Loop Quantum Cosmology, Modified Gravity and Extra Dimensions

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Abstract: Loop quantum cosmology (LQC) is a framework of quantum cosmology based on the quantization of symmetry reduced models following the quantization techniques of loop quantum gravity (LQG). This paper is devoted to reviewing LQC as well as its various extensions including modified gravity and higher dimensions. For simplicity considerations, we mainly focus on the effective theory, which captures main quantum corrections at the cosmological level. We set up the basic structure of Brans–Dicke (BD) and higher dimensional LQC. The effective dynamical equations of these theories are also obtained, which lay a foundation for the future phenomenological investigations to probe possible quantum gravity effects in cosmology. Some outlooks and future extensions are also discussed.

Keywords: loop quantum cosmology; singularity resolution; effective equation

1. Introduction

Loop quantum gravity (LQG) is a quantum gravity scheme that tries to quantize general relativity (GR) with the nonperturbative techniques consistently [1–4]. Many issues of LQG have been carried out in the past thirty years. In particular, among these issues, loop quantum cosmology (LQC), which is the cosmological sector of LQG has received increasing interest and has become one of the most thriving and fruitful directions of LQG [5–9]. It is well known that GR suffers singularity problems and this, in turn, implies that our universe also has an infinitely dense singularity point that is highly unphysical. However, in contrast with GR, LQC is a singularity free theory that stands for one of the most attractive features of this theory. In LQC, the classical cosmological singularity is naturally replaced by a quantum bounce [10,11] and thus avoids the inevitable cosmological singularity in classical GR.

Recently, this non-perturbatively loop quantization procedure has been successfully generalized to the modified gravity theories such as metric $f(\mathcal{R})$ theories [12,13], Brans–Dicke (BD) theory [14] and scalar-tensor theories [15,16]. However, at the level of the full theory of quantum gravity, it is extremely complex. In order get around this complexity, to test the ideas and constructions of the full theory and also to draw some physical predictions, it is desirable to study their symmetry-reduced models, such as cosmological models. Among all scalar-tensor theories of gravity, the simplest one is the so-called Brans–Dicke theory that was introduced by Brans and Dicke in 1961 to modify GR in accordance with Mach's principle [17]. Therefore, in this review, we will focus on the loop quantum cosmology of Brans–Dicke theory.

The cosmological models of classical Brans–Dicke theory were first studied in [18,19]. Since then, many aspects of Brans–Dicke cosmology have been widely investigated in the past decades [20]. The scalar field non-minimally coupled with curvature in Brans–Dicke theory is even expected to account for the dark energy problem [21–26], which has become a topical issue in cosmology [27]. It should be noted that the solar system experiments constrain the coupling constant ω of the

original four-dimensional Brans–Dicke theory to being a very large number [28,29]. For simplicity consideration and consistency with the solar system experiments, in this review, we will only focus on the Brans–Dicke LQC theory [30] with coupling constant $\omega \neq -\frac{3}{2}$.

On the other hand, higher dimensional spacetime and gravity theories are subjects of great interest in the grand unify theories. Historically, the very first higher dimensional gravity theory is the famous five-dimensional Kaluza–Klein theory, which tries to unify the four-dimensional GR and Maxwell theory [31]. Recent theoretical developments such as the string/M theories [32], AdS/CFT correspondence [33], Brane world scenario [34,35], and so on reveal that higher dimensions are preferred by these theories. In the past half century, many aspects of these higher dimensional gravity theories have been extensively studied, particularly on the physical issues related with Black holes and cosmology. In fact, higher dimensional cosmology now is a rather huge and active field with fruitful outputs. For instance, the accelerated expansions of our universe can be naturally explained by some of the higher dimensional cosmological models [23,36]. Hence, one is naturally to ask: is it possible to generalize the structure of LQC to the spacetime dimensions other than four, (particularly in the higher dimensions)?

However, this is not an easy task, essentially because LQG is a quantization scheme based on the connection dynamics, while the $SU(2)$ connection dynamics are only well defined in three and four dimensions (the LQC in $2 + 1$ dimensions has already been constructed in [37]), and, thus, do not have a simple generalization to the higher dimensions. Fortunately, this difficulty has been overcome by Thiemann et al. in a series of papers [38–41]. The main idea of [38] is that in $n + 1$ dimensional GR, in order to obtain a well defined connection dynamics, one should adopt $SO(n + 1)$ connections A_a^{IJ} rather than the speculated $SO(n)$ connections. With these higher dimensional connection dynamics in hand, Thiemann et al. successfully generalize the LQG to arbitrary spacetime dimensions. Based on this higher dimensional LQG, the authors construct the $n + 1$ dimensional LQC, with $n \geq 3$ [42].

This paper is organized as follows: after a brief introduction, we review four-dimensional spatial flat $k = 0$ LQC in Section 2. Then, we generalize LQC to modified gravity and higher dimensional case in Section 3 and 4, respectively. The effective Hamiltonian and modified Friedman equations of these theories are obtained. Conclusions and outlooks are given in the last section.

2. $k = 0$ Loop Quantum Cosmology

2.1. Classical Connection Dynamics

To make this paper self-contained, we first review some basic elements of four-dimensional $k = 0$ loop quantum cosmology [7]. Loop quantum cosmology is a model that quantizes the symmetry reduced cosmological model by following the techniques of LQG as closely as possible. While loop quantum gravity is a quantization scheme of gravity built on the connection dynamics formalism of general relativity, with 3-metric h_{ab}, loop quantum gravity instead uses the densitized triad E_i^a and $SU(2)$ connections A_a^i as the basic building blocks, where these canonical geometric variables are defined as follows:

$$E_i^a = \sqrt{h}e_i^a, \quad A_a^i = \Gamma_a^i + \gamma K_a^i, \tag{1}$$

where e_i^a is the triad such that $h_{ab}e_i^a e_j^b = \delta_{ij}$, $K_i^a \equiv K^{ab}e_b^i$, Γ_a^i is the spin connection determined by E_i^a, and γ is a nonzero real number called Barbero–Immizi parameter.

