clinically, this scenario corresponds to delayed detection, late intervention, and accumulation of untreated dental lesions, leading to higher disease burden and treatment demand. When low-intensity I-OWSDs are introduced, a noticeable moderation in disease progression is observed. The susceptible population declines more slowly, while the exposed and infected compartments grow at reduced rates compared to the uncontrolled case. From a fractional-order perspective, this indicates that memory effects—captured by the fractional derivative—begin to influence system dynamics, smoothing transitions between compartments and introducing partial corrective feedback. Clinically, this scenario reflects intermittent or limited real-time monitoring, enabling earlier detection than conventional care but still allowing some disease progression due to delayed or suboptimal response intensity. In contrast, high-intensity I-OWSD deployment produces a fundamentally different system behavior. The exposed and infected compartments rapidly decay toward near-zero levels, while the susceptible population stabilizes at a lower but controlled equilibrium. Mathematically, this demonstrates enhanced system stability and faster convergence driven by strong feedback control and long-memory effects inherent in the fractional-order framework. The reduced amplitude and damped oscillations across all compartments indicate suppression of

disease persistence and prevention of relapse. Clinically, this corresponds to continuous real-time monitoring of oral health parameters (such as pH, biofilm activity, and enamel demineralization), enabling prompt preventive actions, early treatment, and sustained caries control. Further compartment-wise comparisons in Figure 3 confirm that increased I-OWSD intensity significantly reduces treatment burden and cumulative infection load, while improving recovery outcomes. The treated and recovered compartments show minimal overshoot under high-intensity monitoring, suggesting optimised intervention timing and reduced overtreatment. The I-OWSD compartment itself stabilises, indicating feasible long-term device integration without destabilising the oral health system. Overall, these results highlight that coupling IoT-enabled intraoral wearable systems with fractional-order modelling provides both mathematical robustness—through memory-aware, stable dynamics—and clinical effectiveness, by enabling early detection, personalised intervention, and sustained management of dental caries. The proposed framework supports a shift from reactive dental care to predictive, preventive, and precision oral healthcare.

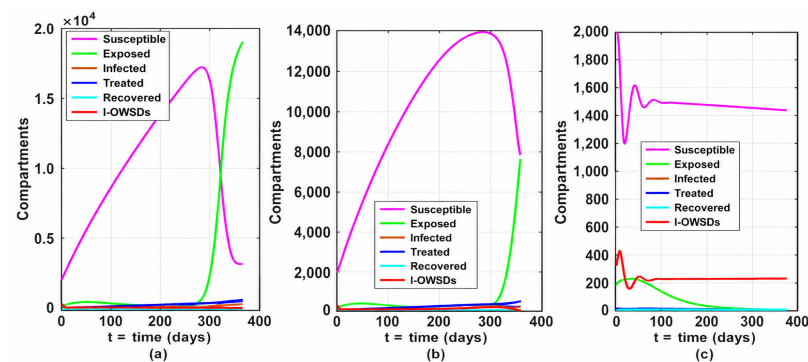


Figure 2. (a–c) Dynamic behavior of dental cavity among the compartments for model system (10) with and without I-OWSDs.

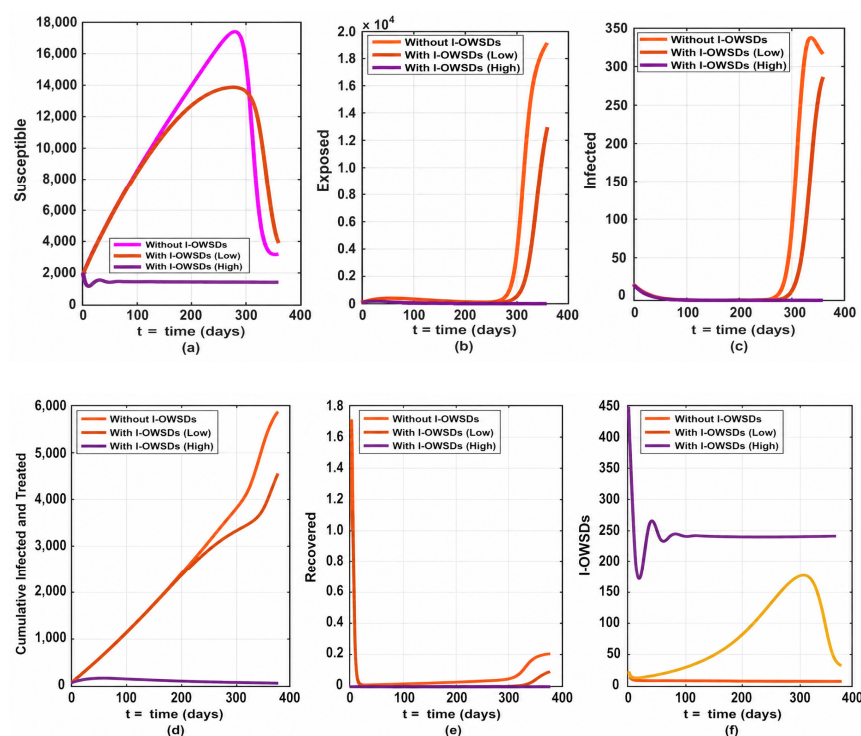


Figure 3. (a–f) Comparing the dynamic behavior of dental cavities among the compartments with and without I-OWSDs.

Figure 4a–f illustrate the influence of varying fractional order $q \in [0.8, 1.0]$ on the dynamics of the susceptible $S(t)$, exposed $E(t)$, infected $I(t)$, consultation and treated $H(t)$, recovered $R(t)$, and I-OWSDs $W(t)$ compartments within the IoT-enabled dental caries model. Across all compartments, decreasing the fractional order introduces pronounced memory effects, resulting in smoother trajectories, reduced oscillations, and delayed convergence toward equilibrium. This behavior reflects the nonlocal property of fractional derivatives, where current system evolution depends on historical states, yielding enhanced stability compared to the integer-order case ($q = 1$). In particular, the infected compartment $I(t)$ decays more gradually for lower q , indicating prolonged disease persistence when past pathological states exert stronger influence. This suggests that patients with long-standing oral hygiene neglect or chronic enamel demineralization may exhibit slower recovery, emphasizing the importance of early detection. The exposed compartment $E(t)$ shows a faster decline at higher fractional orders, while lower q values sustain exposure over longer periods, highlighting how latent caries activity can persist undetected without continuous monitoring. The consultation and treated $H(t)$ and recovered $R(t)$ compartments benefit from higher fractional orders, achieving quicker stabilization, which corresponds clinically to timely remineralization and effective preventive care. The I-OWSDs $W(t)$ converges to a stable level for all q , demonstrating that IoT-enabled monitoring remains robust under varying memory intensities, thereby supporting long-term deployment feasibility.

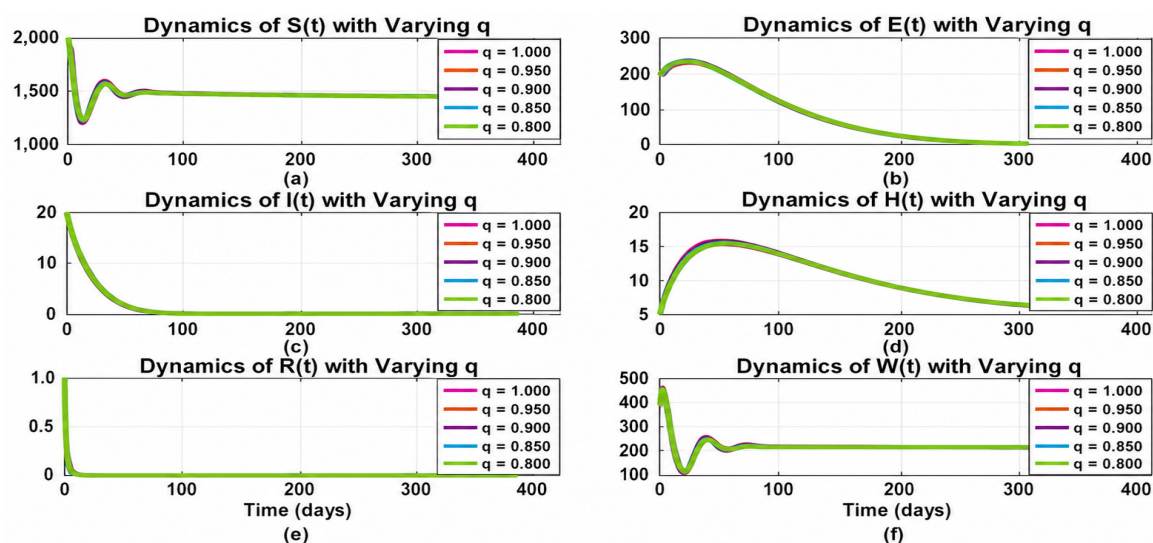


Figure 4. (a–f) Dynamics of S , E , I , H , R , and W with different values of the order q .

Figure 5a–c further examine the infected compartment under different time-delay values ($\tau = 10, 30, 50, 70$) in the absence of intraoral wearable systems. Increasing delay consistently shifts the infection peak forward in time while amplifying its magnitude, particularly for lower fractional orders. The combined effect of delay and fractional memory exacerbates instability, allowing infection to accumulate before intervention occurs. Clinically, this represents delayed diagnosis and postponed dental visits, which significantly worsen caries progression. Lower fractional orders intensify this effect, reflecting patients with prolonged behavioral or biological memory of disease risk. Overall, these results demonstrate that fractional-order dynamics and time-delay effects are critical in accurately characterizing dental caries progression. The integration of IoT-enabled intraoral wearable systems effectively counteracts memory-driven disease persistence and diagnostic delays by enabling continuous monitoring and early intervention. This framework supports a transition from episodic dental care to predictive,

preventive, and precision oral healthcare, grounded in mathematically robust and clinically actionable modeling.

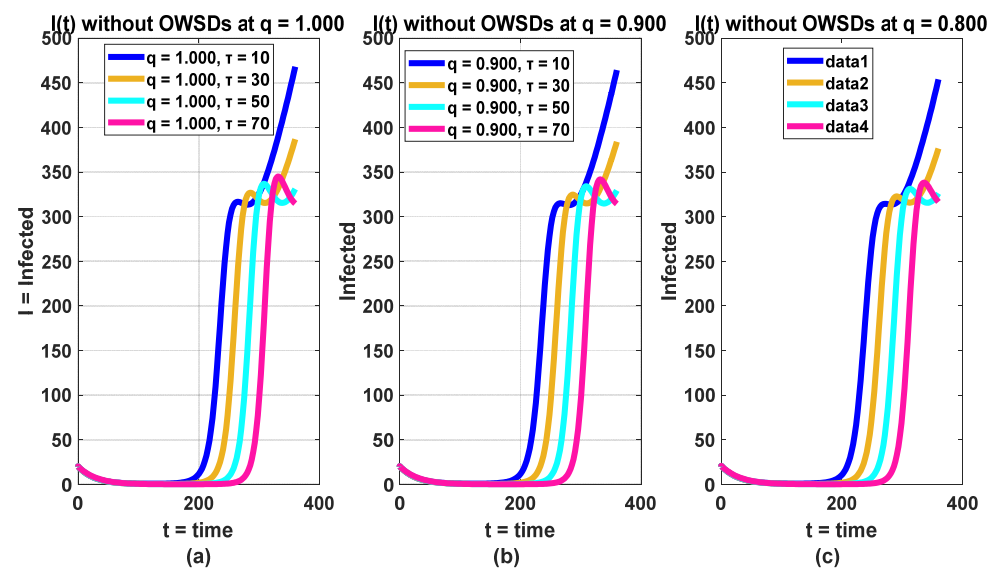


Figure 5. (a–c) Dynamics of the infected at different time-delay values ($\tau = 10, 30, 50, 70$) and $q = 1.000, 0.900$, and 0.800 respectively for (a), (b), and (c).

Figures 6a–c and 7a–c present the combined influence of fractional order $q \in \{0.8, 0.9, 1.0\}$ and diagnostic delay $\tau \in \{30, 70\}$ on the exposed $E(t)$, consultation and treated $H(t)$, and recovered $R(t)$ compartments of the IoT-enabled dental caries model. For $\tau = 30$ (Figure 6), the exposed population $E(t)$ remains low initially but rises sharply after a critical time, with the increase occurring earlier and more steeply for higher fractional-order values. This reflects weaker memory effects at $q \approx 1$, where the system responds more rapidly to changes in exposure dynamics. In contrast, lower fractional orders delay the growth of $E(t)$, indicating that historical oral health states suppress rapid disease escalation.

Clinically, this suggests that individuals with sustained preventive behaviors may experience delayed onset of caries exposure even in the presence of diagnostic lag. The consultation and treated compartment $H(t)$ increase monotonically for all q , with slightly higher trajectories observed at lower fractional orders. This indicates that memory effects enhance cumulative healthcare engagement over time, as prior exposure and symptoms influence future treatment-seeking behavior. From a clinical standpoint, this aligns with the role of IoT-enabled intraoral wearables, which reinforce continuous feedback and promote timely dental consultations despite inherent delays. The recovered compartment $R(t)$ remains minimal initially and increases slowly toward the end of the simulation, with higher q values achieving marginally faster recovery. This implies that rapid system responsiveness facilitates quicker remineralisation and healing once treatment is initiated. For larger delay $\tau = 70$ (Figure 7), the exposed compartment $E(t)$ exhibits a more pronounced and delayed surge across all fractional orders, with higher peaks compared to the $\tau = 30$ case. The interaction between long delays and fractional memory amplifies disease accumulation before intervention, increasing system vulnerability. This corresponds to late diagnosis or prolonged absence of monitoring, during which caries progress unnoticed. Although the consultation and treated compartment $H(t)$ still increases, its growth is shifted further in time, indicating delayed access to care. Recovery $R(t)$ remains suppressed for most of the time horizon, emphasising that delayed intervention significantly prolongs disease burden.

Consequently, these results demonstrate that fractional-order dynamics provide a realistic representation of memory-driven oral health processes, while time delay critically governs disease escalation and recovery. The findings mathematically validate the necessity of incorporating both memory and delay in caries modeling and clinically underscore the importance of IoT-enabled intraoral wearable systems in minimising diagnostic delays, accelerating treatment initiation, and improving long-term oral health outcomes.

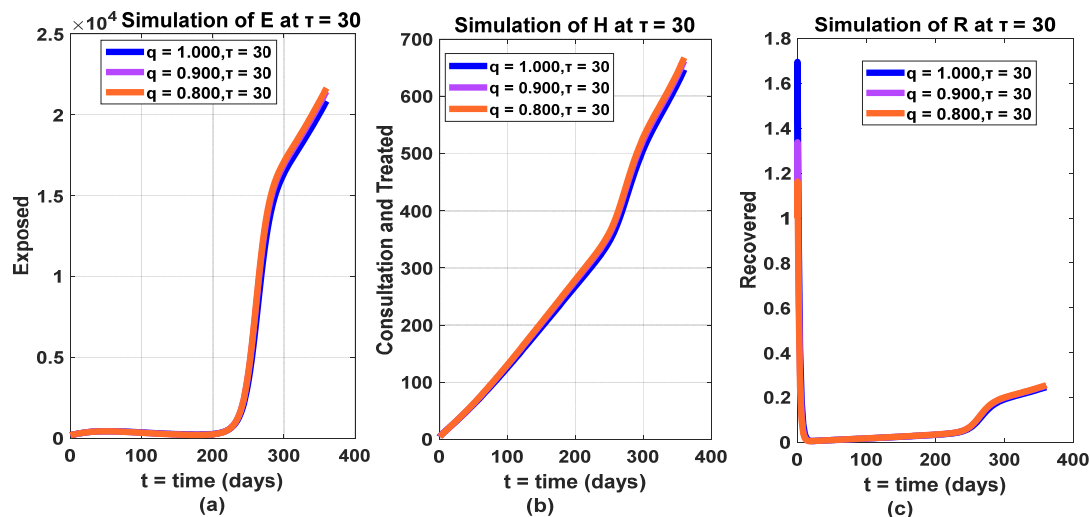


Figure 6. (a–c) Variation in the Exposed $E(t)$, Consultation and Treated $H(t)$ and Recovered $R(t)$ with time delay values ($\tau = 30$) and at Fractional Order values ($q = 0.800, 0.900, 1.000$).

