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Reissner–Nordström Anti-de Sitter Black Holes in Mimetic $F(R)$ Gravity

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Abstract: In this paper, we study under which conditions the Reissner–Nordström anti-de Sitter black hole can be a solution of the vacuum mimetic $F(R)$ gravity with Lagrange multiplier and mimetic scalar potential. As the author demonstrates, the resulting picture in the mimetic $F(R)$ gravity case is a trivial extension of the standard $F(R)$ approach, and in effect, the metric perturbations in the mimetic $F(R)$ gravity case, for the Reissner–Nordström anti-de Sitter black hole metric, at the first order of the perturbed variables are the same at the leading order.

1. Introduction

The striking late 90s observations [1] indicated that the Universe is expanding in an accelerating way. This observation utterly changed our perception of the late-time Universe, with this late-time acceleration being attributed by the scientific community to a negative pressure perfect fluid called dark energy. Modified gravity [2–9] plays an important role towards the consistent modeling of late-time acceleration, since early-time acceleration [10–17] can also be described with the same theoretical framework [18–22]. For an important stream of reviews and research articles on the concept of dark energy, the author refers the reader to [23–28]. The latest observational data coming from the Planck telescope collaboration [29] indicate that the present time Universe consists of ordinary matter ($\Omega_m \sim 4.9\%$), dark energy ($\sim 68.3\%$) and what is perceived as cold dark matter ($\Omega_{DM} \sim 26.8\%$). With regards to the latter, a lot of possible models exist that can explain dark matter, with most of these assuming that dark matter is described by a particle which does not interact with ordinary matter [30,31].

Recently, a quite elegant description of dark matter was given in reference [32], in which the conformal degrees of freedom of the metric in an ordinary Einstein–Hilbert action can actually mimic dark matter. The approach was given the name mimetic dark matter and was further developed later in [33,34]. The applications and implications of the mimetic approach are numerous and have been adopted in many theoretical studies [35–47]. In this paper, the author shall be interested in the vacuum $F(R)$ gravity mimetic approach [36], in which case an ordinary vacuum $F(R)$ gravity is equipped with a scalar potential and a Lagrange multiplier [48,49]. Particularly, the author shall study in detail for which conditions a Reissner–Nordström anti-de Sitter (AdS-RN) black hole can be a solution of a general vacuum mimetic $F(R)$ gravity with Lagrange multiplier and scalar mimetic potential. Notice that with the terminology mimetic, the author refers to the scalar gravitational degrees of freedom, so this does not have to be specified this from now on. The study of Reissner–Nordström solutions for a general vacuum $F(R)$ gravity was performed in [50], where in order for the Reissner–Nordström black hole spacetime to be a solution of the corresponding Einstein

equations, certain constraints should be satisfied. It is obvious that in the case of mimetic vacuum $F(R)$ gravity, the presence of the mimetic potential and the Lagrange multiplier will modify the resulting picture, with regards to the constraints that have to be satisfied. Indeed, as is demonstrated, the general set of constraints have differences to the ordinary vacuum $F(R)$ case. For a similar study but for a Schwarzschild anti-de Sitter black hole in vacuum $F(R)$ gravity, see [51] and also [52]. Moreover, a large number of studies devoted to black hole solutions exist in the context of $F(R)$ gravity, and, for an incomplete list, the author refers to [53–65] and references therein.

The motivation for studying AdS-RN black holes in the context of mimetic $F(R)$ gravity is mainly to see whether these constant curvature black hole solutions, which are solutions in the case of metric $F(R)$ gravity, can also be solutions of mimetic $F(R)$ gravity. This is important for two reasons—firstly, in order to see if the mimetic $F(R)$ gravity context can support such solutions. This issue in some way can judge if the mimetic $F(R)$ gravity context is viable, since the existence of compact gravitational solutions is guaranteed or not. Secondly, if these solutions exist, it is important to see what the differences are in comparison to the standard metric F(R) approach. For example, the perturbations of the RN-AdS black hole in the metric $F(R)$ gravity yields anti-evaporation phenomena; therefore, if the mimetic $F(R)$ solutions are different, this can introduce differences in the perturbations in the case of mimetic $F(R)$. As is demonstrated, for the constant curvature solutions, the metric $F(R)$ case and the mimetic $F(R)$ case yield almost the same equations of motion, with a difference being that the mimetic $F(R)$ and the metric $F(R)$ differ by a constant.

