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An Axiomatic DEA Model for Performance Evaluation of Wireless Sensor Networks with Dependent Desirable and Undesirable Outputs

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Abstract

In most production systems, the objective is to minimize input consumption while maximizing the generation of desirable outputs. However, many production processes also generate undesirable outputs as by-products of desirable outputs. In many real-world systems, undesirable outputs are inherently linked to the production of desirable outputs. Several studies in the Data Envelopment Analysis (DEA) literature have addressed performance evaluation of decision-making units (DMUs) in the presence of undesirable outputs. However, most existing models assume that desirable and undesirable outputs are independent, which may not reflect real production environments. The objective of this study is to model the dependency between desirable and undesirable outputs and to develop a novel DEA framework based on an axiomatic approach. Specifically, the classical axiom of output disposability is decomposed into two separate axioms: disposability of desirable outputs and disposability of undesirable outputs. Based on these axioms, a new production possibility set (PPS) is constructed. The proposed DEA model explicitly incorporates the dependency between desirable and undesirable outputs. A case study involving sensor monitoring systems is presented to demonstrate the applicability of the proposed approach.

Keywords: Data Envelopment Analysis (DEA); efficiency evaluation; undesirable outputs; sensor monitoring systems; axiomatic DEA modeling

1. Introduction

Data Envelopment Analysis (DEA) is a non-parametric mathematical programming technique widely used to evaluate the relative efficiency of decision-making units (DMUs) in various sectors. One of the major challenges in DEA modeling is handling undesirable outputs, which represent unwanted by-products such as pollution, waste, or errors generated during the production process. Several approaches have been proposed in the DEA literature to incorporate undesirable outputs into efficiency analysis. In some early studies, undesirable outputs were ignored during efficiency evaluation [1–3]. In other approaches, undesirable outputs are transformed, treated as inputs, or modified so that they can be incorporated into standard DEA models [4–9].

Several other studies have incorporated undesirable outputs directly into DEA mathematical modeling. Hailu and Veeman [10] provided a DEA model modifying the Banker



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and Maïndiretta model to deal with undesirable outputs directly. On the other hand, it was demonstrated by [11] that the modified axiom proposed by Hailu and Veeman (2001) [10] is inconsistent with traditional production theory principles. To address this, Kuosmanen [12] introduced a DEA approach incorporating the weak disposability assumption alongside non-proportional reduction factors. Furthermore, a slack-based measure (SBM) framework was expanded in [13] to simultaneously account for the reduction of undesirable factors and the growth of desirable ones. It was subsequently verified by [14] that the production technology established by Kuosmanen exhibits convexity while maintaining the joint weak disposability of both clean and undesirable outputs. D'Inverno et al. [15] integrated the analytic hierarchy process (AHP) and non-radial directional distance function considering undesirable outputs. Huang et al. [16] proposed a directional technology distance function that expands desirable factors and contracts undesirable outputs simultaneously. Khoshroo et al. [17] presented a non-radial DEA model for dealing with undesirable outputs. In a related development, a distinct DEA methodology was formulated by Song et al. [18] to optimize operations by minimizing bad outputs and maximizing good ones relative to existing input levels. Tajbakhsh and Shamsi [19] presented a double-bootstrap DEA model for dealing with undesirable factors. Wu et al. [20] proposed an epsilon-based measure DEA model for production efficiency considering undesirable outputs. Shirazi and Mohammadi [21] provided a robust DEA model for efficiency assessment of airlines in the presence of undesirable outputs.

Multiple studies have applied DEA to environmental and energy performance evaluation. For instance, Wu et al. [22] evaluated the eco-efficiency of coal-fired power plants in China using a super-efficiency DEA model. Pan et al. [23] measured carbon emission performance in China's energy market utilizing an improved non-radial directional distance function. Cecchini et al. [24] analyzed environmental efficiency and estimated CO₂ abatement costs in dairy cattle farms using an SBM-DEA model with undesirable outputs. Furthermore, Han et al. [25] developed an improved environmental DEA cross-model based on information entropy for industrial departments, while Liu et al. [26] focused on an emergy-based evaluation for sustainable manufacturing systems. More recently, He and Jie [27] evaluated the eco-efficiency of industrial systems considering regional heterogeneity, Li et al. [28] proposed a three-stage SBM-SE-DEA model for the power generation industry, and Xu et al. [29] provided a comprehensive literature review on DEA-based energy efficiency evaluation.

Reviewing the existing literature on managing bad outputs within DEA frameworks reveals that each study has endeavored to clarify specific dimensions of this analytical challenge. Despite the significant progress in DEA models that incorporate undesirable outputs, most existing studies assume that desirable and undesirable outputs are independent. In many real-world systems, however, undesirable outputs are technologically linked to desirable outputs and cannot be modified independently. Ignoring such dependencies may lead to inaccurate efficiency assessments.

Excluding undesirable factors from the evaluation matrix can lead to distorted and untrustworthy performance metrics [1,2]. Similarly, applying data transformations or treating negative outputs as standard inputs might violate conventional production theory, potentially generating negative data values or misrepresenting input disposability characteristics [6,7,30,31]. Notably, the formulation introduced in [11] possessed a nonlinear structure, and while [12] refined the underlying definitions to present an upgraded model, it similarly retained a nonlinear nature. Moreover, Ref. [14] provided a linear counterpart of the Kuosmanen model and showed that the model provided by [11] suffered from being non-convex.

In many real-world production systems, undesirable outputs are generated as unavoidable by-products of desirable outputs. In such cases, classic DEA models for dealing with undesirable outputs cannot accurately reflect real-world situations. Therefore, it is important to explicitly model the dependency between desirable and undesirable outputs. This means any reduction in undesirable outputs depends on the reduction of desirable outputs. Consequently, an increase in desirable outputs may lead to a corresponding increase in undesirable outputs. Despite its importance, this dependency relationship has received limited attention in the existing DEA literature. To improve the accuracy of the proposed model, an axiomatic approach is developed in which technological dependencies between the desirable and undesirable outputs are defined. This study discusses that reducing undesirable outputs may require simultaneous reductions in desirable outputs.