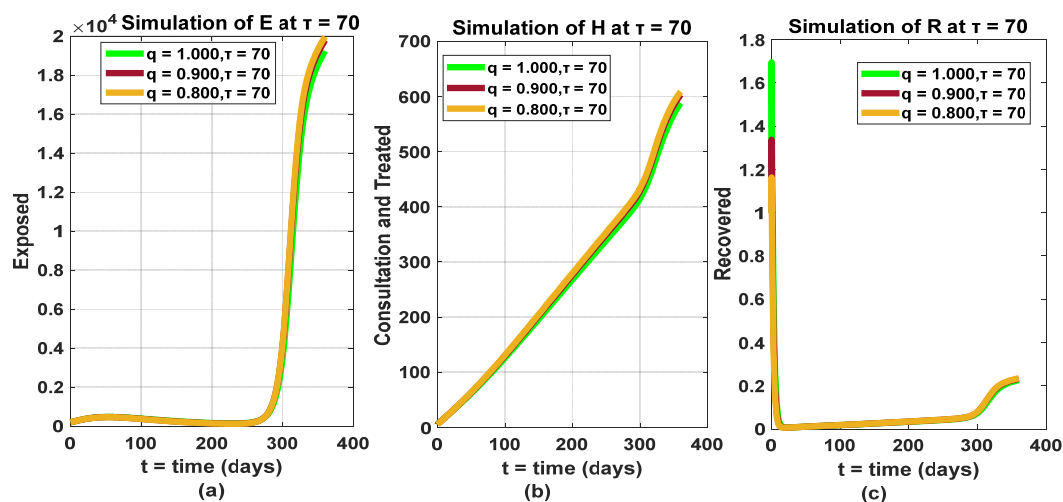


Figure 7. (a–c) Variation in the Exposed $E(t)$, Consultation and Treated $H(t)$ and Recovered $R(t)$ with time delay values ($\tau = 70$) and at Fractional-Order values ($q = 0.800, 0.900, 1.000$).

Table 2 quantitatively demonstrates the impact of I-OWSDs on the dynamics of dental caries by comparing key state variables with and without device implementation. With strong system utilization and sustained monitoring by I-OWSD engagement variable W (67.49%), the susceptible population S shows a 36.25% improvement with I-OWSDs, indicating enhanced preventive awareness and early behavioural modifications that reduce progression toward exposure. A substantial 41.51% reduction in the exposed population E highlights the effectiveness of continuous intraoral monitoring in detecting early pathological changes and limiting disease advancement. The infected population I exhibits a more modest but meaningful improvement of 14.19%, reflecting the inherent

biological persistence of established caries, while still benefiting from earlier detection and intervention. Notably, the consultation and treated population H improves by 25.45%, suggesting that real-time feedback from I-OWSDs facilitates timely dental consultations and treatment uptake. The recovered population R records a 61.47% improvement, underscoring the role of early diagnosis and prompt management in accelerating recovery and remineralization processes.

Table 2. Percentage improvement of the state variables with the use of I-OWSDs.

State Variables	Without I – OWSDs	With I – OWSDs	Percentage Improvement (%)
Susceptible Population (S)	3,211.5425	5,037.8462	36.25
Exposed Population (E)	19,250.2739	11,259.5722	41.51
Infected Population (I)	316.4321	271.5153	14.19
Consultation/Treated Population (H)	588.1962	438.4943	25.45
Recovered Population (R)	0.22631	0.087186	61.47
I-OWSD (W)	13.8698	42.6591	67.49

5.2. System Architecture of the I-OWSD Framework

The results presented in Table 2 provide analytical evidence that integrating I-OWSDs into dental care can significantly enhance early detection, improve treatment efficiency, and support better recovery outcomes. This highlights the potential of a smarter, data-driven approach to dental caries management. Developing a practical system for real-world deployment requires lightweight network. To implement these insights, we established an architecture I-OWSD framework.

As illustrated in Figure 8, the system employs an IoT-based edge inferencing device integrated with AI-assisted APIs, enabling secure real-time monitoring, diagnosis, and remote treatment [46]. Unlike standard AI-assisted diagnostics, which provide predictions solely to support clinician decisions without adaptive behaviour, the proposed I-OWSD system exhibits agency by autonomously integrating multimodal outputs, performing context-aware risk assessments, and generating patient-specific alerts in real time. Edge-level inference and lightweight AI models enable the system to operate autonomously under constrained IoT networks. It processes sensor data locally without continuous cloud connectivity. When connectivity is available, the system synchronizes with cloud platforms for advanced analytics, data logging, and model updates [54]. This hybrid edge–cloud architecture ensures proactive, individualized oral healthcare even in low-power, intermittent, or bandwidth-limited environments, making it particularly suited for underserved regions where access to dental professionals is sporadic.

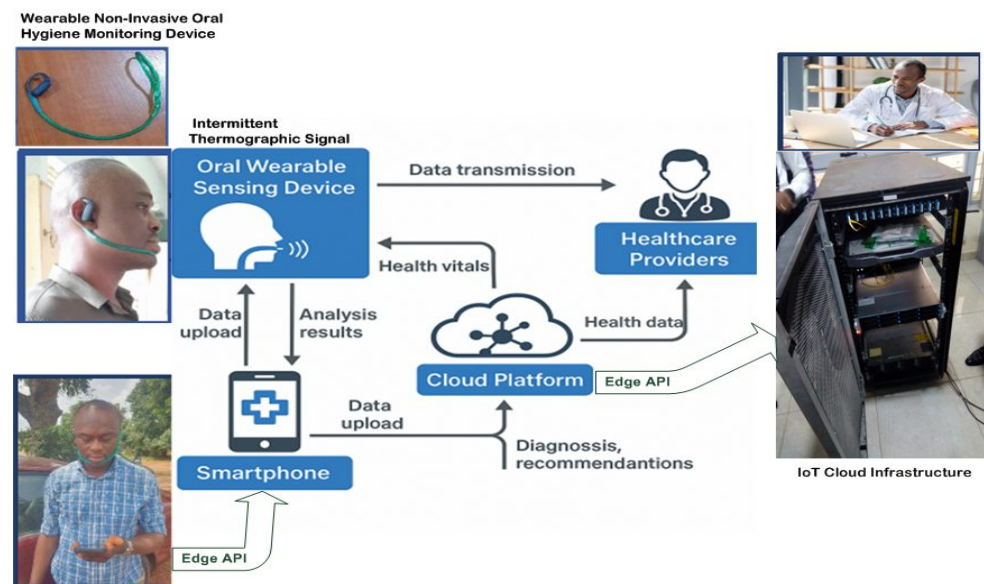


Figure 8. Biocompatible intraoral sensor-interface and user-model diagram showing IoT-integrated I-OWSDs for proactive, data-driven dental care.

The multi-layered integration drives low-power operation within the regulatory and security requirements. Key security measures include the following:

- Device-level safeguards: lightweight encryption (AES-128/256) [56], secure boot, firmware integrity checks, and tamper-evident hardware ensure device authentication and data protection [41].
- Data transmission security: mutual authentication between devices and IoT gateways ensures safe communication.
- Edge and cloud processing: edge computing reduces transmission of sensitive raw data, while cloud communication employs TLS 1.3, role-based access control, audit trails, and logging, to comply with HIPAA and GDPR. Trusted Platform Modules (TPMs) and Secure Elements (SEs) provide cryptographic security and self-preservation.
- Anomaly detection: an AI-powered agent monitors system behaviour in real time, identifying potential vulnerabilities and supporting continuous operational integrity.
- Ethical and Security Considerations: ethical aspects of using Agentic AI within I-OWSDs were addressed, including strategies for alignment, resource optimisation, and environmental adaptability [34–37]. The integrated framework ensures data confidentiality, integrity, and availability across wearable, gateway, and cloud layers, supporting safe and compliant deployment in smart healthcare infrastructures.

5.3. Component-Level Description

The I-OWSD infrastructure consists of the following components:

- I-OWSD Sensors: embedded within intraoral devices (as explained previously in Section 2), these sensors continuously monitor key oral health indicators, (e.g., pH, temperature, and bacterial activity). Each sensor node functions as a proactive agent, capturing real-time data for downstream processing.
- Data Transmission Layer: sensor readings are securely transmitted in real time via IoT communication protocols such as LoRa, Bluetooth, Zigbee, or Wi-Fi to smartphones or dedicated gateways, ensuring low-latency data delivery.

- Edge/Cloud Processing: data can be processed locally on smartphones or transmitted to cloud/edge servers. Prescriptive analytics, combined with the CPCF-based fractional-order model, evaluate disease risk and progression for timely clinical insights.
- Telemedicine Platform: processed information is routed to telemedical agents or dental professionals, enabling remote consultations, intervention planning, and personalized treatment decisions.
- Implicit Feedback Loop: clinicians provide feedback, alerts, and care plans to the patient's device, creating a closed-loop system that supports proactive disease management.
- Global Patient Health Dashboard: patients access their oral health status through API-based dashboards, enabling self-monitoring, preventive behaviour reinforcement, and engagement with care recommendations.

This modular architecture allows seamless integration of IoT sensing, AI analytics, and telemedicine, forming a robust platform for early detection and monitoring of oral diseases.

5.4. AI-Based Intraoral Disease Diagnostics

To validate the I-OWSD system, an AI-assisted diagnostic scheme was developed to classify and predict oral disease states using continuous multimodal data [53]. The workflow includes data preparation, feature extraction, model development, training, and validation. The study utilises a dataset of de-identified thermographic images and reference oral temperatures from over 1000 subjects. These high-quality data provide a foundation for validating I-OWSD AI models, supporting early detection, real-time monitoring, and predictive assessment of oral health.

5.4.1. Dataset Preparation

The dataset comprises synthetically generated thermographic images and simulated biomarker readings emulating intraoral conditions.

- Image modality: thermographic images of size $64 \times 64 \times 3$, capturing intraoral temperature variations.
- Biomarker data: simulated sensor outputs, including pH levels, salivary flow rate, and enamel demineralisation indicators.
- Data partitioning: a total of 1000 samples from a synthetic thermographic intraoral dataset were analysed. The dataset was partitioned into 800 samples for model development and 200 for independent testing. For the Neural Network, the training set was further divided into 640 training and 160 validation samples to support hyperparameter optimization and overfitting control. The Support Vector Machine utilized all 800 development samples for training. The test set remained unseen for unbiased evaluation. Class distribution was balanced across dental caries, gingivitis, oral lesions, and healthy tissue.

5.4.2. Feature Extraction

Multimodal features are extracted from both imaging and sensor data:

- Image-based features: pixel intensity distributions, texture descriptors (e.g., Local Binary Patterns and Gabor filters), and temperature gradients.
- Sensor-based features: statistical measures (mean and standard deviation), temporal variation patterns, and fractional-order-derived indicators capturing memory-dependent disease progression.

A late-fusion strategy was employed to integrate multimodal data by concatenating independently extracted image- and sensor-feature vectors. The image modality produced a 128-dimensional vector, while the sensor modality yielded 32 dimensions. The resulting 160-dimensional representation was used for SVM and Neural Network classification, preserving modality-specific information and enabling effective joint learning across heterogeneous data sources.

The fused-feature vector is mapped to four diagnostic classes: dental caries, gingivitis, oral lesions, and healthy tissue, with Neural Network employing a fully connected softmax layer to generate class probabilities, and prediction is based on the maximum score. The SVM applies kernel-based projection with a one-vs-rest strategy, ensuring consistent, interpretable classification aligned with clinically relevant categories.

5.4.3. Model Development

Two complementary ML models are employed:

- Support Vector Machine (SVM): linear and radial-basis-function kernels for multi-class classification.
- Neural Network (NN): feedforward multilayer perceptron with two hidden layers (128 and 64 neurons) and ReLU activation.

This dual-model approach enables comparative evaluation and supports hybrid decision-making for improved diagnostic robustness.

5.4.4. Training and Optimisation

- Neural Network training: Adam optimiser with a learning rate of 0.001 over 200 epochs.
- Loss function: cross-entropy for multi-class classification.
- Regularisation: dropout (0.3) and L2 weight decay to mitigate overfitting.
- SVM tuning: hyperparameters (C , γ) optimised via grid search using the validation set.

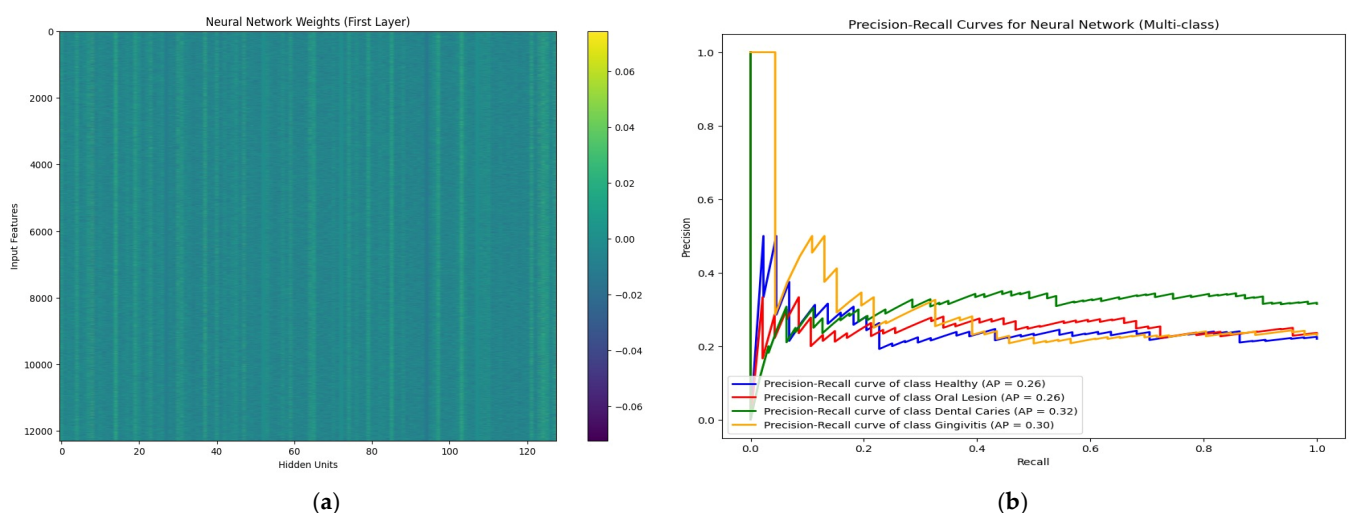
5.5. Validation and Evaluation Metrics

Model performance is assessed using standard classification metrics: accuracy, precision, recall, F1-score, and area under the curve (AUC). A 5-fold cross-validation strategy is applied to ensure robustness and reliable estimation of generalisation performance.

6. Performance Evaluation

6.1. Model Performance Analysis

Figure 8 depicts the proposed I-OWSD framework within a data-driven oral healthcare environment. Its AI-based diagnostic engine is evaluated using the results reported in Tables 3 and 4 and Figure 9a–g. Two complementary ML models are employed. We used SVM and NN to classify intraoral conditions into four categories: dental caries, gingivitis, oral lesions, and healthy tissue. The SVM model achieved an overall accuracy of 24.50%, with a macro-averaged F1-score of 0.24 and a weighted F1-score of 0.25. Class-wise analysis indicates moderate discrimination for class 2 (oral lesions), with an F1-score of 0.36, while other classes exhibit lower performance, reflecting limited separability in the feature space. The relatively balanced macro and weighted averages suggest that the SVM maintains a consistent, albeit modest, performance across all classes without severe bias toward any single category. In contrast, the Neural Network model recorded a slightly lower accuracy of 22.50%, with a macro F1-score of 0.15 and a weighted F1-score of 0.13, indicating weaker generalization.



