Black Holes and Exotic Spinors

J. M. Hoff da Silva [1], C. H. Coronado Villalobos [1] and Roldao da Rocha [2,*]

[1] Departamento de Física e Química, Universidade Estadual Paulista, Guaratinguetá, São Paulo 19060-900, Brazil; hoff@feg.unesp.br (J.M.H.d.S.); ccoronado@feg.unesp.br (C.H.C.V.)

[2] Centro de Matemática, Computação e Cognição, Universidade Federal do ABC, UFABC, 09210-580 Santo André, Brazil

[*] Correspondence: roldao.rocha@ufabc.edu.br

Abstract: Exotic spin structures are non-trivial liftings, of the orthogonal bundle to the spin bundle, on orientable manifolds that admit spin structures according to the celebrated Geroch theorem. Exotic spin structures play a role of paramount importance in different areas of physics, from quantum field theory, in particular at Planck length scales, to gravity, and in cosmological scales. Here, we introduce an in-depth panorama in this field, providing black hole physics as the fount of spacetime exoticness. Black holes are then studied as the generators of a non-trivial topology that also can correspond to some inequivalent spin structure. Moreover, we investigate exotic spinor fields in this context and the way exotic spinor fields branch new physics. We also calculate the tunneling probability of exotic fermions across a Kerr-Sen black hole, showing that the exotic term does affect the tunneling probability, altering the black hole evaporation rate. Finally we show that it complies with the Hawking temperature universal law.

Keywords: exotic spinor; black holes

1. Introduction

Black holes (BHs) are most known in literature by their idiosyncratic role in the appreciation of the physical laws associated to them. In fact, they perform one of those examples of physical phenomena in which plenty of sometimes-fundamental concepts must be employed, in order to be better understood. General Relativity (GR), Quantum Mechanics, and Thermodynamics, are some of the theories, whose basilar concepts relate to the study of BHs.

In essence, BHs are a manifestation of an utterly disturbed spacetime at the point where they are placed. There is nothing new in envisaging BHs in this way, and this shall also be the point of view adopted in this work. The peculiarity here rests upon the fact that, spacetime being a curved base manifold, the very existence of BHs can be faced as generating non-trivial topologies. We delve into this question for the following reason: it is a very well-known fact that only vector/tensorial objects are naturally accommodated in a theory based on general coordinates transformation, as GR. Hence, in order to achieve half-integer representations of the Poincaré group, it is necessary to resort to the tangent bundle concept, which, in turn, is constructed from the union of all tangent spaces to the base manifold. This entire program being fulfilled, then spinors arise quite naturally as elements carrying the aforementioned half-integer representations of the Poincaré group. What is the effect, however, of a given point (or region) where the tangent space, and, consequently, the tangent bundle, cannot be defined? Roughly speaking, it would jeopardize the achievement of irreducible representations of the underlying group. From the GR point of view, at that point, or region, there is no meaning in any geometric quantity, since invariance under general coordinate transformation is lost. A similar effect is shared by representations of the Poincaré group.

To circumvent the problem just outlined, different patches of local coverings are used, avoiding the ill-defined region. Nevertheless, once it is accomplished, the possibility of different, and non-equivalent, spin structures to be defined appears [1,2]. The resulting spinors from elements of those sections are the so-called exotic spinors. Essentially, here we are interested in BH as non-trivial topology generators, in the sense that by their presence, the first homotopy group (and also the first homology group) of the base manifold—the spacetime itself—is rendered non-trivial. We shall, however, pursue this reasoning by giving a complementary account of exotic spinors, based upon the Cartan spinors view [3].

This work is presented as follows: in the next section, we formalize an intuitive approach to understand the existence of exotic spinors. After that, we make a connection to the irreducible Poincaré group representations. Within this context, BHs enter as non-trivial topology generators. In Section 2, we explore, on an argumentative basis, further consequences of BH as exotic spinor providers. Moreover, we compute the tunneling probability of exotic fermions across a Kerr-Sen black hole—that encompasses the Kerr and Schwarzschild BH—showing that the exotic term does affect the tunneling probability, altering the black hole evaporation rate, however still in accordance with the Hawking temperature universal law. In Section 3, we conclude the paper.

2. Topological Effects of a Spacetime Riddled by Black Holes

The relevance of global features of spacetime is huge, in diverse areas of physics. Relativity can be partially incorporated in quantum field theory with the requirement of the Lorentz invariance under the Lorentz group $SL(2,\mathbb{C}) \simeq Spin_+(1,3)$, namely, the spin group associated with Minkowski spacetime, which is the double covering of $SO(1,3)$. The associated Lie algebra merely determines the group local structure, and some surprises occur when either the group is not connected or when it is not simply connected. In fact, the fundamental group of $SO(1,3)$ is given by $\pi_1(SO(1,3)) = \mathbb{Z}$. Hence, local physics suggests the appearance of spin.