In recent years, sensor monitoring systems have become essential components of modern technological infrastructures, including environmental monitoring, industrial automation, smart cities, and healthcare applications. These systems often involve complex trade-offs between multiple performance indicators such as monitoring accuracy, energy consumption, sensing coverage, and measurement errors. In such systems, improving desirable outputs such as sensing quality may simultaneously influence undesirable outputs such as error rates or energy consumption. Therefore, evaluating the efficiency of sensor monitoring systems requires analytical methods capable of handling multiple inputs and both desirable and undesirable outputs. Data Envelopment Analysis provides a suitable framework for such evaluations.

This study addresses this limitation by explicitly modeling the dependency between desirable and undesirable outputs within an axiomatic DEA framework. The proposed approach decomposes the classical disposability axiom into separate axioms for desirable and undesirable outputs and incorporates functional relationships between them. By doing so, the proposed model provides a more realistic representation of production technologies where undesirable outputs are generated as unavoidable by-products of desirable outputs.

The main contributions of this study can be summarized as follows:

1. A new axiomatic DEA framework is proposed that explicitly models the dependency between desirable and undesirable outputs.
2. The traditional output disposability axiom is decomposed into separate axioms for desirable and undesirable outputs.
3. A DEA optimization model is developed to evaluate the performance of decision-making units under dependent output conditions.
4. The applicability of the proposed model is demonstrated using a case study involving sensor monitoring systems.

While the above points outline the methodological contributions, the primary practical motivation of this study is to resolve the inability of conventional DEA models to manage interdependent outputs in network structures. By explicitly modeling the intrinsic links between desirable monitoring operations and measurement error rates, our framework provides a diagnostic tool for real-world sensor networks, enabling managers to identify internal bottlenecks that generate inefficiencies—a critical operational insight that standard black-box DEA models fail to deliver.

The remainder of this paper is organized as follows. Section 2 reviews the basic concepts of Data Envelopment Analysis and existing approaches for handling undesirable outputs. Section 3 presents the proposed axiomatic DEA model that incorporates dependencies between desirable and undesirable outputs. Section 4 describes the case study and illustrates the application of the proposed model to sensor monitoring systems. Finally, Section 5 concludes the paper and outlines potential directions for future research.

2. Preliminaries

2.1. Basic DEA Models

Let DMU_o denote the decision-making unit under evaluation among a set of n decision-making units (DMUs). Define $x_o \in Rm_+$ and $y_o \in Rs_+$ as inputs and outputs of DMU_o . Consider the basic axioms introduced in the CCR model as:

- $\dot{A}1$. Feasibility: For all j : $(x_j, y_j) \in T_c$.
- $\dot{A}2$. Convexity: T_c is a convex set.
- $\dot{A}3$. Constant returns to scale: For all $(x_j, y_j) \in T_c$: $(\alpha x_j, \alpha y_j) \in T_c, \alpha \in R$.
- $\dot{A}4$. Free disposability: If $(x, y) \in T_c, y \geq \dot{y} \geq 0$ and $x \leq \dot{x}$ then $(\dot{x}, \dot{y}) \in T_c$.
- $\dot{A}5$. Closeness: T_c is a closed set.
- $\dot{A}6$. Minimum extrapolation: T_c is the minimal set that satisfies axioms $\dot{A}1$ – $\dot{A}5$.

Regarding the axioms $\dot{A}1$ – $\dot{A}6$, the associated production possibility set (PPS) could be stated as follows.

The most general way to characterize production technology is the production possibility set T , which is defined with a set of semi-positive (x, y) as:

$$T = \{(x, y) | x \geq \sum_{j=1}^n \lambda_j x_j, y \leq \sum_{j=1}^n \lambda_j y_j, \lambda_j \geq 0, j = 1, \dots, n\}$$

The constant returns to scale form of the enveloping problem, which was first introduced by [32], is as follows:

$$\begin{aligned} \min \quad & \theta \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq \theta x_{io}, \quad i = 1, \dots, m, \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro}, \quad r = 1, \dots, s, \\ & \lambda_j \geq 0, \quad j = 1, \dots, n, \end{aligned} \quad (1)$$

The dual of model (1) is to solve model (2) with variables $u \in R^s$ and $v \in R^m$ such that:

$$\begin{aligned} \max \quad & \sum_{r=1}^s u_r y_{ro} \\ \text{s.t.} \quad & \sum_{i=1}^m v_i x_{io} = 1, \\ & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0, \quad j = 1, \dots, n, \\ & u_r \geq 0, v_i \geq 0, \quad r = 1, \dots, s, i = 1, \dots, m. \end{aligned} \quad (2)$$

According to the duality theorem in linear programming [33], the optimal objective values of the primal and dual models are equal whenever a finite optimal solution exists $\sum_{r=1}^s u_r^* y_{ro} = \theta^*$. This optimal value represents the relative efficiency score of the evaluated decision-making unit $DMU_o, o \in \{1, \dots, n\}$.

2.2. Undesirable Outputs in DEA

Färe and Grosskopf were among the first researchers to formally introduce the concepts of null-joint outputs and weak disposability of outputs in DEA models [11]. Suppose there are n decision-making units (DMUs) under evaluation. Let (x, y, v) be the vector consisting of m inputs, s desirable outputs, and l vectors of undesirable outputs.

Definition 1. The desirable output vector ' y ' is Null-Joint with the undesirable outputs ' v ' if $(y, v) \in P(x)$, and $v = 0$ then $y = 0$.

where $P(x)$ denotes the output possibility set. This implies that the generation of desirable outputs inherently and unavoidably leads to the co-production of undesirable ones. Stated differently, under the assumption of null jointness, it is deduced that if a combined output

vector (y, v) is technologically feasible and the undesirable components are entirely absent, then the production of desirable outputs must strictly be zero. Regarding the disposability characteristics, Färe and Grosskopf introduced two distinct assumptions to model output disposability:

Definition 2. *Weak disposability of output:* $(y, v) \in P(x)$ and $0 \leq \theta \leq 1$ imply $(\theta y, \theta v) \in P(x)$.