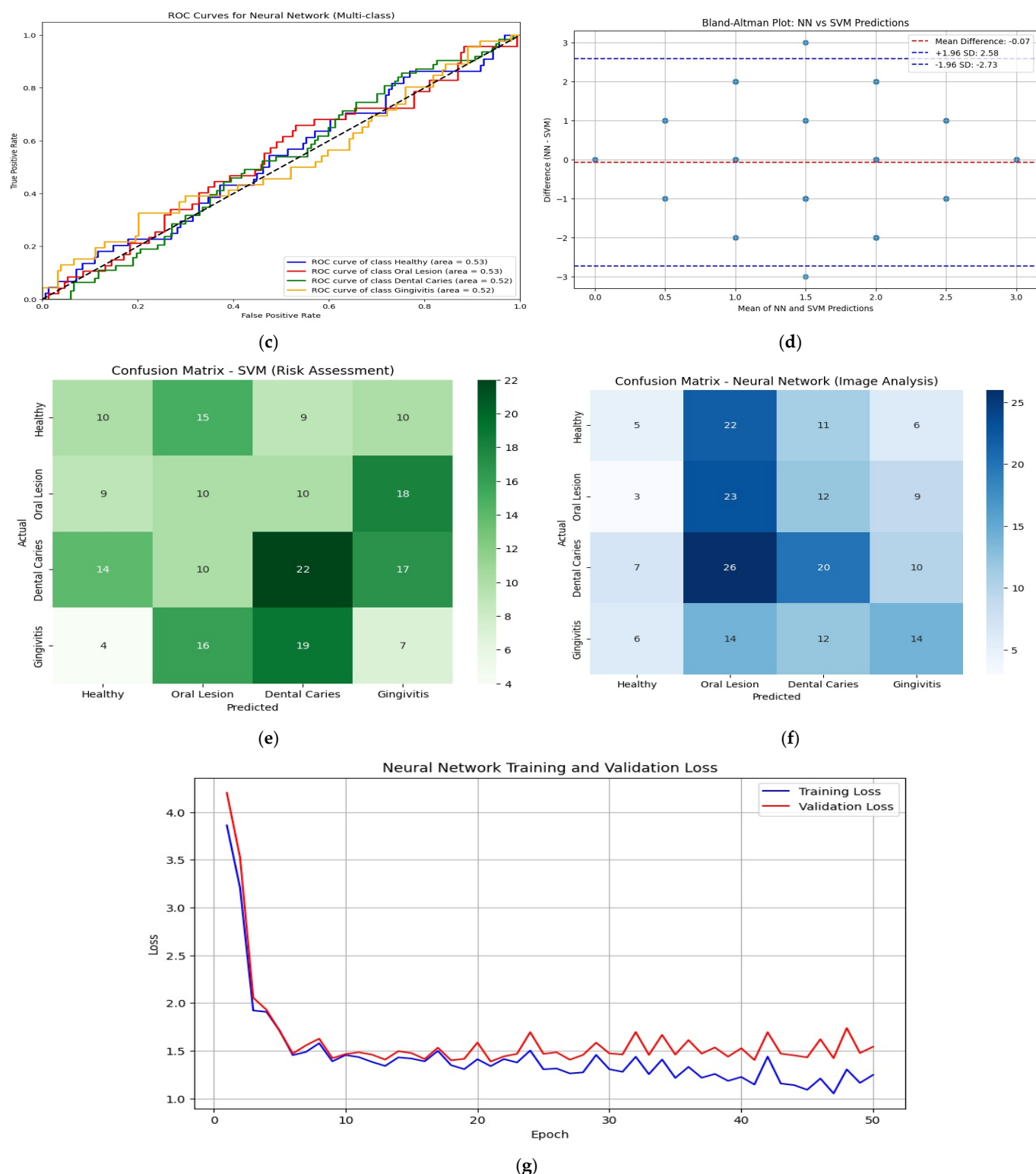


Figure 9. (a) Neural Network Weights (First Layer); (b) Precision Recall Curve for Neural Network (Multi-Class); (c) ROC Curves for Neural Network (Multi-Class); (d) Brand Altman plot: NN vs. SVM. Confusion matrices for (e) Support Vector Machine (SVM) and (f) Neural Network (NN), showing the distribution of predicted versus actual class labels across the four diagnostic categories (dental caries, gingivitis, oral lesions, and healthy tissue) and (g) Neural Network Training and Validity.

Table 3. Dataset Distribution Summary.

Category	Samples
Total Dataset	1000
Training (Total)	800
Neural Network (NN) Training	640
Neural Network (NN) Validation	160
Support Vector Machine (SVM) Training	800
Testing (NN & SVM)	200

Notably, from Table 3, the NN shows a strong recall of 0.75 for class 0, suggesting over-prediction of this class, while failing to correctly identify classes 1 and 2, as reflected by zero precision and recall. Training and validation losses (1.2463 and 1.5410, respectively) indicate overfitting and insufficient feature learning, likely influenced by the synthetic dataset and suboptimal convergence.

Table 4. SVM Classification Results.

	Precision	Recall	F1-Score	Support
0	0.27	0.23	0.25	44
1	0.20	0.21	0.20	47
2	0.37	0.35	0.36	63
3	0.13	0.15	0.14	46
Accuracy			0.24	200
macro avg	0.24	0.24	0.24	200
weighted avg	0.25	0.24	0.25	200

SVM Accuracy: 0.2450.

To ensure a comprehensive and rigorous evaluation beyond overall accuracy, additional performance metrics, such as precision, recall, and F1-score, including macro and weighted averages, were computed to complement accuracy in evaluating multi-class performance, as can be seen in Table 5. Confusion matrices further revealed class-wise misclassification patterns and limited separability, providing robust, consistent evidence of model behaviour and performance limitations.

Table 5. NN Classification Results.

	Precision	Recall	F1-Score	Support
0	0.22	0.75	0.34	44
1	0.00	0.00	0.00	47
2	0.00	0.00	0.00	63
3	0.26	0.26	0.26	46
Accuracy			0.23	200
macro avg	0.12	0.25	0.15	200
weighted avg	0.11	0.23	0.13	200

NN Accuracy: 0.2250.

The relatively low classification accuracy was rigorously verified through repeated experimental trials and cross-validation, confirming the consistency and reliability of the results. The performance limitation is primarily attributed to the use of synthetically generated thermographic and biomarker datasets, which do not adequately capture the variability, noise characteristics, and complex patterns inherent in real intraoral clinical data. Furthermore, the derived feature space exhibits limited separability among the four

diagnostic classes, thereby constraining the discriminative performance of conventional models, including Support Vector Machines (SVMs) and shallow Neural Networks.

Despite these constraints, the results remain reproducible, supporting the validity of the experimental framework. The principal objective of this study is to establish the feasibility of integrating multimodal sensing, fractional-order modelling, and AI-driven diagnostics within the I-OWSD framework, rather than achieving state-of-the-art accuracy. The fused feature vector is mapped to four clinically relevant classes, with classification performed using softmax-based Neural Networks and one-vs-rest kernel SVM, ensuring interpretable and consistent outcomes.

6.2. Hybrid Decision Fusion Strategy

To address the limitations of individual models, a hybrid decision fusion approach is introduced at the orchestration layer. This rule-based consensus mechanism combines predictions from both SVM and NN models, leveraging their complementary strengths; where the NN misclassifies a sample, the SVM prediction refines the final decision. This strategy enhances diagnostic reliability and robustness in handling uncertain or conflicting predictions. Although direct clinical comparisons are not yet available, the proposed scheme is benchmarked against baseline classifiers (SVM-only, NN-only) and existing AI approaches for dental thermography, providing preliminary insight into its effectiveness in early disease detection. The trained models are integrated with an IoT-enabled intraoral wearable sensing system, enabling real-time or near-real-time processing of continuous sensor data streams. This integration supports:

- Early detection alerts for potential lesions.
- Risk stratification for preventive interventions.
- Data logging and transmission for remote clinical-decision support.

6.3. Discussion and Implications

Overall, the results highlight the inherent challenges of thermographic-based oral disease classification, particularly with synthetic datasets that may not capture real-world variability. Despite modest classification performance, the evaluation establishes a proof-of-concept for integrating ML into the I-OWSD framework. The findings validate the feasibility of the proposed intelligent intraoral wearable system with clear pathways for performance improvement, viz.:

- Collection of real clinical datasets for model calibration.
- Implementation of advanced deep learning architectures (e.g., CNNs, hybrid multimodal models).
- Feature enrichment and multimodal data fusion to improve class separability.

These improvements will enhance predictive accuracy, generalisation, and practical applicability of AI-assisted oral healthcare diagnostics.

6.4. Current Limitations

Despite the results obtained, some major limitations were noted when I-OWSD is deployed, including the following:

- The I-OWSD may be constrained by battery life and power efficiency, as continuous sensing and wireless transmission within the oral cavity demand optimised energy management. IoT architectural challenges, including interoperability, data latency, and dependence on stable network connectivity, may affect real-time monitoring, particularly in resource-limited settings [57].
- User comfort and compliance remain critical concerns, as prolonged intraoral device usage may vary across individuals, and is not explicitly modelled.

- Ethical issues related to informed consent, data ownership, and patient autonomy, alongside data privacy and cybersecurity risks, must also be addressed.
- From a modelling standpoint, uncertainties in fractional-order parameter estimation and calibration may arise, due to limited clinical data.

7. Conclusions

This study presents an AI-enabled oral healthcare system. A fractional-order delay-type model (FODM), integrated with I-OWSD, is employed to manage dental caries progression. Using the Caputo–Fabrizio derivative, the model captures memory-dependent dynamics with greater fidelity than conventional integer-order approaches. When integrated with a cloud-enabled telemedicine platform, it supports early detection, continuous monitoring, and remote diagnostics, particularly in resource-limited settings. Analytical validation confirms non-negativity, boundedness, existence, uniqueness, and stability, ensuring mathematical and clinical robustness. Simulation results indicate that at 67.49% adoption, dental caries prevalence may decrease by 14.19%, with 25.45% of users receiving timely treatment, 61.47% reverting to susceptibility, and 36.25% maintaining protective oral health. The combination of biocompatible wearable design, secure data transmission, AI-assisted analytics, and seamless cloud API integration positions I-OWSD as a scalable, preventive, and personalised oral healthcare solution. The system is further validated using neural network weights, precision–recall and ROC curves, Bland–Altman analysis (NN vs. SVM), and confusion matrices for both risk assessment and image analysis. Future work will focus on clinical validation, adaptive parameter learning, and enhanced hardware and security frameworks. Additionally, integrating agent-based ML with advanced IoT-enabled health infrastructures will further improve system intelligence, responsiveness, and scalability.

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Appendix A. Proof of Boundedness of the Solution for the Model

From Equation (11a), we obtain [A1];

$${}^{CPCF}_0D_t^q N_1(t) = {}^{CPCF}_0D_t^q S(t) + {}^{CPCF}_0D_t^q E(t) + {}^{CPCF}_0D_t^q I(t) + {}^{CPCF}_0D_t^q H(t) + {}^{CPCF}_0D_t^q S(t),$$

which gives

$$N_1(t) = \psi - (\Lambda + \mu)N_1,$$

where N_1 is the aggregate number of variables included in the proposed system. We have

$$N_1(t) + (\Lambda + \mu)N_1 \leq \psi \quad (\text{A1})$$

We can easily derive the fact that the solution of the system (13) can be given as follows:

$$N_1(t) = \frac{\psi}{(\Lambda + \mu)} + ce^{(\Lambda + \mu)t}$$

Then, at $t = 0$, we have

$$N_1(t) \leq N(0)e^{(\Lambda + \mu)t} + \frac{\psi}{(\Lambda + \mu)}(1 - e^{(\Lambda + \mu)t})$$

Therefore, when $N(0) \leq \frac{\psi}{(\Lambda + \mu)}$, then $N(t) \leq \frac{\psi}{(\Lambda + \mu)}$ for all $t > 0$. Thus, we obtain the non-negative bounded solution of individuals with oral wearable sensory devices for the model system (8) as

$$\Pi_1 = \left\{ (S(t), E(t), I(t), H(t), R(t)) \in \mathbb{R}_{>0}^5 : 0 \leq N_1 \leq \frac{\psi}{\Lambda + \mu} \right\} \quad (\text{A2})$$

Similarly, for the population of the oral wearable sensory devices, we use system (11b) to obtain the non-negative bounded solution as

$$\Pi_2 = \left\{ W(t) \in \mathbb{R}_+^1 : 0 \leq N_2 \leq \frac{\Delta}{\kappa\delta} \right\} \quad (\text{A3})$$

Appendix B. Proof of Existence and Uniqueness Analysis of the Model

Model system (10) is simplified and presented as system (A4):

$$\left. \begin{aligned} \mathcal{M}_1(t, S, E, I, H, R, W) &= \psi + \xi R + \phi H - (\beta I + \Lambda + vW + \mu + \Delta)S \\ \mathcal{M}_2(t, S, E, I, H, R, W) &= \beta SI - \zeta \beta S(t - \tau)I(t - \tau)e^{-\Lambda\tau} - (\mu + \Lambda + \sigma + \Delta_1 + \gamma W)E \\ \mathcal{M}_3(t, S, E, I, H, R, W) &= \zeta \beta S(t - \tau)I(t - \tau)e^{-\Lambda\tau} - (\mu + \sigma_1 + \Lambda + \phi W + \varpi + \varepsilon + \Delta_2)I \\ \mathcal{M}_4(t, S, E, I, H, R, W) &= \Delta S + \Delta_1 E + (\varepsilon + \Delta_2)I + \chi W - (\Lambda + \mu + \phi + \omega)H \\ \mathcal{M}_5(t, S, E, I, H, R, W) &= \omega H + \sigma E + \sigma_1 I - (\Lambda + \mu + \xi)R \\ \mathcal{M}_6(t, S, E, I, H, R, W) &= \theta + (vS + \gamma E + \phi I)W - (\chi + \delta)W \end{aligned} \right\} \quad (\text{A4})$$

Starting with the function $\mathcal{M}_1(t, S, E, I, H, R, W)$, we subsequently showed that

$$|\mathcal{M}_1(t, S_1, E, I, H, R, W) - \mathcal{M}_1(t, S_2, E, I, H, R, W)|^2 = \left| -\left(\frac{\beta I}{1 + \alpha I} + \Lambda + vW + \mu + \Delta \right) (S_1 - S_2) \right|^2 \quad (\text{A5})$$

Therefore, we obtain the following:

$$\left. \begin{aligned} |\mathcal{M}_1(t, S_1, E, I, H, R, W) - \mathcal{M}_1(t, S_2, E, I, H, R, W)|^2 &= |-(\beta I + \Lambda + vW + \mu + \Delta)(S_1 - S_2)|^2, \\ |\mathcal{M}_1(t, S_1, E, I, H, R, W) - \mathcal{M}_1(t, S_2, E, I, H, R, W)|^2 &= |(\beta I + \Lambda + vW + \mu + \Delta)(S_1 - S_2)|^2, \\ |\mathcal{M}_1(t, S_1, E, I, H, R, W) - \mathcal{M}_1(t, S_2, E, I, H, R, W)|^2 &\leq \{2\beta^2|I|^2 + 2\Lambda^2 + 2v^2|W|^2 + 2\mu^2 + 2\Delta^2\}|(S_1 - S_2)|^2, \\ |\mathcal{M}_1(t, S_1, E, I, H, R, W) - \mathcal{M}_1(t, S_2, E, I, H, R, W)|^2 &\leq \left\{ 2\beta^2 \sup_{0 \leq t \leq t_f} |I|^2 + 2\Lambda^2 + 2v^2 \sup_{0 \leq t \leq t_f} |W|^2 + 2\mu^2 + 2\Delta^2 \right\} |(S_1 - S_2)|^2, \\ |\mathcal{M}_1(t, S_1, E, I, H, R, W) - \mathcal{M}_1(t, S_2, E, I, H, R, W)|^2 &\leq \{2\beta^2|I|_\infty^2 + 2\Lambda^2 + 2v^2|W|_\infty^2 + 2\mu^2 + 2\Delta^2\}|(S_1 - S_2)|^2, \\ |\mathcal{M}_1(t, S_1, E, I, H, R, W) - \mathcal{M}_1(t, S_2, E, I, H, R, W)|^2 &\leq \mathfrak{I}_1|(S_1 - S_2)|^2 \end{aligned} \right\} \quad (\text{A6})$$

where $\mathfrak{I}_1 = 2\{\beta^2|I|_\infty^2 + \Lambda^2 + v^2|W|_\infty^2 + \mu^2 + \Delta^2\}$.