These canonical conjugate pairs satisfy the following commutation relation:

$$\{A_i^a(x), E_b^j(y)\} = 8\pi G\gamma\delta_i^j\delta_b^a\delta(x, y). \tag{2}$$

Moreover, to make sure these connection dynamics are equivalent to the general relativity, we have to impose three constraints as follows [2,4]

$$G_i \;=\; D_a E_i^a, \tag{3}$$

$$V_a \;=\; \frac{1}{\kappa\gamma} F_{ab}^i E_i^b, \tag{4}$$

$$H_{gr} \;=\; \frac{\epsilon^{ijk} E_i^a E_j^b}{2\kappa\sqrt{h}} F_{ab,}{}^k - 2(\gamma^2+1)\frac{E_{[i}^a E_{j]}^b}{2\kappa\sqrt{h}} K_a^i K_b^j, \tag{5}$$

where $\kappa = 8\pi G$.

Now, we consider an isotropic and homogenous $k=0$ Universe. We choose a fiducial Euclidean metric $^o q_{ab}$ on the spatial slice of the isotropic observers and introduce a pair of fiducial orthonormal triad and co-triad as $(^o e_i^a, {}^o\omega_a^i)$, respectively, such that $^o q_{ab} = {}^o\omega_a^i {}^o\omega_b^i$. Then, the physical spatial metric is related to the fiducial one by $q_{ab} = a^{2\,o}q_{ab}$. In the cosmological model, our universe can be described by the Friedman–Robertson–Walker (FRW) metric:

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{6}$$

where a is the scale factor. By using these fiducial orthonormal triad and co-triad $(^o e_i^a, {}^o\omega_a^i)$, the densitized triad and spin connection can be simplified as

$$E_i^a = p V_0^{-\frac{2}{3}} \sqrt{\det(^0 q)}\, {}^o e_i^a, \quad A_b^j = c V_0^{-\frac{1}{3}\,o}\omega_b^j. \tag{7}$$

By using the classical expression and comological line elements, one can easily link (c, p) with scale factor a as $p = a^2$ and $c = \gamma\dot{a}$, respectively. Commutation relation between c and p can be reduced from (2) as follows

$$\{c, p\} = \frac{\kappa\gamma}{3}. \tag{8}$$

For our spatial flat and isotropic cosmological case, Gaussian and diffeomorphism constraints are satisfied automatically. The only remaining part is the Hamiltonian constraint (5). By using the strategy in [5], assume that the matter field is a massless scalar field ϕ. We denote p_ϕ as the conjugate momentum of scalar field ϕ. The commutation relation between ϕ and p_ϕ reads $\{\phi, p_\phi\} = 1$. In the cosmological model, this Hamiltonian, therefore, reduces to

$$H_T \;=\; -\frac{3}{\kappa\gamma^2} c^2 \sqrt{p} + \frac{p_\phi^2}{2p^{3/2}}. \tag{9}$$

In order to yield the dynamic evolution equation, we need to calculate the equation of motion for p, which reads

$$\dot{p} = \{p, H_T\} = \frac{2}{\gamma} c\sqrt{p}. \tag{10}$$

By using the Hamiltonian constraint, we can easily yield Friedman equation

$$H^2 = \left(\frac{\dot{p}}{2p}\right)^2 = \frac{1}{\gamma^2 p} c^2 = \frac{\kappa}{3}\frac{p_\phi^2}{2p^3} = \frac{\kappa}{3}\rho, \tag{11}$$

where $H = \frac{\dot{a}}{a}$ and $\rho = \frac{p_\phi^2}{2p^3}$ are the Hubble parameter and the matter density, respectively.

2.2. Quantum Theory

To quantize the cosmological model rigorously, we first need to construct the quantum kinematical Hilbert space of quantum cosmology. This quantum kinematical Hilbert space constitutes hybridly with the so-called polymer-like quantization for the gravity part, and with Schrodinger representation for the scalar field . The kinematical Hilbert space for the geometry part can be defined as $\mathcal{H}^{gr}_{kin} := L^2(R_{Bohr}, d\mu_H)$, where R_{Bohr} and $d\mu_H$ are, respectively, the Bohr compactification of the real line and the Haar measure on it [5]. On the other hand, for convenience, we choose the Schrodinger representation for the scalar field [8]. Thus, the kinematical Hilbert space for the scalar field part is defined as in the usual quantum mechanics: $\mathcal{H}^{sc}_{kin} := L^2(R, d\mu)$. Hence, the whole Hilbert space of the 3 + 1 dimensional LQC is a direct product, $\mathcal{H}_{kin} = \mathcal{H}^{gr}_{kin} \otimes \mathcal{H}^{sc}_{kin}$. In order to implement the Hamiltonian constraint at the quantum level, there are essentially two quantum effects that we need to consider. The first one is inverse volume corrections, which comes from the classical Hamiltonian constraint that involves the inverse of the determinate of the metric and thus can not be promoted as a well defined operator on the kinematic Hilbert space. In a 3 + 1 dimensional LQG, this difficulty can be overcome through the well known classical identity (Thiemann trick) $\frac{1}{2}\epsilon^{ijk}\frac{\epsilon_{abc}E^b_j E^c_k}{\sqrt{q}} = \frac{1}{\kappa\gamma}\{A^i_a, V\}$ [2], where the q denotes the determinant of the three metric and V is the volume of the fiducial cell. The second quantum effect is called a holonomy correction because the connection is not a well-defined quantum operator and needs to be replaced by holonomy:

$$h^\lambda_{\Box_{jk}} = \cos(\lambda c) + \sin(\lambda c)\tau_{jk}. \tag{12}$$

At the effective level, this is equivalent for replacing $c \mapsto \frac{\sin(\lambda c)}{\lambda}$ [10,11].