This formulation is rooted in the foundational concept of weak disposability within non-parametric activity analysis frameworks, as originally established by Shephard [34]. Within a production system that yields both good and bad outputs—where eliminating the bad ones incurs a penalty or cost—Definition (2) signifies that for a fixed level of inputs x , a reduction in undesirable outputs can only be achieved if there is a corresponding decrease in desirable outputs.

Definition 3. *Strong (free) disposability of output:* $(\bar{y}, \bar{v}) \in P(x)$ and $(y, v) \leq (\bar{y}, \bar{v})$ imply $(y, v) \in P(x)$.

Based on this conceptualization, any output that is unfavorable to the production system can be eliminated, albeit with certain operational consequences. In light of these established definitions, it is evident that strong disposability inherently encompasses weak disposability, whereas the reverse implication is not automatically valid.

3. An Axiomatic Approach for Dealing with Dependent Desirable and Undesirable Outputs

In many production processes, undesirable outputs are generated as secondary by-products alongside desirable outputs. For example, consider electricity generation in a fossil-fuel power plant, where electricity is the desirable output while carbon dioxide (CO₂) emissions represent an undesirable output. In such systems, the amount of CO₂ emissions is directly related to the level of electricity production, indicating a dependency between desirable and undesirable outputs. Although reducing emissions is desirable, such reductions are often technologically linked to reductions in electricity generation.

Another example can be found in service systems, where the number of customer complaints represents an undesirable output while service quality represents a desirable output. In such cases, these outputs are interrelated, meaning that improvements in service quality may lead to reductions in complaints. Therefore, many real-world production systems generate both desirable and undesirable outputs simultaneously.

Several DEA models have attempted to incorporate undesirable outputs into efficiency analysis. For example, Refs. [12,14] proposed models that consider the dependency between undesirable outputs and input consumption. However, these models primarily consider the relationship between undesirable outputs and input consumption rather than the dependency between desirable and undesirable outputs themselves. In contrast, this study explicitly models the technological dependency between desirable and undesirable outputs within the DEA framework. Ignoring this dependency may lead to unrealistic production technologies and inaccurate efficiency evaluations. To address this issue, an axiomatic DEA framework is developed in which technological dependencies between desirable and undesirable outputs are explicitly defined.

Consider a set of n decision-making units (DMUs), each using m inputs to produce s outputs.

The output vector is divided into two subsets: desirable outputs and undesirable outputs. Let the vector of output be (y, v) to show the desirable and undesirable outputs. Consider DMU₀ denoting a unit from a total of n units whose relative efficiency is under

assessment. T_c denotes the production possibility set (PPS). Now, consider the basic axioms of DEA as follows:

B1. Feasibility: For all j : $(x_j, y_j, v_j) \in T_c$.

B2. Convexity: T_c is a convex set.

B3. Variable returns to scale: $\sum_{j=1}^n \lambda_j = 1$.

B4. Disposability of inputs: If $x \in T_c$ and $x \leq \hat{x}$ then $\hat{x} \in T_c$.

B5. Disposability of undesirable outputs: If $(y, v) \in T_c$, $v \leq v'$ then $(y, v') \in T_c$.

B6. Disposability of desirable outputs: If $(y, v) \in T_c$, $y \geq y'$ and $F(v, y')$ then $(y', v) \in T_c$.

B7. Closedness: T_c is a closed set.

B8. Minimum extrapolation: T_c is the minimal set that satisfies axioms **B1–B7**.

The proposed model is formulated under variable returns to scale (VRS), since the DMUs under evaluation differ considerably in size and operational scale, making the constant-returns-to-scale assumption unrealistic in this context.

Three kinds of disposability are defined: disposability of inputs, desirable outputs, and undesirable outputs. By **B4**, the disposability of inputs is considered; larger amounts of inputs belong to the PPS. By **B5**, the disposability of undesirable outputs is considered, with the larger quantity of undesirable outputs belonging to the corresponding PPS. The disposability of desirable outputs has two features that are closely related. The less desirable outputs that satisfy the defined function according to $F(v, y')$ can belong to the corresponding PPS. Thus, the production possibility set T , which is defined with a set of semi-positive (x, y, v) , is as follows:

$$\begin{aligned} T_c &= \{(x, y, v) | x \geq \sum_{j=1}^n \lambda_j x_j, \quad (a) \\ v &\geq \sum_{j=1}^n \lambda_j v_j, \quad (b) \\ y &\leq \sum_{j=1}^n \lambda_j y_j, \quad (c) \\ F(v, y), \quad (d) \\ \lambda_j &\geq 0, j = 1, \dots, n \} \quad (e) \end{aligned} \quad (3)$$

According to the defined T_c constraint, Equation (3a) shows the disposability of input. Equations (3b) and (3c) show the disposability of undesirable and desirable outputs, respectively. Equation (3d) represents the dependency relationship between desirable and undesirable outputs. Equation (3e) ensures the non-negativity of intensity variables.

According to the examples mentioned above, we can consider two types of dependencies between desirable and undesirable outputs. The first is direct dependency, for example, of CO₂ and electricity as the undesirable and desirable outputs. The second is indirect dependency, for example, of complaints and quality as the undesirable and desirable outputs.

Crucially, the essential mathematical distinction between these two scenarios lies in the direction of their operational trade-offs. In a direct dependency, the undesirable output is a strict technological co-product of the desirable output, meaning that reducing the undesirable factor inevitably requires a reduction in the desirable one. Conversely, in an indirect dependency, the relationship is inverse or driven by a shared operational bottleneck; hence, an efficiency improvement allows the decision maker to simultaneously decrease the undesirable output while increasing the desirable performance indicator.

According to the axioms mentioned above, for the cases of direct dependency and indirect dependency, different types of function F should be considered.

In the following, two different relations are considered for the description of the dependency between the desirable and undesirable outputs. The first relation describes a direct dependency, and the second is an indirect dependency.