Before investigating the consequence of the BH presence in a given spacetime, concerning the existence of exotic spinors, let us make some effort to evince what a spinor is, in a context relevant for our purposes, reviewing part of the well-known literature. We shall take advantage of the work carried out by Penrose [4]. We consider a given spacetime, with trivial topology, as the base manifold $\mathcal{M}$. This manifold being curved, we define its tangent bundle as the union of all the tangent spaces to $\mathcal{M}$ at every point $p \in \mathcal{M}$. One can then present, in a summarized way, the underlying idea regarding exotic spin structures that can endow an orientable manifold $\mathcal{M}$. The tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ has structure group $O(n)$. The associated frame bundle has transition functions $g_{ij} : U_i \cap U_j \to O(n)$, where $\{U_k\}$ is a set of open sets in the frame bundle [5]. Moreover, a set of functions $f(U_i, U_j) = \det g_{ij} = \pm 1$ in a Čech chain also defines a cocycle, since $g_{ij}g_{jk}g_{ki} = \mathrm{Id}_{O(n)}$. Hence, it is also an element of the Čech cohomology class, namely, the first Stiefel–Whitney class w_1. By restricting to the connected component of the orthogonal group, the frame bundle transition functions are elements of $SO(n)$. When the frame bundle is lifted, a spin bundle, Spin(n), a spin structure can be thus defined by the covering mapping $\phi : Spin(n) \to SO(n)$. Clearly, the transition functions $\mathring{g}_{ij} \in Spin(n)$ are mapped $\phi(\mathring{g}_{ij}) = g_{ij}$, accordingly. The tangent bundle $T\mathcal{M}$ admits a spin bundle structure if and only if $w_1(\mathcal{M})$ and $w_2(\mathcal{M})$ are trivial. In this sense, a unique transport of spinors exists. The spin structures are in one-to-one correspondence with the elements of the cohomology class $H^1(\mathcal{M}, \mathbb{Z}_2)$. The principal bundle Spin(n) can be associated to a vector bundle with fibers $\mathbb{C}^k$, whose sections are spinors. To any non-trivial spin structure, a non-trivial element of $H^1(\mathcal{M}, \mathbb{Z}_2)$ corresponds. Hence, the sections of the vector bundle are the so-called exotic spinors. When the metric signature is taken into account, any pseudo-Riemmanian manifold of signature (p,q) has a tangent bundle splitting $T\mathcal{M} = (T\mathcal{M})^p \oplus (T\mathcal{M})^q$ into time-like and space-like sub-bundles, respectively. The manifold $\mathcal{M}$ has a spin structure if and only if $w_2(\mathcal{M}) = w_1(T\mathcal{M})^p \cup w_1(T\mathcal{M})^q$ is an element of $H^2(\mathcal{M}, \mathbb{Z}_2)$. There are many distinct spin structures in correspondence with elements in $H^1(\mathcal{M}, \mathbb{Z}_2)$.

Given the pseudo-Riemmanian (or Lorentzian, for the tangent bundle) character of the base manifold, there exists a non-null set of vectors, whose conjugation under the bilinear form vanishes. These are, of course, the light-like vectors. Let v be an arbitrary light-like vector, defining a light cone, and take the intersection of a given (hiper) plane, (T_1, X, Y, Z), where T_1 is constant and the light speed equals the unity. The intersection gives rise to a spherical shell, the so-called Riemann sphere, whose radius is given by T_1 (see Figure 1). Now, consider an injective mapping (the stereographic projection), associating to each point on the sphere a given point in the complex plane, Σ, intersecting the sphere at $Z = 0$. Following this reasoning, it is fairly trivial to see that coordinates (X, Y, Z) on the sphere may be given by a complex number $\beta = X' + iY'$. Figure 2 shows that this mapping may be constructed out from the triangles $P'CN$ and PBN in such a way that (taking $T_1 = 1$ for simplicity)
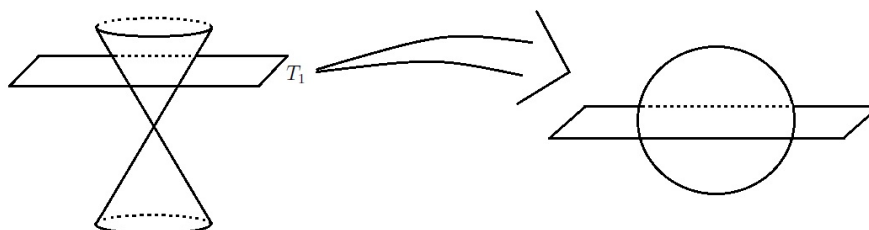
$$\beta = \frac{X + iY}{1 - Z} \tag{1}$$

Figure 1. Interception of the light-cone, leading to the Riemann sphere.
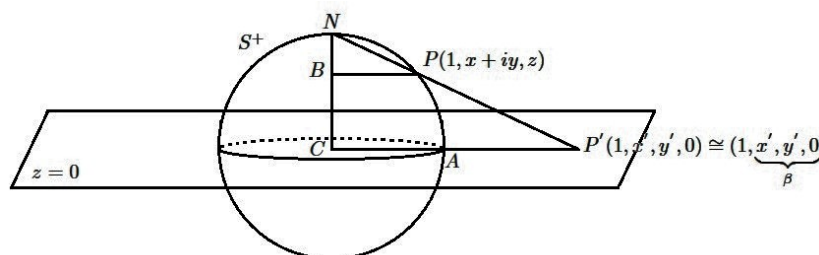

Figure 2. The simple construction of Equation (2).

Now, it is important to remark that $S^2 = \mathbb{R}^2 \cup \{\infty\}$, and therefore, in order to reach the north pole by the projection, it is necessary that $\beta = \infty$. Hence, it is convenient to associate the points on the sphere not to a unique c-number, but rather to a pair (η, ξ), such that $\beta = \xi/\eta$. The pair (η, ξ) are the so-called projective coordinates. The north pole ($\beta = \infty$) now has its description by means of the coordinate:

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2}$$

By accomplishing the previous procedure, we arrive at the most important concept of the viewpoint that we are concerned about here: spinors are given by the projective coordinates of the stereographic projection of a light-cone section into the complex plane. Two remarks are in order: firstly, despite using the light-cone to evince the spinors, this procedure may well be applied to the entire spacetime [4]. Secondly, the Cartan's view of spinors as spacetime generators may be

straightforwardly faced from this construction. In fact, by denoting $\bar{\chi}$, the complex conjugate of a given c-number χ, it can be checked that

$$X = \frac{\xi\bar{\eta} + \eta\bar{\bar{\xi}}}{\xi\bar{\xi} + \eta\bar{\eta}}, \qquad Y = \frac{\xi\bar{\eta} - \eta\bar{\bar{\xi}}}{i(\xi\bar{\xi} + \eta\bar{\eta})}, \qquad Z = \frac{\xi\bar{\bar{\xi}} - \eta\bar{\eta}}{\xi\bar{\xi} + \eta\bar{\eta}} \tag{3}$$

Thus, vector coordinates describing events in the tangent bundle are given by spinors [6]. We remark, by passing, that the conception of spinors as performed here also has a clear connection to the idea of elements carrying irreducible representations of the Poincaré group. In order to see this, take a point, say B, belonging to the null direction towards the causal future obtained from the previous point by $(1, X, Y, Z)(\xi\bar{\xi} + \eta\bar{\eta})/\sqrt{2}$, for instance, and perform a $SL(2,\mathbb{C})$ transformation. It can be readily verified that the quantity

$$\det\left[\begin{pmatrix}\xi\\\eta\end{pmatrix} \otimes (\bar{\xi} \;\; \bar{\eta})\right] \tag{4}$$

is invariant under such a transformation. Going back to the usual coordinate system, we see that this determinant is nothing but the line element in Minkowski space. Therefore, the spinor transformation corresponds to a Lorentz transformation.

The crucial fact we have to face now is the following: in the presence of black holes, many of the previous effort gets lost. Take an arbitrary BH in the base manifold $\mathcal{M}$ (The general argument that we are going to explore in this Section holds, regardless of the number or types of BHs). At the region where the BH is placed, which can conceivably be a point or some extension delimited by the event horizon, the very existence of the BH forbids the appreciation of the tangent bundle as a whole, in such a way that it is not conceivable to ask for representations of the Lorentz group there. Equivalently, at that region, there is no meaning in the concept of spacetime points (see Figure 3).
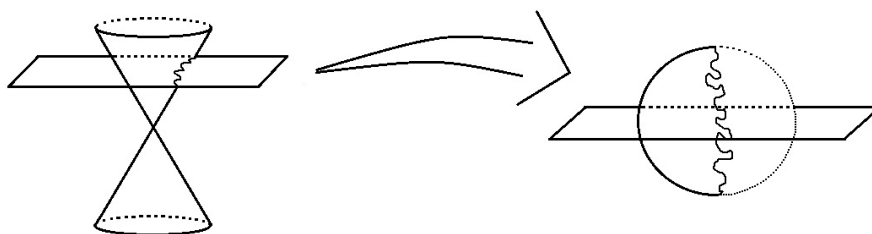

Figure 3. Pictorial effect of the black hole in the spinor structure.

Therefore, one may expect some breaking, in a manner of speaking, of the spinor structure. More precisely, the non-trivial topology, engendered by the BH, is reflected (among other effects) by a non-trivial first homotopy group of the base manifold, *i.e.*, $\pi_1(\mathcal{M}) \neq 0$. On the other hand, the group of homomorphisms of this fundamental group into $\mathbb{Z}_2$ shall label the set of real line bundles on $\mathcal{M}$. This group is nothing but the first cohomology group $H^1(\mathcal{M}, \mathbb{Z}_2)$, and its non-triviality is inherited from $\pi_1(\mathcal{M}) \neq 0$, or, essentially, by the BH existence. Now, inequivalent real line bundles imply inequivalent patching of the local coverings, necessary to circumvent the problematic region (see Figure 3 for a pictorial view of the problem). It means that we shall have inequivalent stereographic projections of (also inequivalent) light-cone sections into complex planes. In other words, different spinors arise. These are the so-called exotic spinors.

It is possible now to complement the picture of exotic spinors by the following construction. Let ρ denote the spinor representation space $(0, 1/2)$ or $(1/2, 0)$ for Weyl spinors, or $(1/2, 0) \oplus (0, 1/2)$, for Dirac spinors. Spinor fields carrying a ρ representation of $SL(2, \mathbb{C})$ are sections of the vector bundle $P_{Spin(1,3)}(\mathcal{M}) \times_\rho \mathbb{C}^4$. The exotic spinors, from the aforementioned construction, are also sections of a

vector bundle. We shall denote it by $\tilde{P}_{Spin(1,3)}(\mathcal{M}) \times_\rho \mathbb{C}^4$ and, accordingly, its spinors by $\tilde{\psi}$. In order to compute the dynamics of these new spinor fields, we shall elaborate on an heuristic approach [7] resulting from the appreciation of the mathematical structure behind [8]: As is well-known, the appreciation of quantum particles (wave functions) motion in curved backgrounds is encoded into the covariant derivative. Usually, in treating with fermions, the covariant derivative is defined in a unique way depending on the Lorentz manifold underlying properties. The point is that the wave functions in question are sections of a vector bundle which, by means of $Spin(1,3)$, is sensitive to different patching, due to the BH presence. In this case, it is necessary to define a covariant derivative for each inequivalent resulting representation. The dynamics associated to exotic spinors shall be very similar to the one associated to usual spinors. In fact, the only difference between the two dynamics comes from the fact that the connection related to $\tilde{\psi}$ must feel the non-trivial cohomology group, which is labeled by an integer number. Actually, the very existence of $\tilde{\psi}$ is conditioned to the BH existence, and therefore it must be reflected in its dynamics. This reasoning enables us to write the exotic Dirac operator $i\gamma^\mu \tilde{\nabla}_\mu$, in terms of the usual one, added by a given term, whose macroscopic effect is to give an integer number arising from the non-trivial topology. Hence,