Therefore, the purpose of this paper is twofold: First, the paper will investigate how the presence of the Lagrange multiplier and of the mimetic potential affects the constraints that need to be satisfied, in order for the AdS-RN metric to be a solution of the vacuum mimetic $F(R)$ gravity. Secondly, the paper will focus on how the aforementioned constraints affect the perturbations of the AdS-RN black hole, studying the problem at first order in the perturbed variables. As is demonstrated, the resulting mimetic $F(R)$ gravity is a trivial extension of the standard $F(R)$ gravity, and in effect, the metric perturbations at first order are the same.

The outline of the paper is as follows: In Section 2, the mimetic vacuum $F(R)$ gravity formalism with Lagrange multiplier and mimetic potential is presented in brief. The reason why the study of the AdS-RN black hole is important is also discussed, and the paper investigates which constraints have to be satisfied, so that the AdS-RN black hole is a solution of the vacuum mimetic $F(R)$ gravity. As is demonstrated, the solutions can be classified into two classes, and only one of these is thoroughly investigated, since the latter leads to the Schwarzschild anti-de Sitter black hole. Finally, the metric perturbations in the mimetic case are briefly discussed. The concluding remarks follow at the end of the paper.

2. Mimetic $F(R)$ Gravity and Reissner–Nordström Black Holes

2.1. The Mimetic $F(R)$ Gravity Theoretical Framework

The mimetic $F(R)$ gravity was first studied in [35], and in the context of mimetic $F(R)$ gravity, the conformal symmetry is actually an internal degree of freedom [32], which is not violated. The mimetic gravity approach was introduced by [32], and in the context of mimetic gravity, the physical metric $g_{\mu\nu}$ that describes our Universe, can be written in terms of an auxiliary scalar degree of freedom, the scalar field ϕ, and also in terms of an auxiliary metric tensor $\hat{g}_{\mu\nu}$, in the following way:

$$g_{\mu\nu} = -\hat{g}^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\hat{g}_{\mu\nu} \tag{1}$$

The equations of motion are obtained by varying the gravitational action with respect to the auxiliary metric $\hat{g}_{\mu\nu}$ and with respect to the extra scalar degree of freedom, instead of varying the action with respect to the physical metric $g_{\mu\nu}$. From Equation (1), it easily follows that

$$g^{\mu\nu}(\hat{g}_{\mu\nu}, \phi)\partial_\mu\phi\partial_\nu\phi = -1 \tag{2}$$

As can be easily verified, the Weyl transformation $\hat{g}_{\mu\nu} = e^{\sigma(x)}g_{\mu\nu}$ leaves Equation (1) invariant, and the auxiliary metric $\hat{g}_{\mu\nu}$ eventually does not appear in the final action. The Jordan frame mimetic $F(R)$ gravity action, equipped with a scalar field potential $V(\phi)$, and a Lagrange multiplier $\lambda(\phi)$, is equal to [35]

$$S = \int d^4x \sqrt{-g} \left(F\left(R(g_{\mu\nu}) \right) - V(\phi) + \lambda \left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 1 \right) \right) \tag{3}$$

In the following, the author shall refer to the auxiliary scalar potential $V(\phi)$, as the mimetic potential. Notice that the mimetic potential in the case of the mimetic gravity and also in the case of the mimetic $F(R)$ gravity is arbitrarily chosen, which enables one to use as a specific choice, which in effect constrains the final form of the Lagrange multiplier and of the $F(R)$ gravity. In Equation (3), it was assumed that no matter fluids are present, and the vacuum mimetic $F(R)$ gravity with mimetic potential and Lagrange multiplier is studied. By varying the action of Equation (3), with respect to the physical metric $g_{\mu\nu}$, the author obtains the following set of equations:

$$\frac{1}{2}g_{\mu\nu}F(R) - R_{\mu\nu}F'(R) + \nabla_\mu\nabla_\nu F'(R) - g_{\mu\nu}\Box F'(R) \tag{4}$$
$$+ \frac{1}{2}g_{\mu\nu}\left(-V(\phi) + \lambda \left(g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + 1 \right) \right) - \lambda\partial_\mu\phi\partial_\nu\phi = 0$$