Under such a replacement, the effective Hamiltonian becomes

$$H_{eff} = -\frac{3}{\kappa\gamma^2}\frac{\sin^2(\bar{\mu}c)}{\bar{\mu}^2}\sqrt{p} + \frac{p_\phi^2}{2p^{3/2}}. \tag{13}$$

Here, we choose $\lambda = \bar{\mu} = \sqrt{\frac{\Delta}{p}}$ [6–8,10], with Δ being the minimal area in loop quantum gravity [2,10]. With this effective Hamiltonian in hand, we can easily derive the modified Friedman equation. To this aim, similar to the classical case, we again calculate the evolution of a canonical variable p

$$\dot{p} = \{p, H_{eff}\} = \frac{2}{\gamma}\frac{\sin(\bar{\mu}c)\cos(\bar{\mu}c)}{\bar{\mu}}\sqrt{p} = \frac{2}{\gamma\sqrt{\Delta}}\sin(\bar{\mu}c)\cos(\bar{\mu}c)p. \tag{14}$$

On the other hand, we can rewrite the effective Hamiltonian as

$$\sin^2(\bar{\mu}c) = \frac{\kappa\gamma^2\Delta}{3}\frac{p_\phi^2}{2p^3} = \frac{\rho}{\rho_c}, \tag{15}$$

where $\rho_c = \frac{3}{\kappa\gamma^2\Delta}$. Therefore, the modified Friedman equation reads

$$H = \left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{\dot{p}}{2p}\right)^2 = \frac{1}{\gamma^2\Delta}\sin^2(\bar{\mu}c)\cos^2(\bar{\mu}c) = \frac{\kappa}{3}\rho(1 - \frac{\rho}{\rho_c}). \tag{16}$$

It is easy to see at the point $\rho = \rho_c$, the Hubble parameter $H = 0$. Moreover, it can be verified that there is a bounce occurring at that point [10].

3. Loop Quantum Cosmology of Modified Gravity

3.1. Classical Theory

Modified gravity now receives more and more attention. In this section, we generalize loop quantum cosmology to modified gravity [14,15,30]. We mainly focus on one of the simplest modified gravity theories, which is the so-called Brans–Dicke theory. The treatment of other more complicated gravity can be found in [15]. The action of Brans–Dicke theory reads [14]

$$S(g) = \frac{1}{2\kappa} \int_{\Sigma} d^4x \sqrt{-g} \left[\phi \mathcal{R} - \frac{\omega}{\phi} g^{\mu\nu} (\partial_\mu \phi) \partial_\mu \phi \right], \tag{17}$$

where ω is the coupling constant of the Brans–Dicke theory.

Though a complicated Hamiltonian analysis, we can rewrite Brans–Dicke theory in the connection dynamics formalism and find that BD theory also admits three constraints just like GR. The Gauss and Diffeomophism constraints keep the same form as in GR and the Hamiltonian constraint of the Brans–Dicke theory reads [14]

$$
\begin{aligned}
H_{BD} &= \frac{\phi}{2\kappa} \left[F_{ab}^j - (\gamma^2 + \frac{1}{\phi^2}) \varepsilon_{jmn} \tilde{K}_a^m \tilde{K}_b^n \right] \frac{\varepsilon_{jkl} E_k^a E_l^b}{\sqrt{h}} \\
&+ \frac{\kappa}{3+2\omega} \left(\frac{(\tilde{K}_a^i E_i^a)^2}{\kappa^2 \phi \sqrt{h}} + 2 \frac{(\tilde{K}_a^i E_i^a)\pi}{\kappa\sqrt{h}} + \frac{\pi^2 \phi}{\sqrt{h}} \right) \\
&+ \frac{\omega}{2\kappa\phi} \sqrt{h}(D_a\phi)D^a\phi + \frac{1}{\kappa}\sqrt{h}D_a D^a \phi,
\end{aligned}
\tag{18}
$$

where $\tilde{K}_a^i \equiv \tilde{K}_{ab} e_i^b$, with $\tilde{K}_{ab}$ being

$$\tilde{K}^{ab} = \phi K^{ab} + \frac{h^{ab}}{2N}(\dot{\phi} - N^c \partial_c \phi), \tag{19}$$

where K^{ab} is the extrinsic curvature of three manifold. The only non-vanishing Poisson bracket between these canonical variables reads

$$\{\tilde{K}_a^j(x), E_k^b(y)\} = \kappa \delta_a^b \delta_k^j \delta(x,y). \tag{20}$$

Moreover, the connection A_a^i defined as

$$A_a^i = \Gamma_a^i + \gamma \tilde{K}_a^i, \tag{21}$$

where Γ_a^i is the spin connection, and γ is a nonzero real number. $F_{ab}^i \equiv 2\partial_{[a} A_{b]}^i + \epsilon^i_{\ kl} A_a^k A_b^l$ is the curvature of A_a^i.

Since the spatial topology is non-compact and the total volume of the spatial manifold is infinite, we introduce an "elemental cell" $\mathcal{V}$ and restrict all integral to $\mathcal{V}$. The homogeneity of the universe guarantees that the whole space information is reflected in this elemental cell. Now, we choose a fiducial Euclidean metric ${}^o q_{ab}$ and introduce a pair of fiducial orthnormal triad and co-triad as $({}^o e_i^a, {}^o \omega_a^i)$, respectively, such that ${}^o q_{ab} = {}^o \omega_a^i {}^o \omega_b^i$. For simplicity, we let the elemental cell $\mathcal{V}$ be a cubic measured by our fiducial metric and denote its volume as V_o. Because our FRW metric is spatially flat, we have $\Gamma_a^i = 0$, and, hence, $A_a^i = \gamma \tilde{K}_a^i$. Via fixing the degrees of freedom of local gauge and diffeomorphism, we finally obtain the connection and densitized triad by symmetrical reduction as [5]:

$$A_a^i = \tilde{c} V_0^{-\frac{1}{3}} {}^o\omega_a^i, \qquad E_j^b = p V_0^{-\frac{2}{3}} \sqrt{\det({}^0 q)} {}^o e_j^b, \tag{22}$$

where $\tilde{c}, p$ are only functions of t. Hence, the phase space of the cosmological model consists of conjugate pairs $(\tilde{c}, p)$ and (ϕ, π). The basic Poisson brackets between them can be simply read as:

$$
\begin{aligned}
\{\tilde{c}, p\} &= \frac{\kappa}{3}\gamma, \\
\{\phi, \pi\} &= 1.
\end{aligned}
\tag{23}
$$