Figure 1 illustrates the case of direct dependency between desirable and undesirable outputs. The figure shows that a reduction in undesirable outputs must be accompanied by a simultaneous reduction in desirable outputs due to technological dependency. Moving from point A to point B represents a feasible adjustment where both outputs are reduced proportionally, while moving from point A to point C is infeasible because undesirable outputs cannot be reduced independently of desirable outputs.

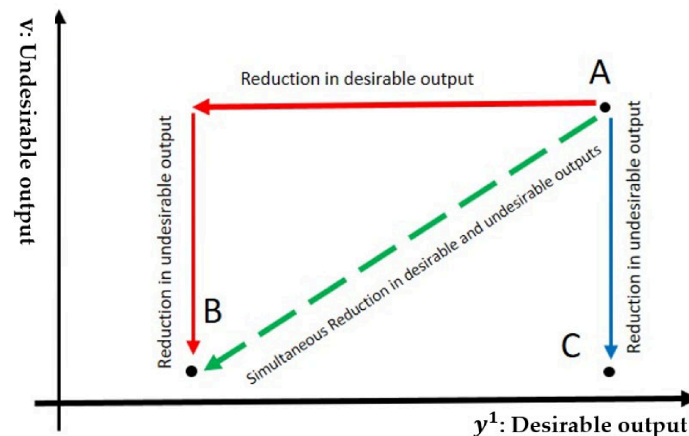


Figure 1. Illustration of direct dependency between desirable and undesirable outputs.

Let v and y^1 be dependent undesirable and desirable outputs, respectively, which have a direct dependency. Thus, for any reduction in undesirable outputs, desirable outputs should also be decreased. This is shown with a green dotted vector in Figure 1. We do not want to consider reductions in undesirable output alone, since undesirable outputs are by-products of desirable outputs. Thus, reaching point C is impractical, as shown with the blue vector. To reduce the production of undesirable outputs, the production of desirable outputs should be reduced as well. This is shown in Figure 1 by the green dotted line, with which the production level decreased from point A to reach point B. The simultaneous production reductions in desirable and undesirable outputs are decomposed in red vectors, as shown in Figure 1.

Figure 2 illustrates an indirect dependency between desirable and undesirable outputs. In this scenario, a reduction in undesirable outputs is associated with an increase in desirable outputs. The movement from point A to point D represents a feasible improvement path where undesirable outputs decrease while desirable outputs increase. The movement from point A to point B represents an infeasible scenario where undesirable outputs are reduced without adjusting desirable outputs.

Let v and y^1 be dependent undesirable and desirable outputs, respectively, which have an indirect dependency. Thus, for any reduction in undesirable outputs, desirable outputs should be increased. This is shown with a green dotted vector in Figure 2. We do not want to consider reductions in undesirable output alone. Since undesirable outputs are by-products of desirable outputs. Thus, reaching point B by considering only the reduction of undesirable output, as shown with the blue vector, is impractical. For any decrease in the production of undesirable outputs, production of desirable outputs should be increased as well. This is shown in Figure 2 by the green dotted line, with which the production level of undesirable output is decreased, and at the same time, the production level of desirable output is increased. This means moving from point A to reach point D. The simultaneous production reductions in undesirable output and the increase in undesirable outputs are decomposed in red vectors, as shown in Figure 2.

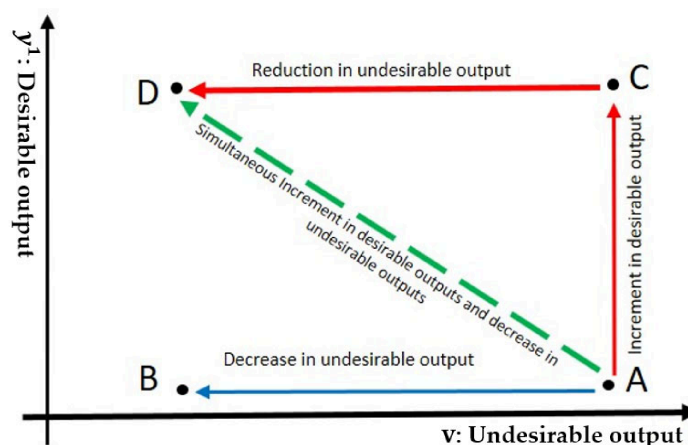


Figure 2. Illustration of indirect dependency between desirable and undesirable outputs.

To define the relation between desirable and undesirable outputs, function F should be defined by decision makers or managers who have in-depth information about the details of relations in an activity or a production process. The only assumption that decision makers should be aware of is that for the simultaneous changes in desirable and undesirable outputs, an increasing function should be considered. Also, considering separate changes in desirable and undesirable outputs and occasions where undesirable outputs are decreased and desirable outputs are increased, decision makers should be aware that for the separate changes in desirable and undesirable outputs, a decreasing function should be considered. Decision makers may define the functional form of F_c based on technological knowledge of the production process. However, nonlinear functions may increase the complexity of the optimization model.

The provided DEA model for performance evaluation is as follows. In model (4), assume v and y^1 as the dependent undesirable and desirable outputs, respectively, for both simultaneous and separate changes. Consider $d = (d_i^x, d_p^v, d_t^{y^1}, d_r^{y^2})$ as a direction used for performing changes in the inputs and outputs. This vector is defined by decision makers, experts, and managers. Let $\delta, \gamma, \alpha, \sigma$ be positive variables for performing changes in the inputs and outputs. Note that the function “ F ” should be an increasing function.

