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Non-Conventional Thermodynamics, Cattaneo's Heat Conduction Law, Thermo-Diffusion Coupling and Variational Formulations

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Abstract

Conventional continuum thermodynamics is characterized by a classical form of the energy law and the second law of thermodynamics in the form of the Clausius–Duhem inequality. This thermodynamic framework fails to capture certain features in material response as, e.g., length scale effects, temperature waves in rigid heat conductors and diffusion phenomena. Typically, such material characteristics are associated with pronounced non-localities in time and space. To tackle these issues, non-conventional thermodynamic approaches might be appropriate. The employment of non-conventional thermodynamics is very attractive, as it enables the extension of the applicability of conventional thermodynamics in a simple way. A specific non-conventional thermodynamic framework has previously been proposed as a generalization of irreversible thermodynamics. Energy supply effects were neglected in this work. However, energy supply terms may become important when discussing thermodynamical consistency of many physical models. The present paper extends the applicability of the proposed non-conventional thermodynamic framework by accounting for energy supply densities and demonstrates its capabilities by addressing Cattaneo's heat conduction law—known for predicting temperature waves—and thermo-diffusion coupling theories. It is shown that, within the adopted thermodynamics, the considered physical models are thermodynamically consistent and that the resulting field theories admit formulations within a variational framework for rate problems; in this sense, the models are properly formulated.

Keywords: non-conventional thermodynamics; Cattaneo's heat conduction law; thermo-diffusion coupling; compressible viscous fluid; thermoelasticity



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1. Introduction

Following Dunn and Serrin [1], p. 96, we refer to classical (or conventional or standard) continuum thermodynamics as a thermodynamic framework consisting of the usual balance equations for mass, momentum, angular momentum and energy, and the Clausius–Duhem inequality. Evidently, ways to arrive at a non-conventional thermodynamics are to modify the classical energy law and/or the classical form of the second law. Mindlin's gradient elasticity (see Mindlin [2]) may be proved thermodynamically consistent in the context of the Clausius–Duhem inequality, but by imposing non-classical boundary conditions. This implies non-classical stress power and, consequently, non-classical energy law. In order

to achieve thermodynamical consistency of the Korteweg fluid, Dunn and Serrin [1] (cf. also Dunn [3]), following an idea of Toupin [4], modified the energy balance equation by assuming the existence of an energy flux vector (interstitial work flux) in addition to the classical heat flux vector. That means that they postulated a non-conventional energy law, but the second law was assumed in the form of the classical Clausius–Duhem inequality. The second law can be modified, while keeping the classical form of the energy law, by amending the entropy flux vector or the entropy supply or both. The first possibility has been proposed by Müller [5] (see also Giorgi and Morro [6] and the references cited there), the second by Iremán and Nguyen [7], and second laws where the absolute temperature is replaced by a function of the absolute temperature might be interpreted to correspond to the third possibility (see, e.g., Green and Laws [8]).

Alber et al. [9] (see also Broese et al. [10]) introduced a non-conventional thermodynamic framework, inspired by the concept of local state of the irreversible thermodynamics. It is known that the classical thermodynamics of homogeneous (uniform) systems in equilibrium states, often called thermostatics, provides the motivation for the thermodynamics of irreversible processes. Absolute temperature and entropy are well-defined quantities, and a Gibbs relation applies in thermostatics. Accordingly, the thermodynamics of irreversible processes assumes that each material point of the body can be locally considered as an isolated classical homogeneous system in equilibrium. Stated otherwise, to each material point, a local equilibrium state is assigned (local equilibrium state), so that the fundamental equations of thermostatics are valid. In particular, absolute temperature and entropy do exist, and a Gibbs relation applies. Generally, the second law is postulated as the production term in the entropy balance law, which is derived from the Gibbs relation for the entropy. This entropy production inequality can generally be different from the standard Clausius–Duhem inequality (see, e.g., the diffusion theory discussed in Section 4.1.1). This approach of the irreversible thermodynamics to non-equilibrium processes has also been applied to cases where internal variables are included in the state functions (cf. Maugin [11], Part 4, Germain et al. [12], and the references cited there). Some non-equilibrium effects in the material response, however, can be captured, e.g., by variables reflecting non-localities in space and/or non-localities in time (induced, e.g., by space and time derivatives of state variables). Conventional irreversible thermodynamics fails conceptually to include, e.g., spatial gradients of state variables as arguments of the energy response functions. To overcome such problems, Alber et al. [9] generalized, on the one hand, the energy balance law by accounting for a general energy flux vector, and, on the other hand, the local equilibrium state by considering equilibrium and non-equilibrium parts of the stress power and the energy flux vector. Different from the conventional irreversible thermodynamics, they did not consider an entropy production inequality as a consequence of the Gibbs relation. Rather, they postulated the second law as an inequality of the Clausius–Duhem type, but not identical to the standard Clausius–Duhem inequality.

Energy supply effects were neglected in Alber et al. [9]. Thus, the aim of the paper is to generalize the non-conventional thermodynamic theory of Alber et al. [9] by accounting additionally for specific energy supply densities, and to illustrate its abilities by discussing the heat conduction law of Cattaneo, which is the most widely known heat conduction model predicting temperature waves in a rigid heat conductor. It should be remarked that the presence of the time derivative of the heat flux vector in Cattaneo’s law causes a non-locality in time. The origin of such time effects is usually attributed to viscosity or/and inertia effects in the thermal material response. In the case of viscosity, the energy response function should depend on the variables modeling the heat conduction law. On the contrary, in the case of inertia, no additional storage term should be included in the energy response function. The thermodynamical consistency of Cattaneo’s law has been addressed by the

extended irreversible thermodynamics (see, e.g., Jou et al. [13], Section 2.3) and the rational thermodynamics (see, e.g., Coleman and Owen [14] and Sabri Öncü and Bryant Moodie [15]). In both cases, the energy response function is assumed to depend additionally on the heat flux vector, which indicates that the origin of the incorporated non-locality in time is attributed to viscosity effects. The case in which inertia effects cause the non-locality in time (no energy storage in the material) is addressed in the present paper within the non-conventional thermodynamic framework. A further objective of the paper is to apply the non-conventional thermodynamics to analyze the structure of thermo-diffusion coupling theories, among which prominent examples are classical diffusion theories (cf. Landau and Lifshitz [16], De Groot and Mazur [17] and Kondepudi and Prigogine [18]) and the diffusion theory of Gurtin et al. [19]. The considered constitutive models are regarded as properly formulated, as they are thermodynamically consistent, and, as shown in the paper, the resulting field theories admit variational formulations within a rate type framework. It is shown that classical diffusion theories within the conventional irreversible thermodynamic framework do not include the diffusion theory of Gurtin et al. as a particular case, nor does the latter include the former. However, both types of theories can be formulated as particular cases of the developed non-conventional thermodynamics.

2. Fundamental Assumptions and Relations

2.1. Notation—Preliminaries

Vectors and second-order tensors are denoted by boldface letters, and all tensorial components are referred to an orthonormal basis systems. Unless explicitly stated, all indices have the range of the integers $(1, 2, 3)$, while summation over repeated indices is implied. Let $\{\bar{\mathbf{e}}_i\}$ be a fixed orthonormal basis in the three-dimensional vector space that we deal with. We write, with respect to this basis, $\mathbf{v}_i \equiv (\mathbf{v})_i$, $\mathbf{w}_i \equiv (\mathbf{w})_i$, $A_{ij} \equiv (\mathbf{A})_{ij}$, $B_{ij} \equiv (\mathbf{B})_{ij}$ and $\mathbb{C}_{ijkl}$ for the components of the vectors $\mathbf{v}$, $\mathbf{w}$, the second-order tensors $\mathbf{A}$, $\mathbf{B}$ and the fourth-order tensor $\mathbb{C}$, respectively. Then, $\mathbf{v} \cdot \mathbf{w} = v_i w_i$ is the internal product between $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{A} \cdot \mathbf{B} = A_{ij} B_{ij}$ is the internal product between $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A}\mathbf{v} = A_{ij} v_j \bar{\mathbf{e}}_i$ is the product between $\mathbf{A}$ and $\mathbf{v}$, $\mathbf{v} \otimes \mathbf{w}$ is the tensorial product between $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{A}\mathbf{B} = A_{ij} B_{ik} \bar{\mathbf{e}}_j \otimes \bar{\mathbf{e}}_k$ is the product between $\mathbf{A}$ and $\mathbf{B}$, and $\mathbb{C}\mathbf{A} = \mathbb{C}_{ijkl} A_{kl} \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j$ is the product of $\mathbb{C}$ with $\mathbf{A}$. Further, we write $\mathbf{1} = \delta_{ij} \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j$ for the second-order identity tensor, where δ_{ij} is the Kronecker delta, and $\text{tr}\{\mathbf{A}\} = A_{ii}$, $\mathbf{A}^D = \mathbf{A} - \frac{1}{3}(\text{tr}\{\mathbf{A}\})\mathbf{1}$, $\mathbf{A}^T$, for the trace, the deviator and the transpose of $\mathbf{A}$, respectively. If the inverse $\mathbf{A}^{-1}$ of $\mathbf{A}$ exists, then we set $(\mathbf{A}^T)^{-1} = \mathbf{A}^{-T}$.

Denote time by t , and assume $\mathcal{R}_R$ and $\mathcal{R}_t$ to be the ranges in Euclidean point space occupied by a material body $\mathcal{B}$ under the reference and the actual (current) configuration, respectively. For simplicity, the reference and the initial configuration (at time $t = 0$) are supposed to coincide. We identify each particle of $\mathcal{B}$ with the position vectors $\mathbf{X}$, $\mathbf{x}$ in $\mathcal{R}_R$ and $\mathcal{R}_t$, respectively, and we set $\mathbf{x} = \chi(\mathbf{X}, t)$ for the motion of $\mathcal{B}$. Let $\{X_i\}$ and $\{x_i\}$ be Cartesian coordinate systems on $\mathcal{R}_R$ and $\mathcal{R}_t$, inducing coordinate bases $\{\mathbf{E}_i\}$ and $\{\mathbf{e}_i\}$, respectively. We set $\mathbf{X} = X_i \mathbf{E}_i$, $\mathbf{x} = x_i \mathbf{e}_i$, so that the deformation gradient tensor becomes $\mathbf{F}(\mathbf{X}, t) = \frac{\partial \chi}{\partial \mathbf{X}} = \text{GRAD } \chi(\mathbf{X}, t) = \frac{\partial \chi_i}{\partial X_j} \mathbf{e}_i \otimes \mathbf{E}_j$. The operator GRAD represents the gradient with respect to $\mathbf{X}$, while the gradient with respect to $\mathbf{x}$ is designated by grad. Similarly, DIV and div represent the divergence operators with respect to $\mathbf{X}$ and $\mathbf{x}$, respectively. Thus, for scalars $f(\mathbf{X}, t)$, vectors $\mathbf{v}(\mathbf{X}, t) = v_i \mathbf{E}_i$ and second-order tensors $\mathbf{A}(\mathbf{X}, t) = A_{ij} \mathbf{E}_i \otimes \mathbf{E}_j$, we have $\text{GRAD } f = \frac{\partial f}{\partial X_i} \mathbf{E}_i$, $\text{GRAD } \mathbf{v} = \frac{\partial v_i}{\partial X_j} \mathbf{E}_i \otimes \mathbf{E}_j$, $\text{DIV } \mathbf{v} = \frac{\partial v_i}{\partial X_i}$, $\text{DIV } \mathbf{A} = \frac{\partial A_{ij}}{\partial X_j} \mathbf{E}_i$ and

$$\text{grad } f = \mathbf{F}^{-T} \text{GRAD } f. \quad (1)$$

We also use the notation $\det \mathbf{F}$ for the determinant of $\mathbf{F}$ and $(\cdot)' = \frac{d}{dt}(\cdot)$ for the material time derivative of $(\cdot)$. Thus, for a field $f(\mathbf{x}, t)$, we have

$$\dot{f} = \frac{\partial f}{\partial t} + (\text{grad } f) \dot{\mathbf{x}}. \quad (2)$$

Consider the differential equation

$$\tau \dot{\mathbf{z}} + \mathbf{z} = k \dot{\mathbf{g}}, \quad (3)$$

with $\mathbf{g} = \mathbf{g}(t)$ and τ, k being constants. For homogeneous initial conditions, the solution of (3) is (cf. Appendix A)

$$\mathbf{z}(t) = \int_0^t \frac{k}{\tau} e^{-\frac{t-\bar{t}}{\tau}} \dot{\mathbf{g}}(\bar{t}) d\bar{t}. \quad (4)$$

It can be seen that the present value $\mathbf{z}(t)$ depends on the past history of $\dot{\mathbf{g}}(\bar{t})$, $\bar{t} < t$, but not on the actual value $\dot{\mathbf{g}}(t)$ (cf. related remarks in Lubliner [20]). The type of differential equations (3) is analogous to the Maxwell model of linear viscoelasticity.

2.2. Thermodynamic Frameworks

First, we review the main equations of conventional thermodynamics, in a manner suitable for our purposes, and subsequently generalize/modify the non-conventional thermodynamics proposed by Alber et al. [9].

2.2.1. Conventional Thermodynamics

The basic equations of conventional continuum thermodynamics are the balance of linear momentum, the usual form of the balance of energy and the Clausius–Duhem inequality. Denote the symmetric Cauchy stress tensor by $\mathbf{T}$, the velocity vector by $\mathbf{v} = \dot{\mathbf{x}}$, the mass density of the body in the actual configuration by ρ , the specific (per unit mass) body force by $\mathbf{b}$, the stress power per unit volume of the body in the actual configuration by $w = \mathbf{T} \cdot \text{grad } \mathbf{v}$, the Eulerian or Cauchy heat flux vector (per unit surface of the body in the actual configuration) by $\mathbf{H}$, the specific heat supply density by h , the absolute temperature by $\theta > 0$, the specific entropy by η , the specific internal energy by e , and the specific free energy by ψ ,

$$\psi := e - \theta \eta. \quad (5)$$

Then, the local forms of the basic equations, expressed with respect to the actual configuration, are given by

$$\text{div } \mathbf{T} + \rho \mathbf{b} = \rho \ddot{\mathbf{x}} \quad (6)$$

for the balance of linear momentum,

$$\rho \dot{e} = w + \rho h - \text{div } \mathbf{H} \quad (7)$$

for the balance of energy, and

$$\rho \dot{\eta} - \rho \frac{h}{\theta} + \text{div} \left(\frac{\mathbf{H}}{\theta} \right) \geq 0 \quad (8)$$

for the Clausius–Duhem inequality. Further equivalent representations of (8) are

$$-\rho \dot{\theta} \eta + w - \rho \dot{\psi} - \frac{1}{\theta} \mathbf{H} \cdot \text{grad } \theta \geq 0 \quad \Leftrightarrow \quad -\rho \dot{\theta} \eta + w - \rho \dot{\psi} + \mathbf{H} \cdot \mathbf{G} \geq 0, \quad (9)$$

where (cf., e.g., Yang et al. [21])

$$\mathbf{G} := -\text{grad} \left(\ln \frac{\theta}{\theta_0} \right), \quad (10)$$

and $\theta_0 = \text{const.}$ is some reference temperature. Assume that a Gibbs relation holds (e.g., within classical irreversible thermodynamics), implying that θ is given as derivative of e with respect to η , and thus that (5) represents ψ as the Legendre transform of e , i.e.,

$$\eta = -\frac{\partial \psi}{\partial \theta}, \quad \theta = \frac{\partial e}{\partial \eta}. \quad (11)$$

(Equation (11) can be established within rational thermodynamics by exploiting the Clausius–Duhem inequality, see, e.g., Coleman and Noll [22].) In the remainder of the paper, it is assumed that ψ is a strictly concave function of θ , implying that e is a strictly convex function of η and that θ is a strictly monotonic, and hence invertible, function of η .

Towards expressing the above relations in the referential form, we denote by $\rho_R := (\det \mathbf{F})\rho$ the mass density of the body in the reference configuration, by $\mathbf{T}_R := (\det \mathbf{F})\mathbf{T}\mathbf{F}^{-T}$ the first Piola–Kirchhoff stress tensor, by $w_R := (\det \mathbf{F})w = \frac{\rho_R}{\rho}w = \mathbf{T}_R \cdot \dot{\mathbf{F}}$ the stress power per unit volume of the body in the reference configuration and by

$$\mathbf{H}_R := (\det \mathbf{F})\mathbf{F}^{-1}\mathbf{H} \quad (12)$$

the referential heat flux vector (per unit surface of the body in the reference configuration). Then, the counterparts with respect to the reference configuration of Equations (6)–(9) (referential forms, cf. any textbook on continuum thermodynamics) are given by

$$\text{DIV } \mathbf{T}_R + \rho_R \mathbf{b} = \rho_R \ddot{\mathbf{x}}, \quad (13)$$

$$\rho_R \dot{e} = w_R + \rho_R h - \text{DIV } \mathbf{H}_R, \quad (14)$$

$$\rho_R \dot{\eta} - \rho_R \frac{h}{\theta} + \text{DIV } \frac{\mathbf{H}_R}{\theta} \geq 0, \quad (15)$$

$$-\rho_R \dot{\theta} \eta + w_R - \rho_R \dot{\psi} - \frac{1}{\theta} \mathbf{H}_R \cdot \text{GRAD } \theta \geq 0 \quad \Leftrightarrow \quad -\rho_R \dot{\theta} \eta + w_R - \rho_R \dot{\psi} + \mathbf{H}_R \cdot \mathbf{G}_R \geq 0, \quad (16)$$

where

$$\mathbf{G}_R := -\text{GRAD} \left(\ln \frac{\theta}{\theta_0} \right) = \mathbf{F}^T \mathbf{G}. \quad (17)$$

2.2.2. Non-Conventional Thermodynamics

In Dunn and Serrin [1] and Dunn [3], an idea of Toupin [4] is revisited, according to which the heat flux vector $\mathbf{H}$ in Equation (7) should be replaced by a more general energy flux vector $\mathbf{q}$. This idea has also been employed in Alber et al. [9] in order to design a non-conventional thermodynamic framework. As mentioned in Section 1, energy supply densities were neglected in [9]. However, when examining thermodynamic consistency, energy supply terms in the energy laws may become important for many physical theories and will be accounted for in the following formulations. We shall generalize Toupin's idea by replacing in the energy law (7) not only $\mathbf{H}$ by $\mathbf{q}$, but also the specific heat supply density h by a more general specific energy supply density q :

$$\rho \dot{e} = w + \rho q - \text{div } \mathbf{q}. \quad (18)$$

Here, as in Section 2.2.1, $w = \mathbf{T} \cdot \text{grad } \mathbf{v}$ indicates the usual form of stress power. Following essentially steps similar to those in Alber et al. [9], in this section we will generalize/modify the work of these authors to develop a non-conventional thermodynamic theory that we will use in the remainder of the paper. We start by establishing the Eulerian formulation of the theory, and we at once notice that Equation (6), being of a purely mechanical nature, remains unchanged,

$$\text{div } \mathbf{T} + \rho \mathbf{b} = \rho \ddot{\mathbf{x}}. \quad (19)$$

Now, let $v_i, i = 1, \dots, n_v$, be components of state variables, and let $\xi_j, j = 1, \dots, n_\xi$ denote components of their space derivatives. Assume that the thermodynamic state of the material at $\mathbf{x}, t$ is described by the specific internal energy density e , together with the variables v_i and ξ_j . We associate with the real state at $\mathbf{x}, t$ a generalized local equilibrium (reference) state. This reference thermodynamic state is defined such that it shares with the real state the same values of internal energy density e , mass density ρ , and variables v_i, ξ_j , but the derivatives ξ_j are treated in the reference state as independent state variables. In this sense, the generalized local equilibrium state is a constitutive extension of the classical local equilibrium state. The inclusion of ξ_j as independent state variables gives rise to additional contributions to the stress power. Accordingly, the stress power w_{eq} , as well as the associated energy supply density q^{eq} and energy flux vector $\mathbf{q}^{\text{eq}}$ entering the energy balance for the generalized local equilibrium state, will in general differ from the corresponding real quantities w, q and $\mathbf{q}$. Furthermore, we assume that this holds in general—not only in the case where the energy response functions depend on ξ_j —and we postulate for the generalized local equilibrium state the energy balance in the form

$$\rho \dot{e} = w^{\text{eq}} + \rho q^{\text{eq}} - \text{div } \mathbf{q}^{\text{eq}}. \quad (20)$$

The generalized local equilibrium framework ensures the existence of an absolute temperature $\theta(\mathbf{x}, t)$ and a specific entropy $\eta(\mathbf{x}, t)$. These quantities are assigned to the real state at $\mathbf{x}, t$ through the associated reference state. More precisely, it is assumed that, if a system characterized by the variables e, v_i and ξ_j were in thermodynamic equilibrium, then temperature θ and entropy η would exist and satisfy the standard Gibbs relations. These relations are then employed to define θ and η for the real state via the generalized local equilibrium representation.