Moreover, by varying the action of Equation (3), now with respect to the auxiliary scalar field ϕ, the author obtains,

$$-2\nabla^\mu(\lambda\partial_\mu\phi) - V'(\phi) = 0 \tag{5}$$

Note that the "prime" in this case denotes differentiation with respect to the auxiliary scalar field, but in the rest of the paper, this notation will be used to denote differentiation with respect to the Ricci scalar, unless differently stated. Finally, upon variation of the action Equation (3) with respect to $\lambda(\phi)$, the result is as follows:

$$g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi = -1 \tag{6}$$

and, by noticing Equation (6), it is observed that this result is identical to the one appearing in Equation (2). In the following sections, the results of this section will be used extensively, in order to study black hole solutions in vacuum mimetic $F(R)$ gravity.

2.2. Motivation for Studying the Reissner–Nordström Black Holes

Before investigation of the AdS-RN black hole solutions in the mimetic $F(R)$ gravity begins, it is necessary to discuss in some detail the motivation to study such black hole solutions. The motivation is twofold, since these black hole solutions have applications in the early Universe [66–68], but also have applications in condensed matter systems, via the holographic principle and phase transitions [69–71].

Particularly, it has been known since the work of Hawking [72], that black holes evaporate, and thus, effectively, their horizon decreases. However, the inverse process for Nariai types black holes is also possible [73]. Actually, this anti-evaporation procedure is triggered by instabilities of the perturbations of the Nariai black hole (see reference [73] for further details on this). The Nariai black holes however, are not black holes that result from the usual gravitational collapse of a star, since these are not asymptotically flat. Therefore, these black holes are relevant for the early Universe only, since these can be primordial black holes of some sort.

It is remarkable that, in the classical $F(R)$ gravity case, anti-evaporation of the Nariai spacetime can occur at the classical level, without any quantum gravity effects being involved [52]. In addition, the Reissner–Nordström black holes are solutions of the $F(R)$ gravity even in the absence of abelian Maxwell fields [74]. As was demonstrated in [50], the AdS-RN black hole is a solution of the $F(R)$ gravity, if certain constraints are satisfied, and in the present work, the study of [50] shall be extended, for the case of vacuum mimetic $F(R)$ gravity. Note that the AdS-RN spacetime is similar to the Nariai black hole, so the results can have relevance to the physics of the early Universe, owing to the instabilities of the AdS-RN black hole, which in effect can be responsible for the anti-evaporation of such massive objects.

As already mentioned, the AdS-RN black hole solutions are relevant for the physics of condensed matter systems, via the holographic principle. Actually, in order to provide a gravitational description of condensed matter phenomena, it is of fundamental importance to find black hole solutions that encompass the physical features of a many-body system and its corresponding phase diagram. Note that the stability of the system depends of course on the field content of the theory. AdS-RN black holes are relevant in condensed systems study, and these result from an Einstein–Maxwell classical theory as the only static solutions that remain stable below a critical temperature [69]. Actually, the ground state of such a system is the extremal AdS-RN black hole, which is the case of black holes studied in this paper. The inclusion of a scalar field in the gravitational action, splits the possible ground states of the system, and the resulting instability of the AdS-RN black hole, actually provides the holographic description of the phase transitions that take place in the dual theory [70,71]. This kind of phase transition is expected to occur in superfluid or superconducting systems, and their study involves linear perturbations of hairy black holes [70,71]. Therefore, the presence of an instability in an $F(R)$ gravity AdS-RN black hole, without the presence of a Maxwell field, is rather intriguing to study, since there might be a possible connection to the condensed matter systems yet to be found.