From the classical line element, we can obtain $\tilde{c} = \gamma(\phi\dot{a} + \dot{\phi}\frac{a}{2}), p = a^2$. Note that by the symmetric reduction, the Gaussian and diffeomorphism constraints are satisfied automatically. Moreover, since we only consider the homogeneous universe model, the last two terms in the Hamiltonian constraint (18) only evolve spatial derivatives, and thus should vanish. Hence, we only need to consider the remaining five terms in the Hamiltonian constraint (18). The reduced Hamiltonian in the cosmological model reads

$$
H_{BD} = -\frac{3\tilde{c}^2\sqrt{|p|}}{\gamma^2\kappa\phi} + \frac{\kappa}{(3+2\omega)\phi|p|^{\frac{3}{2}}}\left(\frac{3\tilde{c}p}{\kappa\gamma} + \pi\phi\right)^2.
\tag{24}
$$

Similar to the last section, by using evolution of p and the Hamiltonian constraint, we can get the Friedman equation as follows [30]:

$$
\begin{aligned}
\left(\frac{\dot{a}}{a} + \frac{\dot{\phi}}{2\phi}\right)^2 &= \frac{1}{\phi^2}\frac{\kappa}{3}\rho_{eff} = \frac{\kappa}{3\phi^2}\left(\frac{\beta\dot{\phi}^2}{4\kappa} + \phi\rho\right) \\
&= \frac{\beta\dot{\phi}^2}{12\phi^2} + \frac{\kappa\rho}{3\phi}.
\end{aligned}
\tag{25}
$$

3.2. Effective Equation

To study the effective theory of loop quantum Brans–Dicke cosmology, we also want to know the effect of matter fields on the dynamical evolution. Hence, we include an extra massless scalar matter field φ into Brans–Dicke cosmology. Then, classically, the total Hamiltonian constraint of the model reads

$$
H = -\frac{3\tilde{c}^2\sqrt{|p|}}{\gamma^2\kappa\phi} + \frac{\kappa}{(3+2\omega)\phi|p|^{\frac{3}{2}}}\left(\frac{3\tilde{c}p}{\kappa\gamma} + \pi\phi\right)^2 + \frac{p_\varphi^2}{2|p|^{\frac{3}{2}}},
\tag{26}
$$

where p_φ is the momentum conjugate to φ. The effective description of LQC is a delicate and topical issue since it may relate the quantum gravity effects to low-energy physics. The effective equations of LQC are being studied from both the canonical perspective [43–46] and the path integral perspective [47–53]. Since the key element in the polymer-like quantization of the previous subsection is to take holonomies rather than connections as basic variables, a heuristic and simple way to get the effective equations is to do the replacement $\tilde{c} \rightarrow \frac{\sin(\bar{\mu}\tilde{c})}{\bar{\mu}}$ [30]. In fact, this replacement is rigorous in Brans–Dicke as well as more general scalar-tensor theories of gravity [14,15,30]. Under this replacement, the effective version of Hamiltonian constraint (26) takes the form

$$
H = -\frac{3\sin^2(\bar{\mu}\tilde{c})\sqrt{|p|}}{\kappa\gamma^2\phi\bar{\mu}^2} + \frac{\kappa}{\beta\phi|p|^{\frac{3}{2}}}\left(\frac{3\sin(\bar{\mu}\tilde{c})p}{\bar{\mu}\kappa\gamma} + \pi\phi\right)^2 + |p|^{\frac{3}{2}}\rho,
\tag{27}
$$

where $\rho = \frac{p_\varphi^2}{2|p|^3}$ by definition is the matter density of massless scalar field. The more detailed study show that the effective Hamiltonian (27) derived by a path integral formalism [30] actually takes the same form. Then, the canonical equations of motion read:

$$\dot{p} = \frac{2\sqrt{|p|}}{\gamma\phi\bar{\mu}}\sin(\bar{\mu}\tilde{c})\cos(\bar{\mu}\tilde{c}) - \frac{2\kappa}{\beta\phi|p|^{\frac{1}{2}}}\left(\frac{3\sin(\bar{\mu}\tilde{c})p}{\bar{\mu}\kappa\gamma} + \pi\phi\right)\cos(\bar{\mu}\tilde{c}), \tag{28}$$

$$\dot{\phi} = \frac{2\kappa}{\beta|p|^{\frac{3}{2}}}\left(\frac{3\sin(\bar{\mu}\tilde{c})p}{\bar{\mu}\kappa\gamma} + \pi\phi\right). \tag{29}$$

In the above calculation, the Poisson brackets (23) have been used. The combination of Equations (28) and (29) gives:

$$\left(\frac{\dot{p}}{2p} + \frac{\dot{\phi}}{2\phi}\right)^2 = \left[\frac{\sin(\bar{\mu}\tilde{c})\cos(\bar{\mu}\tilde{c})}{\gamma\phi\bar{\mu}\sqrt{|p|}} + \frac{\kappa}{\beta\phi|p|^{\frac{3}{2}}}\left(\frac{3\sin(\bar{\mu}\tilde{c})p}{\bar{\mu}\kappa\gamma} + \pi\phi\right)(1 - \cos(\bar{\mu}\tilde{c}))\right]^2$$

$$= \left[\frac{\sin(\bar{\mu}\tilde{c})\cos(\bar{\mu}\tilde{c})}{\gamma\phi\sqrt{\Delta}} + \frac{\dot{\phi}}{2\phi}(1 - \cos(\bar{\mu}\tilde{c}))\right]^2. \tag{30}$$