In addition, we introduce the same free energy density ψ for both the generalized local equilibrium state and the real state by postulating that Equation (5),

$$\psi := e - \theta \eta, \quad (21)$$

holds equally for both systems. We also assume that the energy balance (18) applies to the real material. These equations naturally lead to the introduction of non-equilibrium contributions

$$w^{\text{neq}} := w - w^{\text{eq}}, \quad q^{\text{neq}} := q - q^{\text{eq}}, \quad \mathbf{q}^{\text{neq}} := \mathbf{q} - \mathbf{q}^{\text{eq}}. \quad (22)$$

By subtracting Equation (20) from Equation (18), we then obtain the energy transfer law for the non-equilibrium contributions

$$w^{\text{neq}} = -\rho q^{\text{neq}} + \text{div } \mathbf{q}^{\text{neq}}. \quad (23)$$

If multiple mechanisms act simultaneously, each contributing to non-equilibrium phenomena, it is convenient to introduce additive decompositions of the form

$$w^{\text{neq}} = w_{(1)}^{\text{neq}} + \dots + w_{(N)}^{\text{neq}}, \quad (24)$$

$$q^{\text{neq}} = q_{(1)}^{\text{neq}} + \dots + q_{(N)}^{\text{neq}}, \quad (25)$$

$$\mathbf{q}^{\text{neq}} = \mathbf{q}_{(1)}^{\text{neq}} + \dots + \mathbf{q}_{(N)}^{\text{neq}}. \quad (26)$$

In accordance with the energy transfer relation (23), we postulate that each mechanism separately satisfies an energy transfer law of the form (23), i.e.,

$$\begin{aligned} w_{(1)}^{\text{neq}} &= -\rho q_{(1)}^{\text{neq}} + \text{div } \mathbf{q}_{(1)}^{\text{neq}}, \\ &\vdots \\ w_{(N)}^{\text{neq}} &= -\rho q_{(N)}^{\text{neq}} + \text{div } \mathbf{q}_{(N)}^{\text{neq}}. \end{aligned} \quad (27)$$

This decomposition allows the different non-equilibrium processes to be treated independently at the level of energy exchange, while remaining consistent with the overall balance laws.

To complete the non-conventional thermodynamic framework, an appropriate form of the second law must be specified. We postulate for the real state an entropy production inequality of the Clausius–Duhem type, written as

$$\rho \dot{\eta} - \rho \frac{q^{\text{eq}}}{\theta} + \text{div } \frac{\mathbf{q}^{\text{eq}}}{\theta} \geq 0. \quad (28)$$

Although this expression formally resembles a Clausius–Duhem inequality for the generalized local equilibrium state, it is in fact a constraint on all processes of the real system. The generalized local equilibrium state serves solely as an auxiliary reference state, providing the definitions of the thermodynamic quantities $\theta, \eta, w^{\text{eq}}, q^{\text{eq}}$ and $\mathbf{q}^{\text{eq}}$. It does not represent an independently evolving physical system and is not itself subject to thermodynamic constraints. Rather, it is a constitutive construct used to introduce thermodynamic structure. The entropy production inequality (28) therefore is a constraint for the processes of the real material, expressed in terms of the locally defined quantities $\theta, \eta, q^{\text{eq}}$ and $\mathbf{q}^{\text{eq}}$.

To summarize, the central idea of the proposed framework is to associate the real non-equilibrium state with a generalized local equilibrium state that shares the same internal energy and state variables. This mapping provides an energetically equivalent equilibrium representation, allowing the use of equilibrium thermodynamic relations to define $\theta, \eta, w^{\text{eq}}, q^{\text{eq}}$ and $\mathbf{q}^{\text{eq}}$. These quantities are then assigned to the real system, and the entropy production inequality is formulated in terms of them, thereby constraining the irreversible evolution of the material. Note in passing that temporal non-localities, in addition to the classical ones, can be captured within the present framework through appropriate constitutive assumptions for the non-equilibrium contributions $w^{\text{neq}}, q^{\text{neq}}$ and $\mathbf{q}^{\text{neq}}$; an example of this approach is provided by the discussion of Cattaneo's heat conduction law in Section 3. Moreover, the entropy inequality (28) was treated in [9] as a constraint for both the evolution of the generalized local equilibrium state and that of the real state, in contrast to the approach outlined here.

Further equivalent forms of (28) arise by using (20), (21) and (10):

$$-\rho \dot{\theta} \eta + w^{\text{eq}} - \rho \dot{\psi} - \frac{1}{\theta} \mathbf{q}^{\text{eq}} \cdot \text{grad } \theta \geq 0 \quad \Leftrightarrow \quad -\rho \dot{\theta} \eta + w^{\text{eq}} - \rho \dot{\psi} + \mathbf{q}^{\text{eq}} \cdot \mathbf{G} \geq 0. \quad (29)$$

We recall that the classical Gibbs relations imply the potential formulas

$$\eta = -\frac{\partial \psi}{\partial \theta}, \quad \theta = \frac{\partial e}{\partial \eta}, \quad (30)$$

so that Equation (21) represents a Legendre transformation. Additional potential relations will be derived by examining the second law. As stated in Maugin [11], p. 88, using the classical relations (30) in conjunction with the Clausius–Duhem inequality is common practice when considering specific examples. Then, the only thing left to do is to choose suitable constitutive equations for some or all of the fields $w_{(i)}^{\text{eq}}, q_{(i)}^{\text{eq}}$ and $\mathbf{q}_{(i)}^{\text{eq}}$ (or equivalently, for some or all of the fields $w_{(i)}^{\text{neq}}, q_{(i)}^{\text{neq}}$ and $\mathbf{q}_{(i)}^{\text{neq}}$). That means that all of $w_{(i)}^{\text{eq}}, q_{(i)}^{\text{eq}}$ and $\mathbf{q}_{(i)}^{\text{eq}}$ may be prescribed by constitutive equations but need not be. For example, $\mathbf{q}_{(i)}^{\text{neq}}$ and $q_{(i)}^{\text{neq}}$ can be defined by suitable constitutive equations, while $w_{(i)}^{\text{neq}}$ is determined from (27)_(i). This clarification is a correction of the technical formulations adopted in Alber et al. [9], where, on p. 704, it was suggested that constitutive equations should be prescribed for all fields involved in Equations (46) of [9] (respectively, Equations (27) of the present paper). Evidently, in formulating constitutive equations for the non-equilibrium quantities, new variables, not present in the energy response functions, might be employed, which offers new perspectives for an in-depth and detailed description of material behavior. It is worth noticing that the proposed non-conventional thermodynamics generalizes/modifies, in addition to the notion of local state, the forms of the energy and the second law in the thermodynamics of irreversible processes.

The referential formulation of the theory can be established in a straight-forward way (cf. Section 2.2.1). We have, on the one hand, for the balance of linear momentum

$$\text{DIV } \mathbf{T}_R + \rho_R \mathbf{b} = \rho_R \dot{\mathbf{x}}, \quad (31)$$

for the energy laws

$$\rho_R \dot{e} = w_R + \rho_R q - \text{DIV } \mathbf{q}_R, \quad \rho_R \dot{e} = w_R^{\text{eq}} + \rho_R q^{\text{eq}} - \text{DIV } \mathbf{q}_R^{\text{eq}}, \quad (32)$$

$$w_R^{\text{neq}} = -\rho_R q^{\text{neq}} + \text{DIV } \mathbf{q}_R^{\text{neq}}, \quad (33)$$

$$w_R := (\det \mathbf{F}) w, \quad w_R^{\text{eq}} := (\det \mathbf{F}) w_R^{\text{eq}}, \quad w_R^{\text{neq}} := w_R - w_R^{\text{eq}}, \quad (34)$$

$$q^{\text{neq}} = q - q^{\text{eq}}, \quad (35)$$

$$\mathbf{q}_R := (\det \mathbf{F}) \mathbf{F}^{-1} \mathbf{q}, \quad \mathbf{q}_R^{\text{eq}} := (\det \mathbf{F}) \mathbf{F}^{-1} \mathbf{q}^{\text{eq}}, \quad \mathbf{q}_R^{\text{neq}} := \mathbf{q}_R - \mathbf{q}_R^{\text{eq}}, \quad (36)$$

for the free energy

$$\psi := e - \theta \eta, \quad \eta = -\frac{\partial \psi}{\partial \theta}, \quad \theta = \frac{\partial e}{\partial \eta}, \quad (37)$$

and for the second law

$$-\rho_R \dot{\theta} \eta + w_R^{\text{eq}} - \rho_R \dot{\psi} - \frac{1}{\theta} \mathbf{q}_R^{\text{eq}} \cdot \text{GRAD } \theta \geq 0 \quad \Leftrightarrow \quad -\rho_R \dot{\theta} \eta + w_R^{\text{eq}} - \rho_R \dot{\psi} + \mathbf{q}_R^{\text{eq}} \cdot \mathbf{G}_R \geq 0. \quad (38)$$

On the other hand, for the decompositions of $w_R^{\text{neq}}, q^{\text{neq}}$ and $\mathbf{q}_R^{\text{neq}}$ into several parts, we have

$$w_R^{\text{neq}} = w_{R(1)}^{\text{neq}} + \dots + w_{R(N)}^{\text{neq}}, \quad (39)$$

$$q^{\text{neq}} = q_{(1)}^{\text{neq}} + \dots + q_{R(N)}^{\text{neq}}, \quad (40)$$

$$\mathbf{q}_R^{\text{neq}} = \mathbf{q}_{R(1)}^{\text{neq}} + \dots + \mathbf{q}_{R(N)}^{\text{neq}}, \quad (41)$$

and, in agreement with (33),

$$\begin{aligned} w_{R(1)}^{\text{neq}} &= -\rho_R q_{(1)}^{\text{neq}} + \text{DIV } \mathbf{q}_{R(1)}^{\text{neq}} \\ &\vdots \\ w_{R(N)}^{\text{neq}} &= -\rho_R q_{(N)}^{\text{neq}} + \text{DIV } \mathbf{q}_{R(N)}^{\text{neq}}. \end{aligned} \quad (42)$$

2.2.3. Rigid Heat Conductor

Assume $\mathcal{B}$ to be a rigid heat conductor at rest (cf. Gurtin et al. [19], Part VIII). Since $\mathcal{B}$ does not undergo any deformation, the actual and reference configurations coincide, and we have $\text{GRAD} = \text{grad}$, $\text{Div} = \text{div}$, $\rho_R = \rho = \text{const.}$, $\mathbf{G}_R = \mathbf{G}$, $\mathbf{H}_R = \mathbf{H}$ and $w = 0$. Energy supply and energy flux arise solely from heat and, hence, $N = 1$, $q = h$ and $\mathbf{q} = \mathbf{H}$. Thus,

$$0 = w = w^{\text{eq}} + w^{\text{neq}}, \quad (43)$$

$$h = h^{\text{eq}} + h^{\text{neq}} \quad (44)$$

$$\mathbf{H} = \mathbf{H}^{\text{eq}} + \mathbf{H}^{\text{neq}}. \quad (45)$$

For the rigid heat conductor, Equations (21), (30), (18), (20), (23) and (29) simplify to

$$\psi = e - \theta\eta, \quad \eta = -\frac{\partial\psi}{\partial\eta}, \quad \theta = \frac{\partial e}{\partial\eta} \quad (46)$$

$$\rho\dot{e} = \rho h - \text{div } \mathbf{H}, \quad (47)$$

$$\rho\dot{e} = w^{\text{eq}} + \rho h^{\text{eq}} - \text{div } \mathbf{H}^{\text{eq}}, \quad (48)$$

$$w^{\text{neq}} = -\rho h^{\text{neq}} + \text{div } \mathbf{H}^{\text{neq}}, \quad (49)$$

$$-\rho\dot{\theta}\eta - w^{\text{eq}} - \rho\dot{\psi} - \frac{1}{\theta}\mathbf{H}^{\text{eq}} \cdot \text{grad } \theta \geq 0 \quad \Leftrightarrow \quad -\rho\theta\dot{\eta} + w^{\text{eq}} - \rho\dot{\psi} + \mathbf{H}^{\text{eq}} \cdot \mathbf{G} \geq 0. \quad (50)$$

2.3. Variational Formulation of Rate Type

Variational formulations of physical theories are important since, among other reasons, the lack of a variational formulation can be a signal that something more subtle is going on as, e.g., incorrect constitutive assumptions. Conversely, the existence of a variational formulation is a strong indication of internal mathematical coherence and often of a physically well-structured theory. Hence, it is natural to ask whether the theories established within the non-conventional thermodynamic framework admit a variational formulation.

A suitable method for the variational characterization of thermomechanical problems, hereafter referred to as variational framework (or formulation) of rate type, has been developed in Yang et al. [21]. Romero et al. [23] extended this variational framework to address thermo-mechanical diffusion coupling. Although not stated explicitly in the aforementioned references which variables the independent variations depend on, we take it to mean that they depend only on $\mathbf{x}$ (time-independent). We now summarize the main features of the approach for the case of conventional thermoelasticity and dynamics.

Conventional thermoelasticity is characterized by the balance of linear momentum (13), the stress power $w_R = \mathbf{T}_R \cdot \dot{\mathbf{F}}$, the response functions

$$e = \bar{e}(\mathbf{F}, \eta), \quad \psi = \bar{\psi}(\mathbf{F}, \theta) = \bar{e} - \eta\theta, \quad (51)$$

$$\eta = -\frac{\partial\bar{\psi}}{\partial\theta}, \quad \theta = \frac{\partial\bar{e}}{\partial\eta}, \quad (52)$$

$$\mathbf{T}_R = \bar{\mathbf{T}}_R(\mathbf{F}, \theta) = \rho_R \frac{\partial\bar{\psi}}{\partial\mathbf{F}} = \rho_R \frac{\partial\bar{e}}{\partial\mathbf{F}}, \quad (53)$$

and the thermal dissipation

$$\mathbf{H}_R \cdot \mathbf{G}_R \geq 0, \quad (54)$$

which follows from Equation (16)₂.

Using Equations (51)–(53) in the energy balance (14), we find the heat conduction equation

$$\rho_R \theta \dot{\eta} - \rho_R h_R + \text{DIV } \mathbf{H}_R = 0. \quad (55)$$

A fundamental assumption in Yang et al. [21] (cf. also Yang [24] and Romero et al. [23]) is the existence of a generalized Fourier potential (or simply thermal potential) Φ for $\mathbf{H}_R$. Especially, Φ is supposed to be a convex function of $\mathbf{G}_R$, that depends parametrically on the state function

$$\bar{\theta}(\mathbf{F}, \eta) := \frac{\partial \bar{e}(\mathbf{F}, \eta)}{\partial \eta}, \quad (56)$$

and vanishes at $\mathbf{G}_R = \mathbf{0}$, and whose derivative with respect to $\mathbf{G}_R$ furnishes $\mathbf{H}_R$,

$$\mathbf{H}_R = \frac{\partial \Phi(\mathbf{G}_R; \bar{\theta})}{\partial \mathbf{G}_R}. \quad (57)$$

Note that the assumed properties of Φ ensure the validity of inequality (54), and that (57) includes the commonly used Fourier heat conduction law.

Equations (13) and (55), together with appropriate boundary conditions, are the governing field equations, which have to be solved. The solution approach of Yang et al. is based on a variational formulation and is interpreted to entail the following three main ideas. (1) $\mathbf{T}_R$ and $\mathbf{H}_R$ are considered to be given by the constitutive relations (53) and (57). (2) The variations of velocity, $\delta \mathbf{v}(\mathbf{X})$, and entropy rate, $\delta \dot{\eta}(\mathbf{X})$, are considered to be independent variations of state variables. (3) In addition, for the purposes of the variational formulation, $\delta \theta(\mathbf{X})$ is also considered as an independent variation. Accordingly, Equation (52)₂ is not regarded as a state function for θ but as an additional field equation (besides the two Equations (13) and (55)), and $\bar{\theta}(\mathbf{F}, \eta)$ in Equation (56) should be understood merely as an abbreviation for $\partial \bar{e}(\mathbf{F}, \eta) / \partial \eta$.

