Article

Generalized $f(R, \phi, X)$ Gravity and the Late-Time Cosmic Acceleration

Sebastian Bahamonde [1], **Christian G. Böhmer** [1,*], **Francisco S.N. Lobo** [2,3] **and Diego Sáez-Gómez** [2]

[1] Department of Mathematics and Institute of Origins, University College London, Gower Street, London WC1E 6BT, UK; E-Mail: sebastian.beltran.14@ucl.ac.uk

[2] Instituto de Astrofísica e Ciências do Espaço, Faculdade de Ciências, Universidade de Lisboa, Campo Grande, PT1749-016 Lisboa, Portugal; E-Mails: fslobo@fc.ul.pt (F.S.N.L.); dsgomez@fc.ul.pt (D.S.-G.)

[3] Faculdade de Ciências da Universidade de Lisboa, Edifício C8, Campo Grande, P-1749-016 Lisboa, Portugal

* Author to whom correspondence should be addressed; E-Mail: c.boehmer@ucl.ac.uk; Tel.: +44-(0)20-76793597; Fax: +44-(0)20-73835519.

Academic Editors: Kazuharu Bamba and Sergei D. Odintsov

Abstract: High-precision observational data have confirmed with startling evidence that the Universe is currently undergoing a phase of accelerated expansion. This phase, one of the most important and challenging current problems in cosmology, represents a new imbalance in the governing gravitational equations. Historically, physics has addressed such imbalances by either identifying sources that were previously unaccounted for or by altering the gravitational theory. Several candidates, responsible for this expansion, have been proposed in the literature, in particular dark energy models and modified gravity models, amongst others. Outstanding questions are related to the nature of this so-called "dark energy" that is driving this acceleration, and whether it is due to the vacuum energy or a dynamical field. On the other hand, the late-time cosmic acceleration may be due to modifications of general relativity. In this work, we explore a generalised modified gravity theory, namely $f(R, \phi, X)$ gravity, where R is the Ricci scalar, ϕ is a scalar field and X is a kinetic term. This theory contains a wide range of dark energy and modified gravity models. We considered specific models and applications to the late-time cosmic acceleration.

Keywords: cosmology; modified gravity; dark energy

1. Introduction

A central theme in cosmology is the fact that the Universe is currently undergoing an accelerating expansion [1,2]. In this context, during the last two decades, cosmology has evolved from being mainly a theoretical area of physics to becoming a field supported by observational data. Recent experiments call upon state-of-the-art technology in astronomy and astrophysics to provide detailed information on the contents and history of the Universe, which has led to the measuring of the parameters that describe our Universe with increasing precision. The standard model of cosmology is remarkably successful in accounting for the observed features of the Universe. However, there remain a number of open questions concerning the foundations of that standard model. In particular, we lack a fundamental understanding of the mechanisms underlying the acceleration of the late-time universe. What is the so-called "dark energy" that is driving the cosmic acceleration? Is it a vacuum energy or a dynamical field? Is the acceleration due to modifications of Einstein's theory of general relativity (GR)? How is structure formation affected in these alternative scenarios? What happens to the Universe in the asymptotic future?

The resolution of these fundamental questions, looking beyond the standard theories of gravity and particle physics, is extremely important for theoretical cosmology and theoretical physics as a whole. The standard model of cosmology has favoured dark energy models as fundamental candidates responsible for the cosmic expansion. However, it is clear that these questions involve not only gravity, but also particle physics. String theory provides a synthesis of these two branches of physics and is widely believed to be moving towards a viable quantum gravity theory, given time. One of the key predictions of string theory is the existence of extra spatial dimensions. In the brane-world scenario, motivated by recent developments in string theory, the observed four-dimensional universe is embedded in a higher-dimensional spacetime [3]. The new degrees of freedom belong to the gravitational sector and can be responsible for the late-time cosmic acceleration [4,5]. On the other hand, generalisations of the Einstein–Hilbert Lagrangian, including quadratic Lagrangians, which involve second order curvature invariants, have also been extensively explored [6–11]. An alternative approach was formulated in [12,13]. Curvature-matter couplings have also been extensively analysed [14–20]. While these modified theories of gravity offer an alternative explanation to the standard cosmological model for the expansion history of the Universe [21], it offers a paradigm for nature fundamentally distinct from dark energy models of cosmic acceleration [22], even those that perfectly mimic the same expansion history. It is also fundamental to understand how one may differentiate these modified theories of gravity from dark energy models.