On the other hand, from the effective Hamiltonian constraint (27), we can get

$$-\frac{3\sin^2(\bar{\mu}\tilde{c})}{\kappa\gamma^2\phi\Delta} + \frac{\beta\dot{\phi}^2}{4\kappa\phi} + \rho = 0, \tag{31}$$

which implies

$$\sin^2(\bar{\mu}\tilde{c}) = \frac{\rho_{eff}}{\rho_c}, \tag{32}$$

where $\rho_c = \frac{3}{\gamma^2\Delta\kappa} = \frac{\sqrt{3}}{32\pi^2G^2\gamma^3\hbar}$ and $\rho_{eff} = \frac{\beta\dot{\phi}^2}{4\kappa} + \phi\rho$. Taking Equation (32) into account and $p = a^2$, we can rewrite Equation (30) as:

$$\left(\frac{\dot{a}}{a} + \frac{\dot{\phi}}{2\phi}\right)^2 = \left[\frac{1}{\phi}\sqrt{\frac{\kappa}{3}\rho_{eff}\left(1 - \frac{\rho_{eff}}{\rho_c}\right)} + \frac{\dot{\phi}}{2\phi}\left(1 - \sqrt{1 - \frac{\rho_{eff}}{\rho_c}}\right)\right]^2. \tag{33}$$

This is the effective Friedmann equation of Brans–Dicke cosmology, which contains important quantum correction terms. In addition, we can show that, for a contracting universe, ρ_{eff} monotonically increases while v decreases [30]. Thus, it is easy to see from Equation (33) that, when ρ_{eff} approaches ρ_c, one gets $\cos(\bar{\mu}\tilde{c}) = 1 - \frac{\rho_{eff}}{\rho_c} = 0$. Then, from Equation (28), we can obtain $\dot{p} = 0$. This implies a quantum bounce that would happen at that point for a contracting universe.

The modified Friedman equation has two important limits cases. The first one is when $\phi = 1$, and note that in this case we have $\rho_{eff} = \rho$ and $\dot{\phi} = 0$, Equation (33) returns to the well-known effective Friedmann equation of LQC [10,44] as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3}\rho\left(1 - \frac{\rho}{\rho_c}\right). \tag{34}$$

Second, in the classical regime, i.e., when $\rho_{eff} \ll \rho_c$, we can omit $\frac{\rho_{eff}}{\rho_c}$ terms in Equation (33) to get the classical limit as:

$$\left(\frac{\dot{a}}{a} + \frac{\dot{\phi}}{2\phi}\right)^2 = \frac{1}{\phi^2}\frac{\kappa}{3}\rho_{eff} = \frac{\kappa}{3\phi^2}\left(\frac{\beta\dot{\phi}^2}{4\kappa} + \phi\rho\right)$$

$$= \frac{\beta\dot{\phi}^2}{12\phi^2} + \frac{\kappa\rho}{3\phi}, \tag{35}$$

which is nothing but the classical Friedmann equation of Brans–Dicke cosmology. This, in turn, guarantees our effective theory as a correct classical limit.

4. Loop Quantum Cosmology in Higher Dimensions

Now, we turn to generalizing our results to higher dimensions, and this section is mainly based on [42]. The connection dynamics of n+1 dimensional gravity ($n \geq 3$) with a gauge group $SO(n+1)$ or $SO(1,n)$ is obtained in [38]. The Ashtekar connection formalism of n + 1 dimensional gravity constitutes an $SO(1,n)$ (or $SO(n+1)$) connections A_a^{IJ} and a group value densitized vector π_{IJ}^b defined on an oriented n dimensional manifold S, where $a, b = 1, 2 \ldots n$ are the spatial indices and $I, J = 1, 2, 3 \ldots n$ denotes $SO(1,n)$ group indices. The commutation relation for the canonical conjugate pairs satisfies

$$\{A_{aIJ}(x), \pi^{bKL}(y)\} = 16\pi G\gamma \delta_{[I}^K \delta_{J]}^L \delta_a^b \delta(x, y), \tag{36}$$

where γ is a nonzero real number. Here, the π^{bKL} satisfies the "Simplicity constraint" [38]. Solving this "Simplicity constraint" at the classical level gives us $\pi^{bKL} = 2n^{[K}E^{b|L]} = 2\sqrt{h}h^{ab}n^{[K}e_a^{L]}$, where the spatial metric reads $h_{ab} = e_a^i e_{bi}$, n^K is a normal, which satisfies $e_a^K n_K = 0$ and $n^K n_K = -1$ for $SO(1,n)$ (for the case $SO(n+1)$, $n^K n_K = 1$). Moreover, the densitized vector E_I^a satisfies $hh^{ab} = E_I^a E^{bI}$, where h is the determinant of the n dimensional spatial metric h_{ab}. A_a^{IJ} is an $SO(1,n)$ connection defined as $A_a^{IJ} = \Gamma_a^{IJ} + \gamma K_a^{IJ}$, where Γ_a^{IJ} and K_a^{IJ} are n dimensional spin connection and extrinsic curvature, respectively. Besides the Simplicity constraint, the n+1 dimensional gravity has three constraints similar to 3+1 dimensional general relativity [38,40]:

$$G^{IJ} \quad := \quad \mathcal{D}_a \pi^{aIJ} := \partial_a \pi^{aIJ} + 2A_a^{[I}{}_K \pi^{a|K|J]}, \tag{37}$$

$$V_a \quad = \quad \frac{1}{2\gamma} F_{abIJ} \pi^{bIJ}, \tag{38}$$

$$H_{gr} \quad = \quad \frac{1}{2\kappa\sqrt{h}} \left(F_{abIJ} \pi^{aIK} \pi^b{}_K^J + 4\bar{D}_T^{aIJ}(F^{-1})_{aIJ,bKL} \bar{D}_T^{bKL} - 2(1+\gamma^2)K_{aI}K_{bJ}E^{a[I}E^{b|J]} \right), \tag{39}$$

where $F_{abIJ} \equiv 2\partial_{[a}A_{b]IJ} + 2A_{a[I|K}A_b{}^K{}_{|J]}$ is the curvature of connection A_{aIJ}, and $\bar{D}_T^{aIJ} = \frac{\gamma}{4}F^{aIJ,bKL}\bar{K}_{bKL}^T$, with $\bar{K}_{bKL}^T$ being the transverse and traceless part of extrinsic curvature K_{bKL}. Moreover, we have $[F \cdot F^{-1}]_{bKL}^{aIJ} = \delta_b^a \bar{\eta}_{[K}^I \bar{\eta}_{L]}^J$, with $\bar{\eta}_J^I = \delta_J^I - n^I n_J$.