2.3. General Study of the Solutions

In this section, it shall be investigated under which conditions a static metric with constant curvature and spherical symmetry can be a solution of a general vacuum mimetic $F(R)$ gravity. In the following, for convenience, the notation of reference [50] is adopted. As already stated, it is assumed that the spacetime is described by a spherical symmetric and static metric, $g_{\mu\nu}$ with its line element being of the form,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{7}$$

In Equation (7), the functions $A(r)$ and $B(r)$ are assumed to be smooth and differentiable functions of r, and in addition, $\mathrm{d}\Omega^2$ denotes the metric of unit 2-sphere, that is,

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin(\theta)^2\mathrm{d}\phi^2 \tag{8}$$

It shall be investigated which conditions much hold true in order for black hole solutions in vacuum mimetic $F(R)$ gravity, to exist, which satisfy the following constraints:

$$A(r) = \frac{1}{B(r)}, \quad R = R_0 \tag{9}$$

where R is the Ricci scalar, and R_0 a constant. Therefore, solutions are looked for which lead to a constant scalar curvature and also for which forms of solutions the functions $A(r)$ and $B(r)$ satisfy the constraint Equation (9). The Ricci scalar for the metric Equation (7), is equal to,

$$R = -A''(r) - \frac{4}{r}A'(r) - \frac{2}{r^2}A(r) + \frac{2}{r^2} \tag{10}$$

Notice that for deriving Equation (10), the constraint Equation (9) for the function $A(r)$ was also taken into account, and the "prime" denotes, in this case, differentiation with respect to the radial coordinate r. Since it was assumed that $R = R_0$, the following differential equation is obtained:

$$-A''(r) - \frac{4}{r}A'(r) - \frac{2}{r^2}A(r) + \frac{2}{r^2} = R_0 \tag{11}$$

which can easily be solved to yield

$$A(r) = 1 - \frac{r^2}{12}R_0 + \frac{C_1}{r} + \frac{C_2}{r^2} \tag{12}$$

By setting $C_1 = -M$ and $C_2 = Q$, Equation (12) becomes,

$$A(r) = 1 - \frac{R_0\, r^2}{12} - \frac{M}{r} + \frac{Q}{r^2} \tag{13}$$

Hence, by combining Equations (7) and (13), the metric becomes,

$$ds^2 = -\left(1 - \frac{R_0\, r^2}{12} - \frac{M}{r} + \frac{Q}{r^2}\right)dt^2 + \frac{1}{\left(1 - \frac{R_0\, r^2}{12} - \frac{M}{r} + \frac{Q}{r^2}\right)}dr^2 + r^2 d\Omega^2 \tag{14}$$

which is the Reissner–Nordström anti-de Sitter black hole spacetime. This black hole solution has two event horizons and one cosmological horizon, only in the case $R_0 > 2$, which can be easily found by solving the equation $\frac{1}{g_{rr}} = 0$. However, for notational convenience, it is assumed that the event horizons occur at $r = r_0$ and $r = r_1$, so by choosing the parameters M and Q in the way presented in the Appendix A,

$$A(r) = (1 - \frac{r_0}{r})(1 - \frac{r_1}{r})\left(1 - \frac{(r+r_0)(r+r_1) + r_0^2 + r_1^2}{12}R_0\right) \tag{15}$$

from which it can easily be seen that the two event horizons occur at $r = r_0, r = r_1$ and the cosmological horizon at

$$\left(1 - \frac{(r+r_0)(r+r_1) + r_0^2 + r_1^2}{12}R_0\right) = 0 \tag{16}$$

2.4. Mimetic $F(R)$ Reissner–Nordström Black Holes: A Study of the Solutions

The focus in this section is to investigate under which conditions, the metric of Equation (14) is a solution of the mimetic $F(R)$ gravity equations of motion Equation (4), with action Equation (3). Notice that the conditions appearing in Equation (9) are assumed to hold true. A vacuum mimetic $F(R)$ solution shall be searched for, meaning that only the mimetic potential $V(\phi)$ and the Lagrange multiplier $\lambda(\phi)$ are present, and no matter fluids are assumed to be present. By combining Equations (5) and (6), the mimetic $F(R)$ equation of motion of Equation (4), can be cast as follows:

$$\frac{1}{2}g_{\mu\nu}F(R) - R_{\mu\nu}F'(R) + \nabla_\mu\nabla_\nu F'(R) - g_{\mu\nu}\Box F'(R) + \frac{1}{2}g_{\mu\nu}(-V(\phi)) - \lambda\partial_\mu\phi\partial_\nu\phi = 0 \tag{17}$$

which, for constant scalar curvature, it becomes,

$$\frac{1}{2}g_{\mu\nu}F(R) - R_{\mu\nu}F'(R) - \frac{1}{2}g_{\mu\nu}V(\phi) - \lambda\partial_\mu\phi\partial_\nu\phi = 0 \tag{18}$$