In this context, a promising way to explain the late-time cosmic acceleration is to assume that at large scales, Einstein's theory of GR breaks down, and a more general action describes the gravitational field. Thus, one may generalise the Einstein–Hilbert action by including "quadratic Lagrangians", involving second order curvature invariants. Some of the physical motivations for these modifications of gravity were related to the possibility of a more realistic representation of the gravitational fields near curvature singularities and to create some first order approximation for the quantum theory of gravitational fields. On the other hand, field theories in theoretical physics tend to have Lagrangians quadratic in field strengths, gravity being somewhat different. One may tackle the problem using the metric formalism, which consists of varying the action with respect to the metric, although other alternative approaches

have been considered in the literature, namely the Palatini formalism [23], where the metric and the connections are treated as separate variables; and the metric-affine formalism, where the matter part of the action now depends and is varied with respect to the connection [24]. Recently, a novel approach to modified theories of gravity consists of adding to the Einstein–Hilbert Lagrangian an $f(R)$ term constructed à la Palatini [25]. It was shown that the theory can pass the Solar System observational constraints even if the scalar field is very light. This implies the existence of a long-range scalar field, which is able to modify the cosmological and galactic dynamics, but leaves the Solar System unaffected. These explicit models are consistent with local tests, lead to the late-time cosmic acceleration and also verify the absence of instabilities in perturbations.

In this work, we intend to consider a generalised modified gravity theory, $f(R, \phi, X)$ gravity, where R is the Ricci scalar, ϕ a scalar field and X a kinetic term. This theory contains a wide range of known dark energy and modified gravity models, for instance $f(R)$ gravity models or Galileons. Here, we apply it to the late-time cosmic acceleration. In particular, several cosmological solutions are studied within the framework of these theories, specifically solutions that can provide cosmic acceleration at late times, and even the exact ΛCDM evolution. Reconstruction techniques are implemented in order to obtain the $f(R, \phi, X)$ of the action given a particular Hubble parameter. This provides a way to efficiently check the viability of any gravitational action by just considering a particular cosmological evolution and then analysing the gravitational action. An action of the form $f(R, \phi, X)$ is in fact quite natural, as it removes any assumptions on the underlying theory of gravity with the exception of being second order. We can think of the field ϕ as the effective field controlling the strength of the gravitational force. One should emphasise that normal matter is still coupled minimally to this theory.

This paper is organised in the following manner: In Section 2, we present the general formalism of $f(R, \phi, X)$ gravity. In Section 3, we analyse cosmological applications, in particular to the late-time cosmic acceleration, by considering specific models and reconstructing the corresponding gravitational action. Then, we discuss in the conclusions at Section 4 about the results obtained and possible future considerations of these theories.

2. Generalised Gravity Models: Formalism

Consider the following general action:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} f(R, \phi, X) + L_m \right] , \tag{1}$$

where $f = f(R, \phi, X)$ is a function of the Ricci scalar R, the scalar field ϕ and a kinetic term $X = -(1/2)(\nabla\phi)^2$. The matter Lagrangian is $L_m = L_m(g, \psi)$; we note that matter is minimally coupled to the gravitational sector of the theory, and ψ collectively denotes any matter fields.

The variation of action Equation (1) with respect to the metric $g_{\mu\nu}$ gives the following field equation:

$$F G_{\mu\nu} = \frac{1}{2} \left(f - R F \right) g_{\mu\nu} + \nabla_\nu F_{,\mu} - g_{\mu\nu} \nabla_\alpha \nabla^\alpha F + \frac{1}{2} f_{,X}\, \phi_{,\mu} \phi_{,\nu} + T^{(\mathrm{m})}_{\mu\nu} , \tag{2}$$

while variations with respect to the scalar field ϕ yield:

$$\nabla_\mu \left(f_{,X}\, \phi^{,\mu} \right) + f_{,\phi} = 0 . \tag{3}$$

Here, $F = \partial f / \partial R$ and $T_{\mu\nu}^{(m)}$ is the matter energy-momentum tensor defined by:

$$T_{\mu\nu}^{(m)} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\, L_m)}{\delta(g^{\mu\nu})}\,. \qquad (4)$$

Now, taking into account a flat Friedman–Lemaître–Robertson–Walker (FLRW) metric in spherical coordinates:

$$ds^2 = -dt^2 + a^2(t) \left[dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi \right) \right]\,, \qquad (5)$$

where $a(t)$ is the scale factor, then Equations (2) and (3) take the following form:

$$3FH^2 = f_{,X}X + \frac{1}{2}(FR - f) - 3H\dot{F} + \kappa^2 \rho_{\mathrm{m}}\,, \qquad (6)$$

$$-2F\dot{H} = f_{,X}X + \ddot{F} - H\dot{F} + \kappa^2(\rho_{\mathrm{m}} + p_{\mathrm{m}})\,, \qquad (7)$$

$$0 = \frac{1}{a^3}\left(a^3 \dot{\phi} f_{,X}\right)^{\cdot} - f_{,\phi}\,, \qquad (8)$$

where $H = \dot{a}/a$ is the Hubble parameter and the overdot denotes a derivative with respect to cosmological time t. For simplicity, we will not take into account additional forms of matter.