To elaborate, let V_R be the space range occupied by the material body in $\mathcal{R}_R$, ∂V_R the boundary of V_R and $\mathbf{N}_R$ the unit outward normal vector on ∂V_R . Denote by $\mathbf{t}_R = \mathbf{T}_R \mathbf{N}_R$ the traction vector on ∂V_R and by $H_R = \mathbf{H}_R \cdot \mathbf{N}_R$ the heat flux on ∂V_R , and assume that $\partial V_R = \partial V_R^{\mathbf{v}} \cup \partial V_R^{\mathbf{t}_R} = \partial V_R^{\theta} \cup \partial V_R^{H_R}$, $\partial V_R^{\mathbf{v}} \cap \partial V_R^{\mathbf{t}_R} = \partial V_R^{\theta} \cap \partial V_R^{H_R} = \emptyset$. The prescribed boundary conditions are $\mathbf{v} = \mathbf{v}^0$ on $\partial V_R^{\mathbf{v}}$, $\mathbf{t}_R = \mathbf{t}_R^0$ on $\partial V_R^{\mathbf{t}_R}$, $\theta = \theta^0$ on ∂V_R^{θ} and $H_R = H_R^0$ on $\partial V_R^{H_R}$. As usual, $\delta \mathbf{v} = \mathbf{0}$ on $\partial V_R^{\mathbf{v}}$ and $\delta \theta = 0$ on ∂V_R^{θ} . The variational formulation is suggested by the weak forms of Equations (13), (55) and (52)₂, which may be derived by multiplying Equation (13) by $\delta \mathbf{v}$, Equation (55) by $\frac{\delta \theta}{\theta}$ and Equation (52)₂ by $\delta \dot{\eta}$, integrating over V_R , and using standard manipulations:

$$\int_{V_R} \left[\rho_R \frac{\partial \bar{e}}{\partial \mathbf{F}} \cdot \text{GRAD } \delta \mathbf{v} - \rho_R (\mathbf{b} - \dot{\mathbf{v}}) \cdot \delta \mathbf{v} \right] dV_R - \int_{\partial V_R^{\mathbf{t}_R}} \mathbf{t}_R^0 \cdot \delta \mathbf{v} dS_R = 0, \quad (58)$$

$$\int_{V_R} \left(-\rho_R \dot{\eta} \delta \theta + \rho_R h \frac{\delta \theta}{\theta} - \delta \Phi(\mathbf{G}_R; \bar{\theta}) \right) dV_R - \int_{\partial V_R^{H_R}} H_R^0 \frac{\delta \theta}{\theta} dS_R = 0, \quad (59)$$

$$\int_{V_R} \rho_R \left(\frac{\partial \bar{e}}{\partial \eta} - \theta \right) \delta \dot{\eta} dV_R = 0. \quad (60)$$

In deriving (59), use has been made of the equation

$$-\int_{V_R} (\text{DIV } \mathbf{H}_R) \frac{\delta\theta}{\theta} dV_R = -\int_{\partial V_R} (\mathbf{H}_R \cdot \mathbf{N}_R) \frac{\delta\theta}{\theta} dS_R + \int_{V_R} \mathbf{H}_R \cdot \text{GRAD} \left(\frac{\delta\theta}{\theta} \right) dV_R \quad (61)$$

$$= -\int_{\partial V_R^{H_R}} H_R^0 \frac{\delta\theta}{\theta} dS_R - \int_{V_R} \mathbf{H}_R \cdot \delta \mathbf{G}_R dV_R \quad (62)$$

$$= -\int_{\partial V_R^{H_R}} H_R^0 \frac{\delta\theta}{\theta} dS_R - \int_{V_R} \delta \Phi(\mathbf{G}_R; \bar{\theta}) dV_R. \quad (63)$$

Given that $\delta \mathbf{v}$ does not depend on time t , any virtual change $\delta \dot{\mathbf{v}}$ is meaningless within the present approach, and formally we have $\delta \dot{\mathbf{v}} = \mathbf{0}$. Thus, as in Weinberg et al. [25], by summing the left-hand sides of Equations (58)–(60), we may introduce the functional

$$\begin{aligned} \mathcal{F}(\mathbf{v}, \theta, \dot{\eta}) = & \int_{V_R} \left[\rho_R \dot{e} - \rho_R \theta \dot{\eta} - \Phi - \rho_R (\mathbf{b} - \dot{\mathbf{v}}) \cdot \mathbf{v} + \rho_R h \ln \frac{\theta}{\theta_0} \right] dV_R \\ & - \int_{\partial V_R^{\mathbf{t}_R^0}} \mathbf{t}_R^0 \cdot \mathbf{v} dS_R - \int_{\partial V_R^{H_R}} H_R^0 \left(\ln \frac{\theta}{\theta_0} \right) dS_R, \end{aligned} \quad (64)$$

and consider the stationarity of this functional,

$$\delta \mathcal{F}(\mathbf{v}, \theta, \dot{\eta}) = 0, \quad (65)$$

as a variational counterpart of Equations (13), (55) and (52)₂, provided the specifications for $\mathbf{b}, h, \mathbf{t}_R^0$ and H_R^0 do not depend on $(\mathbf{v}, \theta, \dot{\eta})$. The variational statement (64), (65) has been derived as a necessary condition for the partial differential equations (13), (55) and (52)₂. That this is also a sufficient condition can be proved by using the methods of the calculus of variations.

We conclude by remarking that the acceleration term $-\rho_R \dot{\mathbf{v}}$ in Equation (64) is regarded as a body force. A similar treatment of this term can be found, among others, in Batra [26], Podio-Guidugli [27] and Gurtin et al. [19]. In particular, $\rho_R (\mathbf{b} - \dot{\mathbf{v}})$ is referred to in Gurtin et al. [19], Section 19.4, as a generalized body force.

3. Cattaneo's Heat Conduction Law

Cattaneo's heat conduction law is often invoked in order to account for finite speed of heat propagation (see, e.g., Joseph and Preziosi [28] and the references cited there). Let τ be a relaxation time and k^* a scalar parameter denoting the thermal conductivity. With reference to a rigid heat conductor at rest, Cattaneo's law reads

$$\tau \dot{\mathbf{H}} + \mathbf{H} = -k^* \text{grad } \theta. \quad (66)$$

For small deviations of the temperature $\theta(\mathbf{x}, t)$ from the reference temperature θ_0 , $\text{grad } \theta$ can be thought of as approximating $-\theta_0 \mathbf{G}$ and, hence, by replacing $k^* \theta_0$ with k , Equation (66) as approximating the constitutive law

$$\tau \dot{\mathbf{H}} + \mathbf{H} = k \mathbf{G}. \quad (67)$$

Henceforth, we shall refer to this constitutive equation as Cattaneo's law for a rigid heat conductor. We begin by discussing, within the non-conventional thermodynamic framework,

Cattaneo's law for a rigid heat conductor and then extend the analysis to thermoelastic solids.

3.1. Cattaneo's Law for a Rigid Heat Conductor

We shall prove the thermodynamical consistency of the constitutive law (67) without assuming that the variables involved in this law contribute to the energy stored in the material (cf. related remarks in Section 1).

In Equations (43)–(50), besides $w = 0$, we also set $w^{\text{eq}} = w^{\text{neq}} = 0$, and we specify ψ as a function solely of temperature θ , $\psi = \bar{\psi}(\theta)$. The Legendre transformation in Equation (21) yields

$$\bar{e}(\eta) = \bar{\psi}(\theta) + \theta\eta, \quad \eta = -\frac{\partial \bar{\psi}}{\partial \theta}, \quad \theta = \frac{\partial \bar{e}}{\partial \eta}, \quad (68)$$

so that Equation (47) reduces to the heat conduction equation

$$-\rho\theta\dot{\eta} + \rho h - \text{div } \mathbf{H} = 0, \quad (69)$$

and inequality (50)₂ to

$$\mathbf{H}^{\text{eq}} \cdot \mathbf{G} \geq 0 \quad \Leftrightarrow \quad (\mathbf{H} - \mathbf{H}^{\text{neq}}) \cdot \mathbf{G} \geq 0. \quad (70)$$

A sufficient condition for (70)₂ is

$$\mathbf{H} - \mathbf{H}^{\text{neq}} = k\mathbf{G}, \quad (71)$$

where k is a positive material parameter. (In the general case, k should be replaced by a positive definite second-order tensor). If, in addition, we set $\mathbf{H}^{\text{neq}} = -\tau\dot{\mathbf{H}}$, with τ being a relaxation time, then (71) implies the law (67). That means that Cattaneo's law can be well addressed within the adopted non-conventional thermodynamic framework. Note that, by prescribing $\mathbf{H}^{\text{neq}}$, in the absence of any further constitutive structure, Equation (49) turns into an equation for determining h^{neq} . However, h^{neq} is not needed in the heat conduction equation (69), provided that ρh is prescribed. Thus, we shall omit further discussion at this point.

Before closing this section, it is perhaps of interest to discuss two aspects concerning Cattaneo's heat conduction law. Firstly, it is remarked that the version of Cattaneo's law (66) has been examined in Alber et al. [9] as well in the framework of the non-conventional thermodynamics and without assuming that the variables involved contribute to the energy stored in the material. However, opposite to the approach here, $h = h^{\text{eq}} = h^{\text{neq}} = 0$, $w = 0$ and $w' = (\tau\dot{\mathbf{q}} + \mathbf{q}') \cdot \frac{\text{grad } \theta}{\theta}$, with $\mathbf{q} = \mathbf{H}$, $\mathbf{q}' = \mathbf{H}^{\text{neq}}$ and $w' = w^{\text{neq}} = -w^{\text{eq}}$ have been assumed in [9]. Thermodynamical consistency has been proved to hold, the heat flux vector is determined from Equation (66) and the temperature distribution may be obtained as solution of the differential equation (69). The characteristic feature in this approach is that $\mathbf{q}'$, and therefore w' , too, cannot be determined unambiguously from Equation (49), i.e., from $\text{div } \mathbf{q}' = w' = (\tau\dot{\mathbf{q}} + \mathbf{q}') \cdot \frac{\text{grad } \theta}{\theta}$. This, however, is not a defect of the theory and is analogous to the conditions in the statics of beams. To explain, consider a beam subject to external loads and reaction forces. The external loads may consist of forces given by a physical law (as, e.g., gravitational forces) or by a constitutive law (as, e.g., spring forces). The reaction forces are not given by a specific law but are determined from the global conditions of equilibrium. The local equations of equilibrium are $\text{div } \mathbf{T} + \rho\mathbf{b} = \mathbf{0}$. If rigid body material properties are assumed for the beam, then there is no constitutive law for the stress tensor $\mathbf{T}$, and the internal stress distribution is indeterminable. That means that,

within the rigid body idealization of the beam only the reaction forces can be obtained from the global equilibrium equations. If the internal stress distribution is also to be calculated, the beam should be assumed to be a deformable, e.g., elastic body. Stated differently, the local equilibrium equations must be supplemented by suitable constitutive equations. Quite analogously, the idealization of the rigid heat conductor imposed by the assumptions made in [9] for w' allows one to determine the temperature distribution, but the energy transfer law (49) alone does not suffice to determine unambiguously the fields $\mathbf{q}'$ and w' . In opposite to [9], in our approach, both fields $\mathbf{H}^{\text{neq}}$ and h^{neq} are determinable, namely $\mathbf{H}^{\text{neq}}$ directly from Equation (71) and h^{neq} from Equation (49), provided that $\mathbf{H}$ and $\mathbf{G}$ are known. The second aspect refers to a comparison of Cattaneo's law (67) with the Maxwell model of viscoelasticity, which is described by a differential equation of the form (3), to which Equation (67) also belongs. Green and Naghdi [29] proposed a thermodynamical theory for material bodies, the characteristic feature of which is that it uses as state variables the so-called thermal displacement α and the thermal displacement gradient Γ . Here, slightly modifying the definitions of Green and Naghdi, we set $\alpha(t) := \int_{t_0}^t \ln \frac{\theta(\bar{t})}{\theta_0} d\bar{t} + \alpha(t_0)$ and $\Gamma := -\text{grad } \alpha \Rightarrow \dot{\Gamma} = \mathbf{G}$. Then, according to Equations (3) and (4), the solution of Equation (67) can be expressed in terms of $\mathbf{G}$,

$$\mathbf{H}(t) = \int_0^t \frac{k}{\tau} e^{-\frac{t-\bar{t}}{\tau}} \mathbf{G}(\bar{t}) d\bar{t}. \quad (72)$$

It is, however, emphasized that, in contrast to the Maxwell model of viscoelasticity, and in agreement with the statement at the beginning of this section, our approach to Cattaneo's law is purely dissipative (no contribution to the energy storage in the material is involved).

3.2. Rigid Heat Conductor, Variational Formulation

We show in this section that a variational formulation of rate type exists for the rigid heat conductor in conjunction with Cattaneo's heat conduction law.

Let $\delta\theta$ and $\delta\dot{\eta}$ be time-independent variations, as in Section 2.3. Then, multiply Equation (69) by $\frac{\delta\theta}{\theta}$, integrate the result over $V \equiv V_R$, use the notation from Section 2.3, and use algebraic manipulations as in Section 2.3, to obtain the following weak form of Equation (69):

$$\int_V \left(-\rho\dot{\eta}\delta\theta + \rho h \frac{\delta\theta}{\theta} - \mathbf{H} \cdot \delta\mathbf{G} \right) dV - \int_{\partial V^H} H^0 \frac{\delta\theta}{\theta} dS = 0. \quad (73)$$

As in Section 2.3, we also take the equation

$$\int_V \rho \left(\frac{\partial \bar{\epsilon}}{\partial \eta} - \theta \right) \delta\dot{\eta} dV = 0 \quad (74)$$

into account and sum the two Equations (73) and (74) to obtain

$$\int_V \left[\rho \frac{\partial \bar{\epsilon}}{\partial \eta} \delta\dot{\eta} - \delta(\rho\theta\dot{\eta}) - \mathbf{H} \cdot \delta\mathbf{G} + \delta \left(\rho h \ln \frac{\theta}{\theta_0} \right) \right] dV - \int_{\partial V^H} H^0 \frac{\delta\theta}{\theta} dS = 0. \quad (75)$$

Keeping in mind the solution (72) and the remarks after Equation (4), we may conclude that $\mathbf{H}(\mathbf{X}, t)$ depends on the past history of $\mathbf{G}(\mathbf{X}, \bar{t})$, $\bar{t} < t$, but not on the actual value $\mathbf{G}(\mathbf{X}, t)$, implying that $\delta\mathbf{H} = \mathbf{0}$. This enables us to recast Equation (75) as the variational statement

$$\delta\mathcal{F}(\theta, \dot{\eta}) = 0, \quad (76)$$

with

$$\mathcal{F}(\theta, \dot{\eta}) := \int_V \left[\rho \dot{e} - \rho \theta \dot{\eta} - \mathbf{H} \cdot \mathbf{G} + \rho h \ln \frac{\theta}{\theta_0} \right] dV - \int_{\partial V^H} H^0 \left(\ln \frac{\theta}{\theta_0} \right) dS, \quad (77)$$

which is of the desired type.

3.3. Use of Cattaneo's Law in Thermoelasticity

So far, only rigid bodies at rest have been studied. In this case, the ordinary time derivative is involved. To highlight the effects of objective time derivatives, deformable bodies are analyzed in this section using the example of thermoelasticity. Note that issues concerning objective time derivatives employed in heat conduction law's have also been addressed in the interesting works Morro [30] and Morro [31], but within a thermodynamic context different from the one adopted here.

We focus, within non-conventional thermodynamics, on thermoelasticity combined with Cattaneo's heat conduction law discussed above. As in Section 2.3, the governing equations will be formulated in the referential configuration. Accordingly, Equations (31)–(42) will apply, and in addition, we assume $N = 1$. We set

$$\mathbf{w}_R = \mathbf{w}_R^{\text{eq}} = \mathbf{T}_R \cdot \dot{\mathbf{F}}, \quad \mathbf{w}_R^{\text{neq}} = 0, \quad \mathbf{q}_R = \mathbf{H}_R = \mathbf{H}_R^{\text{eq}} + \mathbf{H}_R^{\text{neq}}, \quad q = h = h^{\text{eq}} + h^{\text{neq}}, \quad (78)$$

so that, in view of Equation (42),

$$0 = -\rho_R h^{\text{neq}} + \text{DIV } \mathbf{H}_R^{\text{neq}}. \quad (79)$$

As in Section 2.3, we assume thermoelasticity to be characterized by the relations

$$e = \bar{e}(\mathbf{F}, \eta), \quad \psi = \bar{\psi}(\mathbf{F}, \theta) = \bar{e} - \eta \theta, \quad (80)$$

$$\eta = \bar{\eta}(\mathbf{F}, \theta) = -\frac{\partial \bar{\psi}}{\partial \theta}, \quad \theta = \frac{\partial \bar{e}}{\partial \eta}, \quad (81)$$

$$\mathbf{T}_R = \rho_R \frac{\partial \bar{\psi}}{\partial \mathbf{F}}. \quad (82)$$

It follows from (32)₁ that

$$-\rho_R \theta \dot{\eta} + \rho_R h - \text{DIV } \mathbf{H}_R = 0, \quad (83)$$

and from (38)₂ that

$$\mathbf{H}_R^{\text{eq}} \cdot \mathbf{G}_R = (\mathbf{H}_R - \mathbf{H}_R^{\text{neq}}) \cdot \mathbf{G}_R \geq 0, \quad (84)$$

with $\mathbf{G}_R$ defined in Equation (17). Since $\mathbf{G}_R$ is a Lagrangian quantity, its time derivative $\dot{\mathbf{G}}_R$ is regarded as the Lagrangian objective time derivative, also referred to as the associated time derivative of $\mathbf{G}_R$. The associated Eulerian objective time derivative of $\mathbf{G}$ is the lower Oldroyd (or lower convected) time derivative $\overset{\Delta}{\mathbf{G}}$; it is derived from Equation (17) as

$$\overset{\Delta}{\mathbf{G}} := \dot{\mathbf{G}} + \mathbf{L}^T \mathbf{G} = \mathbf{F}^{-T} \dot{\mathbf{G}}_R \quad (85)$$

with

$$\mathbf{L} := \dot{\mathbf{F}} \mathbf{F}^{-1} = \text{grad } \mathbf{v}(\mathbf{x}, t). \quad (86)$$

Generalizing the ideas developed in Haupt and Tsakmakis [32] for dual stresses, we shall now introduce an Eulerian counterpart of $\mathbf{H}_R$, and its associated objective time derivative, in a way that is irrespective of material properties. Specifically, we require that the thermal power (per unit volume of the body in the reference configuration)

$$P_{\text{th}} := \mathbf{H}_R \cdot \mathbf{G}_R, \quad (87)$$

and all its time derivatives $\dot{P}_{\text{th}}, \ddot{P}_{\text{th}}, \dots$ be form-invariant with respect to any configuration. We call $\mathbf{H}_R$ as the dual heat flux to $\mathbf{G}_R$ (with respect to P_{th}). The dual heat flux to $\mathbf{G}$ is $\mathbf{H}_w$, and is known as the Kirchhoff or weighted Cauchy flux vector (cf., e.g., Wcisło et al. [33] and the references cited there),

$$P_{\text{th}} = \mathbf{H}_R \cdot \mathbf{F}^T \mathbf{G} = \mathbf{H}_w \cdot \mathbf{G}, \quad \mathbf{H}_w := \mathbf{F} \mathbf{H}_R = (\det \mathbf{F}) \mathbf{H}. \quad (88)$$

Given that

$$\dot{P}_{\text{th}} = \dot{\mathbf{H}} \cdot \mathbf{G}_R + \mathbf{H}_R \cdot \dot{\mathbf{G}}_R, \quad (89)$$

and that the second term on the right-hand side is form-invariant,

$$\mathbf{H}_R \cdot \dot{\mathbf{G}}_R = \mathbf{H}_w \cdot \overset{\Delta}{\dot{\mathbf{G}}}, \quad (90)$$

$\dot{P}_{\text{th}}$ will be form-invariant exactly when the term $\dot{\mathbf{H}}_R \cdot \mathbf{G}_R$ is form-invariant, i.e., when

$$\dot{\mathbf{H}}_R \cdot \mathbf{G}_R = \overset{\nabla}{\mathbf{H}}_w \cdot \mathbf{G}. \quad (91)$$

By virtue of (17) and (88)₂, we can verify that $\overset{\nabla}{\mathbf{H}}_w$ is the upper Oldroyd (or upper convective) time derivative of $\mathbf{H}_w$,

$$\overset{\nabla}{\mathbf{H}}_w := \dot{\mathbf{H}}_w - \mathbf{L} \mathbf{H}_w = \mathbf{F} \dot{\mathbf{H}}_R. \quad (92)$$

Higher-order associated time derivatives can be introduced in a similar manner, but are not needed here.