Now, let us consider the $n + 1$ dimensional isotropic and homogenous $k = 0$ Universe. Its line element is described by the n+1 dimensional Friedmann–Robertson–Walker (FRW) metric

$$ds^2 = -N^2 dt^2 + a^2(t)d\Omega^2, \tag{40}$$

where a is the scale factor, N is the lapse function, and $d\Omega^2$ is the n dimensional spatial section. In the following, for simplicity, we fix $N = 1$. We choose a fiducial Euclidean metric ${}^o q_{ab}$ on the n dimensional spatial slice of the isotropic observers and introduce a pair of fiducial orthnormal bases as $({}^o e_I^a, {}^o \omega_a^I)$ such that ${}^o q_{ab} = {}^o \omega_a^I \omega_b^I$. The physical spatial metric is related to the fiducial one by $q_{ab} = a^{2o}q_{ab}$. Then, the densitized vector can be expressed as $E_I^a = pV_0^{-\frac{n-1}{n}}\sqrt{{}^0 q}{}^o e_I^a$, thus the π^{aIJ} and spin connection A_b^{IJ}, respectively, reduce to

$$\pi^{aIJ} \quad = \quad 2pV_0^{-\frac{n-1}{n}}\sqrt{{}^0 q}n^{[Io}e^{a|J]} = pV_0^{-\frac{n-1}{n}}\sqrt{{}^0 q}{}^o\pi^{aIJ}, \tag{41}$$

$$A_b^{IJ} \quad = \quad 2cV_0^{-\frac{1}{n}o}n^{[Io}\omega_b^{J]} = cV_0^{-\frac{1}{n}o}\Omega_a^{IJ}. \tag{42}$$

In the following, without loss of generality, we will fix the fiducial volume $V_0 = 1$. A straightforward calculation shows

$$p = a^{n-1}, \qquad c = \gamma \dot{a}. \tag{43}$$

These canonical variables satisfy the commutation relation as follows:

$$\{c, p\} = \frac{\kappa \gamma}{n}. \tag{44}$$

For our cosmological case, the Gaussian and diffeomorphism constraints are satisfied automatically. For the Hamiltonian constraint, we first note that, in our cosmological situation, the extrinsic curvature only has the diagonal part. Hence, the transverse traceless part of extrinsic curvature $\bar{K}^T_{bKL}$ is identical to zero. Therefore, the second term of the Hamiltonian constraint is vanishing. Moreover, the spin connection Γ is also zero for our homogenous and isotropic universe. Thus, a simple straightforward calculation shows $KKEE$ term proportional to $F\pi\pi$ term. Combining all of the above ingredients together, the Hamiltonian constraint (39) reduces to

$$H_{gr} = -\frac{1}{2\kappa\gamma^2} F_{abIJ} \frac{\pi^{aIK}\pi^b{}_K{}^J}{\sqrt{h}}. \tag{45}$$

Now, as in the 3 + 1 dimensional LQC, we also consider a minimally coupled massless scalar field ϕ as our matter field. The total Hamiltonian now reads

$$H_{Total} = -\frac{1}{2\kappa\gamma^2} F_{abIJ} \frac{\pi^{aIK}\pi^b{}_K{}^J}{\sqrt{h}} + \frac{p_\phi^2}{2\sqrt{h}}, \tag{46}$$

where the p_ϕ by definition is the conjugate momentum of massless scalar field ϕ. The Poisson bracket between scalar field ϕ and conjugate momentum p_ϕ reads $\{\phi, p_\phi\} = 1$. In the cosmological model that we consider in this paper, this Hamiltonian therefore reduces to

$$H_{Total} = -\frac{n(n-1)}{2\kappa\gamma^2} c^2 p^{\frac{n-2}{n-1}} + \frac{p_\phi^2}{2p^{\frac{n}{n-1}}}. \tag{47}$$

In order to reproduce an n+1 dimensional Friedmann equation with our Hamiltonian (47) and commutation relation (44), we calculate the equation of motion for p, which reads

$$\dot{p} = \{p, H_{Total}\} = \frac{n-1}{\gamma} c p^{\frac{n-2}{n-1}}. \tag{48}$$

By using the Hamiltonian constraint, we successfully reproduce the classical $n+1$ dimensional Friedmann equation

$$\begin{aligned} H^2 &= \left(\frac{\dot{p}}{(n-1)p}\right)^2 = \frac{1}{\gamma^2 p^2} c^2 p^{\frac{2(n-2)}{n-1}} \\ &= \frac{2\kappa}{n(n-1)} \frac{p_\phi^2}{2p^{\frac{2n}{n-1}}} = \frac{2\kappa}{n(n-1)} \rho, \end{aligned} \tag{49}$$

where

$$\begin{aligned} H &= \frac{\dot{a}}{a} \\ \rho &= \frac{p_\phi^2}{2V^2} = \frac{p_\phi^2}{2p^{\frac{2n}{n-1}}} \end{aligned} \tag{50}$$

are the Hubble parameter and the matter density in n+1 dimensions, respectively.