$$i\gamma^\mu \tilde{\nabla}_\mu \mapsto i\left(\gamma^\mu \nabla_\mu + \frac{1}{2\pi i}\xi^{-1}d\xi\right) \tag{5}$$

where the Christoffel symbols and the spin connection are taken into account in ∇, and the Dirac matrices are generically denoted here by γ^μ. The term ξ is a unimodular function [7], $\xi = e^{i\theta(x)}$, implying that $\xi^{-1}d\xi = i\gamma^\mu \partial_\mu \theta(x)$. We shall present a heuristic account of this important term encoding the net macroscopic topological effect. Firstly, by macroscopic, we mean integration upon any closed curve. Therefore,

$$\frac{1}{2\pi i}\oint \xi^{-1}d\xi \in \mathbb{Z} \tag{6}$$

leading to $\xi = \exp(in\theta) \in U(1)$, where $n \in \mathbb{Z}$ and, then, the exterior derivative operator performs the mapping $d : \sec\Lambda^0(T\mathcal{M}) \to \sec\Lambda^1(T\mathcal{M})$. Locally, the topological term gives rise to a novel contribution encoded in the expression

$$\frac{1}{2\pi i}\xi^{-1}d\xi = \frac{n}{2\pi}d\theta \tag{7}$$

Absorbing the numerical coefficient into θ, and remembering that γ^μ serves as a basis for $\sec\Lambda^1(T\mathcal{M})$, we are left with a Dirac operator given by

$$D = i(\gamma^\mu \nabla_\mu + \gamma^\mu \partial_\mu \theta) \tag{8}$$

acting on exotic spinors.

The reader may understand the term coming from topology as a type of additional (spin-dependent) coupling. The vector character of this coupling allows the interpretation of that as a vector-like coupling. Hence, concerning Dirac spinors, the net topological defect could be perceived as a coupling with an extra vector field. Even though this interpretation can be pursued, it is remarkable that these spinors obey such slightly different dynamics. The situation can be summarized as follows: usual Dirac spinors will be annihilated by the Dirac operator (with the mass term, eventually)—setting the spinor field dynamics—and for these spinors, BHs are perceived by quite constraining boundary conditions. Exotic spinors, in turn, are annihilated by the Dirac operator added by the extra (topological) coupling, and their very existence is linked to BH. Within this regard, we shall also report on the profitable branch of research performed by Elko spinors [9–11], since their peculiarities forbid usual interaction terms with vector fields. In fact, mass dimension

3/2 fermions, as the Dirac spinor field, feel the exotic term in Equation (5), induced by inequivalent spinor structures, merely as a shift in the vector potential in the Dirac equation. Indeed, as a one-form field representing an element of the cohomology group $H^1(\mathcal{M}, \mathbb{Z}_2)$, the exotic structure in Equation (5) and any electromagnetic potential in the Dirac equation are physically indistinguishable. Nevertheless, mass dimension-one spinor fields have a completely distinct feature, regarding the interaction with gauge fields, since such kinds of interaction is suppressed by coupling constants of the order of the unification scale. Hence, the exotic term in Equation (5) cannot be absorbed in external electromagnetic vector fields. Then, Elko, and any kind of other mass dimension-one spinor field [12], can, in fact, probe exotic spin structures [13]. Hence, exotic Elko spinors are an excellent tool for probing the underling topology [13,14]. In the next section, we shall enumerate some interesting features associated to the study of exotic spinors, whose non-trivial topology is engendered by BH.

In order to provide a relevant application, the framework heretofore developed can be applied for studying the tunneling methods across black hole horizons and the Hawking radiation spectrum. We implement this by regarding Kerr-Sen axion-dilaton black holes and studying the dark exotic spinors and Dirac exotic spinors tunneling across the horizon. The Kerr-Sen dilaton-axion black hole metric is a solution of the field equations derived from a low energy effective action that rules heterotic string theory [15], describing a black hole with angular momentum J, mass M, charge Q, and magnetic dipole moment μ. In Boyer–Lindquist coordinates, it reads:

$$ds^2 = -\frac{1}{\Sigma}\left(\Delta - a^2\sin^2\theta\right)dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[-\Delta a^2\sin^2\theta + \left(r^2 - 2\beta r - a^2\right)^2\right]d\varphi^2 +$$
$$- \frac{2a\sin^2\theta}{\Sigma}\left[-\Delta + \left(r^2 - 2\beta r - a^2\right)^2\right]dt d\varphi \tag{9}$$

where

$$\Sigma = r^2 - 2\beta r + a^2\cos^2\theta, \qquad \Delta = (r - r_-)(r - r_+) \tag{10}$$

$r_\pm = -\frac{Q^2}{2M} + M \pm \left(-\frac{J^2}{M^2} + \left(M - \frac{Q^2}{2M}\right)^2\right)^{1/2}$ being the coordinate singularities. For $a = 0$, the Kerr-Sen black hole is led to to the Gibbons–Maeda–Garfinkle–Horowitz–Strominger solution, and, for $\beta = 0$, the Kerr black hole is recovered. In particular, for $a = 0$ and $\beta = 0$, the Kerr-Sen black hole is led to the Schwarzschild black hole. This is the main reason to consider the most general Kerr-Sen black hole.