Similarly, it can be proved that E, I, H, R and W are all non-negative, and this is done using the remaining sub-equations of the system (A4). We have:

$$\left. \begin{aligned}
& |\mathcal{M}_2(t, S, E_1, I, H, R, W) - \mathcal{M}_2(t, S, E_2, I, H, R, W)|^2 = |-(\mu + \Lambda + \sigma + \Delta_1 + \gamma W)(E_1 - E_2)|^2 \\
& |\mathcal{M}_2(t, S, E_1, I, H, R, W) - \mathcal{M}_2(t, S, E_2, I, H, R, W)|^2 = |(\mu + \Lambda + \sigma + \Delta_1 + \gamma W)(E_1 - E_2)|^2 \\
& |\mathcal{M}_2(t, S, E_1, I, H, R, W) - \mathcal{M}_2(t, S, E_2, I, H, R, W)|^2 \leq \{2\mu^2 + 2\Lambda^2 + 2\sigma^2 + 2\Delta_1^2 + 2\gamma^2|W|^2\}|(E_1 - E_2)|^2 \\
& \quad |\mathcal{M}_2(t, S, E_1, I, H, R, W) - \mathcal{M}_2(t, S, E_2, I, H, R, W)|^2 \\
& \leq \{2\mu^2 + 2\Lambda^2 + 2\sigma^2 + 2\Delta_1^2 + 2\gamma^2 \sup_{0 \leq t \leq t_f} |W|^2\}|(E_1 - E_2)|^2 \\
& \quad |\mathcal{M}_2(t, S, E_1, I, H, R, W) - \mathcal{M}_2(t, S, E_2, I, H, R, W)|^2 \\
& \leq \{2\mu^2 + 2\Lambda^2 + 2\sigma^2 + 2\Delta_1^2 + 2\gamma^2|W|_\infty^2\}|(E_1 - E_2)|^2 \\
& |\mathcal{M}_2(t, S, E_1, I, H, R, W) - \mathcal{M}_2(t, S, E_2, I, H, R, W)|^2 \leq \mathfrak{I}_2|(E_1 - E_2)|^2 \\
& \quad \text{where } \mathfrak{I}_2 = 2\{\mu^2 + \Lambda^2 + \sigma^2 + \Delta_1^2 + \gamma^2|W|_\infty^2\}.
\end{aligned} \right\} \quad (A7)$$

$$\left. \begin{aligned}
& |\mathcal{M}_3(t, S, E, I_1, H, R, W) - \mathcal{M}_3(t, S, E, I_2, H, R, W)|^2 = |-(\mu + \sigma_1 + \Lambda + \phi W + \varpi + \varepsilon + \Delta_2)(I_1 - I_2)|^2, \\
& |\mathcal{M}_3(t, S, E, I_1, H, R, W) - \mathcal{M}_3(t, S, E, I_2, H, R, W)|^2 = |(\mu + \sigma_1 + \Lambda + \phi W + \varpi + \varepsilon + \Delta_2)(I_1 - I_2)|^2, \\
& |\mathcal{M}_3(t, S, E, I_1, H, R, W) - \mathcal{M}_3(t, S, E, I_2, H, R, W)|^2 \leq \{2\mu^2 + 2\sigma_1^2 + 2\Lambda^2 + 2\phi^2|W|^2 + 2\varpi^2 + 2\varepsilon^2 + 2\Delta_2^2\}|(I_1 - I_2)|^2, \\
& \quad |\mathcal{M}_3(t, S, E, I_1, H, R, W) - \mathcal{M}_3(t, S, E, I_2, H, R, W)|^2 \\
& \leq \{2\mu^2 + 2\sigma_1^2 + 2\Lambda^2 + 2\phi^2 \sup_{0 \leq t \leq t_f} |W|^2 + 2\varpi^2 + 2\varepsilon^2 + 2\Delta_2^2\}|(I_1 - I_2)|^2, \\
& \quad |\mathcal{M}_3(t, S, E, I_1, H, R, W) - \mathcal{M}_3(t, S, E, I_2, H, R, W)|^2 \\
& \leq \{2\mu^2 + 2\sigma_1^2 + 2\Lambda^2 + 2\phi^2|W|_\infty^2 + 2\varpi^2 + 2\varepsilon^2 + 2\Delta_2^2\}|(I_1 - I_2)|^2, \\
& |\mathcal{M}_3(t, S, E, I_1, H, R, W) - \mathcal{M}_3(t, S, E, I_2, H, R, W)|^2 \leq \mathfrak{I}_3|(I_1 - I_2)|^2. \\
& \quad \text{where } \mathfrak{I}_3 = 2\{\mu^2 + \sigma_1^2 + \Lambda^2 + \phi^2|W|_\infty^2 + \varpi^2 + \varepsilon^2 + \Delta_2^2\}.
\end{aligned} \right\} \quad (A8)$$

$$\left. \begin{aligned}
& |\mathcal{M}_4(t, S, E, I, H_1, R, W) - \mathcal{M}_4(t, S, E, I, H_2, R, W)|^2 = |-(\Lambda + \mu + \varphi + \omega)(H_1 - H_2)|^2, \\
& |\mathcal{M}_4(t, S, E, I, H_1, R, W) - \mathcal{M}_4(t, S, E, I, H_2, R, W)|^2 = |(\Lambda + \mu + \varphi + \omega)(H_1 - H_2)|^2, \\
& |\mathcal{M}_4(t, S, E, I, H_1, R, W) - \mathcal{M}_4(t, S, E, I, H_2, R, W)|^2 \leq \{2\Lambda^2 + 2\mu^2 + 2\varphi^2 + 2\omega^2\}|(H_1 - H_2)|^2, \\
& \quad |\mathcal{M}_4(t, S, E, I, H_1, R, W) - \mathcal{M}_4(t, S, E, I, H_2, R, W)|^2 \leq \mathfrak{I}_4|(H_1 - H_2)|^2. \\
& \quad \text{where } \mathfrak{I}_4 = 2\{\Lambda^2 + \mu^2 + \varphi^2 + \omega^2\}.
\end{aligned} \right\} \quad (A9)$$

$$\left. \begin{aligned}
& |\mathcal{M}_5(t, S, E, I, H, R_1, W) - \mathcal{M}_5(t, S, E, I, H, R_2, W)|^2 = |-(\Lambda + \mu + \xi)(R_1 - R_2)|^2, \\
& |\mathcal{M}_5(t, S, E, I, H, R_1, W) - \mathcal{M}_5(t, S, E, I, H, R_2, W)|^2 = |(\Lambda + \mu + \xi)(R_1 - R_2)|^2, \\
& |\mathcal{M}_5(t, S, E, I, H, R_1, W) - \mathcal{M}_5(t, S, E, I, H, R_2, W)|^2 \leq \{2\Lambda^2 + 2\mu^2 + 2\xi^2\}|(R_1 - R_2)|^2, \\
& \quad |\mathcal{M}_5(t, S, E, I, H, R_1, W) - \mathcal{M}_5(t, S, E, I, H, R_2, W)|^2 \leq \mathfrak{I}_5|(R_1 - R_2)|^2. \\
& \quad \text{where } \mathfrak{I}_5 = 2\{\Lambda^2 + \mu^2 + \xi^2\}.
\end{aligned} \right\} \quad (A10)$$

$$\left. \begin{aligned}
& |\mathcal{M}_6(t, S, E, I, H, R, W_1) - \mathcal{M}_6(t, S, E, I, H, R, W_2)|^2 = |-(\chi + \delta)(W_1 - W_2)|^2, \\
& |\mathcal{M}_6(t, S, E, I, H, R, W_1) - \mathcal{M}_6(t, S, E, I, H, R, W_2)|^2 = |(\pi S + \pi\eta E + \pi\eta I + \delta)(W_1 - W_2)|^2 \\
& |\mathcal{M}_6(t, S, E, I, H, R, W_1) - \mathcal{M}_6(t, S, E, I, H, R, W_2)|^2 = \{2\pi^2|S|^2 + 2\pi^2 2\eta^2|E|^2 + 2\pi^2 2\eta^2|I|^2 + 2\delta^2\}|(W_1 - W_2)|^2, \\
& \quad |\mathcal{M}_6(t, S, E, I, H, R, W_1) - \mathcal{M}_6(t, S, E, I, H, R, W_2)|^2 \\
& \leq \{2\pi^2 \sup_{0 \leq t \leq t_f} |S|^2 + 4\pi^2 \eta^2 \sup_{0 \leq t \leq t_f} |E|^2 + 4\pi^2 \eta^2 \sup_{0 \leq t \leq t_f} |I|^2 + 2\delta^2\}|(W_1 - W_2)|^2, \\
& |\mathcal{M}_6(t, S, E, I, H, R, W_1) - \mathcal{M}_6(t, S, E, I, H, R, W_2)|^2 \leq \{2\pi^2|S|_\infty^2 + 4\pi^2 \eta^2|E|_\infty^2 + 4\pi^2 \eta^2|I|_\infty^2 + 2\delta^2\}|(W_1 - W_2)|^2, \\
& |\mathcal{M}_6(t, S, E, I, H, R, W_1) - \mathcal{M}_6(t, S, E, I, H, R, W_2)|^2 \leq \{2\pi^2(|S|_\infty^2 + 2\eta^2(|E|_\infty^2 + |I|_\infty^2)) + 2\delta^2\}|(W_1 - W_2)|^2, \\
& \quad |\mathcal{M}_6(t, S, E, I, H, R, W_1) - \mathcal{M}_6(t, S, E, I, H, R, W_2)|^2 \leq \mathfrak{I}_6|(R_1 - R_2)|^2. \\
& \quad \text{where } \mathfrak{I}_6 = 2\{\pi^2(|S|_\infty^2 + 2\eta^2(|E|_\infty^2 + |I|_\infty^2)) + \delta^2\}.
\end{aligned} \right\} \quad (A11)$$

To confirm the second requirement of the above theorem 2, we double-check the initial condition of each function, and we have the following:

$$\left. \begin{aligned}
& |\mathcal{M}_1(t, S, E, I, H, R, W)|^2 = |\psi + \xi R + \varphi H - (\beta I + \Lambda + \nu W + \mu + \Delta)S|^2, \\
& |\mathcal{M}_1(t, S, E, I, H, R, W)|^2 = |\psi + \xi R + \varphi H + (\beta I + \Lambda + \nu W + \mu + \Delta)S|^2, \\
& \quad |\mathcal{M}_1(t, S, E, I, H, R, W)|^2 \\
& \leq \{2\psi^2 + 2\xi^2|R|^2 + 2\varphi^2|H|^2 + 2\beta^2|I|^2 + 2\Lambda^2 + 2\nu^2|W|^2 + 2\mu^2 + 2\Delta^2\}|S|^2, \\
& \quad |\mathcal{M}_1(t, S, E, I, H, R, W)|^2 \\
& \leq 2\psi^2 + 2\xi^2 \sup_{0 \leq t \leq t_f} |R|^2 + 2\varphi^2 \sup_{0 \leq t \leq t_f} |H|^2 \\
& + (2\beta^2 \sup_{0 \leq t \leq t_f} |I|^2 + 2\Lambda^2 + 2\nu^2 \sup_{0 \leq t \leq t_f} |W|^2 + 2\mu^2 + 2\Delta^2) \sup_{0 \leq t \leq t_f} |S|^2, \\
& \quad |\mathcal{M}_1(t, S, E, I, H, R, W)|^2 \\
& \leq 2\psi^2 + 2\xi^2|R|_\infty^2 + 2\varphi^2|H|_\infty^2 + (2\beta^2|I|_\infty^2 + 2\Lambda^2 + 2\nu^2|W|_\infty^2 + 2\mu^2 + 2\Delta^2)|S|_\infty^2, \\
& |\mathcal{M}_1(t, S, E, I, H, R, W)|^2 \leq 2\{\psi^2 + \xi^2|R|_\infty^2 + \varphi^2|H|_\infty^2\} \left\{ 1 + \frac{(\beta^2|I|_\infty^2 + \Lambda^2 + \nu^2|W|_\infty^2 + \mu^2 + \Delta^2)|S|_\infty^2}{\psi^2 + \xi^2|R|_\infty^2 + \varphi^2|H|_\infty^2} \right\}
\end{aligned} \right\} \quad (A12)$$

With the condition that $\frac{(\beta^2|I|_\infty^2 + \Lambda^2 + \nu^2|W|_\infty^2 + \mu^2 + \Delta^2)|S|_\infty^2}{\psi^2 + \xi^2|R|_\infty^2 + \varphi^2|H|_\infty^2} < 1$

$$\left. \begin{aligned} |\mathcal{M}_2(t, S, E, I, H, R, W)|^2 &= |\beta SI - \zeta \beta S(t - \tau)I(t - \tau)e^{-\Lambda\tau} - (\mu + \Lambda + \sigma + \Delta_1 + \gamma W)E|^2, \\ |\mathcal{M}_2(t, S, E, I, H, R, W)|^2 &= |\beta SI + \zeta \beta S(t - \tau)I(t - \tau)e^{-\Lambda\tau} + (\mu + \Lambda + \sigma + \Delta_1 + \gamma W)E|^2, \\ |\mathcal{M}_2(t, S, E, I, H, R, W)|^2 &\leq 2\beta^2|S|^2|I|^2 + 4\zeta^2\beta^2|S|^2(t - \tau)|I|^2(t - \tau)e^{-\Lambda\tau} + (2\mu^2 + 2\Lambda^2 + 2\sigma^2 + 2\Delta_1^2 + 2\gamma^2|W|^2)|E|^2, \\ |\mathcal{M}_2(t, S, E, I, H, R, W)|^2 &\leq 2\beta^2\sup_{0 \leq t \leq t_f}|S|^2\sup_{0 \leq t \leq t_f}|I|^2 + 4\zeta^2\beta^2\sup_{0 \leq t \leq t_f}|S|^2(t - \tau)\sup_{0 \leq t \leq t_f}|I|^2(t - \tau)e^{-\Lambda\tau} \\ &\quad + (2\mu^2 + 2\Lambda^2 + 2\sigma^2 + 2\Delta_1^2 + 2\gamma^2\sup_{0 \leq t \leq t_f}|W|^2)\sup_{0 \leq t \leq t_f}|E|^2, \\ |\mathcal{M}_2(t, S, E, I, H, R, W)|^2 &\leq 2\beta^2|S|_\infty^2|I|_\infty^2 + 4\zeta^2\beta^2|S|_\infty^2(t - \tau)|I|_\infty^2(t - \tau)e^{-\Lambda\tau} + (2\mu^2 + 2\Lambda^2 + 2\sigma^2 + 2\Delta_1^2 + 2\gamma^2|W|_\infty^2)|E|_\infty^2, \\ |\mathcal{M}_2(t, S, E, I, H, R, W)|^2 &\leq 2\{\beta^2|S|_\infty^2|I|_\infty^2 + 2\zeta^2\beta^2|S|_\infty^2(t - \tau)|I|_\infty^2(t - \tau)e^{-\Lambda\tau}\} \left\{ 1 + \frac{(\mu^2 + \Lambda^2 + \sigma^2 + \Delta_1^2 + \gamma^2|W|_\infty^2)|E|_\infty^2}{\beta^2|S|_\infty^2|I|_\infty^2 + 2\zeta^2\beta^2|S|_\infty^2(t - \tau)|I|_\infty^2(t - \tau)e^{-\Lambda\tau}} \right\} \end{aligned} \right\} \quad (A13)$$

With the condition that $\frac{(\mu^2 + \Lambda^2 + \sigma^2 + \Delta_1^2 + \gamma^2|W|_\infty^2)|E|_\infty^2}{\beta^2|S|_\infty^2|I|_\infty^2 + 2\zeta^2\beta^2|S|_\infty^2(t - \tau)|I|_\infty^2(t - \tau)e^{-\Lambda\tau}} < 1$

$$\left. \begin{aligned} |\mathcal{M}_3(t, S, E, I, H, R, W)|^2 &= |\zeta \beta S(t - \tau)I(t - \tau)e^{-\Lambda\tau} - (\mu + \sigma_1 + \Lambda + \phi W + \varpi + \varepsilon + \Delta_2)I|^2, \\ |\mathcal{M}_3(t, S, E, I, H, R, W)|^2 &= |\zeta \beta S(t - \tau)I(t - \tau)e^{-\Lambda\tau} + (\mu + \sigma_1 + \Lambda + \phi W + \varpi + \varepsilon + \Delta_2)I|^2, \\ |\mathcal{M}_3(t, S, E, I, H, R, W)|^2 &\leq 4\zeta^2\beta^2|S|^2(t - \tau)|I|^2(t - \tau)e^{-\Lambda\tau} + (2\mu^2 + 2\sigma_1^2 + 2\Lambda^2 + 2\phi^2|W|^2 + 2\varpi^2 + 2\varepsilon^2 + 2\Delta_2^2)|I|^2, \\ |\mathcal{M}_3(t, S, E, I, H, R, W)|^2 &\leq 4\zeta^2\beta^2\sup_{0 \leq t \leq t_f}|S|^2(t - \tau)\sup_{0 \leq t \leq t_f}|I|^2(t - \tau)e^{-\Lambda\tau} \\ &\quad + (2\mu^2 + 2\sigma_1^2 + 2\Lambda^2 + 2\phi^2\sup_{0 \leq t \leq t_f}|W|^2 + 2\varpi^2 + 2\varepsilon^2 + 2\Delta_2^2)\sup_{0 \leq t \leq t_f}|I|^2, \\ |\mathcal{M}_3(t, S, E, I, H, R, W)|^2 &\leq 4\zeta^2\beta^2|S|_\infty^2(t - \tau)|I|_\infty^2(t - \tau)e^{-\Lambda\tau} + (2\mu^2 + 2\sigma_1^2 + 2\Lambda^2 + 2\phi^2|W|_\infty^2 + 2\varpi^2 + 2\varepsilon^2 + 2\Delta_2^2)|I|_\infty^2, \\ |\mathcal{M}_3(t, S, E, I, H, R, W)|^2 &\leq 4\zeta^2\beta^2|S|_\infty^2(t - \tau)|I|_\infty^2(t - \tau)e^{-\Lambda\tau} \left\{ 1 + \frac{(\mu^2 + \sigma_1^2 + \Lambda^2 + \phi^2|W|_\infty^2 + \varpi^2 + \varepsilon^2 + 2\Delta_2^2)}{2\zeta^2\beta^2|S|_\infty^2(t - \tau)(t - \tau)e^{-\Lambda\tau}} \right\}. \end{aligned} \right\} \quad (A14)$$