Keeping in mind Equations (17) and (88), we can rewrite (84) as

$$\mathbf{H}_w^{\text{eq}} \cdot \mathbf{G} = (\mathbf{H}_w - \mathbf{H}_w^{\text{neq}}) \cdot \mathbf{G} \geq 0, \quad (93)$$

with $\mathbf{H}_w^{\text{neq}} := \mathbf{F} \mathbf{H}_R^{\text{neq}}$. We shall now establish two possible counterparts of Cattaneo's law (67) in the context of thermoelasticity. The first one is introduced as a sufficient condition for inequality (84) by setting (cf. Equation (71))

$$\mathbf{H}_R - \mathbf{H}_R^{\text{neq}} := k \mathbf{G}_R, \quad \mathbf{H}_R^{\text{neq}} := -\tau \dot{\mathbf{H}}_R, \quad (94)$$

where k can depend on $\frac{\partial \bar{e}}{\partial \eta} =: \bar{\theta}(\mathbf{F}, \eta)$. Altogether, the result is

$$\tau \dot{\mathbf{H}}_R + \mathbf{H}_R = k \mathbf{G}_R, \quad (95)$$

i.e., the same equation as in (67), but formulated in referential terms. According to (4), the solution can be represented as

$$\mathbf{H}_R(t) = \int_0^t \frac{k}{\tau} e^{-\frac{t-\bar{t}}{\tau}} \mathbf{G}_R(\bar{t}) d\bar{t}. \quad (96)$$

The other possibility arises by considering an analogous sufficient condition for inequality (93), i.e.,

$$\mathbf{H}_w - \mathbf{H}_w^{\text{neq}} = k \mathbf{G}, \quad \mathbf{H}_w^{\text{neq}} := -\tau \overset{\nabla}{\mathbf{H}}_w, \quad (97)$$

and, hence,

$$\tau \overset{\nabla}{\mathbf{H}}_w + \mathbf{H}_w = k \mathbf{G}. \quad (98)$$

This is an Eulerian format of Cattaneo's law (67) in the context of thermoelasticity. We can employ Equations (17), (88)₂ and (92) and the right Cauchy-Green deformation tensor

$$\mathbf{C} := \mathbf{F}^T \mathbf{F}, \quad (99)$$

to obtain the pull-back transformation of (98) to the reference configuration,

$$\tau \dot{\mathbf{H}}_R + \mathbf{H}_R = k \mathbf{C}^{-1} \mathbf{G}_R. \quad (100)$$

The solution to this differential equation has the form of Equation (4) and is given by

$$\mathbf{H}_R(t) = \int_0^t \frac{k}{\tau} e^{-\frac{t-\bar{t}}{\tau}} \mathbf{C}^{-1}(\bar{t}) \mathbf{G}_R(\bar{t}) d\bar{t}. \quad (101)$$

It is important to outline that both heat conduction laws (95) and (98) satisfy the principle of material frame indifference.

3.4. Cattaneo's Law, Thermoelasticity, Variational Formulation

The established thermoelasticity coupled with Cattaneo's heat conduction law admits a variational formulation of rate type that represents a combination of the results from Sections 2.3 and 3.2. To be more precise, the field equations under consideration are the balance of linear momentum (31) (respectively, (13)), the heat conduction equation (83), which is identical to Equation (55), and Equation (81)₂, which is the same as Equation (52)₂. Consequently, Equations (64) and (65) apply here in the same way, with the only difference that now $\mathbf{H}_R$ is given by (96) or (101) instead of (57). According to the remarks after Equation (4), for both Equations (96) and (101), $\mathbf{H}_R(t)$ is independent of $\mathbf{G}_R(t)$ and, hence, $\delta \mathbf{H}_R = \mathbf{0}$. This means that Equations (64) and (65) apply here as well, but with Φ being replaced by $\mathbf{H}_R \cdot \mathbf{G}_R$, and the variational statement we seek is given by

$$\delta \mathcal{F}(\mathbf{v}, \theta, \dot{\eta}) = 0 \quad (102)$$

with

$$\begin{aligned} \mathcal{F}(\mathbf{v}, \theta, \dot{\eta}) = & \int_{V_R} \left[\rho_R \dot{e} - \rho_r \theta \dot{\eta} - \rho_R (\mathbf{b} - \dot{\mathbf{v}}) \cdot \mathbf{v} + \rho_R h \left(\ln \frac{\theta}{\theta_0} \right) - \mathbf{H}_R \cdot \mathbf{G}_R \right] dV_R \\ & - \int_{\partial V_R^{tR}} \mathbf{t}_R^0 \cdot \mathbf{v} dS_R - \int_{\partial V_R^{H_R}} H_R^0 \left(\ln \frac{\theta}{\theta_0} \right) dS_R. \end{aligned} \quad (103)$$

4. Thermo-Diffusion Coupling

Diffusion processes are non-equilibrium phenomena, which have been addressed adequately by several non-equilibrium thermodynamic theories. Both classical irreversible thermodynamics and rational thermodynamics have turned out to be suitable for describing such phenomena. We shall discuss in this section two examples of diffusion theories. The first one concerns a compressible, viscous fluid consisting of two components and is outlined in Landau and Lifshitz [16] in the context of classical irreversible thermodynamics. The second example represents a diffusion theory, which is formulated in the framework of rational thermodynamics based on a non-classical energy law. It refers to a thermoelastic body subject to thermo-diffusion process and has been outlined in Gurtin et al. [19]. We shall show that both examples may also be well addressed within the adopted non-conventional thermodynamics. Even more, as we shall see, the assumed

non-conventional thermodynamic framework enables a comparative discussion of the thermodynamic structures of both examples.

4.1. Two-Component Compressible Viscous Fluid

Consider a compressible viscous fluid that consists of a mixture of two components which do not interact chemically. The concentration c of the mixture is defined as the ratio of the mass of one component to the total mass of the fluid in a given volume element. An irreversible change in c is called diffusion (cf. Landau and Lifshitz [16], §58). In the following Section 4.1.1, we provide a brief review of Landau and Lifshitz's thermo-diffusion theory for the two-component fluid, in a manner that is suitable for our aims. In Section 4.1.2, we revisit the theory from the perspective of the adopted non-conventional thermodynamics, and in Section 4.1.3, we present a rate-type variational formulation.

4.1.1. The Diffusion Approach Outlined in Landau and Lifshitz

The main equations of the approach outlined in Landau and Lifshitz [16], §58, are as follows.

There are two balance equations of mass, the balance of the total mass of the fluid

$$\rho = \frac{\rho_R}{\det \mathbf{F}} \quad \Leftrightarrow \quad \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0 \quad \Leftrightarrow \quad \dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0, \quad (104)$$

where $\dot{\rho} = \frac{\partial \rho}{\partial t} + \operatorname{grad} \rho \cdot \mathbf{v}$ has been used (see Equation (2)), and the balance of the mass of the one component

$$\rho \dot{c} = -\operatorname{div} \mathbf{J} \quad \Leftrightarrow \quad \frac{\partial(\rho c)}{\partial t} + \operatorname{div}(\rho c \mathbf{v}) = -\operatorname{div} \mathbf{J}. \quad (105)$$

The vector $\mathbf{J}$ is called the density of the diffusion flux (diffusion flux vector; diffusion supply is neglected). Fluids are preferably described relative to the current configuration, so that the balance of linear momentum (6) or (19) is used. In the work of Landau and Lifshitz, body forces are neglected, and hence, this balance becomes

$$\operatorname{div} \mathbf{T} = \rho \dot{\mathbf{v}}. \quad (106)$$

Moreover, these authors assume the fluid exhibits compressible Newtonian constitutive behavior, i.e.,

$$\mathbf{T} = -p\mathbf{1} + \mathbf{T}^v, \quad p = \bar{p}\left(\frac{1}{\rho}\right), \quad (107)$$

where p is the pressure; the viscous part of the stress tensor $\mathbf{T}^v$ is given by

$$\mathbf{T}^v = \mathbb{C}\mathbf{D} := 2\eta_v \mathbf{D}^D + \zeta_v (\operatorname{tr} \mathbf{D})\mathbf{1}, \quad (108)$$

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T), \quad D_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right). \quad (109)$$

Here, $\mathbb{C}$ is a symmetric, fourth-order isotropic tensor, and $\eta_v, \zeta_v \geq 0$ are shear and bulk viscosities, respectively.

A Gibbs relation

$$de = \theta d\eta + \frac{p}{\rho^2} d\rho + \mu dc = \theta d\eta - p d\left(\frac{1}{\rho}\right) + \mu dc \quad (110)$$

is assumed for the internal energy, implying that

$$e = \bar{e}(\eta, \frac{1}{\rho}, c) \quad (111)$$

and

$$\theta = \frac{\partial \bar{e}}{\partial \eta}, \quad \mu = \frac{\partial \bar{e}}{\partial c}, \quad -p = \frac{\partial \bar{e}}{\partial (\frac{1}{\rho})}. \quad (112)$$

The variable μ , referred to as chemical potential, is defined in Equation (112)₂ as the thermodynamic conjugate of c . The assumed energy law is (see Equation (58.5) in Landau and Lifshitz [16])

$$\frac{\partial}{\partial t} \left(\frac{\rho}{2} \mathbf{v} \cdot \mathbf{v} + \rho e \right) = -\operatorname{div} \left[\rho \mathbf{v} \left(\frac{\mathbf{v} \cdot \mathbf{v}}{2} + \varphi \right) - \mathbf{T}^v \mathbf{v} + \mathbf{H} \right], \quad (113)$$

where φ denotes the specific enthalpy. It is related to the internal energy e via the Legendre transformation $\varphi = e + p \frac{1}{\rho}$. In Appendix B, it is shown that the overall energy principle behind the Formula (113) is the energy law (7) with $h = 0$ and $w = \mathbf{T} \cdot \mathbf{D}$,

$$\rho \dot{e} = w - \operatorname{div} \mathbf{H}. \quad (114)$$

The entropy balance equation is derived on the basis of Equation (110), taking into account Equations (114), (105), (107) and (104)₃ and the identity $\operatorname{div} \mathbf{v} = \operatorname{tr} \mathbf{D}$,

$$\begin{aligned} \rho \theta \dot{\eta} &= \rho \dot{e} + \rho p \left(\frac{1}{\rho} \right)' - \rho \mu \dot{c} \\ &= \mathbf{T} \cdot \mathbf{D} - \operatorname{div} \mathbf{H} - p \frac{\dot{\rho}}{\rho} + \mu \operatorname{div} \mathbf{J} \\ &= -p \operatorname{div} \mathbf{v} + \mathbf{T}^v \cdot \mathbf{D} + p \operatorname{div} \mathbf{v} + \operatorname{div} (\mu \mathbf{J}) - \mathbf{J} \cdot \operatorname{grad} \mu - \operatorname{div} \mathbf{H}, \end{aligned} \quad (115)$$

or (cf. Equation (58.6) in Landau and Lifshitz [16])

$$\rho \theta \dot{\eta} = \mathbf{T}^v \cdot \mathbf{D} - \mathbf{J} \cdot \operatorname{grad} \mu - \operatorname{div} (\mathbf{H} - \mu \mathbf{J}). \quad (116)$$

The latter can be recast (see Appendix B, Equation (A9)) as the entropy balance law

$$\frac{\partial(\rho \eta)}{\partial t} = -\operatorname{div} (\rho \eta \mathbf{v}) - \operatorname{div} \frac{\mathbf{H} - \mu \mathbf{J}}{\theta} + \frac{1}{\theta} \mathbf{T}^v \cdot \operatorname{grad} \mathbf{v} - \frac{1}{\theta} \mathbf{J} \cdot \operatorname{grad} \mu - \frac{(\mathbf{H} - \mu \mathbf{J}) \cdot \operatorname{grad} \theta}{\theta^2}. \quad (117)$$

The last three terms on the right-hand side represent the local specific entropy production or dissipation (see, e.g., Jou et al. [13], Section 1.2, or the comments after Equation (A10) in Appendix B), which must be non-negative. Clearly, the dissipation will be non-negative whenever

$$\mathbf{T}^v \cdot \operatorname{grad} \mathbf{v} \geq 0 \quad (118)$$

and

$$-\mathbf{J} \cdot \operatorname{grad} \mu - (\mathbf{H} - \mu \mathbf{J}) \cdot \frac{\operatorname{grad} \theta}{\theta} \geq 0. \quad (119)$$

By virtue of Equation (108), condition (118) is always satisfied. A sufficient condition for (119) will be examined in the next section.

These are the most important equations in Landau and Lifshitz's theory. It should be remarked that this theory is formulated within irreversible thermodynamics and is consistent with a variant of the second law of thermodynamics that differs from the Clausius–Duhem inequality. Thus, it is to be considered as non-conventional thermodynamic theory. In the next section, we will highlight the thermodynamic structure of this theory within the adopted non-conventional thermodynamic framework.

4.1.2. Landau and Lifshitz's Theory Revisited in the Framework of the Adopted Non-Conventional Thermodynamics

The compressible viscous fluid considered in the theory of Landau and Lifshitz is characterized by the balance laws (104)–(106) and (114), i.e.,

$$\rho = \frac{\rho_R}{\det \mathbf{F}}, \quad \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0 \quad \Leftrightarrow \quad \dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0, \quad (120)$$

$$\rho \dot{c} = -\operatorname{div} \mathbf{J}, \quad (121)$$

$$\operatorname{div} \mathbf{T} = \rho \dot{\mathbf{v}}, \quad (122)$$

$$\rho \dot{e} = w - \operatorname{div} \mathbf{H}, \quad (123)$$

and the constitutive laws (cf. Equations (107)–(109), and the response function (111) of e)

$$\mathbf{T} = -p\mathbf{1} + \mathbf{T}^v, \quad p = \bar{p}\left(\frac{1}{\rho}\right), \quad \mathbf{T}^v = \mathbb{C}\mathbf{D}, \quad (124)$$

$$e = \bar{e}\left(\eta, \frac{1}{\rho}, c\right). \quad (125)$$

Then, from Equations (21) and (30), we find that

$$\bar{\psi}\left(\theta, \frac{1}{\rho}, c\right) = \bar{e}\left(\eta, \frac{1}{\rho}, c\right) - \theta\eta, \quad (126)$$

$$\eta = -\frac{\partial \bar{\psi}}{\partial \theta}, \quad \theta = \frac{\partial \bar{e}}{\partial \eta}. \quad (127)$$

Further, as in the last section, define the chemical potential μ through

$$\mu = \frac{\partial \bar{e}}{\partial c} = \frac{\partial \bar{\psi}}{\partial c}, \quad (128)$$

and assume that

$$N = 1, \quad w = \mathbf{T} \cdot \mathbf{D}, \quad q = q^{\text{eq}} = q^{\text{neq}} = 0, \quad \mathbf{q} = \mathbf{H}. \quad (129)$$

Thus, by virtue of Equations (23) and (29)₂,

$$w^{\text{neq}} = \operatorname{div} \mathbf{q}^{\text{neq}}, \quad (130)$$

$$w^{\text{eq}} - \rho \frac{\partial \bar{\psi}}{\partial(\frac{1}{\rho})} \left(\frac{1}{\rho}\right) - \rho \mu \dot{c} + \mathbf{q}^{\text{eq}} \cdot \mathbf{G} \geq 0. \quad (131)$$

With the help of Equations (22) and (129), the last inequality can be rewritten as

$$\mathbf{T} \cdot \mathbf{D} - w^{\text{neq}} + \frac{\partial \bar{\psi}}{\partial(\frac{1}{\rho})} \frac{\dot{\rho}}{\rho} - \rho \mu \dot{c} + (\mathbf{H} - \mathbf{q}^{\text{neq}}) \cdot \mathbf{G} \geq 0, \quad (132)$$

and, by invoking (120)₃ and the identity $\operatorname{tr} \mathbf{D} = \operatorname{div} \mathbf{v}$, we arrive at

$$\mathbf{T} \cdot \mathbf{D} - \frac{\partial \bar{\psi}}{\partial(\frac{1}{\rho})} \operatorname{tr} \mathbf{D} - w^{\text{neq}} - \rho \mu \dot{c} + (\mathbf{H} - \mathbf{q}^{\text{neq}}) \cdot \mathbf{G} \geq 0. \quad (133)$$

It is worth noticing that, compared to an ordinary compressible viscous fluid, the fluid under consideration obeys an energy response function that additionally depends on c and otherwise must satisfy an additional equation, namely the balance law (121). The term $\rho \mu \dot{c}$ in Equation (133) suggests converting Equation (121) into a power balance law by multiplying it with μ ,

$$\rho \mu \dot{c} = -\mu \operatorname{div} \mathbf{J} = -\operatorname{div}(\mu \mathbf{J}) + \mathbf{J} \cdot \operatorname{grad} \mu. \quad (134)$$

Similar to Romero et al. [23], we introduce a new variable $\mathbf{M}$ by

$$\mathbf{M} := -\text{grad } \mu \quad (135)$$

and rephrase Equation (134) as

$$-\rho\mu\dot{c} - \mathbf{J} \cdot \mathbf{M} = \text{div}(\mu\mathbf{J}). \quad (136)$$

This equation has the same form as Equation (130) and thus suggests setting

$$\mathbf{w}^{\text{neq}} = -\rho\mu\dot{c} - \mathbf{J} \cdot \mathbf{M} \Leftrightarrow \mathbf{w}^{\text{eq}} = \mathbf{w} + \rho\mu\dot{c} + \mathbf{J} \cdot \mathbf{M}, \quad (137)$$

$$\mathbf{q}^{\text{neq}} = \mu\mathbf{J} \Leftrightarrow \mathbf{q}^{\text{eq}} = \mathbf{H} - \mu\mathbf{J}. \quad (138)$$

With these identifications, the energy transfer law (130) becomes equivalent to the power law (136). We may also conclude from (133), (124), (137) and (138) that

$$-p(\text{tr } \mathbf{D}) + \mathbf{T}^v \cdot \mathbf{D} - \frac{\partial \bar{\psi}}{\partial(\frac{1}{\rho})} \text{tr } \mathbf{D} + \mathbf{J} \cdot \mathbf{M} + \mathbf{q}^{\text{eq}} \cdot \mathbf{G} \geq 0. \quad (139)$$

Suppose now that $\mathbf{J}$ and $\mathbf{q}^{\text{eq}}$ are independent of $\mathbf{D}$. Then, it can be shown (cf. Gurtin et al. [19], Section 45.3) that (139) is equivalent to

$$p = -\frac{\partial \bar{\psi}}{\partial(\frac{1}{\rho})} = -\frac{\partial \bar{e}}{\partial(\frac{1}{\rho})}, \quad \mathbf{T}^v \cdot \mathbf{D} \geq 0, \quad (140)$$

$$\mathbf{J} \cdot \mathbf{M} + \mathbf{q}^{\text{eq}} \cdot \mathbf{G} \geq 0, \quad (141)$$

which are the same as Equations (112)₃, (118) and (119), respectively.