Transforming $\phi = \varphi + \frac{a\left(-\Delta + a^2 - 2\beta r + r^2 - \Delta\right)}{\Delta a^2\sin^2\theta - (r^2 - 2\beta r + a^2)^2}t$ yields a similar form for the metric (9),

$$ds^2 = -\frac{\Sigma}{\Delta - a^2\sin^2\theta}dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\Delta\sin^2\theta}{\Sigma}\left[\Delta - a^2\sin^2\theta\right]d\phi^2 \tag{11}$$

The tunnelling process and the Hawking radiation can be derived at the event horizon, by rewriting the above metric near the horizon [16]:

$$ds^2 = -F(r_+)dt^2 + \frac{1}{G(r_+)}dr^2 + \Sigma(r_+)d\theta^2 + \frac{H(r_+)}{\Sigma(r_+)}d\phi^2 \tag{12}$$

where the coefficients F, G, H regard Equation (11) and shall be used, hereupon, for the sake of conciseness.

Now, the tunneling of exotic Elko dark particles shall be studied in detail. Thereafter, exotic Dirac particles across the Kerr-Sen black hole event horizons can be analyzed, based upon a similar framework that we are going to accomplish for Elko spinors $\lambda(p^\mu)$. These are eigenspinors of the charge conjugation operator C, namely, $C\lambda(p^\mu) = \pm\lambda(p^\mu)$. The plus [minus] sign regards

self-conjugate, [anti self-conjugate] spinors, denoted by $\lambda^S(p^\mu)$ [$\lambda^A(p^\mu)$]. Elko dark spinors $\lambda(k^\mu)$, for $k^\mu = (m, \lim_{p\to 0} \mathbf{p}/|\mathbf{p}|)$, read [9]

$$\lambda_\pm^A(k^\mu) = \pm \begin{pmatrix} -\sigma_2 \left(\phi^\mp(k^\mu)\right)^* \\ \phi^\mp(k^\mu) \end{pmatrix}, \qquad \lambda_\pm^S(k^\mu) = \begin{pmatrix} \sigma_2 \left(\phi^\pm(k^\mu)\right)^* \\ \phi^\pm(k^\mu) \end{pmatrix} \tag{13}$$

where eigenspinors of the helicity operator [10]

$$\phi^+(k^\mu) = (\alpha, \beta)^\mathsf{T}, \qquad \phi^-(k^\mu) = (-\beta^*, \alpha^*)^\mathsf{T} \tag{14}$$

where $\alpha = \sqrt{m} \cos\left(\frac{\vartheta}{2}\right) e^{-i\varphi/2}$ and $\beta = \sqrt{m} \sin\left(\frac{\vartheta}{2}\right) e^{+i\varphi/2}$ are adopted. The Weyl spinors $\phi^\pm$ have dual helicity. The boosted terms

$$\lambda_\pm^A(p^\mu) = \sqrt{\frac{E+m}{2m}} \left(1 \pm \frac{p^\mu}{E+m}\right) \lambda_\pm^A, \qquad \lambda_\pm^S(p^\mu) = \sqrt{\frac{E+m}{2m}} \left(1 \mp \frac{p^\mu}{E+m}\right) \lambda_\pm^S \tag{15}$$

are mass dimension one quantum field expansion coefficients (Hereupon, θ denotes the exotic term function in Equation (8) and ϑ the azimuthal angle in spherical coordinates). The exotic Dirac operator in Equation (8) does not annihilate the Elko spinors $\lambda^{S/A}(p^\mu)$. However, it is well known that Elko spinors obey a first order derivative constraint [9,10], which shall be used to make explicit the exotic terms contribution:

$$\gamma^\mu(\nabla_\mu + \partial_\mu \theta)\lambda_\pm^S = \pm i \frac{m}{\hbar} \lambda_\mp^S, \tag{16}$$

$$\gamma^\mu(\nabla_\mu + \partial_\mu \theta)\lambda_\mp^A = \pm i \frac{m}{\hbar} \lambda_\pm^A \tag{17}$$

It is worth emphasizing that dark Elko spinors still satisfy the Klein–Gordon equation. In addition, mass dimension one quantum fields can be thus constructed as [9] (all spinors and operators are, obviously, functions of the momentum p^μ):

$$\mathfrak{f}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2mE}} \sum_\rho \left[b_\rho^\dagger \lambda_\rho^A e^{ip\cdot x} + a_\rho \lambda_\rho^S e^{-ip\cdot x} \right] \tag{18}$$

The creation and annihilation operators $a_\tau, a_\tau^\dagger$ (and b_τ and $b_\tau^\dagger$) are governed by Fermi statistics. Then, dual Elko spinors $\vec{\lambda}_\tau(p^\mu) = \left[\Xi \lambda_\tau(p^\mu)\right]^\dagger \sigma_1 \otimes \mathbb{I}_2$, by denoting $\Xi = \frac{1}{2m}\left(\lambda_-^S \bar{\lambda}_-^S - \lambda_-^A \bar{\lambda}_-^A + \lambda_+^S \bar{\lambda}_+^S - \lambda_+^A \bar{\lambda}_+^A\right)$ an involution, are quite different from the Dirac dual $\bar{\lambda}(p^\mu) = \lambda(p^\mu)^\dagger \gamma^0$. In fact, the quantum field $\mathfrak{f}(x)$ has mass dimension-one, since its quantum adjoint

$$\vec{\mathfrak{f}}(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2mE}} \sum_\tau \left[b_\tau \vec{\lambda}_\tau^A e^{-ip\cdot x} + a_\tau^\dagger \vec{\lambda}_\tau^S e^{ip\cdot x} \right] \tag{19}$$

provides a SIM(2) propagator [9]