With the condition that $\frac{(\mu^2 + \sigma_1^2 + \Lambda^2 + \phi^2|W|_\infty^2 + \varpi^2 + \varepsilon^2 + 2\Delta_2^2)}{2\zeta^2\beta^2|S|_\infty^2(t - \tau)(t - \tau)e^{-\Lambda\tau}} < 1$

$$\left. \begin{aligned} |\mathcal{M}_4(t, S, E, I, H, R, W)|^2 &= |\Delta S + \Delta_1 E + (\varepsilon + \Delta_2)I + \chi W - (\Lambda + \mu + \varphi + \omega)H|^2, \\ |\mathcal{M}_4(t, S, E, I, H, R, W)|^2 &= |\Delta S + \Delta_1 E + (\varepsilon + \Delta_2)I + \chi W + (\Lambda + \mu + \varphi + \omega)H|^2, \\ |\mathcal{M}_4(t, S, E, I, H, R, W)|^2 &\leq 2\Delta^2|S|^2 + 2\Delta_1^2|E|^2 + (2\varepsilon^2 + 2\Delta_2^2)|I|^2 + 2\chi^2|W|^2 + (2\Lambda^2 + 2\mu^2 + 2\varphi^2 + 2\omega^2)|H|^2, \\ |\mathcal{M}_4(t, S, E, I, H, R, W)|^2 &\leq 2\Delta^2\sup_{0 \leq t \leq t_f}|S|^2 + 2\Delta_1^2\sup_{0 \leq t \leq t_f}|E|^2 + (2\varepsilon^2 + 2\Delta_2^2)\sup_{0 \leq t \leq t_f}|I|^2 + 2\chi^2\sup_{0 \leq t \leq t_f}|W|^2 \\ &\quad + (2\Lambda^2 + 2\mu^2 + 2\varphi^2 + 2\omega^2)\sup_{0 \leq t \leq t_f}|H|^2, \\ |\mathcal{M}_4(t, S, E, I, H, R, W)|^2 &\leq 2\Delta^2|S|_\infty^2 + 2\Delta_1^2|E|_\infty^2 + (2\varepsilon^2 + 2\Delta_2^2)|I|_\infty^2 + 2\chi^2|W|_\infty^2 + (2\Lambda^2 + 2\mu^2 + 2\varphi^2 + 2\omega^2)|H|_\infty^2, \\ |\mathcal{M}_4(t, S, E, I, H, R, W)|^2 &\leq (2\Delta^2|S|_\infty^2 + 2\Delta_1^2|E|_\infty^2 + (2\varepsilon^2 + 2\Delta_2^2)|I|_\infty^2 + 2\chi^2|W|_\infty^2) \left\{ 1 + \frac{(\Lambda^2 + \mu^2 + \varphi^2 + \omega^2)|H|_\infty^2}{\Delta^2|S|_\infty^2 + \Delta_1^2|E|_\infty^2 + (\varepsilon^2 + \Delta_2^2)|I|_\infty^2 + \chi^2|W|_\infty^2} \right\}. \end{aligned} \right\} \quad (A15)$$

With the condition that $\frac{(\Lambda^2 + \mu^2 + \varphi^2 + \omega^2)|H|_\infty^2}{\Delta^2|S|_\infty^2 + \Delta_1^2|E|_\infty^2 + (\varepsilon^2 + \Delta_2^2)|I|_\infty^2 + \chi^2|W|_\infty^2} < 1$

$$\left. \begin{aligned} |\mathcal{M}_5(t, S, E, I, H, R, W)|^2 &= |\omega H + \sigma E + \sigma_1 I - (\Lambda + \mu + \xi)R|^2, \\ |\mathcal{M}_5(t, S, E, I, H, R, W)|^2 &= |\omega H + \sigma E + \sigma_1 I + (\Lambda + \mu + \xi)R|^2, \\ |\mathcal{M}_5(t, S, E, I, H, R, W)|^2 &\leq 2\omega^2|H|^2 + 2\sigma^2|E|^2 + 2\sigma_1^2|I|^2 + (2\Lambda^2 + 2\mu^2 + 2\xi^2)|R|^2, \\ |\mathcal{M}_5(t, S, E, I, H, R, W)|^2 &\leq 2\omega^2\sup_{0 \leq t \leq t_f}|H|^2 + 2\sigma^2\sup_{0 \leq t \leq t_f}|E|^2 + 2\sigma_1^2\sup_{0 \leq t \leq t_f}|I|^2 \\ &\quad + (2\Lambda^2 + 2\mu^2 + 2\xi^2)\sup_{0 \leq t \leq t_f}|R|^2, \\ |\mathcal{M}_5(t, S, E, I, H, R, W)|^2 &\leq 2\omega^2|H|_\infty^2 + 2\sigma^2|E|_\infty^2 + 2\sigma_1^2|I|_\infty^2 + (2\Lambda^2 + 2\mu^2 + 2\xi^2)|R|_\infty^2, \\ |\mathcal{M}_5(t, S, E, I, H, R, W)|^2 &\leq (2\omega^2|H|_\infty^2 + 2\sigma^2|E|_\infty^2 + 2\sigma_1^2|I|_\infty^2) \left\{ 1 + \frac{(2\Lambda^2 + 2\mu^2 + 2\xi^2)|R|_\infty^2}{2\omega^2|H|_\infty^2 + 2\sigma^2|E|_\infty^2 + 2\sigma_1^2|I|_\infty^2} \right\} \end{aligned} \right\} \quad (A16)$$

With the condition that $\frac{(2\Lambda^2 + 2\mu^2 + 2\xi^2)|R|_\infty^2}{2\omega^2|H|_\infty^2 + 2\sigma^2|E|_\infty^2 + 2\sigma_1^2|I|_\infty^2} < 1$

$$\begin{aligned}
& \left. \begin{aligned}
& |\mathcal{M}_6(t, S, E, I, H, R, W)|^2 = |\theta + (vS + \gamma E + \phi I)W - (\chi + \delta)W|^2, \\
& |\mathcal{M}_6(t, S, E, I, H, R, W)|^2 = |\theta + (vS + \gamma E + \phi I)W + (\chi + \delta)W|^2 \\
& |\mathcal{M}_6(t, S, E, I, H, R, W)|^2 = 2\theta^2 + (2v^2|S|^2 + 2\gamma^2|E|^2 + 2\phi^2|I|^2)|W|^2 + (2\chi^2 + 2\delta^2)|W|^2, \\
& |\mathcal{M}_6(t, S, E, I, H, R, W)|^2 \\
& \leq 2\theta^2 + \left(2v^2 \sup_{0 \leq t \leq t_f} |S|^2 + 2\gamma^2 \sup_{0 \leq t \leq t_f} |E|^2 + 2\phi^2 \sup_{0 \leq t \leq t_f} |I|^2\right) \sup_{0 \leq t \leq t_f} |W|^2 + (2\chi^2 + 2\delta^2) \sup_{0 \leq t \leq t_f} |W|^2, \\
& |\mathcal{M}_6(t, S, E, I, H, R, W)|^2 \leq 2\theta^2 + (2v^2|S|_\infty^2 + 2\gamma^2|E|_\infty^2 + 2\phi^2|I|_\infty^2)|W|_\infty^2 + (2\chi^2 + 2\delta^2)|W|_\infty^2, \\
& |\mathcal{M}_6(t, S, E, I, H, R, W)|^2 \leq 2\theta^2 + (2v^2|S|_\infty^2 + 2\gamma^2|E|_\infty^2 + 2\phi^2|I|_\infty^2 + 2\chi^2 + 2\delta^2)|W|_\infty^2, \\
& |\mathcal{M}_6(t, S, E, I, H, R, W)|^2 \leq 2\theta^2 \left\{1 + \frac{(2v^2|S|_\infty^2 + 2\gamma^2|E|_\infty^2 + 2\phi^2|I|_\infty^2 + 2\chi^2 + 2\delta^2)|W|_\infty^2}{2\theta^2}\right\}.
\end{aligned} \right\} \quad (A17)
\end{aligned}$$

With the condition that $\frac{(2v^2|S|_\infty^2 + 2\gamma^2|E|_\infty^2 + 2\phi^2|I|_\infty^2 + 2\chi^2 + 2\delta^2)|W|_\infty^2}{2\theta^2} < 1$. Therefore, we observe that the solution to our model system (10) is distinct and appropriate in the given situation:

$$\begin{aligned}
& \left. \begin{aligned}
& \frac{(\beta^2|I|_\infty^2 + \Lambda^2 + v^2|W|_\infty^2 + \mu^2 + \Delta^2)|S|_\infty^2}{\psi^2 + \xi^2|R|_\infty^2 + \varphi^2|H|_\infty^2} \\
& \frac{(\mu^2 + \Lambda^2 + \sigma^2 + \Delta_1^2 + \gamma^2|W|_\infty^2)|E|_\infty^2}{\beta^2|S|_\infty^2|I|_\infty^2 + 2\zeta^2\beta^2|S|_\infty^2(t-\tau)|I|_\infty^2(t-\tau)e^{-\Lambda\tau}} \\
& \frac{(\mu^2 + \sigma_1^2 + \Lambda^2 + \phi^2|W|_\infty^2 + \varpi^2 + \varepsilon^2 + 2\Delta_2^2)}{2\zeta^2\beta^2|S|_\infty^2(t-\tau)(t-\tau)e^{-\Lambda\tau}} \\
& \frac{(\Lambda^2 + \mu^2 + \varphi^2 + \omega^2)|H|_\infty^2}{\Delta^2|S|_\infty^2 + \Delta_1^2|E|_\infty^2 + (\varepsilon^2 + \Delta_2^2)|I|_\infty^2 + \chi^2|W|_\infty^2} \\
& \frac{(\Lambda^2 + \mu^2 + \xi^2)|R|_\infty^2}{\omega^2|H|_\infty^2 + \sigma^2|E|_\infty^2 + \sigma_1^2|I|_\infty^2} \\
& \frac{(v^2|S|_\infty^2 + \gamma^2|E|_\infty^2 + \phi^2|I|_\infty^2 + \chi^2 + \delta^2)|W|_\infty^2}{\theta^2}
\end{aligned} \right\} < 1 \quad (A18)
\end{aligned}$$

Appendix C

3.3. Stability Analysis of the Proposed Model

Let's examine the equations associated with the model system (10) by having $\mathbb{G}^*$ as

Picard $\mathbb{G}^*$ stable, to obtain the following:

$$\begin{aligned}
& \left. \begin{aligned}
& S_{i+1}(t) = S_i(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\psi + \xi R_i + \varphi H_i - (\beta I_i + \Lambda + vW_i + \mu + \Delta)S_i)], \\
& E_{i+1}(t) = E_i(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\beta S_i I_i - \zeta \beta S_i(t-\tau)I_i(t-\tau)e^{-\Lambda\tau} - (\mu + \Lambda + \sigma + \Delta_1 + \gamma W_i)E_i)], \\
& I_{i+1}(t) = I_i(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\zeta \beta S_i(t-\tau)I_i(t-\tau)e^{-\Lambda\tau} - (\mu + \sigma_1 + \Lambda + \phi W_i + \varpi + \varepsilon + \Delta_2)I_i)], \\
& H_{i+1}(t) = H_i(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\Delta S_i + \Delta_1 E_i + (\varepsilon + \Delta_2)I_i + \chi W_i - (\Lambda + \mu + \varphi + \omega)H_i)], \\
& R_{i+1}(t) = R_i(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\omega H_i + \sigma E_i + \sigma_1 I_i - (\Lambda + \mu + \xi)R_i)], \\
& W_{i+1}(t) = W_i(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\theta + (vS_i + \gamma E_i + \phi I_i)W_i - (\chi + \delta)W_i)],
\end{aligned} \right\} \quad (A19)
\end{aligned}$$

where we have the fractional Lagrange multiplier represented as $\mathbb{L} = \frac{q + \zeta(1-q)}{\mathbb{T}(q)Z_1(q) + \zeta\mathbb{T}(q)Z_0(q)}$.

Theorem 4. Consequently, we define the self-map $\mathcal{G}$ as follows:

$$\begin{aligned}
& \left. \begin{aligned}
& \mathcal{G}[S_i] = S_{i+1}(t) = S_i(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\psi + \xi R_i + \varphi H_i - (\beta I_i + \Lambda + vW_i + \mu + \Delta)S_i)], \\
& \mathcal{G}[E_i] = E_{i+1}(t) = E_i(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\beta S_i I_i - \zeta \beta S_i(t-\tau)I_i(t-\tau)e^{-\Lambda\tau} - (\mu + \Lambda + \sigma + \Delta_1 + \gamma W_i)E_i)], \\
& \mathcal{G}[I_i] = I_{i+1}(t) = I_i(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\zeta \beta S_i(t-\tau)I_i(t-\tau)e^{-\Lambda\tau} - (\mu + \sigma_1 + \Lambda + \phi W_i + \varpi + \varepsilon + \Delta_2)I_i)], \\
& \mathcal{G}[H_i] = H_{i+1}(t) = H_i(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\Delta S_i + \Delta_1 E_i + (\varepsilon + \Delta_2)I_i + \chi W_i - (\Lambda + \mu + \varphi + \omega)H_i)], \\
& \mathcal{G}[R_i] = R_{i+1}(t) = R_i(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\omega H_i + \sigma E_i + \sigma_1 I_i - (\Lambda + \mu + \xi)R_i)], \\
& \mathcal{G}[W_i] = W_{i+1}(t) = W_i(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\theta + (vS_i + \gamma E_i + \phi I_i)W_i - (\chi + \delta)W_i)],
\end{aligned} \right\} \quad (A20)
\end{aligned}$$

with $\mathcal{G}^*$ stable in the space $\mathcal{L}^1(x, y)$ if

$$\mathcal{J} = \left. \begin{aligned} & \left\{ 1 + \psi + \xi\lambda_1(q) + \varphi\lambda_2(q) + \beta\lambda_3\Theta_1(q) + \beta\lambda_4\Theta_3(q) + \nu\lambda_5\Theta_1(q) + \nu\lambda_6\Theta_4(q) - (\Lambda + \mu + \Delta)\lambda_7(q) \right\} < 1, \\ & \left\{ 1 + \lambda_8\Theta_1(q) + \beta\lambda_9\Theta_3(q) + \zeta\beta(\lambda_{10}\Theta_1(q) + \lambda_{11}\Theta_3(q))(t - \tau)(t - \tau)e^{-\Lambda\tau} \right. \\ & \quad \left. + \gamma\lambda_{12}\Theta_4(q) + \gamma\lambda_{13}\Theta_2(q) + \Delta_1\lambda_{14}(q) + (\mu + \Lambda + \sigma)\lambda_{15}(q) \right\} < 1, \\ & \left\{ 1 + \zeta\beta(\lambda_{16}\Theta_1(q) + \lambda_{17}\Theta_3(q))(t - \tau)(t - \tau)e^{-\Lambda\tau} + \phi\lambda_{18}\Theta_3(q) + \phi\lambda_{19}\Theta_4(q) \right. \\ & \quad \left. + (\mu + \sigma_1 + \Lambda + \varpi + \varepsilon + \Delta_2)\lambda_{20}(q) \right\} < 1, \\ & \left\{ 1 + \Delta\lambda_{21}(q) + \Delta_1\lambda_{22}(q) + (\varepsilon + \Delta_2)\lambda_{23}(q) + \chi\lambda_{24}(q) + (\Lambda + \mu + \varphi + \omega)\lambda_{25}(q) \right\} < 1, \\ & \left\{ 1 + \omega\lambda_{26}(q) + \sigma\lambda_{27}(q) + \sigma_1\lambda_{28}(q) + (\Lambda + \mu + \xi)\lambda_{29}(q) \right\} < 1, \\ & \left\{ 1 + \theta + \nu\lambda_{30}\Theta_1(q) + \nu\lambda_{31}\Theta_4(q) + \gamma\lambda_{32}\Theta_4(q) + \gamma\lambda_{33}\Theta_2(q) + \phi\lambda_{34}\Theta_3(q) + \phi\lambda_{35}\Theta_4(q) \right. \\ & \quad \left. + (\chi + \delta)\lambda_{36}(q) \right\} < 1, \end{aligned} \right\} \quad (A21)$$