In order to derive an equation for the effective equation of state, we rewrite the field Equations (6) and (7) in the following form:

$$3H^2 = \kappa^2 \left(\rho_{\mathrm{m}} + \rho_{\mathrm{DE}} \right)\,, \qquad (9)$$

$$-2\dot{H} = \kappa^2 \left(\rho_{\mathrm{m}} + p_{\mathrm{m}} + \rho_{\mathrm{DE}} + p_{\mathrm{DE}} \right)\,, \qquad (10)$$

where the quantities ρ_{DE} and p_{DE} are defined by:

$$\kappa^2 \rho_{\mathrm{DE}} = f_{,X}X + \frac{1}{2}(FR - f) - 3H\dot{F} + 3H^2(1 - F)\,, \qquad (11)$$

$$\kappa^2 p_{\mathrm{DE}} = \ddot{F} + 2H\dot{F} - \frac{1}{2}(FR - f) - \left(2\dot{H} + 3H^2\right)(1 - F)\,. \qquad (12)$$

One may easily show that the these effective dark energy components satisfy the usual conservation equation, *i.e.*,

$$\dot{\rho}_{\mathrm{DE}} + 3H\left(\rho_{\mathrm{DE}} + p_{\mathrm{DE}}\right) = 0\,. \qquad (13)$$

This is expected since the left-hand sides of Equations (9) and (10) are equivalent to the general relativistic field equations containing the Einstein tensor, which satisfies the twice contracted Bianchi identities.

Let us define the dark energy equation of state given by $\omega_{\mathrm{DE}} \equiv p_{\mathrm{DE}}/\rho_{\mathrm{DE}}$, which by using Equations (9) and (10) takes the form:

$$\omega_{\mathrm{DE}} = -\frac{2\dot{H} + 3H^2 + \kappa^2 p_{\mathrm{m}}}{3H^2 - \kappa^2 \rho_{\mathrm{m}}}\,. \qquad (14)$$

In order to specify the matter content, let us choose the equation of state $p_{\mathrm{m}} = w\rho_{\mathrm{m}}$. In turn, we can write the dark energy equation of state as:

$$\omega_{\mathrm{DE}} = -\frac{1 + \frac{2}{3}\frac{\dot{H}}{H^2} + w\frac{\kappa^2 \rho_{\mathrm{m}}}{3H^2}}{1 - \frac{\kappa^2 \rho_{\mathrm{m}}}{3H^2}} = \frac{\omega_{\mathrm{eff}}}{1 - \tilde{\Omega}_m}\,, \qquad (15)$$

where:

$$\omega_{\text{eff}} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2} - w\tilde{\Omega}_m, \qquad \tilde{\Omega}_m = \frac{\kappa^2 \rho_m}{3H^2} = F\,\Omega_m\,. \tag{16}$$

In the following, we will consider the above equations and seek a solution corresponding to an accelerated expansion.

3. Cosmological Applications: Late-Time Cosmic Acceleration

Let us consider some specific forms of the action (1) and reconstruct particular cosmological solutions capable of reproducing the late-time cosmic acceleration. We are considering pure de Sitter solutions, also power law solutions, and finally, we are reconstructing an exact ΛCDM model. Note that this kind of cosmological solution has already been considered in the literature within modified gravity theories, especially in $f(R)$ gravity and Gauss–Bonnet gravity. In particular, power-law solutions were found in [26,27], while exact the ΛCDM model was reconstructed in [28–30]. In order to simplify the calculations, we restrict to a matter Lagrangian with a pressure-less fluid, as natural when analysing late-time cosmology.

3.1. Brans–Dicke Type Models

Here, we consider a Brans–Dicke-type action with $f(R, X, \phi) = \gamma(X, \phi)R$, where the coupling to the Ricci curvature also includes the kinetic term of the scalar field,

$$S = \int d^4 x \sqrt{-g}\left[\frac{1}{2\kappa^2}\gamma(X, \phi)R + L_{\text{m}}\right]\,. \tag{17}$$

Then, by considering an homogeneous and isotropic metric Equation (5), the FLRW Equations (6) and (7) are expressed as follows:

$$3\gamma(X, \phi)H^2 = \gamma_{,X}XR - 3H\frac{d}{dt}\gamma(X(t), \phi(t)) + \kappa^2 \rho_m\,,$$

$$-2\gamma(X, \phi)\dot{H} = \gamma_{,X}XR + \frac{d^2}{dt^2}\gamma(X(t), \phi(t)) - H\frac{d}{dt}\gamma(X(t), \phi(t)) + \kappa^2(\rho_{\text{m}} + p_{\text{m}})\,. \tag{18}$$