By contracting Equation (18) with the metric $g^{\mu\nu}$, the following equation is received,

$$2F(R) - RF'(R) - 2V(\phi) - \lambda(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 0 \tag{19}$$

which, in view of Equation (6), leads to the following equation,

$$2F(R) - RF'(R) - 2V(\phi) + \lambda(\phi) = 0 \tag{20}$$

By solving Equation (20), with respect to $F(R)$, the following result is obtained:

$$F(R) = \frac{RF'(R)}{2} + V(\phi) - \frac{\lambda(\phi)}{2} \tag{21}$$

By substituting Equation (21) in Equation (18), the following result is easily received:

$$\frac{1}{2} g_{\mu\nu} \left(\frac{RF'(R)}{2} + V(\phi) - \frac{\lambda(\phi)}{2} \right) - R_{\mu\nu} F'(R) - \frac{1}{2} g_{\mu\nu} V(\phi) - \lambda \partial_\mu \phi \partial_\nu \phi = 0 \tag{22}$$

which after some simple algebraic manipulations, becomes,

$$\left(\frac{g_{\mu\nu} R}{4} - R_{\mu\nu} \right) F'(R) - g_{\mu\nu} \left(\frac{\lambda(\phi)}{4} + \lambda(\phi) \partial_\mu \phi \partial^\mu \phi \right) = 0 \tag{23}$$

Finally, by using Equation (6), the Equation (23) can be cast in the following form:

$$\left(\frac{g_{\mu\nu} R}{4} - R_{\mu\nu} \right) F'(R) - g_{\mu\nu} \left(\frac{3\lambda(\phi)}{4} \right) = 0 \tag{24}$$

By looking at Equation (24), it is concluded that, since the first term is independent of ϕ, the above equation can hold true in the following two cases:

- Case I: Both the first and second term are equal to zero, that is,

$$\left(\frac{g_{\mu\nu} R}{4} - R_{\mu\nu} \right) F'(R) = 0, \ \ g_{\mu\nu} \left(\frac{3\lambda(\phi)}{4} \right) = 0 \tag{25}$$

- Case II: Both the first term are equal to the same constant, but with opposite signs, that is,

$$\left(\frac{g_{\mu\nu} R}{4} - R_{\mu\nu} \right) F'(R) = \Gamma, \ \ g_{\mu\nu} \left(\frac{3\lambda(\phi)}{4} \right) = -\Gamma \tag{26}$$

where Γ is a positive real matrix.

In the following section, the two cases listed above shall be analyzed in detail, and the consequences corresponding to each of the two cases will also be discussed in detail. Since it corresponds to the physical problem that interests the author most, Case I will be first.

2.4.1. Case I

For this scenario, it is analyzed, in some detail, what the constraints Equation (25) imply for the vacuum mimetic $F(R)$ gravity model at hand. Firstly, the second constraint in Equation (25) can be true only if $\lambda(\phi) = 0$, and by using Equation (5), it is easily concluded that the mimetic potential is, $V(\phi) = \Lambda$, with Λ some arbitrary real constant, which shall be assumed to be positive, without loss of generality. Hence, the allowed values of the mimetic potential $V(\phi)$ and of the Lagrange multiplier $\lambda(\phi)$, are given below:

$$\lambda(\phi) = 0, \ \ V(\phi) = \Lambda \tag{27}$$

Note that Equation (27) results if it is demanded that the spherical symmetric metric with constant curvature of Equation (14) is a solution of the vacuum mimetic $F(R)$ gravity with mimetic potential and Lagrange multiplier. The first constraint in Equation (25) is a bit more involved, so the