Proof. There exists a fixed point $\mathcal{G}$, so we can find $(m, n) \in (\mathbb{N} \times \mathbb{N}) \hat{\mathfrak{D}}$ for each:

$$\left. \begin{aligned} \mathcal{G}[S_b] - \mathcal{G}[S_a] &= S_{i+1}(t) = S_b(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\psi + \xi R_b + \varphi H_b - (\beta I_b + \Lambda + \nu W_b + \mu + \Delta)S_b)] \\ &\quad - S_a(t) - \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\psi + \xi R_a + \varphi H_a - (\beta I_a + \Lambda + \nu W_a + \mu + \Delta)S_a)], \\ \mathcal{G}[E_b] - \mathcal{G}[E_a] &= E_{i+1}(t) = S_b(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\psi + \xi R_b + \varphi H_b - (\beta I_b + \Lambda + \nu W_b + \mu + \Delta)S_b)] \\ &\quad - S_a(t) - \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\psi + \xi R_a + \varphi H_a - (\beta I_a + \Lambda + \nu W_a + \mu + \Delta)S_a)], \\ \mathcal{G}[I_b] - \mathcal{G}[I_a] &= I_{i+1}(t) = I_b(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\zeta\beta S_b(t - \tau)I_b(t - \tau)e^{-\Lambda\tau} - (\mu + \sigma_1 + \Lambda + \phi W_b + \varpi + \varepsilon + \Delta_2)I_b)] \\ &\quad - I_a(t) - \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\zeta\beta S_a(t - \tau)I_a(t - \tau)e^{-\Lambda\tau} - (\mu + \sigma_1 + \Lambda + \phi W_a + \varpi + \varepsilon + \Delta_2)I_a)], \\ \mathcal{G}[H_b] - \mathcal{G}[H_a] &= H_{i+1}(t) = H_b(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\Delta S_b + \Delta_1 E_b + (\varepsilon + \Delta_2)I_b + \chi W_b - (\Lambda + \mu + \varphi + \omega)H_b)] \\ &\quad - H_a(t) - \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\Delta S_a + \Delta_1 E_a + (\varepsilon + \Delta_2)I_a + \chi W_a - (\Lambda + \mu + \varphi + \omega)H_a)], \\ \mathcal{G}[R_b] - \mathcal{G}[R_a] &= R_{i+1}(t) = R_b(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\omega H_b + \sigma E_b + \sigma_1 I_b - (\Lambda + \mu + \xi)R_b)] \\ &\quad - R_a(t) - \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\omega H_a + \sigma E_a + \sigma_1 I_a - (\Lambda + \mu + \xi)R_a)], \\ \mathcal{G}[W_b] - \mathcal{G}[W_a] &= W_{i+1}(t) = W_b(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\theta + (\nu S_b + \gamma E_b + \phi I_b)W_b - (\chi + \delta)W_b)] \\ &\quad - W_a(t) - \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\theta + (\nu S_a + \gamma E_a + \phi I_a)W_a - (\chi + \delta)W_a)]. \end{aligned} \right\} \quad (A22)$$

By applying the norm to Equation (A22), and without loss of generality, we obtain the following:

$$\left. \begin{aligned} \|\mathcal{G}[S_b] - \mathcal{G}[S_a]\| &= \left\| \begin{aligned} & S_b(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\psi + \xi R_b + \varphi H_b - (\beta I_b + \Lambda + \nu W_b + \mu + \Delta)S_b)] \\ & - S_a(t) - \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\psi + \xi R_a + \varphi H_a - (\beta I_a + \Lambda + \nu W_a + \mu + \Delta)S_a)] \end{aligned} \right\|, \\ \|\mathcal{G}[E_b] - \mathcal{G}[E_a]\| &= \left\| \begin{aligned} & E_b(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\beta S_b I_b - \zeta\beta S_b(t - \tau)I_b(t - \tau)e^{-\Lambda\tau} - (\mu + \Lambda + \sigma + \Delta_1 + \gamma W_b)E_b)] \\ & - E_a(t) - \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\beta S_a I_a - \zeta\beta S_a(t - \tau)I_a(t - \tau)e^{-\Lambda\tau} - (\mu + \Lambda + \sigma + \Delta_1 + \gamma W_a)E_a)] \end{aligned} \right\|, \\ \|\mathcal{G}[I_b] - \mathcal{G}[I_a]\| &= \left\| \begin{aligned} & I_b(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\zeta\beta S_b(t - \tau)I_b(t - \tau)e^{-\Lambda\tau} - (\mu + \sigma_1 + \Lambda + \phi W_b + \varpi + \varepsilon + \Delta_2)I_b)] \\ & - I_a(t) - \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\zeta\beta S_a(t - \tau)I_a(t - \tau)e^{-\Lambda\tau} - (\mu + \sigma_1 + \Lambda + \phi W_a + \varpi + \varepsilon + \Delta_2)I_a)] \end{aligned} \right\|, \\ \|\mathcal{G}[H_b] - \mathcal{G}[H_a]\| &= \left\| \begin{aligned} & H_b(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\Delta S_b + \Delta_1 E_b + (\varepsilon + \Delta_2)I_b + \chi W_b - (\Lambda + \mu + \varphi + \omega)H_b)] \\ & - H_a(t) - \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\Delta S_a + \Delta_1 E_a + (\varepsilon + \Delta_2)I_a + \chi W_a - (\Lambda + \mu + \varphi + \omega)H_a)] \end{aligned} \right\|, \\ \|\mathcal{G}[R_b] - \mathcal{G}[R_a]\| &= \left\| \begin{aligned} & R_b(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\omega H_b + \sigma E_b + \sigma_1 I_b - (\Lambda + \mu + \xi)R_b)] \\ & - R_a(t) - \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\omega H_a + \sigma E_a + \sigma_1 I_a - (\Lambda + \mu + \xi)R_a)] \end{aligned} \right\|, \\ \|\mathcal{G}[W_b] - \mathcal{G}[W_a]\| &= \left\| \begin{aligned} & W_b(t) + \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\theta + (\nu S_b + \gamma E_b + \phi I_b)W_b - (\chi + \delta)W_b)] \\ & - W_a(t) - \mathcal{B}^{-1}[\mathbb{L}\mathcal{B}(\theta + (\nu S_a + \gamma E_a + \phi I_a)W_a - (\chi + \delta)W_a)] \end{aligned} \right\|. \end{aligned} \right\} \quad (A23)$$

By the imposition of triangular inequality on Equation (A23), we obtain the following:

$$\left. \begin{aligned} \|\mathcal{G}[S_b] - \mathcal{G}[S_a]\| &\leq \|S_b(t) - S_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} & \psi + \|\xi(R_b - R_a)\| + \|\varphi(H_b - H_a)\| \\ & + \|\beta I_b(S_b - S_a)\| + \|\beta S_b(I_b - I_a)\| \\ & + \|\nu W_b(S_b - S_a)\| + \|\nu S_b(W_b - W_a)\| + \|(\Lambda + \mu + \Delta)(S_b - S_a)\| \end{aligned} \right\} \right], \\ \|\mathcal{G}[E_b] - \mathcal{G}[E_a]\| &\leq \|E_b(t) - E_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} & \|\beta(1 + \zeta(t - \tau)(t - \tau)e^{-\Lambda\tau})S_b(I_b - I_a)\| \\ & + \|\beta(1 + \zeta(t - \tau)(t - \tau)e^{-\Lambda\tau})I_b(S_b - S_a)\| \\ & + \|(\mu + \Lambda + \sigma)(E_b - E_a)\| + \|\Delta_1(E_b - E_a)\| \\ & + \|\gamma W_b(E_b - E_a)\| + \|\gamma E_b(W_b - W_a)\| \end{aligned} \right\} \right], \\ \|\mathcal{G}[I_b] - \mathcal{G}[I_a]\| &\leq \|I_b(t) - I_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} & \|\beta\zeta(t - \tau)(t - \tau)e^{-\Lambda\tau}S_b(I_b - I_a)\| + \|\beta\zeta(t - \tau)(t - \tau)e^{-\Lambda\tau}I_b(S_b - S_a)\| \\ & + \|\phi W_b(I_b - I_a)\| + \|\phi I_b(W_b - W_a)\| \\ & + \|(\mu + \sigma_1 + \Lambda + \varpi + \varepsilon + \Delta_2)(I_b - I_a)\| \end{aligned} \right\} \right], \\ \|\mathcal{G}[H_b] - \mathcal{G}[H_a]\| &\leq \|H_b(t) - H_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} & \|\Delta(S_b - S_a)\| + \|\Delta_1(E_b - E_a)\| + \|(\varepsilon + \Delta_2)(I_b - I_a)\| \\ & + \|\chi(W_b - W_a)\| + \|(\Lambda + \mu + \varphi + \omega)(H_b - H_a)\| \end{aligned} \right\} \right], \\ \|\mathcal{G}[R_b] - \mathcal{G}[R_a]\| &\leq \|R_b(t) - R_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} & \|\omega(H_b - H_a)\| + \|\sigma(E_b - E_a)\| + \|\sigma_1(I_b - I_a)\| \\ & + \|(\Lambda + \mu + \xi)(R_b - R_a)\| \end{aligned} \right\} \right], \\ \|\mathcal{G}[W_b] - \mathcal{G}[W_a]\| &\leq \|W_b(t) - W_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} & \theta + \|\nu S_b(W_b - W_a)\| + \|\nu W_b(S_b - S_a)\| \\ & + \|\gamma E_b(W_b - W_a)\| + \|\gamma W_b(E_b - E_a)\| \\ & + \|\phi I_b(W_b - W_a)\| + \|\phi W_b(I_b - I_a)\| + \|(\chi + \delta)(W_b - W_a)\| \end{aligned} \right\} \right]. \end{aligned} \right\} \quad (A24)$$

Considering that the solutions found play a similar role, we have the following conclusion:

$$\|S_b(t) - S_a(t)\| = \|E_b(t) - E_a(t)\| = \|I_b(t) - I_a(t)\| = \|H_b(t) - H_a(t)\| = \|R_b(t) - R_a(t)\|, \quad (A25)$$

$$\|W_b(t) - W_a(t)\|.$$

At this point, we substitute system (A25) into (A24) to obtain the relationship as in system (A26) below:

$$\left. \begin{aligned} \|G[S_b] - G[S_a]\| &\leq \|S_b(t) - S_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} &\psi + \|\xi(S_b - S_a)\| + \|\varphi(S_b - S_a)\| \\ &+ \|\beta I_b(S_b - S_a)\| + \|\beta S_b(S_b - S_a)\| \\ &+ \|\nu W_b(S_b - S_a)\| + \|\nu S_b(S_b - S_a)\| + \|(\Lambda + \mu + \Delta)(S_b - S_a)\| \end{aligned} \right\} \right], \\ \|G[E_b] - G[E_a]\| &\leq \|E_b(t) - E_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} &\|\beta(1 + \zeta(t - \tau)(t - \tau)e^{-\Lambda\tau})S_b(E_b - E_a)\| \\ &+ \|\beta(1 + \zeta(t - \tau)(t - \tau)e^{-\Lambda\tau})I_b(E_b - E_a)\| \\ &+ \|(\mu + \Lambda + \sigma)(E_b - E_a)\| + \|\Delta_1(E_b - E_a)\| \\ &+ \|\gamma W_b(E_b - E_a)\| + \|\gamma E_b(E_b - E_a)\| \end{aligned} \right\} \right], \\ \|G[I_b] - G[I_a]\| &\leq \|I_b(t) - I_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} &\|\beta\zeta(t - \tau)(t - \tau)e^{-\Lambda\tau}S_b(I_b - I_a)\| + \|\beta\zeta(t - \tau)(t - \tau)e^{-\Lambda\tau}I_b(I_b - I_a)\| \\ &+ \|\phi W_b(I_b - I_a)\| + \|\phi I_b(I_b - I_a)\| \\ &+ \|(\mu + \sigma_1 + \Lambda + \varpi + \varepsilon + \Delta_2)(I_b - I_a)\| \end{aligned} \right\} \right], \\ \|G[H_b] - G[H_a]\| &\leq \|H_b(t) - H_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} &\|\Delta(H_b - H_a)\| + \|\Delta_1(H_b - H_a)\| + \|(\varepsilon + \Delta_2)(H_b - H_a)\| \\ &+ \|\chi(H_b - H_a)\| + \|(\Lambda + \mu + \varphi + \omega)(H_b - H_a)\| \end{aligned} \right\} \right], \\ \|G[R_b] - G[R_a]\| &\leq \|R_b(t) - R_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} &\|\omega(R_b - R_a)\| + \|\sigma(R_b - R_a)\| \\ &+ \|\sigma_1(R_b - R_a)\| + \|(\Lambda + \mu + \xi)(R_b - R_a)\| \end{aligned} \right\} \right], \\ \|G[W_b] - G[W_a]\| &\leq \|W_b(t) - W_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} &\theta + \|\nu S_b(W_b - W_a)\| + \|\nu W_b(W_b - W_a)\| \\ &+ \|\gamma E_b(W_b - W_a)\| + \|\gamma W_b(W_b - W_a)\| \\ &+ \|\phi I_b(W_b - W_a)\| + \|\phi W_b(W_b - W_a)\| + \|(\chi + \delta)(W_b - W_a)\| \end{aligned} \right\} \right]. \end{aligned} \right. \quad (A26)$$

System (A25) is simplified to obtain the following:

$$\left. \begin{aligned} \|G[S_b] - G[S_a]\| &\leq \|S_b(t) - S_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} &\psi + \|\xi(S_b - S_a)\| + \|\varphi(S_b - S_a)\| + \beta\|I_b\|\|(S_b - S_a)\| + \beta\|S_b\|\|(S_b - S_a)\| \\ &+ \nu\|W_b\|\|(S_b - S_a)\| + \nu\|S_b\|\|(S_b - S_a)\| + (\Lambda + \mu + \Delta)\|(S_b - S_a)\| \end{aligned} \right\} \right], \\ \|G[E_b] - G[E_a]\| &\leq \|E_b(t) - E_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} &\beta(1 + \zeta(t - \tau)(t - \tau)e^{-\Lambda\tau})\|S_b\|\|(E_b - E_a)\| \\ &+ \beta(1 + \zeta(t - \tau)(t - \tau)e^{-\Lambda\tau})\|I_b\|\|(E_b - E_a)\| + (\mu + \Lambda + \sigma)\|(E_b - E_a)\| \\ &+ \Delta_1\|(E_b - E_a)\| + \gamma\|W_b\|\|(E_b - E_a)\| + \gamma\|E_b\|\|(E_b - E_a)\| \end{aligned} \right\} \right], \\ \|G[I_b] - G[I_a]\| &\leq \|I_b(t) - I_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} &\beta\zeta(t - \tau)(t - \tau)e^{-\Lambda\tau}\|S_b\|\|(I_b - I_a)\| \\ &+ \beta\zeta(t - \tau)(t - \tau)e^{-\Lambda\tau}\|I_b\|\|(I_b - I_a)\| + \phi\|W_b\|\|(I_b - I_a)\| \\ &+ \phi\|I_b\|\|(I_b - I_a)\| + (\mu + \sigma_1 + \Lambda + \varpi + \varepsilon + \Delta_2)\|(I_b - I_a)\| \end{aligned} \right\} \right], \\ \|G[H_b] - G[H_a]\| &\leq \|H_b(t) - H_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} &\|\Delta(H_b - H_a)\| + \Delta_1\|(H_b - H_a)\| + (\varepsilon + \Delta_2)\|(H_b - H_a)\| \\ &+ \chi\|(H_b - H_a)\| + (\Lambda + \mu + \varphi + \omega)\|(H_b - H_a)\| \end{aligned} \right\} \right], \\ \|G[R_b] - G[R_a]\| &\leq \|R_b(t) - R_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} &\omega\|(R_b - R_a)\| + \sigma\|(R_b - R_a)\| + \sigma_1\|(R_b - R_a)\| \\ &+ (\Lambda + \mu + \xi)\|(R_b - R_a)\| \end{aligned} \right\} \right], \\ \|G[W_b] - G[W_a]\| &\leq \|W_b(t) - W_a(t)\| + \mathcal{B}^{-1} \left[\mathbb{L}\mathcal{B} \left\{ \begin{aligned} &\theta + \nu\|S_b\|\|(W_b - W_a)\| + \nu\|W_b\|\|(W_b - W_a)\| + \gamma\|E_b\|\|(W_b - W_a)\| \\ &+ \gamma\|W_b\|\|(W_b - W_a)\| + \phi\|I_b\|\|(W_b - W_a)\| + \phi\|W_b\|\|(W_b - W_a)\| \\ &+ (\chi + \delta)\|(W_b - W_a)\| \end{aligned} \right\} \right]. \end{aligned} \right. \quad (A27)$$