Additionally, combining both equations, a differential equation solely of $\gamma(X(t), \phi(t)) = \gamma(t)$ is obtained:

$$\frac{d^2\gamma(t)}{dt^2} + 2H\frac{d\gamma(t)}{dt} + (3H^2 + 2\dot{H})\gamma(t) = 0\,. \tag{19}$$

Then, Equation (19) can be solved by assuming a particular Hubble parameter, and action Equation (17) can be reconstructed by assuming a particular form of $\gamma(X, \phi)$, which has to satisfy:

$$\gamma_{,X}X = \frac{3\gamma H^2 + 3H\dot{\gamma} - \kappa^2 \rho_{\text{m}}}{R}\,, \tag{20}$$

As an example for illustrating the reconstruction method, here we consider the usual form of a scalar field Lagrangian,

$$\gamma(X, \phi) = X - V(\phi)\,, \tag{21}$$

where $V(\phi)$ is a function of the scalar field to be determined. Then, by redefining the scalar field to coincide with the cosmic time $\phi = t$ and using Equations (20) and (21), the kinetic term and the scalar potential are fully reconstructed:

$$
\begin{aligned}
X(\phi) &= \frac{3\gamma H^2 + 3H\dot\gamma - \kappa^2 \rho_0 \mathrm{e}^{-3\int d\phi H}}{R}, \\
V(\phi) &= X(\phi) - \gamma(\phi).
\end{aligned}
\tag{22}
$$

where we have used the continuity Equation $\dot\rho_{\mathrm m} + 3H\rho_{\mathrm m} = 0$.

Let us now consider several cosmological solutions. Firstly, we assume a pure de Sitter solution,

$$
H = H_0,
\tag{23}
$$

where H_0 is a constant. Then, the Equation (19) leads to:

$$
\frac{d^2\gamma(t)}{dt^2} + 2H_0\frac{d\gamma(t)}{dt} + 3H_0^2\gamma(t) = 0.
\tag{24}
$$

which is the equation of a damped harmonic oscillator, whose general solution is given by:

$$
\gamma(t) = \mathrm{e}^{-H_0 t} C \cos\sqrt{2}H_0(t - t_0),
\tag{25}
$$

where $\{C, t_0\}$ are integration constants. Then, assuming Equation (21), the gravitational action is reconstructed by the expressions Equation (22), where the kinetic term and the scalar potential turn out:

$$
\begin{aligned}
X(\phi) &= -\frac{C}{2\sqrt{2}}\mathrm{e}^{-H_0\phi}\sin\sqrt{2}H_0(\phi - \phi_0) - \frac{\kappa^2\rho_0}{12H_0^2}\mathrm{e}^{-3H_0\phi}, \\
V(\phi) &= -\frac{C}{4}\mathrm{e}^{-H_0\phi}\left[4\cos\sqrt{2}H_0(\phi - \phi_0) + \sqrt{2}\sin\sqrt{2}H_0(\phi - \phi_0)\right] - \frac{\kappa^2\rho_0}{12H_0^2}\mathrm{e}^{-3H_0\phi},
\end{aligned}
\tag{26}
$$

The interesting point of this action is that it leads to a constant Hubble parameter in the presence of dust matter and a scalar field, but without a cosmological constant, contrary to standard general relativity.

Another important class of solutions in cosmology is the power-law-type expansion, whose Hubble parameter is given as follows:

$$
H = \frac{n}{t} \quad\rightarrow\quad a(t) = a_0 t^n.
\tag{27}
$$

By applying the same procedure as above, Equation (19) turns out to be:

$$
t^2\frac{d^2\gamma(t)}{dt^2} + 2nt\frac{d\gamma(t)}{dt} + n(3n - 2)\gamma(t) = 0,
\tag{28}
$$

which is a Cauchy–Euler equation whose general solution is given by:

$$
\gamma(t) = C_1 t^{p+q} + C_2 t^{p-q},
\tag{29}
$$

where C_1 and C_2 are two constants of integration, and we defined:

$$
p = \frac{1 - 2n}{2}, \qquad q = \frac{\sqrt{1 + 4n - 8n^2}}{2}.
\tag{30}
$$

By assuming Equation (21) as above, the kinetic term and the scalar potential are obtained:

$$
\begin{aligned}
X(\phi) &= \frac{1}{4(2n-1)}\left[C_1(1+2q)\phi^{p+q} + C_2(1-2q)\phi^{p-q} - \frac{1}{3N}2\kappa^2\rho_0\phi^{2-3n}\right], \\
V(\phi) &= \frac{1}{4(2n-1)}\left[C_1(5-8n+2q)\phi^{p+q} + C_2(5-8n-2q)\phi^{p-q} - \frac{1}{3N}2\kappa^2\rho_0\phi^{2-3n}\right].
\end{aligned} \tag{31}
$$

Hence, the gravitational action capable of reproducing the power-law solutions Equation (27) is obtained.