author explicitly calculates the expression in order to have a clear picture of the implications that this constraint generates. By using the metric of Equation (14), the first constraint explicitly reads,

$$\begin{pmatrix} \frac{Q\left(-12Q+r\left(12M-12r+r^3R_0\right)\right)}{12r^6} & 0 & 0 & 0 \\ 0 & -\frac{12Q}{r^2\left(-12Q+r\left(12M-12r+r^3R_0\right)\right)} & 0 & 0 \\ 0 & 0 & -\frac{Q}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{Q\sin(\theta)^2}{r^2} \end{pmatrix} F'(R_0) = 0 \quad (28)$$

which means that either $Q = 0$ or $F'(R_0) = 0$. Therefore, the following two scenarios exist:

- Scenario I: This scenario corresponds to $Q \neq 0$, and therefore the following constraints correspond to this scenario,

$$F'(R_0) = 0, \ \ V(\phi) = \Lambda, \ \ \lambda(\phi) = 0 \quad (29)$$

- Scenario II: This scenario corresponds to $Q = 0$ and it is described by the following constraints,

$$Q = 0, \ \ V(\phi) = \Lambda, \ \ \lambda(\phi) = 0 \quad (30)$$

Notice that in Scenario I, by using Equation (21), the author gets, $F(R_0) = \Lambda$, while in Scenario II, the author obtains $F(R_0) = \frac{R_0 F'(R_0)}{2} + \Lambda$. In Table 1, the results for the Scenarios I and II are included. In conclusion, the resulting picture of the mimetic $F(R)$ gravity, yields different results in comparison to the non-mimetic $F(R)$ gravity case studied in [50]. Particularly, in the present paper, the requirement that the AdS-RN black hole is a solution of the mimetic $F(R)$ gravitational system, results in many different cases for which this can be true, in comparison to the only cases $Q = 0$ or $F'(R_0) = 0$ corresponding to the case studied in the ordinary $F(R)$ gravity of reference [50]. This is easily explained, since the presence of the mimetic potential and of the Lagrange multiplier offers more freedom in the resulting set of equations that need to be satisfied, namely Equation (23).

Table 1. The Scenarios I and II for the Mimetic $F(R)$ Gravity Reissner–Nordström anti-de Sitter Black Hole.

Scenario	Constraints			
Scenario I	$F'(R_0) = 0,$	$V(\phi) = \Lambda,$	$\lambda(\phi) = 0,$	$F(R_0) = \Lambda$
Scenario II	$F'(R_0) \neq 0,$	$V(\phi) = \Lambda,$	$\lambda(\phi) = 0,$	$F(R_0) = \frac{R_0 F'(R_0)}{2} + \Lambda$

Before closing this section, it should be discussed whether there exists another solution to Equation (24), different from the one described by Case I. It is worth discussing and studying this in detail, so by using the metric of Equation (14) and inserting this in Equation (24), the following set of equations is obtained, which is quoted in matrix form,

$$\begin{pmatrix} \frac{\left(-12Q+r\left(12M-12r+r^3R_0\right)\right)\left(r^4\lambda+QF'(R_0)\right)}{12r^6} & 0 & 0 & 0 \\ 0 & -\frac{12\left(r^4\lambda+QF'(R_0)\right)}{r^2\left(-12Q+r\left(12M-12r+r^3R_0\right)\right)} & 0 & 0 \\ 0 & 0 & r^2\lambda - \frac{QF'(R_0)}{r^2} & 0 \\ 0 & 0 & 0 & \frac{\sin(\theta)^2\left(r^4\lambda-QF'(R_0)\right)}{r^2} \end{pmatrix} = 0 \quad (31)$$

It is easy to see that the following two equations must simultaneously be satisfied, so that Equation (31) holds true:

$$r^4\lambda + QF'(R_0) = 0, \quad (32)$$
$$r^4\lambda - QF'(R_0) = 0$$

The system of Equation (32) has as a solution what Case I describes, that is, $\lambda = 0$ and $QF'(R_0) = 0$,; consequently, this validates this paper's claim that both terms of Equation (24) must independently be equal to zero.