Thus, the innovative model is as follows:

$$\begin{aligned} \max \quad & \sum_{i=1}^m \alpha_i + \sum_{p=1}^k \delta_p + \sum_{t=1}^l \gamma_t + \sum_{r=1}^s \sigma_r \\ \text{s.t.} \quad & (x_{io} - \alpha_i d_i^x, v_{po} - \delta_p d_p^v, y_{to} + \gamma_t d_t^{y^1}, y_{ro} + \sigma_r d_r^{y^2}) \leq F(y^1) \in T_c \end{aligned} \quad (4)$$

Now, according to the definition T_c as mentioned above, we can formulate the model as follows:

$$\begin{aligned} \max \quad & \rho^1 = \sum_{i=1}^m \alpha_i + \sum_{p=1}^k \delta_p + \sum_{t=1}^l \gamma_t + \sum_{r=1}^s \sigma_r \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} - \alpha_i d_i^x, & i = 1, \dots, m, & (a) \\ & \sum_{j=1}^n \lambda_j v_{pj} \leq v_{po} - \delta_p d_p^v, & p = 1, \dots, k, & (b) \\ & \sum_{j=1}^n \lambda_j y_{tj} \geq y_{to} + \gamma_t d_t^{y^1}, & t = 1, \dots, l, & (c) \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} + \sigma_r d_r^{y^2}, & r = 1, \dots, s, & (d) \\ & F(\sum_{p=1}^k \sum_{j=1}^n \lambda_j v_{pj}, \sum_{r=1}^s \sum_{j=1}^n \lambda_j y_{rj}) = 0, & (e) \\ & \sum_{j=1}^n \lambda_j = 1, & (f) \\ & \alpha_i \geq 0, \delta_p \geq 0, \gamma_t \geq 0, \sigma_r \geq 0 & \forall i, p, t, r, & (g) \\ & \lambda_j \geq 0, & j = 1, \dots, n, & (h) \end{aligned} \quad (5)$$

Consider model (5); the aim is to maximize the sum of decision variables. In constraints (a), (b), (c), and (d), non-radial changes for the inputs, desirable independent outputs, and dependent desirable and undesirable outputs are considered. The vector of direction $d = (d_i^x, d_p^y, d_t^{y^1}, d_r^y)$ is defined by the decision makers and managers. In case of $d = (x_{io}, v_{po}, y_{to}, y_{ro})$, d is a vector of natural direction. With the help of positive variables, $\alpha, \delta, \gamma, \sigma$ inputs, dependent undesirable and desirable outputs, and independent desirable outputs are changed until reaching the efficient frontier.

It is known that the dependent undesirable outputs (v) have a relation to the desirable outputs (y^1). Therefore, the production of undesirable output (v) can be considered as a function of the production of desirable output y^1 . Let us assume $F(v, y^1)$. It is rational that any changes in the production of desirable output y^1 affect the production of undesirable output (v).

In practical applications, the functional form of F in constraint (e) is determined by the decision maker (DM) based on the technical characteristics and physical structure of the process under evaluation. Relying on engineering standards or empirical data, the DM configures the behavior of this function to be increasing for simultaneous changes in outputs and decreasing for isolated changes. Although this function can generally possess a nonlinear structure, linear approximations are practically preferred to maintain computational simplicity and the linear programming nature of the model. As a concrete example, in sensor monitoring systems, the desirable output is 'network monitoring quality' (y) and the undesirable output is 'monitoring error rate' (v). In this scenario, the DM can define the function F as a direct linear relationship of the form $v - \alpha y = 0$, where α represents the maximum allowable error rate as a fraction of the monitoring quality. This linear formulation not only reflects the realism of the monitoring system but also guarantees the computational efficiency of the mathematical programming model.

To be clearer in the analysis of model (5), when dealing with dependent desirable and undesirable outputs, a feasible scenario can be considered by a decision maker (DM). According to the scenarios mentioned, the following cases can happen after solving model (4). Consider the following theorems to better explain these scenarios. Consider model (4) and let y^1 be an undesirable output that is a by-product of the desirable output y .

Considering the natural direction, we have model (6) as follows.

$$\begin{aligned}
 \max \quad & \rho^2 = \frac{1}{4} \left(\frac{1}{m} \sum_{i=1}^m \alpha_i + \frac{1}{k} \sum_{p=1}^k \delta_p + \frac{1}{l} \sum_{t=1}^l \gamma_t + \frac{1}{s} \sum_{r=1}^s \sigma_r \right) \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} - \alpha_i x_{io}, & i = 1, \dots, m, & (a) \\
 & \sum_{j=1}^n \lambda_j v_{pj} \leq v_{po} - \delta_p v_{po}, & p = 1, \dots, k, & (b) \\
 & \sum_{j=1}^n \lambda_j y_{tj} \geq y_{to} + \gamma_t y_{to}, & t = 1, \dots, l, & (c) \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} + \sigma_r y_{ro}, & r = 1, \dots, s, & (d) \\
 & F\left(\sum_{p=1}^k \sum_{j=1}^n \lambda_j v_{pj}, \sum_{r=1}^s \sum_{j=1}^n \lambda_j y_{rj}\right) = 0, & & (e) \\
 & \sum_{j=1}^n \lambda_j = 1, & & (f) \\
 & \alpha_i \geq 0, \delta_p \geq 0, \gamma_t \geq 0, \sigma_r \geq 0 & \forall i, p, t, r, & (g) \\
 & \lambda_j \geq 0, & j = 1, \dots, n, & (h)
 \end{aligned} \tag{6}$$

Intuitively, the structure of models (4) through (6) simulates real-world systemic trade-offs, contrasting with classical DEA models that treat the reduction of undesirable outputs and the expansion of desirable outputs as entirely separate, independent processes. Under this framework, since the undesirable output is an inseparable by-product of the desirable performance, the model ensures that projecting inefficient units toward the efficient frontier accounts for the joint, simultaneous interactions of both outputs. This approach allows decision makers to establish performance targets that are technologically realistic and operationally achievable.

Theorem 1. It is proved that: $\alpha_i^* \leq 1, \delta_p^* \leq 1, \gamma_r^* \leq 1$, and $\sigma_r^* \leq 1$.