Before closing this section, it is useful, for the purposes of the next section, to introduce a Newtonian potential

$$\tilde{\omega}(\mathbf{D}) = \frac{1}{2} \mathbf{D} \cdot \mathbb{C} \mathbf{D} = \frac{1}{2} \zeta_v (\text{tr } \mathbf{D})^2 + \eta_v \mathbf{D} \cdot \mathbf{D}, \quad (142)$$

with $\mathbb{C}$ being the fourth-order isotropic tensor defined in Equation (108), a thermo-diffusion potential

$$\tilde{\Phi}(\mathbf{M}, \mathbf{G}) := \frac{1}{2} (\alpha_{11} \mathbf{M} \cdot \mathbf{M} + 2\alpha_{12} \mathbf{M} \cdot \mathbf{G} + \alpha_{22} \mathbf{G} \cdot \mathbf{G}), \quad (143)$$

with $\alpha_{11}, \alpha_{12}, \alpha_{22}$ being material parameters, and to set

$$\mathbf{T}^v = \frac{\partial \tilde{\omega}(\mathbf{D})}{\partial \mathbf{D}} = \mathbb{C} \mathbf{D}, \quad (144)$$

$$\mathbf{J} = \frac{\partial \tilde{\Phi}(\mathbf{M}, \mathbf{G})}{\partial \mathbf{M}} = \alpha_{11} \mathbf{M} + \alpha_{12} \mathbf{G}, \quad (145)$$

$$\mathbf{q}^{\text{eq}} = \frac{\partial \tilde{\Phi}(\mathbf{M}, \mathbf{G})}{\partial \mathbf{G}} = \alpha_{12} \mathbf{M} + \alpha_{22} \mathbf{G}. \quad (146)$$

Then, if $\tilde{\Phi}$ is positive semidefinite, and since $\tilde{\omega}$ is positive semidefinite (by virtue of $\eta_v, \zeta_v \geq 0$), inequalities (140)₂ and Equation (141) will always be satisfied,

$$\mathbf{T}^v \cdot \mathbf{D} = 2\tilde{\omega} \geq 0, \quad \mathbf{J} \cdot \mathbf{M} + \mathbf{q}^{\text{eq}} \cdot \mathbf{G} = 2\tilde{\Phi} \geq 0. \quad (147)$$

Equations (145) and (146) are nothing but the well-known Onsager reciprocal relations for the diffusion flux $\mathbf{J}$ and the heat flux $\mathbf{q}^{\text{eq}}$, with $\alpha_{11}, \alpha_{12}, \alpha_{22}$ being material parameters

(phenomenological coefficients) representing the coefficients of a symmetric block matrix $\mathcal{A}$. We have expressed these relations in terms of the potential $\tilde{\Phi}$, as this is an appropriate representation for the variational principle to be established in the next section. It is shown in Appendix C that $\tilde{\Phi}$, or equivalently $\mathcal{A}$, will be positive semidefinite whenever the Sylvester criterion (non-negativeness of all principal minors of $\mathcal{A}$),

$$\alpha_{11} \geq 0, \quad \det \mathcal{A} = \alpha_{11}\alpha_{22} - \alpha_{12}^2 \geq 0, \quad (148)$$

applies.

Turning back to the energy law (18), and in view of (129)₃, (129)₄ and (138)₂ we have

$$\rho \dot{e} = w - \operatorname{div} \mathbf{q} \quad (149)$$

$$= w - \operatorname{div} (\mu \mathbf{J}) - \operatorname{div} (\mathbf{H} - \mu \mathbf{J}) \quad (150)$$

$$= w - \operatorname{div} (\mu \mathbf{J}) - \operatorname{div} \mathbf{q}^{\text{eq}}, \quad (151)$$

or, with regard to Equations (136) and (129)₂,

$$\rho \dot{e} = \mathbf{T} \cdot \mathbf{D} + \rho \mu \dot{c} + \mathbf{J} \cdot \mathbf{M} - \operatorname{div} \mathbf{q}^{\text{eq}}. \quad (152)$$

By substituting Equations (124)_{1,2}, (140)₁, (128), (127) into (152), it is not difficult to derive the heat conduction equation

$$-\rho \theta \dot{\eta} + \mathbf{T}^v \cdot \mathbf{D} + \mathbf{J} \cdot \mathbf{M} - \operatorname{div} \mathbf{q}^{\text{eq}} = 0. \quad (153)$$

4.1.3. Variational Formulation of the Landau and Lifshitz's Theory

It has been argued in Yang et al. [21] that terms such as the term $\mathbf{T}^v \cdot \mathbf{D}$ in Equation (153), which is related to the potential law (144), do not allow the weak forms of the field equations to be rewritten directly as a variational statement. The same holds true for the term $\mathbf{J} \cdot \mathbf{M}$, which is related to the potential law (145). To overcome this problem, Yang et al. used an integrating factor in the potential laws. With reference to the notation of our paper, their approach can be reinterpreted as follows. We begin with the thermomechanical formulation of the theory, in which Equation (127)₂ has the status of a state function for θ and the potential function for $\mathbf{T}^v$ is given by $\tilde{\omega}(\mathbf{D})$. Hence, by using the abbreviation

$$\bar{\theta}(\eta, \frac{1}{\rho}, c) := \frac{\partial \bar{e}(\eta, \frac{1}{\rho}, c)}{\partial \eta}, \quad (154)$$

we have $1 = \theta/\theta \equiv \theta/\bar{\theta}$ and, therefore, $\tilde{\omega}(\mathbf{D}) \equiv \tilde{\omega}(\frac{\theta}{\bar{\theta}}\mathbf{D})$. When proceeding to elaborate a variational formulation of the theory, θ is no longer treated as being a priori prescribed by Equation (127)₂, which, for the purposes of the variational formulation, has to be regarded as an additional field equation. Accordingly, $\theta/\bar{\theta}$, with $\bar{\theta}$ being given in Equation (154), becomes a factor that induces a time rescaling of the material time derivatives involved in $\mathbf{D} = \frac{1}{2}(\dot{\mathbf{F}}\mathbf{F}^{-1} + \mathbf{F}^{-\text{T}}\dot{\mathbf{F}}^{\text{T}})$. Since, as suggested in Yang et al., the presence of $\theta/\bar{\theta}$ in the potential functions makes it possible to establish a variational statement of the field equations, $\theta/\bar{\theta}$ has been called an integrating factor. It should be noted that Yang et al. regard $\bar{\theta}$ as a second temperature in the theory, which differs from our interpretation here. A single temperature in both the thermomechanical and the variational formulation of the theory had already been assumed in the work of Romero et al. [23]. The variational formulation in the latter relies upon kinetic potentials for thermodynamic forces, which are introduced on the basis of the so-called free entropy (for more details see Romero et al. [23] and the references cited there).

We now discuss an alternative approach to accomplish the variational formulation, which is along the lines of Yang et al. and resembles the one proposed by Romero et al., but is motivated differently. This approach is used in the rest of the paper. For the purposes of the variational formulation, we assume that $\delta\theta(\mathbf{X})$ is an independent variation, that Equation (127)₂ constitutes an additional field equation, and that in Equation (154) $\bar{\theta}$ is merely an abbreviation for $\partial\bar{e}/\partial\eta$. We then define new potentials as follows:

$$\bar{\omega}(\mathbf{D}) = \bar{\omega}\left(\frac{\theta\mathbf{D}}{\theta}\right) =: \hat{\omega}(\theta\mathbf{D}, \theta), \quad \bar{\omega}(\theta\mathbf{D}; h) := \hat{\omega}(\theta\mathbf{D}, h), \quad (155)$$

$$\bar{\Phi}(\mathbf{M}, \mathbf{G}) = \bar{\Phi}\left(\frac{\mathbf{M}}{\theta}, \mathbf{G}\right) =: \hat{\Phi}\left(\frac{\mathbf{M}}{\theta}, \mathbf{G}, \theta\right), \quad \bar{\Phi}\left(\frac{\mathbf{M}}{\theta}, \mathbf{G}; h\right) := \hat{\Phi}\left(\frac{\mathbf{M}}{\theta}, \mathbf{G}, h\right), \quad (156)$$

where $h = h(\mathbf{X}, t)$ is arbitrary. The partial derivatives of $\bar{\omega}, \hat{\omega}, \bar{\Phi}, \hat{\Phi}$ and $\bar{\Phi}$ satisfy the relations

$$\frac{\partial\bar{\omega}(\mathbf{D})}{\partial\mathbf{D}} = \frac{\partial\hat{\omega}(\theta\mathbf{D}, \theta)}{\partial(\theta\mathbf{D})}\theta = \frac{\partial\bar{\omega}(\theta\mathbf{D}; \theta)}{\partial(\theta\mathbf{D})}, \quad (157)$$

$$\mathbf{J} = \frac{\partial\bar{\Phi}(\mathbf{M}, \mathbf{G})}{\partial\mathbf{M}} = \frac{\partial\hat{\Phi}(\frac{\mathbf{M}}{\theta}, \mathbf{G}, \theta)}{\partial(\frac{\mathbf{M}}{\theta})} \frac{1}{\theta} = \frac{\partial\bar{\Phi}(\frac{\mathbf{M}}{\theta}, \mathbf{G}; \theta)}{\partial(\frac{\mathbf{M}}{\theta})} \frac{1}{\theta}, \quad (158)$$

$$\mathbf{q}^{\text{eq}} = \frac{\partial\bar{\Phi}(\mathbf{M}, \mathbf{G})}{\partial\mathbf{G}} = \frac{\partial\hat{\Phi}(\frac{\mathbf{M}}{\theta}, \mathbf{G}, \theta)}{\partial\mathbf{G}} = \frac{\partial\bar{\Phi}(\frac{\mathbf{M}}{\theta}, \mathbf{G}; \theta)}{\partial\mathbf{G}}, \quad (159)$$

where use has been made of the notation, e.g., $\frac{\partial\bar{\omega}(\theta\mathbf{D}; \theta)}{\partial(\theta\mathbf{D})} = \frac{\partial\bar{\omega}(\theta\mathbf{D}; h)}{\partial(\theta\mathbf{D})}|_{h=\theta}$.

Following Romero et al. [23], we consider $\delta\mathbf{v}(\mathbf{x}), \delta\dot{\eta}(\mathbf{x}), \delta\theta(\mathbf{x}), \delta\dot{c}(\mathbf{x})$ and $\delta\mu(\mathbf{x})$ as the independent variations and Equations (122), (153), (121), (127)₂ and (128) as the governing field equations. Let V be the space range occupied by the material body in $\mathcal{R}_t$, ∂V the boundary of V and $\mathbf{N}$ the outward unit normal vector on ∂V . Similar to Section 2.3, we define the boundary conditions $\mathbf{v} = \mathbf{v}^0$ on $\partial V^{\mathbf{v}}$, $\mathbf{t} := \mathbf{T}\mathbf{N} = \mathbf{t}^0$ on $\partial V^{\mathbf{t}}$, $\theta = \theta^0$ on ∂V^{θ} , $q_b := \mathbf{q}^{\text{eq}} \cdot \mathbf{N} = q_b^0$ on ∂V^{q_b} , $\mu = \mu^0$ on ∂V^{μ} and $J := \mathbf{J} \cdot \mathbf{N} = J^0$ on ∂V^J , with $\partial V = \partial V^{\mathbf{v}} \cup \partial V^{\mathbf{t}} = \partial V^{\theta} \cup \partial V^{q_b} = \partial V^{\mu} \cup \partial V^J$, $\partial V^{\mathbf{v}} \cap \partial V^{\mathbf{t}} = \partial V^{\theta} \cap \partial V^{q_b} = \partial V^{\mu} \cap \partial V^J = \emptyset$, $\delta\mathbf{v} = \mathbf{0}$ on $\partial V^{\mathbf{v}}$, $\delta\theta = 0$ on ∂V^{θ} and $\delta\mu = 0$ on ∂V^{μ} . The weak forms of the field equations are derived by multiplying Equation (122) by $\delta\mathbf{v}$, Equation (153) by $\frac{\delta\theta}{\theta}$, Equation (121) by $\delta\mu$, Equation (127)₂ by $\delta\dot{\eta}$, Equation (128) by $\delta\dot{c}$, integrating the results over V and employing the divergence theorem:

$$\int_V \mathbf{T} \cdot \delta\mathbf{D} \, dV + \int_V \rho \dot{\mathbf{v}} \cdot \delta\mathbf{v} \, dV - \int_{\partial V^{\mathbf{t}}} \mathbf{t}^0 \cdot \delta\mathbf{v} \, dS = 0, \quad (160)$$

$$\int_V \left(-\rho \dot{\eta} \delta\theta + \mathbf{T}^{\mathbf{v}} \cdot \mathbf{D} \frac{\delta\theta}{\theta} + \mathbf{J} \cdot \mathbf{M} \frac{\delta\theta}{\theta} - \mathbf{q}^{\text{eq}} \cdot \delta\mathbf{G} \right) dV - \int_{\partial V^{q_b}} q_b^0 \frac{\delta\theta}{\theta} \, dS = 0, \quad (161)$$

$$\int_V (-\rho \dot{c} \delta\mu - \mathbf{J} \cdot \delta\mathbf{M}) dV - \int_{\partial V^J} J^0 \delta\mu \, dS = 0, \quad (162)$$

$$\int_V \rho \left(\frac{\partial\bar{e}}{\partial\eta} - \theta \right) \delta\dot{\eta} \, dV = 0, \quad (163)$$

$$\int_V \rho \left(\frac{\partial\bar{e}}{\partial c} - \mu \right) \delta\dot{c} \, dV = 0. \quad (164)$$

It is shown in Appendix D that rearranging terms in these equations and summing the results leads to the variational statement of rate type

$$\delta\mathcal{F}(\mathbf{v}, \theta, \dot{\eta}, \mu, \dot{c}) = 0, \quad (165)$$

with

$$\begin{aligned} \mathcal{F} = & \int_V \left[\rho \dot{e} - \rho \theta \dot{\eta} - \rho \mu \dot{c} - \Phi \left(\frac{\mathbf{M}}{\theta}, \mathbf{G}; \theta \right) + \bar{\omega}(\theta \mathbf{D}; \theta) + \rho \dot{\mathbf{v}} \cdot \mathbf{v} \right] dV - \int_{\partial V^t} \mathbf{t}^0 \cdot \mathbf{v} dS \\ & - \int_{\partial V^{q_b}} q_b^0 \ln \frac{\theta}{\theta_0} dS - \int_{\partial V^I} \mu J^0 dS. \end{aligned} \quad (166)$$

4.2. Single Species Diffusion Through a Thermoelastic Material

Now, we consider the diffusion of a simple species through a thermoelastic material. As in the case of the fluid mixture, it is assumed that the diffusing species does not chemically interact with the thermoelastic material. Unlike, however, the fluid mixture discussed in Section 4.1, the general approach now is not to assume two distinct materials existing at the same point, but rather to consider a single material, which can be diffused by a species, similar to how heat diffuses in heat conduction (see Baek and Srinivasa [34]). There are several studies that deal with diffusion in solids, cf., e.g., the works Baek and Srinivasa [34], Govindjee and Simo [35], Part XII in Gurtin et al. [19], Romero et al. [23], Chester and Anand [36], Templet and Steigmann [37], Wilmers and Bargmann [38], de Anda Salazar et al. [39], and the references cited therein. As an example, we will sketch in the next section the diffusion theory outlined in Gurtin et al. [19], Part XII. A reformulation of this theory within the framework of the adopted non-conventional thermodynamics is provided in Section 4.2.2. Simplified dissipation potentials, as suggested by Romero et al. [23], are examined in Section 4.2.3, a corresponding variational formulation of rate type for this special case is given in Section 4.2.4, and a comparison of the thermodynamical structures of the thermo-diffusion coupling theories is elaborated in Section 4.2.5. Finally, we shall sketch in Section 4.3 how to extend the theory to account for Cattaneo's heat conduction law.

4.2.1. The Diffusion Approach Outlined in Gurtin et al.

For a thermoelastic body diffused by a single species, the mass balance (104)₁, applies,

$$\rho = \frac{\rho_R}{\det \mathbf{F}}. \quad (167)$$

Note, however, that ρ now refers to the mass density of the elastic body. If we denote by c the mass fraction of the species, then ρc is the species density (per unit volume of the body in the actual configuration) and the local species balance of mass reads (see Gurtin et al. [19], Section 61)

$$\rho \dot{c} = \rho j - \operatorname{div} \mathbf{J}. \quad (168)$$

Here, $\mathbf{J}$ and j are the species flux and the specific species supply, respectively. The two equations (168) and (105)₁, have the same form, but they are not equal because the densities ρ are different. In solid mechanics, the referential description is commonly preferred as a suitable framework, so we will express all relations in referential form. Thus, the referential counterpart of Equation (168) is (cf. Equation (65.8) in Gurtin et al. [19])

$$\rho_R \dot{c} = \rho_R j - \operatorname{DIV} \mathbf{J}_R, \quad (169)$$

where $\mathbf{J}_R := (\det \mathbf{F}) \mathbf{F}^{-1} \mathbf{J}$.