$$S(x - x') = i \left\langle \left| T\left(\mathfrak{f}(x)\vec{f}(x')\right) \right| \right\rangle = -\lim_{\epsilon \to 0^-} \int \frac{d^4 p}{(2\pi)^4} e^{ip^\mu(x_\mu - x'_\mu)} \frac{\mathcal{G}(\varphi) + \mathbb{I}}{p^2 - m^2 - i\epsilon} \tag{20}$$

for $\mathcal{G}(\varphi) = \begin{pmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{pmatrix} \otimes \sigma_1$. Hence, the Hawking radiation from tunneling dark Elko spinors can be derived for the metric (11), by defining the tetrads [16]

$$
\begin{array}{ll}
\gamma^t = \frac{1}{\sqrt{F(r_+)}} \mathbb{I}_2 \otimes \sigma_1 & \gamma^\vartheta = \frac{-i}{\sqrt{\Sigma(r_+)}} \sigma_2 \otimes \sigma_2 \\
\gamma^r = -i\sqrt{G(r_+)}\, \sigma_1 \otimes \sigma_2 & \gamma^z = -i\sqrt{\frac{\Sigma(r_+)}{H(r_+)}} \sigma_3 \otimes \sigma_2
\end{array}
\tag{21}
$$

Dark Elko spinors can be written as

$$
\lambda_+^S = (-i\beta^*, i\alpha^*, \alpha, \beta)^\mathsf{T} \exp\left(\frac{i}{\hbar}\tilde{I}\right), \qquad \lambda_+^A = (i\alpha, i\beta, -\beta^*, \alpha^*)^\mathsf{T} \exp\left(\frac{i}{\hbar}\tilde{I}\right)
\tag{22}
$$

$$
\lambda_-^S = (-i\alpha, -i\beta, -\beta^*, \alpha^*)^\mathsf{T} \exp\left(\frac{i}{\hbar}\tilde{I}\right), \qquad \lambda_-^A = (-i\beta^*, i\alpha^*, -\alpha, -\beta)^\mathsf{T} \exp\left(\frac{i}{\hbar}\tilde{I}\right)
\tag{23}
$$

and $\tilde{I} = \tilde{I}(t, r, \vartheta, z)$ is the classical action of the theory [16]. Equations (22) and (23) can be replaced in the system Equation (16), with the covariant derivative $\nabla_\mu = \partial_\mu + \frac{1}{8}i\Gamma_\mu^{\rho\sigma}[\gamma_\rho, \gamma_\sigma]$, where γ^ρ denote the Clifford bundle generators. Let us denote by $\lambda [\overset{\diamond}{\lambda}]$ the spinor on the left- (right)- hand side of Equations (16) and (17). Consequently, Equation (16) yields $\gamma^\mu(\nabla_\mu + \partial_\mu\theta)\lambda = i\frac{m}{\hbar}\overset{\diamond}{\lambda}$. The WKB approximation prescription, for $\tilde{I} = I + \mathcal{O}(\hbar)$, implies that

$$
(I_\mu + \partial_\mu\theta)\gamma^\mu\lambda = im\overset{\diamond}{\lambda} + \mathcal{O}(\hbar)
\tag{24}
$$

where $I_\rho \equiv \frac{\partial I}{\partial x^\rho}$. The leading order terms in Equation (24) and the usual *ansatz* $I(t, r, \vartheta, \varphi) = (j\Omega - \omega)t + j\varphi + \Theta(\vartheta) + W(r)$ is employed, for ω denoting the energy, whereas and j stands for the magnetic quantum number [17]. The angular function $j\varphi + \Theta(\vartheta)$ is the same for both the incoming and outgoing tunneling situations, and its contribution vanishes for the computation of the tunneling probability. Therefore, angular functions can be disregarded [17].

Equation (16) reads:

$$
\begin{cases}
\sqrt{G(r_+)}W'\alpha^* - \frac{(\omega - j\Omega_H + \dot\theta_+)}{\sqrt{F(r_+)}}\beta^* = \beta^* m \\
\sqrt{G(r_+)}W'\beta + \frac{(\omega - j\Omega_H + \dot\theta_+)}{\sqrt{F(r_+)}}\alpha = \alpha m
\end{cases}
\tag{25}
$$

$$
\begin{cases}
\sqrt{G(r_+)}W'\alpha \mp \frac{(\omega - j\Omega_H + \dot\theta_+)}{\sqrt{F(r_+)}}\beta = \mp\beta m \\
\sqrt{G(r_+)}W'\beta^* \mp \frac{(\omega - j\Omega_H + \dot\theta_+)}{\sqrt{F(r_+)}}\alpha^* = \pm\alpha^* m
\end{cases}
\tag{26}
$$

(where $\dot\theta$ denotes the time derivative of the exotic term θ) and can be combined to yield equations that determine the spinors $\lambda_\pm^S$. Correspondingly, the systems below regard equations for α and β, whereas Equation (28) is related to the coefficients α^* and β^* of the Elko spinors [16]:

$$
\lambda_+^S : \begin{cases}
W_I(r) = \pm \int \frac{m^2 F - (\omega - j\Omega_H + \dot\theta_+)}{(FG)^{1/2}} dr \\
W_{II}(r) = \pm \int \frac{(m\sqrt{F} + \omega - j\Omega_H + \dot\theta_+)}{(FG)^{1/2}} dr
\end{cases}
\tag{27}
$$

$$
\lambda_-^S : \quad W_{III}(r) = \pm \int \frac{(m\sqrt{F} - \omega + j\Omega_H + \dot\theta_+)}{(FG)^{1/2}} dr \quad \text{and} \quad W_{IV}(r) = iW_{II}(r)
\tag{28}
$$

Solving the equations yields

$$
W_{II}(r) = \pm i\pi \frac{r_+^2 - 2\beta r_+ + a^2}{2(r_+ + m)}(\omega - J\Omega_H + \dot\theta_+)
\tag{29}
$$