In n + 1 dimensional quantum gravity, the $n - 1$ dimensional area operator is quantized just like its counterparts in 3 + 1 dimensions, and the discrete spectrum of this $n - 1$ dimensional quantum operator reads [40]

$$\Delta_n = \kappa \hbar \gamma \sum_I \sqrt{I(I + n - 1)} = 8\pi \gamma (\ell_{\rm p})^{n-1} \sum_I \sqrt{I(I + n - 1)}, \tag{51}$$

where I is an integer and $\ell_{\rm p} = \sqrt[n-1]{G\hbar}$ is the Planck length with G and $\hbar$ being the Newton's and Planck's constants in $n + 1$ dimensions [40]. Now, similar to the last two sections, at the effective level, the quantum Hamiltonian can be simply obtained again from a replacement [42]

$$c \rightsquigarrow \frac{\sin(\bar{\mu} c)}{\bar{\mu}}, \tag{52}$$

where $\bar{\mu} = \left(\frac{\Delta_n}{|p|}\right)^{\frac{1}{n-1}}$, with Δ_n being a minimum nonzero eigenvalue of the n dimensional area operator [7].

Under such a replacement, the quantum effective Hamiltonian reads

$$H_{eff} = -\frac{n(n - 1)}{2\kappa \gamma^2} \frac{\sin^2(\bar{\mu} c)}{\bar{\mu}^2} p^{\frac{n-2}{n-1}} + \frac{p_\phi^2}{2p^{\frac{n}{n-1}}}. \tag{53}$$

Now, we are ready to derive the physical evolution equation of the $n + 1$ dimensional Universe, and the most important one is, of course, the modified Friedmann equation. To this aim, combining the effective Hamiltonian constraint H_{eff} (53) with the symplectic structure of $n + 1$ dimensional loop quantum cosmology, one can easily obtain equations of motion for the p as

$$\dot{p} = \{p, H_{eff}\} = \frac{n - 1}{\gamma} \frac{\sin(\bar{\mu} c) \cos(\bar{\mu} c)}{\bar{\mu}} p^{\frac{n-2}{n-1}}. \tag{54}$$

By using Equation (54), and recalling that $p = a^{n-1}$, it is easy to see that

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{\dot{p}}{(n-1)p}\right)^2 = \frac{1}{\gamma^2 (\Delta_n)^{\frac{2}{n-1}}} \sin^2(\bar{\mu} c) \cos^2(\bar{\mu} c). \tag{55}$$

Note that the right-hand side of the above equation evolves $\sin^2(\bar{\mu} c) \cos^2(\bar{\mu} c)$. It therefore strongly hints to us that it might have some relation with the effective Hamiltonian constraint (53). A simple and straightforward calculation shows us that this is true. In fact, the effective Hamiltonian constraint $H_{eff} = 0$ can be rewritten as the following compact form

$$\sin^2(\bar{\mu} c) = \frac{2\kappa \gamma^2 (\Delta_n)^{\frac{2}{n-1}} \rho}{n(n - 1)} = \frac{\rho}{\rho_c}. \tag{56}$$

Here, we define $\rho_c = \frac{n(n-1)}{2\kappa \gamma^2 (\Delta_n)^{\frac{2}{n-1}}}$ as the $n + 1$ dimensional critical matter density, which actually is the upper bound of the matter density. With this equation in hand, the modified Friedmann equation can be easily obtained as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{2\kappa}{n(n - 1)} \rho \left(1 - \frac{\rho}{\rho_c}\right). \tag{57}$$

5. Conclusions

In this paper, we review the four-dimensional LQC as well as its extension to modified gravity and higher dimensions. We give a detailed construction of the effective Hamiltonian and use that effective Hamiltonian to obtain the modified Friedman equation that represents the evolution of the universe. We start from a 3+1 dimensional case, and then generalize it to the higher dimensional case as well as a modified gravity case. Our result shows that the heuristic replacement $c \rightarrowtail \frac{\sin(\bar{\mu}c)}{\bar{\mu}}$ is actually works quite general. In all cases we consider here, the modified Friedman equation contains a corrections term $\frac{\rho}{\rho_c}$. Therefore, in the classical regime, we have $\frac{\rho}{\rho_c} \to 0$. This, in turn, implies the corrected classical limit, while, in the Planck regime, the ratio $\frac{\rho}{\rho_c}$ approaches 1. Hence, in all these theories, one can naturally expect the existence of a quantum bounce. A more detailed discussion can be found in [30,54].

There are several directions to further extend our work. The first one is to incorporate the dimension reduction mechanism with our higher dimensional LQC model. Getting a four-dimensional cosmology theory and comparing this theory with the observation data will be very interesting and might find some higher dimensional quantum effects. Second, for modified gravity LQC, the perturbation theories, especially in the Jordan frame, are yet to be carried out. Moreover, we can compare the perturbation theory in the Jordan frame and in the Einstein frame to see whether there is a difference or not. Third, we can combine modified gravity and higher dimensional LQC to construct higher dimensional modified LQC, since in the classical case, some of the higher dimensional modified cosmological models can naturally explain the accelerated expansions of our current Universe [23,36]. It will be very interesting to investigate the quantum effective of these models. We hope to explore these delicate and important issues in the near future.

Finding observational signals are always one of the most important issues of LQC. In particular: does current CMB data already contain some quantum gravity effects? In the 3+1 dimensional LQC of GR, the effects of LQC will slightly modify the form of Mukhanov–Sasaki equation and therefore give different values of some physical quantities such as the power spectrum, the tensor to scalar ratio, and so on [55,56]. In the higher dimensional LQC model as well as the LQC of modified gravity, we expect the corresponding Mukhanov–Sasaki equation will contain some new terms and give different predictions that can be verified/falsified by the experimental data.

This paper aims to provide a short introduction to effective theories of LQC and its extensions. It should be mentioned that, because of length limitation, many aspects of LQC are not covered. These topics include but are not limited to inflationary scenarios in LQC [57], Starobinski types model [58,59], the value of Immirzi parameters in higher dimensions [60], linking LQC with LQG [61,62], more detailed discussion on singularity issues [63,64], and so on.