Furthermore, because these are convergent sequences, S_b, E_b, I_b and W_b are bounded. However, we can obtain alternate positive constants $\Theta_1, \Theta_2, \Theta_3$, and Θ_4 for every t , such that

$$\|S_b\| < \Theta_1, \|E_b\| < \Theta_2, \|I_b\| < \Theta_3, \|W_b\| < \Theta_4 \quad (A28)$$

$$\left. \begin{aligned} \|G[S_b] - G[S_a]\| &\leq \left\{ \begin{aligned} &1 + \psi + \xi\lambda_1(q) + \varphi\lambda_2(q) + \beta\lambda_3\Theta_1(q) + \beta\lambda_4\Theta_3(q) \\ &+ \nu\lambda_5\Theta_1(q) + \nu\lambda_6\Theta_4(q) - (\Lambda + \mu + \Delta)\lambda_7(q) \end{aligned} \right\} \|S_b - S_a\|, \\ \|G[E_b] - G[E_a]\| &\left\{ \begin{aligned} &1 + \lambda_8\Theta_1(q) + \beta\lambda_9\Theta_3(q) + \zeta\beta(\lambda_{10}\Theta_1(q) + \lambda_{11}\Theta_3(q))(t - \tau)(t - \tau)e^{-\Lambda\tau} \\ &+ \gamma\lambda_{12}\Theta_4(q) + \gamma\lambda_{13}\Theta_2(q) + \Delta_1\lambda_{14}(q) \\ &+ (\mu + \Lambda + \sigma)\lambda_{15}(q) \end{aligned} \right\} \|E_b - E_a\|, \\ \|G[I_b] - G[I_a]\| &\left\{ \begin{aligned} &1 + \zeta\beta(\lambda_{16}\Theta_1(q) + \lambda_{17}\Theta_3(q))(t - \tau)(t - \tau)e^{-\Lambda\tau} + \phi\lambda_{18}\Theta_3(q) \\ &+ \phi\lambda_{19}\Theta_4(q) + (\mu + \sigma_1 + \Lambda + \varpi + \varepsilon + \Delta_2)\lambda_{20}(q) \end{aligned} \right\} \|I_b - I_a\|, \\ \|G[H_b] - G[H_a]\| &\left\{ \begin{aligned} &1 + \Delta\lambda_{21}(q) + \Delta_1\lambda_{22}(q) + (\varepsilon + \Delta_2)\lambda_{23}(q) + \chi\lambda_{24}(q) \\ &+ (\Lambda + \mu + \varphi + \omega)\lambda_{25}(q) \end{aligned} \right\} \|H_b - H_a\|, \\ \|G[R_b] - G[R_a]\| &\left\{ \begin{aligned} &1 + \omega\lambda_{26}(q) + \sigma\lambda_{27}(q) + \sigma_1\lambda_{28}(q) + (\Lambda + \mu + \xi)\lambda_{29}(q) \end{aligned} \right\} \|R_b - R_a\|, \\ \|G[W_b] - G[W_a]\| &\left\{ \begin{aligned} &1 + \theta + \nu\lambda_{30}\Theta_1(q) + \nu\lambda_{31}\Theta_4(q) + \gamma\lambda_{32}\Theta_4(q) + \gamma\lambda_{33}\Theta_2(q) \\ &+ \phi\lambda_{34}\Theta_3(q) + \phi\lambda_{35}\Theta_4(q) + (\chi + \delta)\lambda_{36}(q) \end{aligned} \right\} \|W_b - W_a\|, \end{aligned} \right. \quad (A29)$$

where $\lambda_i(q), i = 1, 2, 3, \dots, 36$ are functions of $\mathcal{B}^{-1}[\mathcal{B}]$.

The mapping $\mathbb{G}^*$ therefore has a fixed point. Therefore, we show that the previous theorem 3 is correct, and that $\mathbb{G}^*$ holds true. Assuming that Equations (A28) and (A29) are true, we also prove

$$\begin{aligned} & \mathcal{J}(0, 0, 0, 0, 0, 0), \\ & \left. \begin{aligned} & \{1 + \psi + \xi\lambda_1(q) + \varphi\lambda_2(q) + \beta\lambda_3\Theta_1(q) + \beta\lambda_4\Theta_3(q) + \nu\lambda_5\Theta_1(q) + \nu\lambda_6\Theta_4(q) - (\Lambda + \mu + \Delta)\lambda_7(q)\} < 1, \\ & \left\{ \begin{aligned} & 1 + \lambda_8\Theta_1(q) + \beta\lambda_9\Theta_3(q) + \zeta\beta(\lambda_{10}\Theta_1(q) + \lambda_{11}\Theta_3(q))(t - \tau)(t - \tau)e^{-\Lambda\tau} \\ & + \gamma\lambda_{12}\Theta_4(q) + \gamma\lambda_{13}\Theta_2(q) + \Delta_1\lambda_{14}(q) + (\mu + \Lambda + \sigma)\lambda_{15}(q) \end{aligned} \right\} < 1, \\ & \left\{ \begin{aligned} & 1 + \zeta\beta(\lambda_{16}\Theta_1(q) + \lambda_{17}\Theta_3(q))(t - \tau)(t - \tau)e^{-\Lambda\tau} + \phi\lambda_{18}\Theta_3(q) + \phi\lambda_{19}\Theta_4(q) \\ & + (\mu + \sigma_1 + \Lambda + \varpi + \varepsilon + \Delta_2)\lambda_{20}(q) \end{aligned} \right\} < 1, \\ & \{1 + \Delta\lambda_{21}(q) + \Delta_1\lambda_{22}(q) + (\varepsilon + \Delta_2)\lambda_{23}(q) + \chi\lambda_{24}(q) + (\Lambda + \mu + \varphi + \omega)\lambda_{25}(q)\} < 1, \\ & \{1 + \omega\lambda_{26}(q) + \sigma\lambda_{27}(q) + \sigma_1\lambda_{28}(q) + (\Lambda + \mu + \xi)\lambda_{29}(q)\} < 1, \\ & \left\{ \begin{aligned} & 1 + \theta + \nu\lambda_{30}\Theta_1(q) + \nu\lambda_{31}\Theta_4(q) + \gamma\lambda_{32}\Theta_4(q) + \gamma\lambda_{33}\Theta_2(q) + \phi\lambda_{34}\Theta_3(q) + \phi\lambda_{35}\Theta_4(q) \\ & + (\chi + \delta)\lambda_{36}(q) \end{aligned} \right\} < 1, \end{aligned} \right\} \quad (\text{A30}) \\ & \mathbb{G}^* \text{ is Picard } \mathbb{G}^* \text{ stable, since it satisfies all the requirements in theorem 4.} \end{aligned}$$

Appendix D. Proof of the Laplace Transform of the CPCF Operator for the Variables S, E, I, H, R and W

$$\begin{aligned} [\text{}^{CPCF}_0 D_t^\varrho S(t)] &= \mathcal{B} \left[S(t) \times \frac{\mathcal{U}(q)Z_1(q)}{1 - q} \exp\left(-\frac{q}{1 - q}t\right) \right] + \mathcal{B} \left[S'(t) \times \frac{\mathcal{U}(q)Z_0(q)}{1 - q} \exp\left(-\frac{q}{1 - q}t\right) \right] \\ &= \frac{\mathcal{U}(q)Z_1(q)}{1 - q} \mathcal{B}[S(t)] \mathcal{B} \left[\exp\left(-\frac{q}{1 - q}t\right) \right] + \frac{\mathcal{U}(q)Z_0(q)}{1 - q} \mathcal{B}[S'(t)] \mathcal{B} \left[\exp\left(-\frac{q}{1 - q}t\right) \right] \\ &= \frac{\mathcal{U}(q)Z_1(q)}{1 - q} \mathcal{B}[S(t)] \times \frac{1}{\varsigma + \frac{q}{1 - q}} + \frac{\mathcal{U}(q)Z_0(q)}{1 - q} (\varsigma \mathcal{B}[S(t)] - S(0)) \times \frac{1}{\varsigma + \frac{q}{1 - q}} \end{aligned} \quad (\text{A31})$$

$$\begin{aligned} &= \left(\frac{\mathcal{U}(q)Z_1(q)}{q + \varsigma(1 - q)} \right) \mathcal{B}[S(t)] + \left(\frac{\varsigma \mathcal{U}(q)Z_0(q)}{q + \varsigma(1 - q)} \right) \mathcal{B}[S(t)] - \left(\frac{\mathcal{U}(q)Z_0(q)}{q + \varsigma(1 - q)} \right) S(0). \\ [\text{}^{CPCF}_0 D_t^\varrho E(t)] &= \mathcal{B} \left[E(t) \times \frac{\mathcal{U}(q)Z_1(q)}{1 - q} \exp\left(-\frac{q}{1 - q}t\right) \right] + \mathcal{B} \left[E'(t) \times \frac{\mathcal{U}(q)Z_0(q)}{1 - q} \exp\left(-\frac{q}{1 - q}t\right) \right] \\ &= \frac{\mathcal{U}(q)Z_1(q)}{1 - q} \mathcal{B}[E(t)] \mathcal{B} \left[\exp\left(-\frac{q}{1 - q}t\right) \right] + \frac{\mathcal{U}(q)Z_0(q)}{1 - q} \mathcal{B}[E'(t)] \mathcal{B} \left[\exp\left(-\frac{q}{1 - q}t\right) \right] \\ &= \frac{\mathcal{U}(q)Z_1(q)}{1 - q} \mathcal{B}[E(t)] \times \frac{1}{\varsigma + \frac{q}{1 - q}} + \frac{\mathcal{U}(q)Z_0(q)}{1 - q} (\varsigma \mathcal{B}[E(t)] - S(0)) \times \frac{1}{\varsigma + \frac{q}{1 - q}} \\ &= \left(\frac{\mathcal{U}(q)Z_1(q)}{q + \varsigma(1 - q)} \right) \mathcal{B}[E(t)] + \left(\frac{\varsigma \mathcal{U}(q)Z_0(q)}{q + \varsigma(1 - q)} \right) \mathcal{B}[E(t)] - \left(\frac{\mathcal{U}(q)Z_0(q)}{q + \varsigma(1 - q)} \right) E(0). \end{aligned} \quad (\text{A32})$$

Similarly, the Laplace transform of the CPCF operator for the variables I, H, R and W as in (14) can be proved using the same procedure as in (A31) and (A32), to obtain the remaining sub-equations of system (14).

Appendix E. Proof of the Numerical Scheme of the Model

From theorem 6, we have

$$\begin{aligned} \mathcal{B}\{S(t)\} &= \frac{S_0}{\varsigma + \frac{Z_1(q)}{Z_0(q)}} + \frac{q}{\mathcal{U}(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)} \right)^i \varsigma^{-i-1} \left(\psi + \xi \mathcal{B}\{R\} + \varphi \mathcal{B}\{H\} - \beta \mathcal{B}\{SI\} - \nu \mathcal{B}\{SW\} \right. \\ &\quad \left. - (\Lambda + \mu + \Delta) \mathcal{B}\{S\} \right) \\ &\quad + \frac{1 - q}{\mathcal{U}(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)} \right)^i \varsigma^{-i} \left(\psi + \xi \mathcal{B}\{R\} + \varphi \mathcal{B}\{H\} - \beta \mathcal{B}\{SI\} - \nu \mathcal{B}\{SW\} \right. \\ &\quad \left. - (\Lambda + \mu + \Delta) \mathcal{B}\{S\} \right) \end{aligned}$$

$$\begin{aligned}
\mathcal{E}\{E(t)\} &= \frac{E_0}{\varsigma + \frac{Z_1(q)}{Z_0(q)}} + \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)} \right)^i \varsigma^{-i-1} \left(\beta \mathcal{E}\{SI\} - \zeta \beta \mathcal{E}\{SI\}(t-\tau)(t-\tau)e^{-\Lambda\tau} - \Delta_1 \mathcal{E}\{E\} \right. \\
&\quad \left. - \gamma \mathcal{E}\{EW\} - (\mu + \Lambda + \sigma) \mathcal{E}\{E\} \right) \\
&\quad + \frac{1-q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)} \right)^i \varsigma^{-i} \left(\beta \mathcal{E}\{SI\} - \zeta \beta \mathcal{E}\{SI\}(t-\tau)(t-\tau)e^{-\Lambda\tau} - \Delta_1 \mathcal{E}\{E\} \right. \\
&\quad \left. - \gamma \mathcal{E}\{EW\} - (\mu + \Lambda + \sigma) \mathcal{E}\{E\} \right) \\
\mathcal{I}\{I(t)\} &= \frac{I_0}{\varsigma + \frac{Z_1(q)}{Z_0(q)}} + \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)} \right)^i \varsigma^{-i-1} \left(\zeta \beta \mathcal{E}\{SI\}(t-\tau)(t-\tau)e^{-\Lambda\tau} - \phi \mathcal{E}\{IW\} \right. \\
&\quad \left. - (\mu + \sigma_1 + \Lambda + \varpi + \varepsilon + \Delta_2) \mathcal{E}\{I\} \right) \\
&\quad + \frac{1-q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)} \right)^i \varsigma^{-i} \left(\zeta \beta \mathcal{E}\{SI\}(t-\tau)(t-\tau)e^{-\Lambda\tau} - \phi \mathcal{E}\{IW\} \right. \\
&\quad \left. - (\mu + \sigma_1 + \Lambda + \varpi + \varepsilon + \Delta_2) \mathcal{E}\{I\} \right) \quad (A33) \\
\mathcal{H}\{H(t)\} &= \frac{H_0}{\varsigma + \frac{Z_1(q)}{Z_0(q)}} + \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)} \right)^i \varsigma^{-i-1} \left(\Delta \mathcal{E}\{S\} + \Delta_1 \mathcal{E}\{E\} + (\varepsilon + \Delta_2) \mathcal{E}\{I\} + \chi \mathcal{E}\{W\} \right. \\
&\quad \left. - (\Lambda + \mu + \varphi + \omega) \mathcal{E}\{H\} \right) \\
&\quad + \frac{1-q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)} \right)^i \varsigma^{-i} \left(\Delta \mathcal{E}\{S\} + \Delta_1 \mathcal{E}\{E\} + (\varepsilon + \Delta_2) \mathcal{E}\{I\} + \chi \mathcal{E}\{W\} \right. \\
&\quad \left. - (\Lambda + \mu + \varphi + \omega) \mathcal{E}\{H\} \right) \\
\mathcal{R}\{R(t)\} &= \frac{R_0}{\varsigma + \frac{Z_1(q)}{Z_0(q)}} + \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)} \right)^i \varsigma^{-i-1} (\omega \mathcal{E}\{H\} + \sigma \mathcal{E}\{E\} + \sigma_1 \mathcal{E}\{I\} - (\Lambda + \mu + \xi) \mathcal{E}\{R\}) \\
&\quad + \frac{1-q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)} \right)^i \varsigma^{-i} (\omega \mathcal{E}\{H\} + \sigma \mathcal{E}\{E\} + \sigma_1 \mathcal{E}\{I\} - (\Lambda + \mu + \xi) \mathcal{E}\{R\}) \\
\mathcal{W}\{W(t)\} &= \frac{W_0}{\varsigma + \frac{Z_1(q)}{Z_0(q)}} + \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)} \right)^i \varsigma^{-i-1} (\theta + (\nu \mathcal{E}\{SW\} + \gamma \mathcal{E}\{EW\} + \phi \mathcal{E}\{IW\}) - (\chi + \delta) \mathcal{E}\{W\}) \\
&\quad + \frac{1-q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)} \right)^i \varsigma^{-i} (\theta + (\nu \mathcal{E}\{SW\} + \gamma \mathcal{E}\{EW\} + \phi \mathcal{E}\{IW\}) - (\chi + \delta) \mathcal{E}\{W\})
\end{aligned}$$