The WKB approximation asserts that the tunneling rate is given by $\Gamma \propto \exp(-2\,\text{Im}\,I)$, where I denotes the classical action for the path [17]. The imaginary part of the action yields

$$\text{Im}\,I_{\pm} = \pm i\pi \frac{\Delta_+}{2(r_+ + m)}(\omega - J\Omega_H + \dot{\theta}_+) \tag{30}$$

Hence, the resulting tunneling probability reads [17]

$$\Gamma \;\; = \;\; \frac{P_{(\text{emission})}}{P_{(\text{absorption})}} = \frac{e^{-2\text{Im}I_+}}{e^{-2\text{Im}I_-}} = -2\pi \frac{r_+^2 - 2\beta r_+ + a^2}{(r_+ + m)}(\omega - J\Omega_H + \dot{\theta}_+) \tag{31}$$

It is worth mentioning that the exotic term does affect the tunneling probability in Equation (31), having consequences for the black hole evaporation and, hence, its lifetime. In addition, despite affecting the tunneling probability, the Hawking universal law holds for the exotic formulation. In fact, the temperature of the Kerr-Sen dilaton-axion black hole reads:

$$T_H = \frac{1}{2\pi} \frac{(r_+ + m)}{r_+^2 - 2\beta r_+ + a^2} \tag{32}$$

and does not take into account the exotic term, being compatible with the standard framework that does not encompass exotic terms [16].

Finally, the analysis for the exotic Dirac spinor can be similarly implemented. We do not aim to be redundant with repetitive derivations; however, it is worth pointing out our results. Equations (31) and (32), obtained in the context of exotic Elko spinors, are the same for exotic Dirac spinors, being of course the θ terms, added by a genuine (not exotic) vectorial field term.

3. Conclusions

Beyond aspects of gravity, exotic spin structures are widely employed in physics. In fact, the path integral in quantum field theories presupposes an average over all possible paths for a field and, furthermore, can take into account an average of spin connections, when multiply connected spacetimes are involved [18,19]. Exotic spin structures play important roles in superconductivity, in particular in what regards the Cooper pairing phenomena and the Joseph current [7], and in the t'Hooft setup to study quark confinement. Exotic spinor fields are also very useful in the study of vacuum polarization effects, where exotic spinors provide a causal photon propagation. Standard spinor fields provide a non-causal photon propagation. Hence, far beyond being a mere formal construct, the employment of exotic spinor structures in physics better paves the way, besides interesting interpretation of physics [13], to derive new possibilities that encompass a framework that is simultaneously more general and more useful as a stage for prominent applications.

In this article, we have provided a novel aspect on black hole physics, by introducing black holes as sources of non-trivial topology. In addition, we further analyzed exotic fermions in this framework. The study of black holes in exotic manifolds brings a different point of view in General Relativity. In fact, an infinite family of non-diffeomorphic manifolds can have the same trivial $\mathbb{R}^4$ topology [5,20]. It can lead to the existence of metrics that, although being identical to standard black hole metrics, are not globally diffeomorphic [5,20,21]. Moreover, Hawking radiation of black holes brought a paradigm shift to the classical theory of black holes a couple of decades ago. Tunneling methods and Hawking radiation of Elko matter were comprehensively studied in Refs. [16,22]. To study these phenomena, Quantum Field Theory plays a prominent role in the black hole background, and the Wick rotation to Euclidean time is usually employed [23–25]. In fact, since the Lorentzian spacetime metric is indefinite, it cannot be employed to generate the locally Euclidean

topology of certain spacetime manifolds. This is one of the main reasons to flip the metric, from the Lorentzian to the Euclidean, signature. In this setup, the 4D Schwarzschild black hole has metric

$$ds^2 = \left(1 - \frac{2GM}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega^2 \tag{33}$$

where τ denotes the Euclidean, Wick-rotated, time. This black hole has several interesting features, as the horizon is smoothed out, appearing as a conical singularity, accordingly [26]. Relevant to our discussion in the previous sections is studying the role of different spin structures in what concerns the existence of fermions, in the black hole background. In fact, the horizon has central importance, by means of the Hawking radiation as a tunneling effect of particles across the horizon. Although the metric (33) has topology $\mathbb{R}^2 \times S^2$, asymptotically when $r \to \infty$, it has topology $\mathbb{R}^3 \times S^1$. In this regime, the analysis regarding fermions in BH backgrounds becomes quite interesting. Indeed, Kaluza-Klein vacua are related to the asymptotic value of the metric, and can be expanded in Fourier modes. Hence, although the Euclidean time has S^1 periodicity, fermions going around S^1 acquire a phase $\psi \mapsto e^{i\theta(x)}\psi$, that is a recognizable measure of spin structure. There is a construction that makes a unique spin structure at the black hole horizon and also induces a unique spin structure at infinity.

Having constructed the appropriate framework, we further applied it, studying and analyzing tunneling methods across Kerr-Sen black holes for exotic Elko and Dirac spinors. We proved that the Kerr-Sen black hole evaporation rate is altered by the presence of exotic terms, which has paramount consequences for the black hole lifetime. Moreover, the tunneling probability is altered by the exotic topology; however, the Hawking universal law still holds, being universal even in the exotic framework. The results are the same for exotic Dirac spinors. Finally, the analysis for the exotic Dirac spinor can be similarly implemented. We do not aim to be redundant with repetitive derivations; however, it is worth pointing out our results. Equations (31) and (32), obtained in the context of exotic Elko spinors, are the same for exotic Dirac spinors. It is worth emphasizing that all of the formulæ employed to derive the tunneling probability, from Equation (25) including Equation (31) itself, regard the time derivative of the exotic term, revealing the same standards as in reference [13]. It seems that, physically, the value of the exotic function $\theta(x)$ cannot be evaluated or probed, but just its time derivative, regarding the black hole topological evolution.