Finally, let us consider the case of exact ΛCDM and explore whether a gravitational action of the type Equation (17) is capable of reproducing such a kind of expansion. In order to simplify the calculations, here we use the red shift $1 + z = \frac{1}{a}$ as the independent variable instead of the cosmic time t. The cosmological evolution of ΛCDM model can be expressed by the following Hubble parameter:

$$
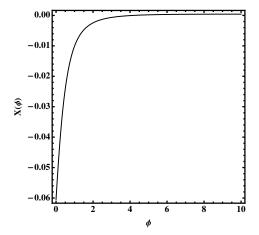H(z) = H_0\sqrt{\Omega_m(1+z)^3 + 1 - \Omega_m}, \tag{32}
$$

where $\Omega_m = \frac{\rho_0}{3H_0^2/\kappa^2}$ and H_0 is the Hubble parameter evaluated at $z = 0$. Then, the Equation (19) can be rewritten in terms of the red shift as,

$$
(1+z)^2H^2\gamma'' + \left[(1+z)^2HH' - (1+z)H^2\right]\gamma' + \left[3H^2 - 2(1+z)HH'\right]\gamma = 0, \tag{33}
$$

where the primes denote derivatives with respect to the red shift. Hence, the Equation (33) can be solved as $\gamma(X, \phi) = \gamma(z)$. Particularly, for the ΛCDM model, the Equation (33) leads to:

$$
2(1+z)^2\left[\Omega_m z(z^2 + 3z + 3) + 1\right]\gamma''(z) + (1+z)\left[\Omega_m(z^3 + 3z^2 + 3z + 3) - 2\right]\gamma'(z) + 6(1-\Omega_m)\gamma(z) = 0. \tag{34}
$$

Unfortunately, this equation cannot be solved exactly, but numerical resources are required. In order to illustrate a particular case, we solve the Equation (34) by assuming $\Omega_m = 0.31$, provided by the last Planck data [31]. The kinetic term and the scalar potential are depicted in Figure 1. Note that we have assumed the initial conditions $\gamma(z = 10) = 1$ and $\gamma'(z = 10) = 0$, so that GR is recovered at large red shifts. As expected, the dynamics of the scalar field become important at small red shifts while remaining virtually constant at large ones.
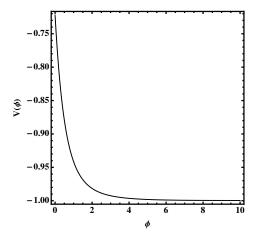

Figure 1. The kinetic term (left panel) and scalar potential (right panel) for the ΛCDM model Equation (32) with $\Omega_m = 0.31$.

3.2. Minimally-Coupled Field Model

Let us now consider another different type of gravitational action Equation (1), where the scalar field is minimally coupled. We choose $f(R, X, \phi) = \alpha(R) + \gamma(X, \phi)$, so that:

$$S = \int dx^4 \sqrt{-g} \left[\alpha(R) + \gamma(X, \phi) + L_{\mathrm{m}}\right] . \tag{35}$$

This action can be seen as an $f(R)$-like action with a k-essence-like fluid in the presence of $f(R)$ gravity. Then, the FLRW Equations (6) and (7) turn out to be:

$$\begin{aligned}
3\alpha_{,R} H^2 &= \gamma_{,X} X + \frac{1}{2}\left(\alpha_{,R} R - \alpha - \gamma\right) - 3H\dot{\alpha}_{,R} + \kappa^2 \rho_m , \\
-2\alpha_{,R} \dot{H} &= \gamma_{,X} X + \ddot{\alpha}_{,R} - H\dot{\alpha}_{,R} + \kappa^2 \rho_m ,
\end{aligned} \tag{36}$$

which can be combined, so that an equation for $\gamma(X, \phi) = \gamma(t)$ is obtained,

$$\gamma(t) = -\alpha + \left(R - 6H^2 - 4\dot{H}\right)\alpha_{,R} - \left(4H\dot{R} + 2\ddot{R}\right)\alpha_{,RR} - 2\dot{R}^2 \alpha_{,RRR} . \tag{37}$$

Then, by specifying $\alpha(R)$ and providing a particular Hubble evolution $H(t)$, the function $\gamma(t)$ is determined. The complete function $\gamma(X, \phi)$ is provided by assuming a particular form of the scalar field Lagrangian that satisfy:

$$\gamma_{,X} X = \frac{1}{2}\alpha + \left(3H^2 - \frac{1}{2}R\right)\alpha_{,R} + 3H\dot{\alpha}_{,R} - \kappa^2 \rho_m + \frac{1}{2}\gamma . \tag{38}$$