2.4.2. Case II

In this case, the constraints appearing in Equation (26) must hold true. By observing Equation (26), it can immediately be seen that neither $F'(R_0)$ and Q can be zero, so, for this case,

$$F'(R_0) \neq 0, \ Q \neq 0 \tag{33}$$

Let it be investigated whether the constraints of Equation (26) can hold true, starting off with the second constraint, namely $g_{\mu\nu}\left(\frac{3\lambda(\phi)}{4}\right) = -\Gamma$, which by using the metric Equation (14), explicitly reads,

$$\begin{pmatrix} \left(\frac{Q}{r^2} - \frac{M}{r} + \frac{1}{12}\left(12 - r^2 R_0\right)\right)\lambda & 0 & 0 & 0 \\ 0 & -\frac{\lambda}{\frac{Q}{r^2} - \frac{M}{r} + \frac{1}{12}\left(12 - r^2 R_0\right)} & 0 & 0 \\ 0 & 0 & -r^2\lambda & 0 \\ 0 & 0 & 0 & -r^2\lambda\sin(\theta)^2 \end{pmatrix} = \Gamma \tag{34}$$

and it can easily be seen that the only constant solution for λ is $\lambda = 0$, and therefore $\Gamma = 0$. Thereby, since $\Gamma = 0$, the first constraint of Equation (26) is satisfied when $Q = 0$ or $F'(R_0) = 0$, hence ending up with the first case, namely Case I.

As has been demonstrated, only Case I leads to a black hole solution for the vacuum mimetic $F(R)$ gravity with Lagrange multiplier and mimetic potential. The requirement that an AdS-RN black hole is a constant curvature solution of the mimetic $F(R)$ gravity results in a certain number of constraints, which are different from the ordinary $F(R)$ gravity case studied in reference [50]. Therefore, it is natural to ask if these new conditions that the mimetic $F(R)$ gravity imposes, can have an effect on the perturbations of the AdS-RN black hole. This is easy to answer, because the mimetic $F(R)$ solution, which is,

$$F'(R_0) = 0, \ V(\phi) = \Lambda, \ \lambda(\phi) = 0, \ F(R_0) = \Lambda \tag{35}$$

is a trivial variant of the ordinary $F(R)$, with the mimetic $F(R)$ being different to the ordinary $F(R)$, up to a constant. The only difference is that $F(R_0) \neq 0$, which however does not alter the perturbation equations at first order. Therefore, the perturbations in the two cases are the same, and therefore in the mimetic $F(R)$ case, the resulting equations are [50]:

$$F''(R_0)\left[-\frac{1}{\cosh^x}\delta R + \tanh x\frac{\partial\delta R}{\partial x} + \frac{\partial^2\delta R}{\partial x^2}\right] = 0 \tag{36}$$

$$F''(R_0)\left[\frac{1}{\cosh^x}\delta R + \delta\ddot{R} + \tanh x\frac{\partial\delta R}{\partial x}\right] = 0 \tag{37}$$

$$F''(R_0)\left[\frac{\partial\delta\dot{R}}{\partial x} + \tanh x\delta\dot{R}\right] = 0 \tag{38}$$

$$F''(R_0)\left[\delta R + \cosh^2 x\left(-\delta\ddot{R} + \frac{\partial^2\delta R}{\partial x^2}\right)\right] = 0 \tag{39}$$

where only first order terms were kept and the following were also used:

$$R = 2e^{-2\rho[x,t]}\left(e^{2(\rho[x,t]+\varphi[x,t])}M_-^2 - 3M_+^2\left(\frac{\partial\varphi}{\partial t}\right)^2 - M_+^2\frac{\partial^2\rho}{\partial t^2} + 2M_+^2\frac{\partial^2\varphi}{\partial t^2} + 3M_+^2\left(\frac{\partial\varphi}{\partial x}\right)^2 + M_+^2\frac{\partial^2\rho}{\partial x^2} - 2M_+^2\frac{\partial^2\varphi}{\partial x^2}\right) \tag{40}$$

Clearly, since the perturbation equations are the same, the same anti-evaporation phenomena that occurred for the ordinary $F(R)$ AdS-RN black holes [50] will also hold true in the mimetic $F(R)$ case.