Consider that the scheme gives the outcomes as an infinite series:

$$S(t) \sum_{g=0}^{\infty} S_g, E(t) \sum_{g=0}^{\infty} E_g, I(t) \sum_{g=0}^{\infty} I_g, H(t) \sum_{g=0}^{\infty} H_g, R(t) \sum_{g=0}^{\infty} R_g, W(t) \sum_{g=0}^{\infty} W_g \quad (A34)$$

where the non-linear expression can be described as

$$\begin{aligned}
SI &= \sum_{g=0}^{\infty} \mathbb{Q}_g, \mathbb{Q}_g = \frac{1}{g!} \frac{d^g}{d\Gamma^g} \left(\sum_{z=0}^g \Gamma^z S_z \sum_{z=0}^g \Gamma^z I_z \right) \Big|_{\Gamma=0}, \\
SW &= \sum_{g=0}^{\infty} \mathbb{X}_g, \mathbb{X}_g = \frac{1}{g!} \frac{d^g}{d\Gamma^g} \left(\sum_{z=0}^g \Gamma^z S_z \sum_{z=0}^g \Gamma^z W_z \right) \Big|_{\Gamma=0}, \\
EW &= \sum_{g=0}^{\infty} \mathbb{X}_g, \mathbb{Y}_g = \frac{1}{g!} \frac{d^g}{d\Gamma^g} \left(\sum_{z=0}^g \Gamma^z E_z \sum_{z=0}^g \Gamma^z W_z \right) \Big|_{\Gamma=0}, \\
IW &= \sum_{g=0}^{\infty} \mathbb{Z}_g, \mathbb{Z}_g = \frac{1}{g!} \frac{d^g}{d\Gamma^g} \left(\sum_{z=0}^g \Gamma^z I_z \sum_{z=0}^g \Gamma^z W_z \right) \Big|_{\Gamma=0}.
\end{aligned} \quad (A35)$$

$g = 1, 2, 3, \dots$

Substituting systems (A34) and (A35) into (A33), we obtain

$$\begin{aligned}
\mathcal{L}\left\{\sum_{g=0}^{\infty} S(t)\right\} &= \frac{S_0}{\varsigma + \frac{Z_1(q)}{Z_0(q)}} + \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \varsigma^{-i-1} \left(\begin{aligned} &\psi + \xi \mathcal{L}\left\{\sum_{g=0}^{\infty} R_g\right\} + \varphi \mathcal{L}\left\{\sum_{g=0}^{\infty} H_g\right\} - \beta \mathcal{L}\left\{\sum_{g=0}^{\infty} \mathbb{Q}_g\right\} \\ &- \nu \mathcal{L}\left\{\sum_{g=0}^{\infty} \mathbb{X}_g\right\} - (\Lambda + \mu + \Delta) \mathcal{L}\left\{\sum_{g=0}^{\infty} S_g\right\} \end{aligned} \right) \\
&+ \frac{1-q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \varsigma^{-i} \left(\begin{aligned} &\psi + \xi \mathcal{L}\left\{\sum_{g=0}^{\infty} R_g\right\} + \varphi \mathcal{L}\left\{\sum_{g=0}^{\infty} H_g\right\} - \beta \mathcal{L}\left\{\sum_{g=0}^{\infty} \mathbb{Q}_g\right\} \\ &- \nu \mathcal{L}\left\{\sum_{g=0}^{\infty} \mathbb{X}_g\right\} - (\Lambda + \mu + \Delta) \mathcal{L}\left\{\sum_{g=0}^{\infty} S_g\right\} \end{aligned} \right) \\
\mathcal{L}\left\{\sum_{g=0}^{\infty} E(t)\right\} &= \frac{E_0}{\varsigma + \frac{Z_1(q)}{Z_0(q)}} + \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \varsigma^{-i-1} \left(\begin{aligned} &\beta \mathcal{L}\left\{\sum_{g=0}^{\infty} \mathbb{Q}_g\right\} - \zeta \beta \mathcal{L}\left\{\sum_{g=0}^{\infty} \mathbb{Q}_g\right\} (t-\tau)(t-\tau)e^{-\Lambda\tau} \mathcal{L}\left\{\sum_{g=0}^{\infty} E_g\right\} \\ &- \gamma \mathcal{L}\left\{\sum_{g=0}^{\infty} \mathbb{Y}_g\right\} - (\Delta_1 + \mu + \Lambda + \sigma) \mathcal{L}\left\{\sum_{g=0}^{\infty} E_g\right\} \end{aligned} \right) \\
&+ \frac{1-q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \varsigma^{-i} \left(\begin{aligned} &\beta \mathcal{L}\left\{\sum_{g=0}^{\infty} \mathbb{Q}_g\right\} - \zeta \beta \mathcal{L}\left\{\sum_{g=0}^{\infty} \mathbb{Q}_g\right\} (t-\tau)(t-\tau)e^{-\Lambda\tau} \mathcal{L}\left\{\sum_{g=0}^{\infty} E_g\right\} \\ &- \gamma \mathcal{L}\left\{\sum_{g=0}^{\infty} \mathbb{Y}_g\right\} - (\Delta_1 + \mu + \Lambda + \sigma) \mathcal{L}\left\{\sum_{g=0}^{\infty} E_g\right\} \end{aligned} \right) \\
\mathcal{L}\left\{\sum_{g=0}^{\infty} I(t)\right\} &= \frac{I_0}{\varsigma + \frac{Z_1(q)}{Z_0(q)}} + \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \varsigma^{-i-1} \left(\begin{aligned} &\zeta \beta \mathcal{L}\left\{\sum_{g=0}^{\infty} \mathbb{Q}_g\right\} (t-\tau)(t-\tau)e^{-\Lambda\tau} - \phi \mathcal{L}\left\{\sum_{g=0}^{\infty} \mathbb{Z}_g\right\} \\ &- (\mu + \sigma_1 + \Lambda + \varpi + \varepsilon + \Delta_2) \mathcal{L}\left\{\sum_{g=0}^{\infty} I_g\right\} \end{aligned} \right) \\
&+ \frac{1-q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \varsigma^{-i} \left(\begin{aligned} &\zeta \beta \mathcal{L}\left\{\sum_{g=0}^{\infty} \mathbb{Q}_g\right\} (t-\tau)(t-\tau)e^{-\Lambda\tau} - \phi \mathcal{L}\left\{\sum_{g=0}^{\infty} \mathbb{Z}_g\right\} \\ &- (\mu + \sigma_1 + \Lambda + \varpi + \varepsilon + \Delta_2) \mathcal{L}\left\{\sum_{g=0}^{\infty} I_g\right\} \end{aligned} \right) \\
\mathcal{L}\left\{\sum_{g=0}^{\infty} H(t)\right\} &= \frac{H_0}{\varsigma + \frac{Z_1(q)}{Z_0(q)}} + \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \varsigma^{-i-1} \left(\begin{aligned} &\Delta \mathcal{L}\left\{\sum_{g=0}^{\infty} S_g\right\} + \Delta_1 \mathcal{L}\left\{\sum_{g=0}^{\infty} E_g\right\} + (\varepsilon + \Delta_2) \mathcal{L}\left\{\sum_{g=0}^{\infty} I_g\right\} + \chi \mathcal{L}\left\{\sum_{g=0}^{\infty} W_g\right\} \\ &- (\Lambda + \mu + \varphi + \omega) \mathcal{L}\left\{\sum_{g=0}^{\infty} H_g\right\} \end{aligned} \right) \\
&+ \frac{1-q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \varsigma^{-i} \left(\begin{aligned} &\Delta \mathcal{L}\left\{\sum_{g=0}^{\infty} S_g\right\} + \Delta_1 \mathcal{L}\left\{\sum_{g=0}^{\infty} E_g\right\} + (\varepsilon + \Delta_2) \mathcal{L}\left\{\sum_{g=0}^{\infty} I_g\right\} + \chi \mathcal{L}\left\{\sum_{g=0}^{\infty} W_g\right\} \\ &- (\Lambda + \mu + \varphi + \omega) \mathcal{L}\left\{\sum_{g=0}^{\infty} H_g\right\} \end{aligned} \right) \\
\mathcal{L}\left\{\sum_{g=0}^{\infty} R(t)\right\} &= \frac{R_0}{\varsigma + \frac{Z_1(q)}{Z_0(q)}} + \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \varsigma^{-i-1} \left(\begin{aligned} &\omega \mathcal{L}\left\{\sum_{g=0}^{\infty} H_g\right\} + \sigma \mathcal{L}\left\{\sum_{g=0}^{\infty} E_g\right\} \\ &+ \sigma_1 \mathcal{L}\left\{\sum_{g=0}^{\infty} I_g\right\} - (\Lambda + \mu + \xi) \mathcal{L}\left\{\sum_{g=0}^{\infty} R_g\right\} \end{aligned} \right) \\
&+ \frac{1-q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \varsigma^{-i} \left(\begin{aligned} &\omega \mathcal{L}\left\{\sum_{g=0}^{\infty} H_g\right\} + \sigma \mathcal{L}\left\{\sum_{g=0}^{\infty} E_g\right\} \\ &+ \sigma_1 \mathcal{L}\left\{\sum_{g=0}^{\infty} I_g\right\} - (\Lambda + \mu + \xi) \mathcal{L}\left\{\sum_{g=0}^{\infty} R_g\right\} \end{aligned} \right)
\end{aligned}
\tag{A36}$$

$$\begin{aligned} \mathcal{L}\left\{\sum_{g=0}^{\infty} W(t)\right\} &= \frac{W_0}{\varsigma + \frac{Z_1(q)}{Z_0(q)}} + \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \varsigma^{-i-1} \left(\begin{aligned} &\theta + \left(\nu \mathcal{L}\left\{\sum_{g=0}^{\infty} X_g\right\} + \gamma \mathcal{L}\left\{\sum_{g=0}^{\infty} Y_g\right\} + \phi \mathcal{L}\left\{\sum_{g=0}^{\infty} Z_g\right\} \right) \\ &-(\chi + \delta) \mathcal{L}\left\{\sum_{g=0}^{\infty} W_g\right\} \end{aligned} \right) \\ &+ \frac{1-q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \varsigma^{-i} \left(\begin{aligned} &\theta + \left(\nu \mathcal{L}\left\{\sum_{g=0}^{\infty} X_g\right\} + \gamma \mathcal{L}\left\{\sum_{g=0}^{\infty} Y_g\right\} + \phi \mathcal{L}\left\{\sum_{g=0}^{\infty} Z_g\right\} \right) \\ &-(\chi + \delta) \mathcal{L}\left\{\sum_{g=0}^{\infty} W_g\right\} \end{aligned} \right) \end{aligned}$$

After applying the inverse Laplace transform on both sides of (A36), we finally obtain the following iterative solutions:

$$\begin{aligned} S_{g+1}(t) &= S_0 \exp\left(-\frac{Z_1(q)}{Z_0(q)} t\right) + \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \frac{t^i}{\mathfrak{L}(j+1)} \mathcal{L}^{-1}\left\{ \begin{aligned} &\psi + \xi \mathcal{L}\{R\} + \varphi \mathcal{L}\{H\} - \beta \mathcal{L}\{SI\} - \nu \mathcal{L}\{SW\} \\ &-(\Lambda + \mu + \Delta) \mathcal{L}\{S\} \end{aligned} \right\} \\ &+ \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \frac{t^{i-1}}{\mathfrak{L}(j+1)} \mathcal{L}^{-1}\left\{ \begin{aligned} &\psi + \xi \mathcal{L}\{R\} + \varphi \mathcal{L}\{H\} - \beta \mathcal{L}\{SI\} - \nu \mathcal{L}\{SW\} \\ &-(\Lambda + \mu + \Delta) \mathcal{L}\{S\} \end{aligned} \right\} \\ E_{g+1}(t) &= E_0 \exp\left(-\frac{Z_1(q)}{Z_0(q)} t\right) + \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \frac{t^i}{\mathfrak{L}(j+1)} \mathcal{L}^{-1}\left\{ \begin{aligned} &\beta \mathcal{L}\{SI\} - \zeta \beta \mathcal{L}\{SI\}(t-\tau)(t-\tau)e^{-\Lambda\tau} - \Delta_1 \mathcal{L}\{E\} \\ &-\gamma \mathcal{L}\{EW\} - (\mu + \Lambda + \sigma) \mathcal{L}\{E\} \end{aligned} \right\} \\ &+ \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \frac{t^{i-1}}{\mathfrak{L}(j+1)} \mathcal{L}^{-1}\left\{ \begin{aligned} &\beta \mathcal{L}\{SI\} - \zeta \beta \mathcal{L}\{SI\}(t-\tau)(t-\tau)e^{-\Lambda\tau} - \Delta_1 \mathcal{L}\{E\} \\ &-\gamma \mathcal{L}\{EW\} - (\mu + \Lambda + \sigma) \mathcal{L}\{E\} \end{aligned} \right\} \\ I_{g+1}(t) &= I_0 \exp\left(-\frac{Z_1(q)}{Z_0(q)} t\right) + \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \frac{t^i}{\mathfrak{L}(j+1)} \mathcal{L}^{-1}\left\{ \begin{aligned} &\zeta \beta \mathcal{L}\{SI\}(t-\tau)(t-\tau)e^{-\Lambda\tau} - \phi \mathcal{L}\{IW\} \\ &-(\mu + \sigma_1 + \Lambda + \varpi + \varepsilon + \Delta_2) \mathcal{L}\{I\} \end{aligned} \right\} \\ &+ \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \frac{t^{i-1}}{\mathfrak{L}(j+1)} \mathcal{L}^{-1}\left\{ \begin{aligned} &\zeta \beta \mathcal{L}\{SI\}(t-\tau)(t-\tau)e^{-\Lambda\tau} - \phi \mathcal{L}\{IW\} \\ &-(\mu + \sigma_1 + \Lambda + \varpi + \varepsilon + \Delta_2) \mathcal{L}\{I\} \end{aligned} \right\} \\ H_{g+1}(t) &= H_0 \exp\left(-\frac{Z_1(q)}{Z_0(q)} t\right) + \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \frac{t^i}{\mathfrak{L}(j+1)} \mathcal{L}^{-1}\left\{ \begin{aligned} &\Delta \mathcal{L}\{S\} + \Delta_1 \mathcal{L}\{E\} + (\varepsilon + \Delta_2) \mathcal{L}\{I\} + \chi \mathcal{L}\{W\} \\ &-(\Lambda + \mu + \varphi + \omega) \mathcal{L}\{H\} \end{aligned} \right\} \\ &+ \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \frac{t^{i-1}}{\mathfrak{L}(j+1)} \mathcal{L}^{-1}\left\{ \begin{aligned} &\Delta \mathcal{L}\{S\} + \Delta_1 \mathcal{L}\{E\} + (\varepsilon + \Delta_2) \mathcal{L}\{I\} + \chi \mathcal{L}\{W\} \\ &-(\Lambda + \mu + \varphi + \omega) \mathcal{L}\{H\} \end{aligned} \right\} \\ R_{g+1}(t) &= R_0 \exp\left(-\frac{Z_1(q)}{Z_0(q)} t\right) + \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \frac{t^i}{\mathfrak{L}(j+1)} \mathcal{L}^{-1}\left\{ \begin{aligned} &\omega \mathcal{L}\{H\} + \sigma \mathcal{L}\{E\} + \sigma_1 \mathcal{L}\{I\} \\ &-(\Lambda + \mu + \xi) \mathcal{L}\{R\} \end{aligned} \right\} \\ &+ \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \frac{t^{i-1}}{\mathfrak{L}(j+1)} \mathcal{L}^{-1}\left\{ \begin{aligned} &\omega \mathcal{L}\{H\} + \sigma \mathcal{L}\{E\} + \sigma_1 \mathcal{L}\{I\} \\ &-(\Lambda + \mu + \xi) \mathcal{L}\{R\} \end{aligned} \right\} \\ W_{g+1}(t) &= W_0 \exp\left(-\frac{Z_1(q)}{Z_0(q)} t\right) + \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \frac{t^i}{\mathfrak{L}(j+1)} \mathcal{L}^{-1}\left\{ \begin{aligned} &\theta + (\nu \mathcal{L}\{SW\} + \gamma \mathcal{L}\{EW\} + \phi \mathcal{L}\{IW\}) \\ &-(\chi + \delta) \mathcal{L}\{W\} \end{aligned} \right\} \\ &+ \frac{q}{u(q)Z_0(q)} \sum_{i=0}^{\infty} \left(-\frac{Z_1(q)}{Z_0(q)}\right)^i \frac{t^{i-1}}{\mathfrak{L}(j+1)} \mathcal{L}^{-1}\left\{ \begin{aligned} &\theta + (\nu \mathcal{L}\{SW\} + \gamma \mathcal{L}\{EW\} + \phi \mathcal{L}\{IW\}) \\ &-(\chi + \delta) \mathcal{L}\{W\} \end{aligned} \right\} \end{aligned} \quad (A37)$$

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