Proof. According to the constraints of model (6), we have:

$$\begin{aligned} x_{io} - \alpha_i^* x_{io} &\geq 0, \forall i \\ v_{po} - \delta_p^* v_{po} &\geq 0, \forall p \\ y_{to} + \gamma_t^* y_{to} &\geq 0, \forall t \\ y_{ro} + \sigma_r^* y_{ro} &\geq 0, \forall r \end{aligned} \quad (7)$$

Thus, it is rational that:

$$\begin{aligned} x_{io} &\geq \alpha_i^* x_{io}, \rightarrow \alpha_i^* \leq 1 \forall i \\ v_{po} &\geq \delta_p^* v_{po}, \rightarrow \delta_p^* \leq 1 \forall p \\ y_{to} &\geq \gamma_t^* y_{to}, \rightarrow \gamma_t^* \leq 1 \forall t \\ y_{ro} &\geq \sigma_r^* y_{ro}, \rightarrow \sigma_r^* \leq 1 \forall r \end{aligned} \quad (8)$$

□

Theorem 2. It is proved that: $\rho^{*2} \leq 1$.

Proof. According to Theorem 1, in any optimal solution we have:

$$\alpha_i^* \leq 1, \forall i; \delta_p^* \leq 1, \forall p; \gamma_t^* \leq 1, \forall t; \sigma_r^* \leq 1, \forall r \quad (9)$$

Therefore:

$$\begin{aligned} \sum_{i=1}^m \alpha_i^* &\leq m \rightarrow \frac{1}{m} \sum_{i=1}^m \alpha_i^* \leq 1 \\ \sum_{p=1}^k \delta_p^* &\leq k \rightarrow \frac{1}{k} \sum_{p=1}^k \delta_p^* \leq 1 \\ \sum_{t=1}^l \gamma_t^* &\leq l \rightarrow \frac{1}{l} \sum_{t=1}^l \gamma_t^* \leq 1 \\ \sum_{r=1}^s \sigma_r^* &\leq s \rightarrow \frac{1}{s} \sum_{r=1}^s \sigma_r^* \leq 1 \end{aligned} \quad (10)$$

Now, considering the objective function of model (5), it is clear that:

$$\begin{aligned} \left(\frac{1}{m} \sum_{i=1}^m \alpha_i^* + \frac{1}{k} \sum_{p=1}^k \delta_p^* + \frac{1}{l} \sum_{t=1}^l \gamma_t^* + \frac{1}{s} \sum_{r=1}^s \sigma_r^* \right) &\leq 4 \\ \frac{1}{4} \left(\frac{1}{m} \sum_{i=1}^m \alpha_i^* + \frac{1}{k} \sum_{p=1}^k \delta_p^* + \frac{1}{l} \sum_{t=1}^l \gamma_t^* + \frac{1}{s} \sum_{r=1}^s \sigma_r^* \right) &\leq 1 \end{aligned} \quad (11)$$

That is $\rho^{*2} \leq 1$. □

Theorem 3. All constraints of model (6) are binding at any optimal solution.

Proof. By contradiction, suppose that all constraints of model (6) are not binding at an optimal solution. So, there exists at least one index for which the corresponding constraint holds an inequality. In this case, there exists a value for which the constraint holds in equality. Thus, the corresponding objective function is better than that of the optimal objective function. This contradicts optimality. Therefore, all constraints of model (6) must be binding at any optimal solution. □

Theorem 4. It is proved that there exists at least one $i \in \{1, \dots, m\}$ in constraint $\sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} - \alpha_i^* x_{io}$, $i = 1, \dots, m$ for which $\alpha_i^* \neq 1$.

Proof. According to Theorem 3, all the constraints of model (6) are binding at any optimal solution. Therefore, we have $\sum_{j=1}^n \lambda_j x_{ij} = x_{io} - \alpha_i^* x_{io}$ for all i . By contradiction, suppose

that there exists an index $i \in \{1, \dots, m\}$ for which $\alpha_i^* = 1$; thus $\sum_{j=1}^n \lambda_j x_{ij} = 0$ for all i . The input vector is semi-positive; therefore, it implies that $\sum_{j=1}^n \lambda_j = 0$. This means $\lambda = (\lambda_1, \dots, \lambda_n) = (0, \dots, 0)$, which contradicts $\sum_{j=1}^n \lambda_j = 1$. $\square$

Definition 4. According to the optimal solution of model (6), consider the following index for obtaining the relative efficiency score for DMUo under evaluation.

$$\beta^* = \frac{1 - \rho^{*2}}{1 + \rho^{*2}} \quad (12)$$

Theorem 5. It is proved that $0 < \beta^* \leq 1$.

Proof. As stated in Theorem 3, there exist at least one input index for which $\alpha_i^* \neq 1$; therefore, $\rho^{*2} < 1$; therefore, $1 - \rho^{*2} > 0$. Of course, $1 + \rho^{*2} \geq 1$; thus, $\beta^* > 0$. In the case that $\rho^{*2} = 0$, $\beta^* = 1$. Therefore $0 < \beta^* \leq 1$. $\square$

Definition 5. According to relation (11), DMUo is efficient if $\beta^* = 1$ and is inefficient if $\beta^* < 1$.

In case of non-discretionary inputs, we formulate model (6) as follows:

$$\begin{aligned} \max \quad & \rho^2 = \frac{1}{4} \left(\frac{1}{m} \sum_{i=1}^m \alpha_i + \frac{1}{k} \sum_{p=1}^k \delta_p + \frac{1}{l} \sum_{t=1}^l \gamma_t + \frac{1}{s} \sum_{r=1}^s \sigma_r \right) \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io}, & i = 1, & (a) \\ & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} - \alpha_i x_{io}, & i = 2, \dots, m & \\ & \sum_{j=1}^n \lambda_j v_{pj} \leq v_{po} - \delta_p v_{po}, & p = 1, \dots, k, & (b) \\ & \sum_{j=1}^n \lambda_j y_{tj} \geq y_{to} + \gamma_t y_{to}, & t = 1, \dots, l & (c) \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} + \sigma_r y_{ro}, & r = 1, \dots, s, & (d) \\ & F\left(\sum_{p=1}^k \sum_{j=1}^n \lambda_j v_{pj}, \sum_{r=1}^s \sum_{j=1}^n \lambda_j y_{rj}\right) = 0, & (e) \\ & \sum_{j=1}^n \lambda_j = 1, & (f) \\ & \alpha_i \geq 0, \delta_p \geq 0, \gamma_t \geq 0, \sigma_r \geq 0 & \forall i, p, t, r, & (g) \\ & \lambda_j \geq 0, & j = 1, \dots, n, & (h) \end{aligned}$$

4. Case Study

To illustrate the applicability of the proposed DEA model, a case study involving sensor monitoring systems is considered. In this study, five sensor monitoring systems are considered as decision-making units (DMUs). Each monitoring system utilizes three types of resources, which are considered as input variables in the DEA model.