Besides the conventional power of external forces, Gurtin et al. assumed an additional power contribution in the energy law, arising from the energy carried by species transport. They postulated the existence of chemical potential μ as a primitive quantity (in contrast to

the definition (112)₂), and that the supply j and the flux $\mathbf{J}$ contribute to the global form of the energy law by the power (see Equation (64)₁ in Gurtin et al. [19])

$$\int_V \rho \mu j \, dV - \int_{\partial V} \mu \mathbf{J} \cdot \mathbf{N} \, dS. \quad (170)$$

Localizing the global energy balance and taking into account Equation (169), they derived its local form, which, using the notation introduced in Section 2.2.1, has the referential expression

$$\begin{aligned} \rho_R \dot{e} &= \mathbf{T}_R \cdot \dot{\mathbf{F}} + \rho_R \mu j - \text{DIV} (\mu \mathbf{J}_R) + \rho_R h - \text{DIV} \mathbf{H}_R \\ &= \mathbf{T}_R \cdot \dot{\mathbf{F}} + \rho_R \mu \dot{c} + \mathbf{J}_R \cdot \mathbf{M}_R + \rho_R h - \text{DIV} \mathbf{H}_R. \end{aligned} \quad (171)$$

Here, $\mathbf{M}_R := \mathbf{F}^T \mathbf{M}$ and $\mathbf{M}$ is defined as in Equation (135). Equation (171) is the referential description of the energy balance (64.3) in Gurtin et al. [19]. Further, the second law is postulated in the standard form of the Clausius–Duhem inequality, as given in Equation (15), or equivalently,

$$\rho_R \theta \dot{\eta} - \rho_R h + \text{DIV} \mathbf{H}_R + \mathbf{H}_R \cdot \mathbf{G}_R \geq 0. \quad (172)$$

Substitution of $-\rho_R h + \text{DIV} \mathbf{H}_R$ from Equation (171) into inequality (172) yields (cf. Equation (64.6) in Gurtin et al. [19])

$$\rho_R \theta \dot{\eta} - \rho_R \dot{e} + \mathbf{T}_R \cdot \dot{\mathbf{F}} + \rho_R \mu \dot{c} - \mathbf{J}_R \cdot \mathbf{M}_R + \mathbf{H}_R \cdot \mathbf{G}_R \geq 0. \quad (173)$$

In terms of the free energy ψ defined in Equation (5), inequality (173) becomes

$$-\rho_R \eta \dot{\theta} - \rho_R \dot{\psi} + \mathbf{T}_R \cdot \dot{\mathbf{F}} + \rho_R \mu \dot{c} + \mathbf{J}_R \cdot \mathbf{M}_R + \mathbf{H}_R \cdot \mathbf{G}_R \geq 0. \quad (174)$$

So far, no specific material properties have been assumed, i.e., Equations (171) and (173) or (174) hold regardless, e.g., of the particular constitutive form of e or ψ . Now, assume the constitutive functions

$$\psi = \bar{\psi}(\mathbf{F}, c, \theta), \quad (175)$$

$$\mathbf{T}_R = \bar{\mathbf{T}}_R(\mathbf{F}, c, \theta), \quad (176)$$

$$\mu = \bar{\mu}(\mathbf{F}, c, \theta), \quad (177)$$

$$\mathbf{J}_R = \bar{\mathbf{J}}_R(\mathbf{F}, c, \theta, \mathbf{M}_R, \mathbf{G}_R), \quad (178)$$

$$\mathbf{H}_R = \bar{\mathbf{H}}_R(\mathbf{F}, c, \theta, \mathbf{M}_R, \mathbf{G}_R). \quad (179)$$

Then, application of the Coleman–Noll procedure to inequality (174) implies the thermodynamic restrictions

$$\mathbf{T}_R = \rho_R \frac{\partial \bar{\psi}}{\partial \mathbf{F}}, \quad \eta = -\frac{\partial \bar{\psi}}{\partial \theta}, \quad \mu = \frac{\partial \bar{\psi}}{\partial c}, \quad (180)$$

and

$$\mathbf{J}_R \cdot \mathbf{M}_R + \mathbf{H}_R \cdot \mathbf{G}_R \geq 0. \quad (181)$$

Consequently, e becomes the Legendre transform of ψ and has the form $e = \bar{e}(\mathbf{F}, c, \eta)$. Further,

$$\mathbf{T}_R = \rho_R \frac{\partial \bar{e}}{\partial \mathbf{F}}, \quad \mu = \frac{\partial \bar{e}}{\partial c}, \quad \theta = \frac{\partial \bar{e}}{\partial \eta} \quad (182)$$

and the energy law (171) reduces to the heat conduction equation

$$-\rho_R \theta \dot{\eta} + \mathbf{J}_R \cdot \mathbf{M}_R + \rho_R h - \text{DIV} \mathbf{H}_R = 0. \quad (183)$$

Together with the balance of linear momentum (cf. Equation (13)),

$$\text{DIV } \mathbf{T}_R + \rho_R \mathbf{b} = \rho_R \ddot{\mathbf{x}}, \quad (184)$$

these are the main features of the diffusion theory of Gurtin et al., which are of interest for our work. In Section 4.2.4, we shall examine simple sufficient conditions for the satisfaction of inequality (181).

It should be remarked that, as mentioned in Gurtin et al. [19], p. 377, the assumptions (175)–(179) violate the principle of equipresence. With this exception, the theory of Gurtin et al. can be regarded as an approach within the framework of rational thermodynamics, though with a non-classical energy law. Thus, it is to be considered as a non-conventional thermodynamic theory.

4.2.2. Revisiting Gurtin et al.'s Theory Within the Adopted Non-Conventional Thermodynamics

Assume that the constitutive Equations (175)–(179) and the Legendre transformation

$$\bar{\psi}(\mathbf{F}, c, \theta) = \bar{e}(\mathbf{F}, c, \eta) - \theta \eta, \quad \eta = -\frac{\partial \bar{\psi}}{\partial \theta}, \quad \theta = \frac{\partial \bar{e}}{\partial \eta}, \quad (185)$$

hold. Further, postulate the mass balance (169)

$$\rho_R \dot{c} = \rho_R j - \text{DIV } \mathbf{J}_R, \quad (186)$$

and, irrespective of whether or not the chemical potential μ is a primitive quantity, set

$$\mu = \frac{\partial \bar{e}}{\partial c} = \frac{\partial \bar{\psi}}{\partial c}. \quad (187)$$

Also, in a manner quite similar to the derivation of Equation (136), we can derive from Equation (186) the power balance law

$$-\rho_R \mu \dot{c} - \mathbf{J}_R \cdot \mathbf{M}_R = -\rho_R \mu j + \text{DIV } (\mu \mathbf{J}_R). \quad (188)$$

Now, choose in Equations (31)–(42) $N = 1$,

$$\mathbf{w}_R = \mathbf{T}_R \cdot \dot{\mathbf{F}}, \quad (189)$$

$$q = h + \mu j, \quad q^{\text{eq}} = h, \quad q^{\text{neq}} = \mu j, \quad (190)$$

$$\mathbf{q}_R = \mathbf{H}_R + \mu \mathbf{J}_R, \quad \mathbf{q}_R^{\text{eq}} = \mathbf{H}_R, \quad \mathbf{q}_R^{\text{neq}} = \mu \mathbf{J}_R, \quad (191)$$

and identify Equation (33) with Equation (188), implying that

$$\mathbf{w}_R^{\text{neq}} = -\rho_R \mu \dot{c} - \mathbf{J}_R \cdot \mathbf{M}_R \Leftrightarrow \mathbf{w}_R^{\text{eq}} = \mathbf{w}_R + \rho_R \mu \dot{c} + \mathbf{J}_R \cdot \mathbf{M}_R. \quad (192)$$

Substitution of Equations (189)–(191) into Equation (32)₁ yields

$$\rho_R \dot{e} = \mathbf{T}_R \cdot \dot{\mathbf{F}} + \rho_R (h + \mu j) - \text{DIV } (\mathbf{H}_R + \mu \mathbf{J}_R), \quad (193)$$

which is nothing but the energy balance law (171)₁. Next, we recast inequality (38)₂ by using Equations (189)–(191) and (192)₂ as follows

$$-\rho_R \eta \dot{\theta} + \mathbf{w}_R + \rho_R \mu \dot{c} + \mathbf{J}_R \cdot \mathbf{M}_R - \rho_R \dot{\psi} + \mathbf{H}_R \cdot \mathbf{G}_R \geq 0. \quad (194)$$

Granted the potential relations (185)₂ and (187), application of the Coleman–Noll procedure to the last inequality implies the elasticity law (182)₁ and the inequality (181). All other relations are the same as in the previous section. This means that the adopted non-conventional thermodynamics provides an appropriate framework for formulating the diffusion theory of Gurtin et al.

4.2.3. Simplified Dissipation Potentials Suggested by Romero et al.

The dissipation inequality (181) can be satisfied by processing in a manner quite similar to that used in section 4.1.2. However, to keep the effort in the next section as small as possible, we shall assume simple sufficient conditions, as suggested by Romero et al. [23]. Accordingly, we assume the existence of convex thermal and diffusion potentials $\tilde{\Phi}_{\text{th}}(\mathbf{G}_R; \bar{\theta}, \bar{\mu})$ and $\tilde{\Phi}_{\text{d}}(\mathbf{M}_R; \bar{\theta}, \bar{\mu})$, that depend parametrically on

$$\bar{\theta}(\mathbf{F}, c, \eta) := \frac{\partial \bar{e}}{\partial \eta}, \quad \bar{\mu}(\mathbf{F}, c, \eta) := \frac{\partial \bar{e}}{\partial c}, \quad (195)$$

are subject to the conditions $\tilde{\Phi}_{\text{th}}(\mathbf{0}; \bar{\theta}, \bar{\mu}) = \tilde{\Phi}_{\text{d}}(\mathbf{0}; \bar{\theta}, \bar{\mu}) = 0$, and whose derivatives with respect to $\mathbf{G}_R$ and $\mathbf{M}_R$ furnish $\mathbf{H}_R$ and $\mathbf{J}_R$,

$$\mathbf{H}_R = \frac{\partial \tilde{\Phi}_{\text{th}}(\mathbf{G}_R; \bar{\theta}, \bar{\mu})}{\partial \mathbf{G}_R}, \quad \mathbf{J}_R = \frac{\partial \tilde{\Phi}_{\text{d}}(\mathbf{M}_R; \bar{\theta}, \bar{\mu})}{\partial \mathbf{M}_R}. \quad (196)$$

In this way, the dissipation inequality is fulfilled in every admissible process. For the purposes of the variational formulation in the next section, we additionally introduce the following potentials analogous to those in Equations (156) and (158):

$$\tilde{\Phi}_{\text{d}}(\mathbf{M}_R; \bar{\theta}, \bar{\mu}) = \tilde{\Phi}_{\text{d}}\left(\frac{\mathbf{M}_R}{\bar{\theta}}; \bar{\theta}, \bar{\mu}\right) =: \hat{\Phi}_{\text{d}}\left(\frac{\mathbf{M}_R}{\bar{\theta}}; \theta; \bar{\theta}, \bar{\mu}\right), \quad (197)$$

$$\tilde{\Phi}_{\text{d}}\left(\frac{\mathbf{M}_R}{\bar{\theta}}; h, \bar{\theta}, \bar{\mu}\right) := \hat{\Phi}_{\text{d}}\left(\frac{\mathbf{M}_R}{\bar{\theta}}; h; \bar{\theta}, \bar{\mu}\right), \quad (198)$$

$$\mathbf{J}_R = \frac{\partial \tilde{\Phi}_{\text{d}}(\mathbf{M}_R; \bar{\theta}, \bar{\mu})}{\partial \mathbf{M}_R} = \frac{\partial \hat{\Phi}_{\text{d}}\left(\frac{\mathbf{M}_R}{\bar{\theta}}; \theta; \bar{\theta}, \bar{\mu}\right)}{\partial \left(\frac{\mathbf{M}_R}{\bar{\theta}}\right)} \frac{1}{\bar{\theta}} = \frac{\partial \hat{\Phi}_{\text{d}}\left(\frac{\mathbf{M}_R}{\bar{\theta}}; \theta; \bar{\theta}, \bar{\mu}\right)}{\partial \left(\frac{\mathbf{M}_R}{\bar{\theta}}\right)} \frac{1}{\bar{\theta}}. \quad (199)$$

4.2.4. Variational Formulation of the Diffusion Theory of Gurtin et al.

The approach we will use here is essentially the same as in the work of Romero et al. [23]. For completeness, however, we briefly recall it, as it will be needed in Section 4.3, where the coupling with Cattaneo's law is examined. Besides the boundary conditions in Section 2.3, it is also assumed that $\mu = \mu^0$ on ∂V_R^μ and $J_R = \mathbf{J}_R \cdot \mathbf{N} = J_R^0$ on ∂V_R^J , with $\partial V_R = \partial V_R^\mu \cup \partial V_R^J$ and $\partial V_R^\mu \cap \partial V_R^J = \emptyset$. The independent variations are now $\delta \mathbf{v}(\mathbf{X})$, $\delta \dot{\eta}(\mathbf{X})$, $\delta \theta(\mathbf{X})$, $\delta \mu(\mathbf{X})$ and $\delta \dot{c}(\mathbf{X})$ with $\delta \mu = 0$ on ∂V_R^μ . The field equations we are concerned with are Equations (184), (183), (169), (182)₃ and (182)₂. The weak forms of these equations are obtained by multiplying them by $\delta \mathbf{v}$, $\frac{\delta \theta}{\bar{\theta}}$, $\delta \mu$, $\delta \dot{\eta}$ and $\delta \dot{c}$, respectively, and then integrating over V_R and using standard manipulations:

$$\int_{V_R} [\mathbf{T}_R \cdot \delta \mathbf{F} - \rho_R (\mathbf{b} - \dot{\mathbf{v}}) \cdot \delta \mathbf{v}] dV_R - \int_{\partial V_R^{\mathbf{t}_R}} \mathbf{t}_R^0 \cdot \delta \mathbf{v} dS_R = 0, \quad (200)$$

$$\int_{V_R} \left(-\rho_R \dot{\eta} \delta \theta + \mathbf{J}_R \cdot \mathbf{M}_R \frac{\delta \theta}{\theta} + \rho_R h \frac{\delta \theta}{\theta} - \mathbf{H}_R \cdot \delta \mathbf{G}_R \right) dV_R - \int_{\partial V_R^{H_R}} H_R^0 \frac{\delta \theta}{\theta} dS_R = 0, \quad (201)$$

$$\int_{V_R} (-\rho_R \dot{c} \delta \mu + \rho_R j \delta \mu - \mathbf{J}_R \cdot \delta \mathbf{M}_R) dV_R - \int_{\partial V_R^{J_R}} J_R^0 \delta \mu dS_R = 0, \quad (202)$$

$$\int_{V_R} \rho_R \left(\frac{\partial \bar{e}}{\partial \eta} - \theta \right) \delta \dot{\eta} dV_R = 0, \quad (203)$$

$$\int_{V_R} \rho_R \left(\frac{\partial \bar{e}}{\partial c} - \mu \right) \delta \dot{c} dV_R = 0. \quad (204)$$

Addition of these equations, keeping in mind Equation (180)₁, yields

$$\begin{aligned} & \int_{V_R} \left[\delta(\rho_R \dot{e}) - \delta(\rho_R \theta \dot{\eta}) - \delta(\rho_R \mu \dot{c}) - \delta(\rho_R (\mathbf{b} - \dot{\mathbf{v}}) \cdot \mathbf{v}) + \delta \left(\rho_R h \ln \frac{\delta \theta}{\theta} \right) \right. \\ & \quad \left. + \delta(\rho_R \mu j) + \mathbf{J}_R \cdot \mathbf{M}_R \frac{\delta \theta}{\theta} - \mathbf{J}_R \cdot \delta \mathbf{M}_R - \mathbf{H}_R \cdot \delta \mathbf{G}_R \right] dV_R \\ & \quad - \int_{\partial V_R^{\mathbf{t}_R}} \mathbf{t}_R^0 \cdot \delta \mathbf{v} dS_R - \int_{\partial V_R^{H_R}} H_R^0 \frac{\delta \theta}{\theta} dS_R - \int_{\partial V_R^{J_R}} J_R^0 \delta \mu dS_R = 0. \end{aligned} \quad (205)$$

Quite similar to Equations (A22) and (A23), by invoking Equations (196)–(199), one can derive the identity

$$\mathbf{J}_R \cdot \mathbf{M}_R \frac{\delta \theta}{\theta} - \mathbf{J}_R \cdot \delta \mathbf{M}_R - \mathbf{H}_R \cdot \delta \mathbf{G}_R = -\delta \Phi_d \left(\frac{\mathbf{M}_R}{\theta}; \theta, \bar{\theta}, \bar{\mu} \right) - \delta \Phi_{th}(\mathbf{G}_R; \bar{\theta}, \bar{\mu}). \quad (206)$$

With regard to this result, Equation (205), and therefore the diffusion theory of Gurtin et al., becomes equivalent to the variational statement

$$\delta \mathcal{F}(\mathbf{v}, \theta, \dot{\eta}, \mu, \dot{c}) = 0, \quad (207)$$

with

$$\begin{aligned} \mathcal{F} := & \int_{V_R} \left[\rho_R \dot{e} - \rho_R \theta \dot{\eta} - \rho_R \mu \dot{c} - \Phi_d \left(\frac{\mathbf{M}_R}{\theta}; \theta, \bar{\theta}, \bar{\mu} \right) - \Phi_{th}(\mathbf{G}_R; \bar{\theta}, \bar{\mu}) - \rho_R (\mathbf{b} - \dot{\mathbf{v}}) \cdot \mathbf{v} + \rho_R \mu j + \rho_R h \ln \frac{\theta}{\theta_0} \right] dV_R \\ & - \int_{\partial V_R^{\mathbf{t}_R}} \mathbf{t}_R^0 \cdot \delta \mathbf{v} dS_R - \int_{\partial V_R^{H_R}} H_R^0 \frac{\delta \theta}{\theta} dS_R - \int_{\partial V_R^{J_R}} J_R^0 \delta \mu dS_R. \end{aligned} \quad (208)$$

4.2.5. Comparison of the Two Thermo-Diffusion Coupling Theories

We have outlined two thermo-diffusion coupling theories, according to Landau and Lifshitz and Gurtin et al., respectively. The first concerns a compressible viscous fluid and was originally formulated within the framework of classical non-equilibrium thermodynamics. The second pertains to a thermoelastic solid and was originally formulated in the context of rational continuum thermodynamics with a non-classical energy balance

law. Both theories are thermodynamically consistent and have been shown to possess variational formulations of the rate type. It has also been demonstrated that both can be well formulated within the framework of the non-conventional thermodynamics adopted in the paper. From the point of view of the latter, and irrespective of the different definitions of the mass densities ρ and of the constitutive assumptions made in the two theories, the main differences lie in the assumed forms of the energy laws, and especially in the forms of the energy flux vectors, the energy supply densities and the stress powers, and the related boundary conditions. For example, expressed in the spatial description, the specific assumptions for the respective energy flux vectors are as follows.

$$\text{Theory of Landau and Lifshitz: } \mathbf{q} := \mathbf{H}, \quad \mathbf{q}^{\text{neq}} = \mu \mathbf{J}, \quad \mathbf{q}^{\text{eq}} = \mathbf{H} - \mu \mathbf{J}. \quad (209)$$

$$\text{Theory of Gurtin et al.: } \mathbf{q} := \mathbf{H} + \mu \mathbf{J}, \quad \mathbf{q}^{\text{neq}} = \mu \mathbf{J}, \quad \mathbf{q}^{\text{eq}} = \mathbf{H}. \quad (210)$$

We recognize from these equations that the two theories postulate the same energy flux vector $\mathbf{q}^{\text{neq}}$ but differ in their assumptions concerning the energy flux vectors $\mathbf{q}$ and $\mathbf{q}^{\text{eq}}$. This observation holds irrespective of the specific material properties, i.e., whether fluid or solid, and clearly reveals, from the perspective of the adopted thermodynamic framework, the differences in the thermodynamic structure of the two approaches. Which of the two approaches is most suitable clearly depends on the available experimental data.