We intend to keep on analyzing new possibilities that can arise in this framework studied here. As mentioned, these matters are open questions concerning the interplay between BH physics and exotic spinors. We finish by pointing out a little paradigm shift concerning the non-trivial topology investigated here. It is well known that unusual sources, such as spinning cosmic strings, may generate non-trivial spacetime regions. However, it is still a source over the spacetime [27]. In our approach, spacetime itself, because it is riddled by Black Holes, is non-trivial. This little difference can be somewhat relegated to a semantic discussion. However, an important difference is that in reference [27], the source is shown to be unstable while the non-trivial topology generators in our manuscript (the black holes) are physical.

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References

1.　Geroch, R.P. Spinor Structure of Space-Times in General Relativity I. *J. Math. Phys.* **1968**, *9*, 1739–1744.
2.　Geroch, R.P. Spinor Structure of Space-Times in General Relativity II. *J. Math. Phys.* **1970**, *11*, 343–348.
3.　Cartan, E. *The Theory of Spinors*, 1st ed.; Dover Publications: New York, NY, USA, 1966; pp. 43–77.
4.　Penrose, R.; MacCallum, M.A.H. Twistor theory: An approach to the quantisation of fields and space-time. *Phys. Rep.* **1972**, *6*, 241–316.
5.　Brans, C.H. Localized exotic smoothness. *Class. Quant. Grav.* **1994**, *11*, 1785–1792.
6.　O'Donnell, P. *Introduction to 2-Spinors in General Relativity*, 1st ed.; World Scientific: Singapore, 2003.
7.　Petry, H.R. Exotic spinors in superconductivity. *J. Math. Phys.* **1979**, *20*, 231–240.
8.　Greub, W.; Halperin, S.; Vanstone, R. *Connection, Curvature and Cohomology*, 1st ed.; Academic Press: London, UK, 1972; pp. 176–228.
9.　Ahluwalia, D.V. A Lorentz Covariant Local Theory of Fermions With Mass Dimension One. 2016, arXiv:1601.03188.
10.　Ahluwalia, D.V.; Grumiller, D. Spin-half fermions with mass dimension one: Theory, phenomenology, and dark matter. *J. Cosmol. Astropart. Phys.* **2005**, *2005*, 012.
11.　Ahluwalia, D.V.; Grumiller, D. Dark matter: A spin one-half fermion field with mass dimension one? *Phys. Rev. D* **2005**, *72*, 067701.
12.　Coronado Villalobos, C.H.; Hoff da Silva, J.M.; da Rocha, R. Questing mass dimension 1 spinor fields. *Eur. Phys. J. C* **2015**, *75*, 266.
13.　Da Rocha, R.; Bernardini, A.E.; Hoff da Silva, J.M. Exotic dark spinor fields. *J. High Energy Phys.* **2011**, *2011*, 110.
14.　Da Rocha, R.; Hoff da Silva, J.M.; Bernardini, A.E. ELKO spinor fields as a tool for probing exotic topological spacetime features. *Int. J. Mod. Phys. Conf. Ser.* **2011**, *3*, 133–142.
15.　Sen, A. Rotating charged black hole solution in heterotic string theory. *Phys. Rev. Lett.* **1992**, *69*, 1006–1009.
16.　Cavalcanti, R.T.; da Rocha, R. Dark Spinors Hawking Radiation in String Theory Black Holes. *Adv. High Energy Phys.* **2016**, *2016*, 4681902.
17.　Kraus, P.; Wilczek, F. Self-Interaction Correction to Black Hole Radiance. *Nucl. Phys. B* **1995**, *433*, 403–420.
18.　Avis, S.J.; Isham, C.J. Lorentz gauge invariant vacuum functionals for quantized spinor fields in non-simply connected space-times. *Nucl. Phys. B* **1979**, *156*, 441–455.
19.　Hawking, S.W. Spacetime foam. *Nucl. Phys. B* **1978**, *144*, 349–362.
20.　Asselmeyer-Maluga, T.; Brans, C.H. How to include fermions into General relativity by exotic smoothness. *Gen. Rel. Grav.* **2015**, *47*, 30.
21.　Friedman, J.L.; Sorkin, R.D. Spin 1/2 from Gravity. *Phys. Rev. Lett.* **1980**, *44*, 1100–1103.
22.　Da Rocha, R.; Hoff da Silva, J.M. Hawking Radiation from Elko Particles Tunnelling across Black Strings Horizon. *Europhys. Lett.* **2014**, *107*, 50001.
23.　Parikh, M.K.; Wilczek, F. Hawking radiation as tunneling. *Phys. Rev. Lett.* **2000**, *85*, 5042–5045.
24.　Akhmedov, E.T. Subtleties in the quasi-classical calculation of Hawking radiation. *Int. J. Mod. Phys. D* **2008**, *17*, 2453–2458.
25.　Akhmedova, V.; Pilling, T.; de Andrea, G.; Singletonc, D. Temporal contribution to gravitational WKB-like calculations. *Phys. Lett. B* **2008**, *666*, 269–271.
26.　Nastase, H. *An Introduction to the AdS/CFT Correspondence*; Cambridge University Press: Cambridge, UK, 2015.
27.　Mazur, P.O. Spinning Cosmic Strings and Quantization of Energy. *Phys. Rev. Lett.* **1986**, *57*, 929–932.