In order to reconstruct some particular solutions, we are considering the following $\alpha(R)$,

$$\alpha(R) = R + aR^2 , \tag{39}$$

where a is a constant free parameter with the appropriate dimensions. This gravitational action is a very well known one in the literature, as it is capable of reproducing inflation [32]. As above, we start by considering a pure de Sitter solution Equation (23), $H = H_0$. In such a case, the expression of $\gamma(t)$ in Equation (37) yields:

$$\gamma = -6H_0^2 . \tag{40}$$

We can assume again $\gamma(X, \phi) = X - V(\phi)$ for illustrating the full reconstruction of the action Equation (35) and redefine the scalar field as $\phi = t$. Then, the kinetic term $X(\phi)$ and the scalar potential $V(\phi)$ turn out to be:

$$X(\phi) = X_0 e^{-3H_0\phi} , \qquad V(\phi) = X_0 e^{-3H_0\phi} + 6H_0^2 , \tag{41}$$

where $X_0 = -\kappa^2 \rho_0$.

In the same way as above, power-law solutions can be easily reconstructed. By assuming $H(t) = \frac{n}{t}$, the expression Equation (37) leads to:

$$\gamma(t) = 2n\frac{18a(4 - 11n + 6n^2) + (2 - 3n)t^2}{t^4} , \tag{42}$$

while the kinetic term and the scalar potential yield:

$$
\begin{aligned}
X(\phi) &= X_0\phi^{-3n} + 2n\frac{a(36-72n)+\phi^2}{\phi^4}, \\
V(\phi) &= X_0\phi^{-3n} + \frac{-36an(2+n(-7+6n))+2n(-1+3n)\phi^2}{\phi^4},
\end{aligned}
\tag{43}
$$

where we have again assumed $\gamma(X,\phi) = X - V(\phi)$ for illustrative purposes.

Finally, let us reconstruct the gravitational action for the ΛCDM model. In such a case, the Hubble parameter can be expressed as follows:

$$
H(t) = \sqrt{\frac{\Lambda}{3}}\tanh\left[\frac{1}{2}\sqrt{3\Lambda}(t-t_0)\right],
\tag{44}
$$

where Λ and t_0 are constants. Then, by using the Equation (37), the expression for $\gamma(t)$ yields:

$$
\gamma(t) = -2\Lambda - 9a\Lambda^2\operatorname{sech}^4\left[\frac{1}{2}\sqrt{3\Lambda}(t-t_0)\right].
\tag{45}
$$

While assuming $\gamma(X,\phi) = X - V(\phi)$, the kinetic term and the scalar potential result:

$$
\begin{aligned}
X(\phi) &= -9a\Lambda^2\operatorname{sech}^4\left[\frac{1}{2}\sqrt{3\Lambda}(\phi-t_0)\right], \\
V(\phi) &= 2\Lambda.
\end{aligned}
\tag{46}
$$

Hence, the scalar potential reduces to a cosmological constant, while the dynamics of the scalar field compensates the extra terms in the gravitational sector Equation (39).

Another interesting set of cosmological models are the so-called scaling solutions, which are characterized by:

$$
\frac{\rho}{\rho_m} = C,
\tag{47}
$$

where C is a constant and ρ is the energy density of an additional perfect fluid. We can rewrite the first FLRW Equation (36) as follows:

$$
3\frac{\alpha_{,R}}{\kappa^2}H^2 = \rho + \rho_m,
\tag{48}
$$

where:

$$
\rho = \frac{1}{\kappa^2}\left[\gamma_{,X}X + \frac{1}{2}\left(\alpha_{,R}R - \alpha - \gamma\right) - 3H\dot{\alpha}_{,R}\right],
\tag{49}
$$

which leads to the constraint equation:

$$
\frac{1}{\kappa^2}\left[\gamma_{,X}X + \frac{1}{2}\left(\alpha_{,R}R - \alpha - \gamma\right) - 3H\dot{\alpha}_{,R}\right] - C\rho_{m0}e^{-3\int dt H(t)} = 0.
\tag{50}
$$

By considering the Hilbert–Einstein action, $\alpha(R) = R$, together with a kind of k-essence field that satisfies $\rho/\rho_m = C$, the Equation (48) can be easily solved by using the continuity equation leading to:

$$
a(t) \propto t^{2/3} \qquad H(t) = \frac{2}{3t},
\tag{51}
$$

while the Equation (50) leads to:

$$
\gamma_{,X}X - \frac{1}{2}\gamma = C\frac{\kappa^2\rho_0}{t^2}.
\tag{52}
$$

Additionally, the second FLRW Equation (36) provides an additional condition over the scalar field sector:

$$\gamma_{,X} X = \frac{4 - 3\kappa^2 \rho_0}{3t^2} \,, \tag{53}$$

which together with Equation (52) allows one to reconstruct the scalar field Lagrangian once a particular form of γ is assumed. In order to illustrate the procedure, let us consider the usual quintessence field $\gamma = X(\phi) - V(\phi)$. Then, the kinetic term and the scalar potential yield:

$$
\begin{aligned}
X(\phi) &= \frac{4 - 3\kappa^2 \rho_0}{3\phi^2} \,, \\
V(\phi) &= \frac{\kappa^2 \rho_0 (6C + 1) - 4}{3\phi^2} \,,
\end{aligned}
\tag{54}
$$

where once again we have redefined the scalar field as the cosmic time, $\phi = t$. Note that the potential Equation (54) is very well known as one of the possible scalar potentials that provides scaling solutions within quintessence models, others are the exponential ones $V(\phi) \propto e^{\phi}$; see [33,34].