4.3. Diffusion Coupled with Cattaneo's Heat Conduction Law

Finally, we will briefly outline an example for the adopted non-conventional thermodynamic framework with $N = 2$. It is a combination of the diffusion theory of Gurtin et al. and the Cattaneo heat conduction law as discussed above.

Assume that Equations (175)–(179) and (185)–(188) hold, that $N = 2$, and

$$h = h^{\text{eq}} + h^{\text{neq}}, \quad (211)$$

$$\mathbf{H}_R = \mathbf{H}_R^{\text{eq}} + \mathbf{H}_R^{\text{neq}}, \quad \mathbf{H}_R^{\text{neq}} = -\tau \dot{\mathbf{H}}_R, \quad (212)$$

$$w_R = \mathbf{T}_R \cdot \dot{\mathbf{F}}, \quad w_R^{\text{eq}} = \mathbf{T}_R \cdot \dot{\mathbf{F}} + \rho_R \mu \dot{c} + \mathbf{J}_R \cdot \mathbf{M}_R, \quad w_{R(1)}^{\text{neq}} = -\rho_R \mu \dot{c} - \mathbf{J}_R \cdot \mathbf{M}_R, \quad w_{R(2)}^{\text{neq}} = 0, \quad (213)$$

$$q = h + \mu j, \quad q^{\text{eq}} = h^{\text{eq}}, \quad q_{(1)}^{\text{neq}} = \mu j, \quad q_{(2)}^{\text{neq}} = h^{\text{neq}}, \quad (214)$$

$$\mathbf{q}_R = \mathbf{H}_R + \mu \mathbf{J}_R, \quad \mathbf{q}_R^{\text{eq}} = \mathbf{H}_R^{\text{eq}}, \quad \mathbf{q}_{R(1)}^{\text{neq}} = \mu \mathbf{J}_R, \quad \mathbf{q}_{R(2)}^{\text{neq}} = \mathbf{H}_R^{\text{neq}}. \quad (215)$$

Then, Equation (42)₁,

$$w_{R(1)}^{\text{neq}} = -\rho_R q_{(1)}^{\text{neq}} + \text{DIV } \mathbf{q}_{R(1)}^{\text{neq}} \Leftrightarrow -\rho_R \mu \dot{c} - \mathbf{J}_R \cdot \mathbf{M}_R = -\rho_R \mu j + \text{DIV } (\mu \mathbf{J}_R), \quad (216)$$

corresponds to Equation (188) and Equation (42)₂,

$$w_{R(2)}^{\text{neq}} = \rho_R q_{(2)}^{\text{neq}} + \text{DIV } \mathbf{q}_{R(2)}^{\text{neq}} \Leftrightarrow 0 = -\rho_R h^{\text{neq}} + \text{DIV } \mathbf{H}_R^{\text{neq}}, \quad (217)$$

corresponds to Equation (79). Furthermore, by appealing to Equations (211)–(215), we conclude from Equation (32)₁, that

$$\begin{aligned} \rho_R \dot{e} &= w_R + \rho_R (q^{\text{eq}} + q_{(1)}^{\text{neq}} + q_{(2)}^{\text{neq}}) - \text{DIV } (\mathbf{q}_R^{\text{eq}} + \mathbf{q}_{R(1)}^{\text{neq}} + \mathbf{q}_{R(2)}^{\text{neq}}) \\ &= w_R + \rho_R (h + q_{(1)}^{\text{neq}}) - \text{DIV } (\mathbf{H}_R + \mathbf{q}_{R(1)}^{\text{neq}}) \\ &= w_R + \rho_R \mu j - \text{DIV } (\mu \mathbf{J}_R) + \rho_R h - \text{DIV } \mathbf{H}_R, \end{aligned} \quad (218)$$

and by using Equations (216)₂ and (213)₁,

$$\rho_R \dot{e} = \mathbf{T}_R \cdot \dot{\mathbf{F}} + \rho_R \mu \dot{c} + \mathbf{J}_R \cdot \mathbf{M}_R + \rho_R h - \text{DIV } \mathbf{H}_R, \quad (219)$$

which corresponds to Equation (171)₂.

Making use of Equations (213)₂ and (215), we can rewrite inequality (38)₂ in the form

$$-\rho_R \eta \dot{\theta} + \mathbf{T}_R \cdot \dot{\mathbf{F}} + \rho_R \mu \dot{c} + \mathbf{J}_R \cdot \mathbf{M}_R - \rho_R \dot{\psi} + \mathbf{H}_R^{\text{eq}} \cdot \mathbf{G}_R \geq 0, \quad (220)$$

which corresponds to inequality (194). By recalling Equations (185)₂ and (187), assuming that $\mathbf{H}_R^{\text{eq}}$ does not depend on $\dot{\mathbf{F}}$, and applying the Coleman–Noll procedure, we may derive from inequality (220) the thermodynamic restrictions

$$\mathbf{T}_R = \rho_R \frac{\partial \bar{\psi}}{\partial \mathbf{F}}, \quad (221)$$

$$\mathbf{J}_R \cdot \mathbf{M}_R + \mathbf{H}_R^{\text{eq}} \cdot \mathbf{G}_R \geq 0. \quad (222)$$

Inserting Equation (221) into Equation (219) results in the heat conduction equation (183),

$$-\rho_R \theta \dot{\eta} + \mathbf{J}_R \cdot \mathbf{M}_R + \rho_R h - \text{DIV } \mathbf{H}_R = 0. \quad (223)$$

Sufficient conditions for inequality (222) are

$$\mathbf{J}_R = \frac{\partial \bar{\Phi}_d(\mathbf{M}_R; \bar{\theta}, \bar{\mu})}{\partial \mathbf{M}_R}, \quad (224)$$

with $\bar{\Phi}_d$ being the diffusion potential introduced in Equation (193), and (cf. Equation (94))

$$\mathbf{H}_R^{\text{eq}} = k \mathbf{G}_R. \quad (225)$$

By additionally postulating $\mathbf{H}_R^{\text{eq}} := \mathbf{H}_R - \tau \dot{\mathbf{H}}_R$, we arrive at Cattaneo's heat conduction law (95),

$$\tau \dot{\mathbf{H}}_R + \mathbf{H}_R = k \mathbf{G}_R. \quad (226)$$

Together with the balance of linear momentum (184), these equations provide the referential description of the Gurtin et al.'s diffusion theory coupled with the Cattaneo's heat conduction model.

Finally, it is not difficult to see, that the established theory can be expressed in variational form by statement (207) and (208), with $\bar{\Phi}_d$ as defined in Equation (198), but with $-\bar{\Phi}_{\text{th}}$ replaced by $-\mathbf{H}_R \cdot \mathbf{G}_R$ (cf. Equation (103)),

$$\delta \mathcal{F}(\mathbf{v}, \theta, \dot{\eta}, \mu, \dot{c}) = 0, \quad (227)$$

$$\begin{aligned} \mathcal{F} := & \int_{V_R} \left[\rho_R \dot{e} - \rho_R \theta \dot{\eta} - \rho_R \mu \dot{c} - \bar{\Phi}_d \left(\frac{\mathbf{M}_R}{\theta}; \theta, \bar{\theta}, \bar{\mu} \right) - \mathbf{H}_R \cdot \mathbf{G}_R - \rho_R (\mathbf{b} - \dot{\mathbf{v}}) \cdot \mathbf{v} + \rho_R \mu \dot{j} + \rho_R h \ln \frac{\theta}{\theta_0} \right] dV_R \\ & - \int_{\partial V_R^{\text{tR}}} \mathbf{t}_R^0 \cdot \delta \mathbf{v} dS_R - \int_{\partial V_R^{\text{H}_R}} H_R^0 \frac{\delta \theta}{\theta} dS_R - \int_{\partial V_R^{\text{J}_R}} J_R^0 \delta \mu dS_R. \end{aligned} \quad (228)$$

5. Concluding Remarks

A previously developed non-conventional thermodynamic framework has been extended to also account for energy supply terms. The new framework has been employed to discuss Cattaneo's heat conduction law and to analyze diffusion theories for fluids and

solids. Constitutive models, which are thermodynamically consistent and admit variational formulations, are regarded as properly formulated. From the analysis provided in the paper, we may draw the following conclusions.

1. The non-conventional thermodynamic framework is capable of incorporating Cattaneo's heat conduction model in such a way that the thermal variables involved in the model do not contribute to the energy stored in the material. If deformable bodies are concerned, then aspects of objective time derivatives become relevant. These issues can be treated independently of specific material properties by elaborating the concept of associated objective time derivatives. The resulting field theories for rigid heat conductors and thermoelastic solids can be expressed equivalently by variational formulations of rate type. That means that the considered approaches to Cattaneo's heat conduction model are properly formulated.
2. The diffusion theory of Landau and Lifshitz is expressible in the framework of the adopted non-conventional thermodynamics, possesses a variational formulation of rate type and, hence, is properly formulated. The same is true for the diffusion theory developed by Gurtin, Fried and Anand.
3. Landau and Lifshitz's diffusion theory does not include the diffusion theory of Gurtin, Fried and Anand as a particular case, nor does the latter include the former. The main reason is that, unlike Landau and Lifshitz's theory, the theory of Gurtin, Fried and Anand is based on a non-classical energy law. However, both theories can be formulated as particular cases of the developed non-conventional thermodynamics. This offers the possibility to compare the structures of both diffusion theories from the perspective of this thermodynamics. It turns out that, e.g., both diffusion theories assume the same non-equilibrium energy flux vector, but differ in their definitions of equilibrium, and therefore in the resulting total energy flux vectors.

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Appendix A

Multiply both sides of Equation (3) by $\frac{1}{\tau}e^{\frac{t}{\tau}}$ to obtain

$$\dot{\mathbf{z}}e^{\frac{t}{\tau}} + \frac{1}{\tau}\mathbf{z}e^{\frac{t}{\tau}} = \frac{k}{\tau}\dot{\mathbf{g}}e^{\frac{t}{\tau}}, \quad (\text{A1})$$

which can be rearranged as

$$\frac{d}{dt}(\mathbf{z}(t)e^{\frac{t}{\tau}}) = \frac{k}{\tau}\dot{\mathbf{g}}e^{\frac{t}{\tau}}. \quad (\text{A2})$$

After integration from 0 to t , and assuming the homogeneous initial condition $\mathbf{z}(0) = \mathbf{0}$, we obtain the result (4).

Appendix B

First, we prove that Formula (113) is equivalent to the energy balance law (114). To this end, use $\varphi = e + p\frac{1}{\rho}$, $\mathbf{T} = -p\mathbf{1} + \mathbf{T}^v$ and $\mathbf{T} \cdot \text{grad } \mathbf{v} = \mathbf{T} \cdot \mathbf{D}$ into Formula (113), to obtain

$$\begin{aligned}\frac{\partial}{\partial t} \left(\rho \frac{\mathbf{v} \cdot \mathbf{v}}{2} + \rho e \right) &= -\text{div} \left[\rho \mathbf{v} \frac{\mathbf{v} \cdot \mathbf{v}}{2} + \rho \mathbf{v} \left(e + p\frac{1}{\rho} \right) - \mathbf{T}^v \mathbf{v} + \mathbf{H} \right] \\ &= -\text{div} \left(\rho \mathbf{v} \frac{\mathbf{v} \cdot \mathbf{v}}{2} \right) - \text{div} (\rho e \mathbf{v}) + \text{div} (\mathbf{T}^v \mathbf{v}) - \text{div} \mathbf{H} \\ &= -\text{div} \left(\rho \mathbf{v} \frac{\mathbf{v} \cdot \mathbf{v}}{2} \right) - \text{div} (\rho e \mathbf{v}) + \mathbf{v} \cdot \text{div} \mathbf{T} + \mathbf{T} \cdot \mathbf{D} - \text{div} \mathbf{H},\end{aligned}\quad (\text{A3})$$

or equivalently,

$$\left[\frac{\partial}{\partial t} \left(\rho \frac{\mathbf{v} \cdot \mathbf{v}}{2} \right) + \text{div} \left(\rho \mathbf{v} \frac{\mathbf{v} \cdot \mathbf{v}}{2} \right) \right] + \left[\frac{\partial}{\partial t} (\rho e) + \text{div} (\rho e \mathbf{v}) \right] = \mathbf{v} \cdot \text{div} \mathbf{T} + \mathbf{T} \cdot \mathbf{D} - \text{div} \mathbf{H}. \quad (\text{A4})$$

The two terms in square brackets on the left-hand side can be rewritten with the help of the mass balance (104), the definition of the material time derivative (2), and the balance of linear momentum (106):

$$\begin{aligned}\frac{\partial}{\partial t} \left(\rho \frac{\mathbf{v} \cdot \mathbf{v}}{2} \right) + \text{div} \left(\rho \mathbf{v} \frac{\mathbf{v} \cdot \mathbf{v}}{2} \right) &= \frac{\partial \rho}{\partial t} \frac{\mathbf{v} \cdot \mathbf{v}}{2} + \rho \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \text{div} \left(\rho \mathbf{v} \frac{\mathbf{v} \cdot \mathbf{v}}{2} \right) \\ &= -\frac{\mathbf{v} \cdot \mathbf{v}}{2} \text{div} (\rho \mathbf{v}) + \rho \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \frac{\mathbf{v} \cdot \mathbf{v}}{2} \text{div} (\rho \mathbf{v}) + \rho \mathbf{v} \cdot \text{grad} \frac{\mathbf{v} \cdot \mathbf{v}}{2} \\ &= \rho \mathbf{v} \cdot \left[\frac{\partial \mathbf{v}}{\partial t} + (\text{grad } \mathbf{v}) \mathbf{v} \right] \\ &= \mathbf{v} \cdot \text{div} \mathbf{T},\end{aligned}\quad (\text{A5})$$

$$\begin{aligned}\frac{\partial}{\partial t} (\rho e) + \text{div} (\rho e \mathbf{v}) &= \frac{\partial \rho}{\partial t} e + \rho \frac{\partial e}{\partial t} + e \text{div} (\rho \mathbf{v}) + \rho \mathbf{v} \cdot \text{grad } e \\ &= \rho \left(\frac{\partial e}{\partial t} + (\text{grad } e) \cdot \mathbf{v} \right) \\ &= \rho \dot{e}.\end{aligned}\quad (\text{A6})$$

Substitution of (A5) and (A6) into (A4), and keeping in mind that $\mathbf{w} = \mathbf{T} \cdot \mathbf{D}$, leads to (114).

Next, we verify Equation (117) by recasting the identity

$$\frac{\partial(\rho\eta)}{\partial t} = \frac{\partial \rho}{\partial t} \eta + \rho \frac{\partial \eta}{\partial t} = \frac{\partial \rho}{\partial t} \eta + \rho(\dot{\eta} - \mathbf{v} \cdot \text{grad } \eta). \quad (\text{A7})$$

Inserting Equations (104)₂ and (116) into the last equation gives

$$\begin{aligned}\frac{\partial(\rho\eta)}{\partial t} &= -\eta \text{div} (\rho \mathbf{v}) + \frac{1}{\theta} \mathbf{T}^v \cdot \text{grad } \mathbf{v} - \frac{1}{\theta} \mathbf{J} \cdot \text{grad } \mu - \frac{1}{\theta} \text{div} (\mathbf{H} - \mu \mathbf{J}) - \rho \mathbf{v} \cdot \text{grad } \eta \\ &= -\text{div} (\rho \eta \mathbf{v}) + \frac{1}{\theta} \mathbf{T}^v \cdot \text{grad } \mathbf{v} - \frac{1}{\theta} \mathbf{J} \cdot \text{grad } \mu - \frac{1}{\theta} \text{div} (\mathbf{H} - \mu \mathbf{J}),\end{aligned}\quad (\text{A8})$$

which leads to Equation (117),

$$\begin{aligned}\frac{\partial(\rho\eta)}{\partial t} &= -\text{div} (\rho \eta \mathbf{v}) - \text{div} \frac{\mathbf{H} - \mu \mathbf{J}}{\theta} + \frac{1}{\theta} \mathbf{T}^v \cdot \text{grad } \mathbf{v} \\ &\quad - \frac{1}{\theta} \mathbf{J} \cdot \text{grad } \mu - \frac{(\mathbf{H} - \mu \mathbf{J}) \cdot \text{grad } \theta}{\theta^2}.\end{aligned}\quad (\text{A9})$$

Let now V_0 be a fixed control volume in the region occupied by the deformed body. Since V_0 does not depend on time, integration of Equation (117) (respectively Equation (A9)) over V_0 , and use of the divergence theorem, yields

$$\begin{aligned} \frac{d}{dt} \int_{V_0} \rho \eta \, dV_0 &= \int_{V_0} \frac{\partial(\rho \eta)}{\partial t} \, dV_0 \\ &= - \int_{\partial V_0} \left[\rho \eta \mathbf{v} + \frac{\mathbf{H} - \mu \mathbf{J}}{\theta} \right] \cdot \mathbf{N} \, dS_0 \\ &\quad + \int_{V_0} \left[\frac{1}{\theta} \mathbf{T}^v \cdot \text{grad } \mathbf{v} - \frac{1}{\theta} \mathbf{J} \cdot \text{grad } \mu - \frac{(\mathbf{H} - \mu \mathbf{J}) \cdot \text{grad } \theta}{\theta^2} \right] dV_0. \end{aligned} \quad (\text{A10})$$

The surface integral on the right-hand side represents the entropy flux over the boundary surface, entropy supply is not present, and the volume integral on the right-hand side accounts for the dissipation (entropy production).