Before closing this section, it is noted that the condition $F'(R_0) = 0$ simplifies a lot of the equations in the mimetic case too; hence, if $F'(R_0) \neq 0$, it is possible that the perturbation equations in the mimetic and ordinary $F(R)$ gravity cases, are in fact different. This may be the case in the Schwarzschild de Sitter black hole, which was studied in the context of the $F(R)$ gravity in reference [51]. The author hopes to address this issue in a future work.

Another interesting task is to investigate what happens if the metric Equation (14) is not assumed to be the background metric. This study is interesting since if Equation (21) is integrated with respect to R, the following will be obtained:

$$F(R) = -\frac{\Lambda}{2} + c_1 R^2 + V(\phi) \tag{41}$$

with c_1 being an arbitrary integration constant. If $c_1 = 0$, this coincides with the case I studied previously, while $c_1 \neq 0$, this describes the case II. Hence, it is possible to obtain the cases studied by using more general metrics, not just the one of Equation (14). This issue, however, should be carefully studied, and the author defers this task to a future work, since this exceeds the purposes of this work.

3. Conclusions

In this paper, the AdS-RN black hole was studied, in the context of mimetic $F(R)$ gravity with Lagrange multiplier and mimetic potential. As was demonstrated, imposing the condition that the AdS-RN black hole is a solution of the mimetic $F(R)$ gravity results in some constraints, which are different in comparison to the ones corresponding to the ordinary $F(R)$ gravity case. As demonstrated, the mimetic case is a trivial extension of the ordinary $F(R)$ gravity case, and therefore the resulting perturbations equations are not affected.

Motivated by the fact that the reason for having the same perturbations equations for the mimetic and ordinary $F(R)$ AdS-RN black hole, is the constraint $F'(R_0) = 0$, it is interesting and tempting to study black holes for which the constraint $F'(R_0) = 0$ no longer applies. This, for example, could be the case for the charged black hole studied in [50], or in the case of the Schwarzschild-de Sitter black hole, studied in [51]. It would therefore be quite interesting to study these cases too, which the author hopes to address in a future publication.

Another interesting possibility for further study is related to primordial black holes, since the black holes studied in this paper are expected to be primordial black holes. Recently, an interesting study related to non-Gaussianities and the formation of supermassive black holes was performed in reference [75]. It is interesting to see if these non-Gaussianities could have some possible effect on the evolution of primordial black holes. This, for example, could be combined with the study of gravitational memory of the primordial black holes [76], an effect which could have an imprint on the primordial black holes at the time of their formation. For a recent study on gravitational memory in the context of $F(R)$ gravity, see [77].

Finally, since the instability of the AdS-RN black hole provides the holographic description of the phase transitions that occur in the dual condensed matter theory [70,71] of an Einstein–Maxwell gravitational theory, it would be interesting to see if there is any possible connection between the $F(R)$ AdS-RN black hole, and the condensed matter systems, like superfluid or superconducting systems. Notice that the most interesting feature is that, in the context of $F(R)$ gravity, no abelian Maxwell fields are needed in order for the AdS-RN black hole to be a solution. The author hopes to address some of these issues in the future.

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Appendix A. The Parameters *M* and *Q* in Terms of r_0 and r_1

Here, the exact form of the parameters *M* and *Q* appearing in the AdS-RN black hole metric of Equation (14) is quoted. Particularly, the parameter *Q* is assumed to have the following form:

$$Q = r_0 r_1 \left(1 - \frac{\left(r_0^2 + r_1^2 + r_0 r_1 \right) R_0}{12} \right) \tag{A1}$$

and the parameter *M* is assumed to be,

$$M = (r_0 + r_1) \left(1 - \frac{\left(r_0^2 + r_1^2 \right) R_0}{12} \right) \tag{A2}$$

By choosing the mass parameter *M* and the charge parameter *Q* as in Equations (A1) and (A2), the two horizons of the AdS-RN black hole occur at r_0 and r_1, and the function $A(r)$ can take the form appearing in Equation (15).

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