4. Conclusions

In this work, we explored a generalised modified gravity theory, namely $f(R, \phi, X)$ gravity, where R is the Ricci scalar, ϕ a scalar field and X a kinetic term. This theory contains a wide range of dark energy and modified gravity models, particularly higher derivative models, both in the gravitational sector and also in the matter one. Some models within this action include $f(R)$ modified gravity and Galileons, which have been widely studied in the literature. We have then considered some specific Hubble parameters, specifically those that reproduce late-time cosmic acceleration and a realistic cosmological evolution, and the corresponding $f(R, \phi, X)$ action has been reconstructed.

Using the reconstruction technique studied here, the gravitational action can be easily obtained once some restrictions are assumed on the kind of function $f(R, \phi, X)$. We note that this action carries several extra degrees of freedom, so a particular cosmological solution does not provide a unique action; a degeneracy problem as in every dark energy model. However, we suggest some simple actions that reproduce late-time acceleration and even the exact ΛCDM behaviour. Future works should be devoted to studying the viability of the different actions within the $f(R, \phi, X)$ gravity framework. Besides the cosmological evolution, one should study cosmological perturbation theory with the view of understanding the growth of structure. Another interesting point may arise when analysing screening-like mechanisms, similar to the chameleon mechanism in this framework. Note also that the cases of the gravitational action Equation (17) studied here stand for a minority of the wide class of models included in the action Equation (17), and some interesting ones to be considered in future papers may be of the type $f_1(R) f_2(X, \phi)$. Taking into account various observational facts about cosmology, it is hoped that one can restrict the form of the action $f(R, \phi, X)$ and provide a clearer way for explaining the late-time acceleration and even its unification with the inflationary paradigm.

Acknowledgements

Author Contributions

All of the authors have substantially contributed to the present work.

Conflicts of Interest

The authors declare no conflict of interest.

References

1. Perlmutter, S.; Aldering, G.; Goldhaber, G.; Knop, R.A.; Nugent, P.; Castro, P.G.; Deustua, S.; Fabbro, S.; Goobar, A.; Groom, D.E.; *et al* . Measurements of Omega and Lambda from 42 high red shift supernovae. *Astrophys. J.* **1999**, *517*, 565–586.
2. Riess, A.G.; Filippenko, A.V.; Challis, P.; Clocchiattia, A.; Diercks, A.; Garnavich, P.M.; Gilliland, R.L.; Hogan, C.J.; Jha, S.; Kirshner, R.P.; *et al*. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *Astron. J.* **1998**, *116*, 1009–1038.
3. Maartens, R. Brane world gravity. *Living Rev. Relativ.* **2004**, *7*, doi:10.12942/lrr-2010-5.
4. Dvali, G.R.; Gabadadze, G.; Porrati, M. 4-D gravity on a brane in 5-D Minkowski space. *Phys. Lett. B* **2000**, *485*, 208–214.
5. De Rham, C.; Hofmann, S.; Khoury, J.; Tolley, A.J. Cascading Gravity and Degravitation. *J. Cosmol. Astropart. Phys.* **2008**, *2008*, 011.
6. Sotiriou, T.P.; Faraoni, V. $f(R)$ Theories Of Gravity. *Rev. Mod. Phys.* **2010**, *82*, 451–497.
7. De Felice, A.; Tsujikawa, S. $f(R)$ theories. *Living Rev. Relativ.* **2010**, *13*, 3.
8. Capozziello, S.; De Laurentis, M. Extended Theories of Gravity. *Phys. Rep.* **2011**, *509*, 167–321.
9. Lobo, F.S.N. The Dark side of gravity: Modified theories of gravity. **2008**, arXiv:0807.1640[gr-qc].
10. Nojiri, S.; Odintsov, S.D. Unified cosmic history in modified gravity: From $F(R)$ theory to Lorentz non-invariant models. *Phys. Rep.* **2011**, *505*, 59–144.
11. De la Cruz-Dombriz, A.; Saez-Gomez, D. Black holes, cosmological solutions, future singularities, and their thermodynamical properties in modified gravity theories. *Entropy* **2012**, *14*, 1717–1770.
12. Boehmer, C.G.; Tamanini, N. A New Approach to Modifying Theories of Gravity. *Found. Phys.* **2013**, *43*, 1478–1488.
13. Boehmer, C.G.; Tamanini, N.; Wright, M. On galaxy rotation curves from a new approach to modified gravity. **2014**, arXiv:gr-qc/1403.4110.