Appendix C

In order to make the present paper self-contained, we derive conditions for the coefficients in the Onsager relations (145) and (146).

Let $M, G \in \mathbb{R}^3$ be the component forms of $\mathbf{M}$ and $\mathbf{G}$, respectively, and $\Lambda = \begin{pmatrix} M \\ G \end{pmatrix} \in \mathbb{R}^6$. Then $\tilde{\Phi}$ in Equation (143) may be rewritten as

$$\tilde{\Phi} = \frac{1}{2} \Lambda \cdot \mathbb{A} \Lambda, \quad (\text{A11})$$

where $\mathbb{A} \in \mathbb{R}^6 \times \mathbb{R}^6$ is the block matrix

$$\mathbb{A} = \begin{pmatrix} \alpha_{11} & 0 & 0 & \alpha_{12} & 0 & 0 \\ 0 & \alpha_{11} & 0 & 0 & \alpha_{12} & 0 \\ 0 & 0 & \alpha_{11} & 0 & 0 & \alpha_{12} \\ \alpha_{12} & 0 & 0 & \alpha_{22} & 0 & 0 \\ 0 & \alpha_{12} & 0 & 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{12} & 0 & 0 & \alpha_{22} \end{pmatrix}. \quad (\text{A12})$$

Denoting by $\mathcal{I}_3 \in \mathbb{R}^3 \times \mathbb{R}^3$ the 3×3 identity matrix, we can represent $\mathbb{A}$ also in the form

$$\mathbb{A} = \begin{pmatrix} \alpha_{11} \mathcal{I}_3 & \alpha_{12} \mathcal{I}_3 \\ \alpha_{12} \mathcal{I}_3 & \alpha_{22} \mathcal{I}_3 \end{pmatrix}. \quad (\text{A13})$$

Evidently, the eigenvalues of $\mathbb{A}$ are the same as the eigenvalues of $\mathcal{A} \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$\mathcal{A} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{pmatrix}, \quad (\text{A14})$$

but with multiplicity 3. These are

$$\lambda_{1,2} = \frac{\alpha_{11} + \alpha_{22} \pm \sqrt{(\alpha_{11} - \alpha_{22})^2 + 4\alpha_{12}^2}}{2}, \quad \lambda_1 > \lambda_2. \quad (\text{A15})$$

Since the eigenvalues of a symmetric matrix are real, positive definiteness of $\mathcal{A}$ is equivalent to

$$2\lambda_2 = \alpha_{11} + \alpha_{22} - \sqrt{(\alpha_{11} - \alpha_{22})^2 + 4\alpha_{12}^2} > 0 \quad (\text{A16})$$

$$\Leftrightarrow \alpha_{11}^2 + 2\alpha_{11}\alpha_{22} + \alpha_{22}^2 > \alpha_{11}^2 - 2\alpha_{11}\alpha_{22} + \alpha_{22}^2 + 4\alpha_{12}^2 \quad (\text{A17})$$

$$\Leftrightarrow \alpha_{11}\alpha_{22} > \alpha_{12}^2 \quad (\text{A18})$$

$$\Leftrightarrow \alpha_{11}\alpha_{22} > 0 \quad \text{and} \quad \lambda_1 + \lambda_2 = \alpha_{11} + \alpha_{22} > 0. \quad (\text{A19})$$

That means, $\tilde{\Phi}$ will be positive definite whenever $\alpha_{11} > 0$, $\alpha_{22} > 0$ and $\alpha_{11}\alpha_{22} > \alpha_{12}^2$, which is equivalent to Sylvester's criterion

$$\alpha_{11} > 0, \quad \det \mathcal{A} = \alpha_{11}\alpha_{22} - \alpha_{12}^2 > 0. \quad (\text{A20})$$

If only positive semidefiniteness is required, then $>$ should be replaced by $\geq$.

Appendix D

We shall prove that Equations (160)–(164) lead to the variational statement (165) and (166).

Addition of Equations (160)–(164) furnishes

$$\begin{aligned} \int_V \left[\rho \frac{\partial \bar{e}}{\partial \eta} \delta \dot{\eta} + \rho \frac{\partial \bar{e}}{\partial c} \delta \dot{c} - \delta(\rho \theta \dot{\eta}) - \delta(\rho \mu \dot{c}) + \mathbf{J} \cdot \mathbf{M} \frac{\delta \theta}{\theta} - \mathbf{J} \cdot \delta \mathbf{M} - \mathbf{q}^{\text{eq}} \cdot \delta \mathbf{G} + \mathbf{T} \cdot \delta \mathbf{D} \right. \\ \left. + \mathbf{T}^v \cdot \mathbf{D} \frac{\delta \theta}{\theta} + \rho \dot{\mathbf{v}} \cdot \delta \mathbf{v} \right] dV - \int_{\partial V^t} \mathbf{t}^0 \cdot \delta \mathbf{v} dS - \int_{\partial V^{qb}} q_b^0 \frac{\delta \theta}{\theta} dS - \int_{\partial V^f} J^0 \delta \mu dS = 0. \end{aligned} \quad (\text{A21})$$

The term $\mathbf{J} \cdot \mathbf{M} \frac{\delta \theta}{\theta}$ can be rewritten by substituting ansatz (145) and repeatedly applying Equation (158):

$$\begin{aligned} \mathbf{J} \cdot \mathbf{M} \frac{\delta \theta}{\theta} &= \frac{\partial \tilde{\Phi}(\mathbf{M}, \mathbf{G})}{\partial \mathbf{M}} \cdot \mathbf{M} \frac{\delta \theta}{\theta} = \frac{\partial \tilde{\Phi}(\frac{\mathbf{M}}{\theta}, \mathbf{G}; \theta)}{\partial (\frac{\mathbf{M}}{\theta})} \cdot \mathbf{M} \frac{\delta \theta}{\theta^2} \\ &= - \frac{\partial \tilde{\Phi}(\frac{\mathbf{M}}{\theta}, \mathbf{G}; \theta)}{\partial (\frac{\mathbf{M}}{\theta})} \cdot \mathbf{M} \delta \left(\frac{1}{\theta} \right) \\ &= - \frac{\partial \tilde{\Phi}(\frac{\mathbf{M}}{\theta}, \mathbf{G}; \theta)}{\partial (\frac{\mathbf{M}}{\theta})} \cdot \delta \left(\frac{\mathbf{M}}{\theta} \right) + \frac{\partial \tilde{\Phi}(\frac{\mathbf{M}}{\theta}, \mathbf{G}; \theta)}{\partial (\frac{\mathbf{M}}{\theta})} \cdot \frac{\delta \mathbf{M}}{\theta} \\ &= - \frac{\partial \tilde{\Phi}(\frac{\mathbf{M}}{\theta}, \mathbf{G}; \theta)}{\partial (\frac{\mathbf{M}}{\theta})} \cdot \delta \left(\frac{\mathbf{M}}{\theta} \right) + \frac{\partial \tilde{\Phi}(\mathbf{M}, \mathbf{G})}{\partial \mathbf{M}} \cdot \delta \mathbf{M} \\ &= - \frac{\partial \tilde{\Phi}(\frac{\mathbf{M}}{\theta}, \mathbf{G}; \theta)}{\partial (\frac{\mathbf{M}}{\theta})} \cdot \delta \left(\frac{\mathbf{M}}{\theta} \right) + \mathbf{J} \cdot \delta \mathbf{M}. \end{aligned} \quad (\text{A22})$$

If we add to this equation the terms $-\mathbf{J} \cdot \delta \mathbf{M}$ and $-\mathbf{q}^{\text{eq}} \cdot \delta \mathbf{G} = - \frac{\partial \tilde{\Phi}(\frac{\mathbf{M}}{\theta}, \mathbf{G}; \theta)}{\partial \mathbf{G}} \cdot \delta \mathbf{G}$ (see Equations (158) and (159)), then we find that

$$\mathbf{J} \cdot \mathbf{M} \frac{\delta \theta}{\theta} - \mathbf{J} \cdot \delta \mathbf{M} - \mathbf{q}^{\text{eq}} \cdot \delta \mathbf{G} = - \delta \tilde{\Phi} \left(\frac{\mathbf{M}}{\theta}, \mathbf{G}; \theta \right). \quad (\text{A23})$$

In order to rewrite the term $\mathbf{T} \cdot \delta \mathbf{D} + \mathbf{T}^v \cdot \mathbf{D} \frac{\delta \theta}{\theta}$, we first invoke the decomposition (124)₁ and the potential relations (140) and (157), to obtain

$$\begin{aligned}
\mathbf{T} \cdot \delta \mathbf{D} + \mathbf{T}^v \cdot \mathbf{D} \frac{\delta \theta}{\theta} &= -p \operatorname{tr}(\delta \mathbf{D}) + \mathbf{T}^v \cdot \delta \mathbf{D} + \mathbf{T}^v \cdot \mathbf{D} \frac{\delta \theta}{\theta} \\
&= \frac{\partial \bar{e}(\eta, \frac{1}{\rho}, c)}{\partial(\frac{1}{\rho})} \operatorname{tr} \delta \mathbf{D} + \frac{\partial \bar{\omega}(\mathbf{D})}{\partial \mathbf{D}} \cdot \delta \mathbf{D} + \frac{\partial \bar{\omega}(\mathbf{D})}{\partial \mathbf{D}} \cdot \mathbf{D} \frac{\delta \theta}{\theta} \\
&= \frac{\partial \bar{e}(\eta, \frac{1}{\rho}, c)}{\partial(\frac{1}{\rho})} \operatorname{tr} \delta \mathbf{D} + \frac{\partial \bar{\omega}(\theta \mathbf{D}; \theta)}{\partial(\theta \mathbf{D})} \cdot (\theta \delta \mathbf{D}) + \frac{\partial \bar{\omega}(\theta \mathbf{D}; \theta)}{\partial(\theta \mathbf{D})} \cdot (\mathbf{D} \delta \theta) \\
&= \frac{\partial \bar{e}(\eta, \frac{1}{\rho}, c)}{\partial(\frac{1}{\rho})} \operatorname{tr} \delta \mathbf{D} + \delta \bar{\omega}(\theta \mathbf{D}; \theta).
\end{aligned} \tag{A24}$$

We recall from Equation (120) that $(\frac{1}{\rho})' = \frac{1}{\rho} \operatorname{tr} \mathbf{D} \Rightarrow \delta(\frac{1}{\rho})' = \frac{1}{\rho} \operatorname{tr} \delta \mathbf{D}$ and, hence,

$$\mathbf{T} \cdot \delta \mathbf{D} + \mathbf{T}^v \cdot \mathbf{D} \frac{\delta \theta}{\theta} = \delta \left(\frac{\partial \bar{e}}{\partial(\frac{1}{\rho})} \left(\frac{1}{\rho} \right)' \right) + \delta \bar{\omega}(\theta \mathbf{D}; \theta). \tag{A25}$$

By inserting Equations (A22)–(A25) into (A21), we conclude that

$$\begin{aligned}
&\int_V \left[\delta(\rho \dot{\mathbf{e}}) - \delta(\rho \theta \dot{\eta}) - \delta(\rho \mu \dot{c}) - \delta \Phi \left(\frac{\mathbf{M}}{\theta}, \mathbf{G}; \theta \right) + \delta \bar{\omega}(\theta \mathbf{D}; \theta) + \rho \dot{\mathbf{v}} \cdot \delta \mathbf{v} \right] dV \\
&- \int_{\partial V^t} \mathbf{t}^0 \cdot \delta \mathbf{v} dS - \int_{\partial V^q_b} q_b^0 \frac{\delta \theta}{\theta} dS - \int_{\partial V^f} J^0 \delta \mu dS = 0
\end{aligned} \tag{A26}$$

which is nothing but Equations (165) and (166).

References

1. Dunn, J.E.; Serrin, J. On the thermomechanics of interstitial working. *Arch. Ration. Mech. Anal.* **1985**, *88*, 955–133.
2. Mindlin, R.D. Second gradient of strain and surface-tension in linear elasticity. *Int. J. Solids Struct.* **1965**, *1*, 417–438.
3. Dunn, J. Interstitial working and a nonclassical continuum thermodynamics. In *New Perspectives in Thermodynamics*; Springer: Berlin/Heidelberg, Germany, 1986; pp. 187–222.
4. Toupin, R. Elastic materials with couple-stresses. *Arch. Ration. Mech. Anal.* **1962**, *11*, 385–414.
5. Müller, I. On the entropy inequality. *Arch. Ration. Mech. Anal.* **1967**, *26*, 118–141.
6. Giorgi, C.; Morro, A. On the second law of thermodynamics in continuum physics. *Thermo* **2024**, *4*, 273–294.
7. Iremán, P.; Nguyen, Q.S. Using the gradients of temperature and internal parameters in continuum thermodynamics. *Comptes Rendus Méc.* **2004**, *332*, 249–255.
8. Green, A.; Laws, N. On a global entropy production inequality. *Q. J. Mech. Appl. Math.* **1972**, *25*, 1–11.
9. Alber, H.D.; Hutter, K.; Tsakmakis, C. Nonconventional thermodynamics, indeterminate couple stress elasticity and heat conduction. *Contin. Mech. Thermodyn.* **2016**, *28*, 699–719.
10. Broese, C.; Tsakmakis, C.; Beskos, D. Mindlin's Micro-structural and Gradient Elasticity Theories and Their Thermodynamics. *J. Elast.* **2016**, *125*, 87–132. <https://doi.org/10.1007/s10659-016-9572-7>.
11. Maugin, G.A. *The Thermomechanics of Nonlinear Irreversible Behaviours*; World Scientific: Singapore, 1999; Volume 27.
12. Germain, P.; Nguyen, Q.S.; Suquet, P. Continuum thermodynamics. *J. Appl. Mech.* **1983**, *50*, 1010–1020.
13. Jou, D.; Casas-Vázquez, J.; Lebon, G. Extended irreversible thermodynamics. *Rep. Prog. Phys.* **1988**, *51*, 1105.
14. Coleman, B.; Owen, D. On the nonequilibrium behavior of solids that transport heat by second sound. *Comput. Math. Appl.* **1983**, *9*, 527–546.
15. Sabri Öncü, T.; Brynat Moodie, A. On the constitutive relations for second sound in elastic solids. *Arch. Ration. Mech. Anal.* **1992**, *121*, 87–99.
16. Landau, L.D.; Lifshitz, E.M. *Fluid Mechanics*; Elsevier: Amsterdam, The Netherlands, 1987; Volume 6.
17. De Groot, S.R.; Mazur, P. *Non-Equilibrium Thermodynamics*; Courier Corporation: Washington, DC, USA, 2013.
18. Kondepudi, D.; Prigogine, I. *Modern Thermodynamics: From Heat Engines to Dissipative Structures*; John Wiley & Sons: New York, NY, USA, 2014.

19. Gurtin, M.E.; Fried, E.; Anand, L. *The Mechanics and Thermodynamics of Continua*; Cambridge University Press: Cambridge, UK, 2010.
20. Lubliner, J. On fading memory in materials of evolutionary type. *Acta Mech.* **1969**, *8*, 75–81.
21. Yang, Q.; Stainier, L.; Ortiz, M. A variational formulation of the coupled thermo-mechanical boundary-value problem for general dissipative solids. *J. Mech. Phys. Solids* **2006**, *54*, 401–424.
22. Coleman, B.D.; Noll, W. The thermodynamics of elastic materials with heat conduction and viscosity. In *The Foundations of Mechanics and Thermodynamics: Selected Papers*; Springer: Berlin/Heidelberg, Germany, 1974; pp. 145–156.
23. Romero, I.; Andrés, E.M.; Ortiz-Toranzo, Á. Variational updates for general thermo–chemo–mechanical processes of inelastic solids. *Comput. Methods Appl. Mech. Eng.* **2021**, *385*, 114013.
24. Yang, Q. *Thermomechanical Variational Principles for Dissipative Materials with Application to Strain Localization in Bulk Metallic Glasses*; California Institute of Technology: Pasadena, CA, USA, 2004.
25. Weinberg, K.; Mota, A.; Ortiz, M. A variational constitutive model for porous metal plasticity. *Comput. Mech.* **2006**, *37*, 142–152.
26. Batra, G. On a principle of virtual work for thermo-elastic bodies. *J. Elast.* **1989**, *21*, 131–146.
27. Podio-Guidugli, P. Inertia and invariance. *Ann. Mat. Pura Appl.* **1997**, *172*, 103–124.
28. Joseph, D.D.; Preziosi, L. Heat waves. *Rev. Mod. Phys.* **1989**, *61*, 41.
29. Green, A.; Naghdi, P. A demonstration of consistency of an entropy balance with balance of energy. *Z. Angew. Math. Phys. ZAMP* **1991**, *42*, 159–168.
30. Morro, A. A thermodynamic approach to rate equations in continuum physics. *J. Phys. Sci. Appl* **2017**, *7*, 15–23.
31. Morro, A. Objective equations of heat conduction in deformable bodies. *Mech. Res. Commun.* **2022**, *125*, 103979.
32. Haupt, P.; Tsakmakis, C. Stress tensors associated with deformation tensors via duality. *Arch. Mech.* **1996**, *48*, 347–384.
33. Wcisło, B.; Pamin, J.; Menzel, A. On spatial vs referential isotropic Fourier’s law in finite deformation thermomechanics. *Eng. Trans.* **2023**, *71*, 111–140.
34. Baek, S.; Srinivasa, A. Diffusion of a fluid through an elastic solid undergoing large deformation. *Int. J. Non-Linear Mech.* **2004**, *39*, 201–218.
35. Govindjee, S.; Simo, J.C. Coupled stress-diffusion: Case II. *J. Mech. Phys. Solids* **1993**, *41*, 863–887.
36. Chester, S.A.; Anand, L. A thermo-mechanically coupled theory for fluid permeation in elastomeric materials: Application to thermally responsive gels. *J. Mech. Phys. Solids* **2011**, *59*, 1978–2006.
37. Templet, G.; Steigmann, D. On the theory of diffusion and swelling in finitely deforming elastomers. *Math. Mech. Complex Syst.* **2013**, *1*, 105–128.
38. Wilmers, J.; Bargmann, S. A continuum mechanical model for the description of solvent induced swelling in polymeric glasses: Thermomechanics coupled with diffusion. *Eur. J. Mech.-A/Solids* **2015**, *53*, 10–18.
39. de Anda Salazar, J.; Heuzé, T.; Stainier, L. Multifield variational formulations of diffusion initial boundary value problems. *Contin. Mech. Thermodyn.* **2021**, *33*, 563–589.

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