The input variables include:

- Sensor deployment cost (i_1), representing the financial investment required to install and maintain the sensing infrastructure.
- Energy consumption of the sensing system (i_2), reflecting the operational energy required for sensor operation and data transmission.
- Number of sensor nodes (i_3), representing the scale of the monitoring network.

Each monitoring system generates two outputs that characterize its operational performance. The first output, o_1 , represents the monitoring quality, which measures the accuracy and reliability of the sensor network in detecting environmental conditions. The second output, o_2 , represents the measurement error rate, which is considered an undesirable output because it reflects inaccuracies in sensor measurements. In practice, these outputs are technologically related. Improvements in monitoring quality typically lead to reductions in measurement errors, indicating a dependency between desirable and undesirable outputs.

Table 1 presents the input and output data for the five sensor monitoring systems analyzed in this study. To model the relationship between the desirable and undesirable outputs, the decision maker assumes that the level of measurement error should remain relatively small compared to the monitoring quality. Specifically, to represent the dependency between outputs, it is assumed that the undesirable output should remain relatively small compared with the desirable output. In this example, the undesirable output is assumed to be approximately 6% of the desirable output, reflecting a realistic balance between sensing accuracy and measurement error. In addition, the natural direction is considered for the directional vector in model (6), meaning that the components of vector d are set exactly equal to the observed input and output values of the specific DMU under evaluation ($d = (x_{io}, v_{po}, y_{to}, y_{ro})$). While the function F can theoretically assume a nonlinear form to capture complex operational environments, a linear approximation is deployed in this case study to maintain computational tractability and align with the decision maker's specified threshold.

Table 1. Input and output data for the five sensor monitoring systems considered as decision-making units (DMUs).

DMUs	Sensor Deployment Cost (i_1)	Energy Consumption of Sensor System (i_2)	Number of Sensor Nodes (i_3)	Quality of Monitoring (o_1)	Measurement Error Rate (o_2)
DMU1	30.00	1500	3000	123,451.00	11,469.00
DMU2	13.00	1800	3200	176,102.00	13,889.00
DMU3	80.00	2000	3500	291,710.00	12,922.00
DMU4	13.00	1600	3100	194,230.00	11,625.00
DMU5	46.00	1700	3300	102,361.00	11,912.00

Specifically, to represent the dependency relationship, the function F in constraint (e) of model (6) is explicitly defined as $\sum_{j=1}^n \lambda_j v_{pj} - 0.06 \sum_{j=1}^n \lambda_j y_{tj} = 0$, where the undesirable output is restricted to approximately 6% of the desirable output. The proposed linear programming models were coded and solved using the GAMS version 52 software environment utilizing the CPLEX solver to obtain the exact target and benchmark values.

Table 2 presents the efficiency results obtained after solving the proposed DEA model for the five DMUs. The value α^* indicates how much the input and desirable output should be adjusted in a radial way, while n^* and p^* represent the suggested decrease or increase in the undesirable output. According to the results, DMU3 and DMU4 operate on the efficient frontier because no adjustments in inputs or outputs are required. This means that these two units already lie on the efficient frontier according to the proposed model. The other units require some improvements to reach an efficient level of performance.

For DMU1, the model suggests increasing both desirable and undesirable outputs proportionally. This result reflects the dependency relationship between outputs and indicates that improvements in monitoring quality may be accompanied by increases in measurement error. For DMU2, the model suggests reducing the undesirable output while maintaining the current level of desirable output. Finally, DMU5 requires both a radial adjustment ($\alpha^* = 33$) and a reduction in the undesirable output ($n^* = 287$) in order to improve its performance. Overall, the results demonstrate that the proposed DEA model provides meaningful efficiency improvement recommendations while accounting for the dependency between desirable and undesirable outputs. This capability is particularly important

in sensor monitoring systems where performance improvements in sensing accuracy may be inherently linked to changes in measurement errors or other undesirable outputs.

Table 2. Efficiency improvement recommendations obtained from the proposed DEA model.

<i>DMUs</i>	Radial Changes for Input and Desirable Output (α^*)	Reduction in Undesirable Output (n^*)	The Number of Increments for Undesirable Output (p^*)
DMU1	17.00	0.00	156.00
DMU2	0.00	264.00	0.00
DMU3	0.00	0.00	0.00
DMU4	0.00	0.00	0.00
DMU5	33.00	287.00	0.00

T_D (Desirable Output Target): Represents the optimal projected value of the desirable output determined by the mathematical model for each unit to achieve the efficient frontier.

T_{UD} (Undesirable Output Target): Represents the revised, optimal level of the undesirable output (either a controlled decrease or an adjusted increase based on the direct dependency relationship) required to reach the efficient status.

$\frac{T_D - D}{D}$ (Desirable Output Alteration Ratio): Quantifies the percentage of adjustment or growth required from the current desirable output level (D) to reach its designated target (T_D).

$\frac{T_{UD} - UD}{UD}$ (Undesirable Output Alteration Ratio): Quantifies the percentage of reduction or adjustment required from the current undesirable output level (UD) to align with its designated target (T_{UD}).

The detailed analysis presented in Section 4 strictly builds upon comparing these optimal targets (T_D and T_{UD}) with the baseline values (D and UD) to reveal the precise, step-by-step improvement path for each decision-making unit (such as analyzing the final $\frac{T_{UD}}{T_D}$ ratios to verify optimized error tolerances).