14. Allemandi, G.; Borowiec, A.; Francaviglia, M.; Odintsov, S.D. Dark energy dominance and cosmic acceleration in first order formalism. *Phys. Rev. D* **2005**, *72*, 063505.

15. Bertolami, O.; Boehmer, C.G.; Harko, T.; Lobo, F.S.N. Extra force in $f(R)$ modified theories of gravity. *Phys. Rev. D* **2007**, *75*, 104016.

16. Harko, T.; Koivisto, T.S.; Lobo, F.S.N. Palatini formulation of modified gravity with a nonminimal curvature-matter coupling. *Mod. Phys. Lett. A* **2011**, *26*, 1467–1480.

17. Harko, T.; Lobo, F.S.N. Generalized curvature-matter couplings in modified gravity. *Galaxies* **2014**, *2*, 410–465.

18. Harko, T.; Lobo, F.S.N.; Nojiri, S.; Odintsov, S.D. $f(R,T)$ gravity. *Phys. Rev. D* **2011**, *84*, 024020.

19. Haghani, Z.; Harko, T.; Lobo, F.S.N.; Sepangi, H.R.; Shahidi, S. Further matters in space-time geometry: $f(R,T,R_{\mu\nu}T^{\mu\nu})$ gravity. *Phys. Rev. D* **2013**, *88*, 044023.

20. Odintsov, S.D.; Sáez-Gómez, D. $f(R,T,R_{\mu}\nu T^{\mu}\nu)$ gravity phenomenology and ΛCDM universe. *Phys. Lett. B* **2013**, *725*, 437–444.

21. Carroll, S.M.; Duvvuri, V.; Trodden, M.; Turner, M.S. Is cosmic speed-up due to new gravitational physics? *Phys. Rev. D* **2004**, *70*, 043528.

22. Copeland, E.J.; Sami, M.; Tsujikawa, S. Dynamics of dark energy. *Int. J. Mod. Phys. D* **2006**, *15*, 1753–1936.

23. Olmo, G.J. Palatini Approach to Modified Gravity: $f(R)$ Theories and Beyond. *Int. J. Mod. Phys. D* **2011**, *20*, 413–462.

24. Sotiriou, T.P.; Liberati, S. Metric-affine $f(R)$ theories of gravity. *Ann. Phys.* **2007**, *322*, 935–966.

25. Harko, T.; Koivisto, T.S.; Lobo, F.S.N.; Olmo, G.J. Metric-Palatini gravity unifying local constraints and late-time cosmic acceleration. *Phys. Rev. D* **2012**, *85*, 084016.

26. Goheer, N.; Larena, J.; Dunsby, P.K.S. Power-law cosmic expansion in $f(R)$ gravity models. *Phys. Rev. D* **2009**, *80*, 061301.

27. Myrzakulov, R.; Sáez-Gómez, D.; Tureanu, A. On the ΛCDM Universe in $f(G)$ gravity. *Gen. Relativ. Gravit.* **2011**, *43*, 1671–1684.

28. De la Cruz-Dombriz, A.; Dobado, A. A $f(R)$ gravity without cosmological constant. *Phys. Rev. D* **2006**, *74*, 087501.

29. Elizalde, E.; Myrzakulov, R.; Obukhov, V.V.; Sáez-Gómez, D. LambdaCDM epoch reconstruction from $F(R,G)$ and modified Gauss-Bonnet gravities. *Class. Quant. Grav.* **2010**, *27*, 095007.

30. Dunsby, P.K.S.; Elizalde, E.; Goswami, R.; Odintsov, S.; Sáez-Gómez, D. On the LCDM Universe in $f(R)$ gravity. *Phys. Rev. D* **2010**, *82*, 023519.

31. Ade, P.A.R.; Aghanim, N.; Arnaud, M.; Ashdown, M.; Aumont, J.; Baccigalupi, C.; Banday, A.J.; Barreiro, R.B.; Bartlett, J.G.; Bartolo, N.; *et al.* Planck 2015 results. XIII. Cosmological parameters. **2015**, arXiv:astro-ph.CO/1502.01589.

32. Starobinsky, A.A. A New Type of Isotropic Cosmological Models Without Singularity. *Phys. Lett. B* **1980**, *91*, 99–102.

33. Tsujikawa, S.; Sami, M. A Unified approach to scaling solutions in a general cosmological background. *Phys. Lett. B* **2004**, *603*, 113–123.

34. Copeland, E.J.; Lee, S.J.; Lidsey, J.E.; Mizuno, S. Generalised cosmological scaling solutions. *Phys. Rev. D* **2005**, *71*, 023526.