According to Table 3, DMU_3 and DMU_4 are assessed to be performed efficiently. No changes in inputs and outputs are suggested for these two DMUs. Now consider DMU_1 . The ratio of the “current undesirable output” to the “current desirable output” is 0.09. This means that, according to the produced amounts of the desirable output, the production of undesirable output is less than the defined level (0.07), which is very satisfactory for the performance of DMU_1 . Also, the ratio of “amount of desirable output’s alterations” to its current level of “amount of desirable output”, i.e., $\frac{T_D - D}{D}$, is 57%. Also, the ratio of “amount of undesirable output’s alterations” to its current level of “amount of undesirable output”, i.e., $\frac{T_{UD} - UD}{UD}$, is 13%. Thus, according to the obtained optimal solutions in Table 3, DMU_1 is suggested to increase its undesirable and desirable outputs. Note that the suggested increase in desirable outputs is greater than that in undesirable ones. By considering increments only for the undesirable output, the corresponding ratio, i.e., $\frac{T_{UD}}{D}$, will be 0.09, which is not satisfactory. Moreover, mere increases in desirable output will turn the corresponding ratio, i.e., $\frac{UD}{T_D}$, to 0.09, which is also not satisfactory. Thus, implementing both alterations in desirable and undesirable outputs will guide DMU_1 to reach the efficient status. After applying the optimal solutions, the ratio of “increased undesirable output” to “increased desirable output”, i.e., $\frac{T_{UD}}{T_D}$, is 0.06; that is evidently suitable. Now consider DMU_2 ; for this unit, the undesirable output is suggested to be decreased, and the desirable output is suggested to be increased. Note that the ratio of the current “undesirable output” to the current “desirable output” equals 0.08, which does not satisfy the constraint (e). Consider the ratio of “reduced undesirable outputs” to the current level of “undesirable outputs”; this scenario cannot be satisfying, since the ratio, i.e., $\frac{T_{UD}}{D}$, is equal to 0.08. The

ratio of “increased desirable output” to the current level of “desirable output”, i.e., $\frac{T_{UD}}{D}$, is equal to 0.071. Thus, both desirable and undesirable output must be adjusted for the DMU to reach an efficient status. Note that the ratio of “reduced undesirable outputs” to “increased desirable output”, i.e., $\frac{T_{UD}}{T_D}$, is 0.07. Please consider that, in this case, the ratio of the “amount of desirable output’s alterations” to its current level of “desirable output”, i.e., $\frac{T_D-D}{D}$, is 1%, and the “amount of undesirable output’s alterations” to the current level of “undesirable output”, i.e., $\frac{T_{UD}-UD}{UD}$, is −0.15%. In Table 3, T_D shows the introduced benchmark for desirable output, and T_{UD} shows the introduced benchmark for undesirable output.

Table 3. Detailed analysis of efficiency results and benchmark values obtained from the proposed DEA model.

DMUs	T_D	T_{UD}	$\frac{UD}{D}$	$\frac{T_{UD}}{T_D}$	$\frac{T_{UD}}{D}$	$\frac{UD}{T_D}$	$\frac{T_D-D}{D}$	$\frac{T_{UD}-UD}{UD}$
DMU1	194,213.00	11,625.00	0.09	0.06	0.09	0.09	0.57	0.0013
DMU2	194,230.00	11,625.00	0.08	0.07	0.08	0.071	0.10	−0.0015
DMU3	291,710.00	12,922.00	0.04	0.04	0.04	0.04	0.00	0.0000
DMU4	194,230.00	11,625.00	0.06	0.06	0.06	0.06	0.00	0.0000
DMU5	194,197.00	11,625.00	0.12	0.06	0.11	0.10	0.90	−0.0028

Crucially, it should be emphasized that the primary contribution of this study lies in its methodological development and the formulation of the novel Network DEA framework. Consequently, the empirical application presented in this section serves as a rigorous numerical and practical illustration to demonstrate the mathematical feasibility, tractability, and step-by-step implementation of the proposed model within a real-world sensor context. While this focused case study perfectly satisfies the illustrative objectives of the paper, we acknowledge that expanding the sample size in future empirical extensions would further enhance the discriminatory power and broader generalizability of the insights, strictly aligning with classical DEA sample size heuristics ($n \geq \max\{m \times s, 3(m + s)\}$).

5. Conclusions

In many production systems, the objective is to minimize input consumption while maximizing the generation of desirable outputs. However, many production processes also generate undesirable outputs such as pollution, waste, or measurement errors as unavoidable by-products of desirable outputs. Since the generation of undesirable outputs is often unavoidable, it is important to incorporate them into efficiency evaluation models rather than ignoring them.

Data Envelopment Analysis (DEA) has become a widely used mathematical optimization tool for evaluating the efficiency of production systems with multiple inputs and outputs. In many real-world systems, both desirable and undesirable outputs are produced simultaneously, and the relationship between them may significantly affect efficiency assessment.

This study addressed this issue by proposing a new DEA framework that explicitly considers the dependency between desirable and undesirable outputs. Within an axiomatic DEA framework, the traditional output disposability axiom was decomposed into separate axioms for desirable and undesirable outputs. Based on these axioms, a DEA optimization model was developed to evaluate the efficiency of decision-making units when desirable and undesirable outputs are technologically dependent.

Unlike conventional DEA frameworks that overlook the technological dependencies between variables or mischaracterize the operational nature of undesirable outputs, the model developed in this study successfully bridges the gap between theoretical axioms and practical system behavior. Through the newly introduced PPS, the model serves as an effective decision-making tool in empirical contexts, allowing for a realistic assessment of efficiency and offering clear managerial insights for waste reduction and process optimization without compromising desirable outputs.

The proposed model is particularly relevant for sensor monitoring systems, where improvements in sensing quality may influence measurement errors or other undesirable outputs. The case study demonstrated how the proposed DEA model can provide meaningful efficiency improvement recommendations for such systems.

Future research may extend the proposed model by incorporating stochastic factors, dynamic efficiency evaluation, or larger real-world datasets from sensor networks and Internet-of-Things (IoT) monitoring systems. In addition, integrating machine learning techniques with DEA models could further improve efficiency evaluation and decision-making in complex sensing environments.

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