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References

1. Rovelli, C. *Quantum Gravity*; Cambridge University Press: Cambridge, UK, 2004.
2. Thiemann, T. *Modern Canonical Quantum General Relativity*; Cambridge University Press: Cambridge, UK, 2007.
3. Ashtekar, A.; Lewandowski, J. Background independent quantum gravity: A status report. *Class. Quantum Gravity* **2004**, *21*, R53.
4. Han, M.; Huang, W.; Ma, Y. Fundamental structure of loop quantum gravity. *Int. J. Mod. Phys. D* **2007**, *16*, 1397–1474.
5. Ashtekar, A.; Bojowald, M.; Lewandowski, J. Mathematical structure of loop quantum cosmology. *Adv. Theor. Math. Phys.* **2003**, *7*, 233–268.
6. Bojowald, M. Loop quantum cosmology. *Living Rev. Relat.* **2005**, *8*, 11.

7.　Ashtekar, A. Loop quantum cosmology: An overview. *Gen. Relat. Gravity* **2009**, *41*, 707–741.

8.　Ashtekar, A.; Singh, P. Loop quantum cosmology: A status report. *Class. Quantum Gravity* **2011**, *28*, 213001.

9.　Banerjee, K.; Calcagni, G.; Martin-Benito, M. Introduction to loop quantum cosmology. *Symmetry Integr. Geom. Methods Appl.* **2012**, *8*, 016.

10.　Ashtekar, A.; Pawlowski, T.; Singh, P. Quantum nature of the big bang: Improved dynamics. *Phys. Rev. D* **2006**, *74*, 084003.

11.　Ashtekar, A.; Corichi, A.; Singh, P. Robustness of key features of loop quantum cosmology. *Phys. Rev. D* **2008**, *77*, 024046.

12.　Zhang, X.; Ma, Y. Extension of loop quantum gravity to $f(R)$ theories. *Phys. Rev. Lett.* **2011**, *106*, 171301.

13.　Zhang, X.; Ma, Y. Loop quantum $f(R)$ theories. *Phys. Rev. D* **2011**, *84*, 064040.

14.　Zhang, X.; Ma, Y. Loop quantum Brans-Dicke theory. *J. Phys. Conf. Ser.* **2012**, *360*, 012055.

15.　Zhang, X.; Ma, Y. Nonperturbative loop quantization of scalar-tensor theories of gravity. *Phys. Rev. D* **2011**, *84*, 104045.

16.　Ma, Y. Extension of loop quantum gravity to metric theories beyond general relativity. *J. Phys. Conf. Ser.* **2012**, *360*, 012006.

17.　Brans, C.; Dicke, R.H. Mach's principle and a relativistic theory of gravitation. *Phys. Rev.* **1961**, *124*, 925.

18.　Greenstein, G.S. Brans–Dicke cosmology, I. *Astrophys. Lett.* **1968**, *1*, 139–141.

19.　Greenstein, G.S. Brans–Dicke cosmology, II. *Astrophys. Space Sci.* **1968**, *2*, 155–165.

20.　Anderson, J.D.; Morris, J.R. Brans–Dicke theory and the Pioneer anomaly. *Phys. Rev. D* **2012**, 86, 064023.

21.　Benerjee, N.; Pavon, D. Cosmic acceleration without quintessence. *Phys. Rev. D* **2001**, *63*, 043504.

22.　Sen, S.; Sen, A.A. Late time acceleration in Brans–Dicke cosmology. *Phys. Rev. D* **2001**, *63*, 124006.

23.　Qiang, L.; Ma, Y.; Han, M.; Yu, D. 5-dimensional Brans–Dicke theory and cosmic acceleration. *Phys. Rev. D* **2005**, *71*, 061501.

24.　Das, S.; Banerjee, N. Brans–Dicke scalar field as a chameleon. *Phys. Rev. D* **2008**, *78*, 043512.

25.　Felice, A.D.; Tsujikawa, S. Generalized Brans–Dicke theories. *J. Cosmol. Astropart. Phys.* **2010**, *2010*, 024.

26.　Bisabr, Y. Cosmic acceleration in Brans–Dicke cosmology. *Gen. Relat. Gravity* **2012**, *44*, 427–435.

27.　Friemann, J.; Turner, M.; Huterer, D. Dark energy and the accelerating Universe. *Ann. Rev. Astron. Astrophys.* **2008**, *46*, 385–432.

28.　Will, C.M. The confrontation between general relativity and experiment. *Living Rev. Relativ.* **2006**, *9*, 3.

29.　Will, C.M. *Theory and Experiment in Gravitational Physics*; Cambridge University Press: Cambridge, UK, 1993.

30.　Zhang, X.; Ma, Y. Loop quantum Brans–Dicke cosmology. *Phys. Rev. D* **2013**, *87*, 084024.

31.　Appelquist, T.; Chodos, A.; Freund, P.G.O. *Modern Kaluza-Klein Theories (Frontiers in Physics)*; Addison-Wesley: Reading, MA, USA, 1986.

32.　Polchinski, J. *String Theory*; Cambridge University Press: Cambridge, UK, 1998.

33.　Maldacena, J.M. TASI 2003 Lectures on AdS/CFT. In Proceedings of the Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2003): Recent Trends in String Theory, Boulder, CO, USA, 1–27 June 2003.

34.　Randall, L.; Sundrum, R. A Large Mass Hierarchy from a Small Extra Dimension. *Phys. Rev. Lett.* **1999**, *83*, 3370–3373.

35.　Randall, L.; Sundrum, R. An Alternative to Compactification. *Phys. Rev. Lett.* **1999**, *83*, 4690–4693.

36.　Mohammedi, N. Dynamical compactification, standard cosmology, and the accelerating universe. *Phys. Rev. D* **2002**, *65*, 104018.

37.　Zhang, X. Loop quantum cosmology in 2 + 1 dimension. *Phys. Rev. D* **2014**, *90*, 124018.

38.　Bodendorfer, N.; Thiemann, T.; Thurn, A. New variables for classical and quantum gravity in all dimensions I. Hamiltonian analysis. *Class. Quantum Gravity* **2013**, *30*, 045001.

39.　Bodendorfer, N.; Thiemann, T.; Thurn, A. New variables for classical and quantum gravity in all dimensions II. Lagrangian analysis. *Class. Quantum Gravity* **2013**, *30*, 045002.

40.　Bodendorfer, N.; Thiemann, T.; Thurn, A. New variables for classical and quantum gravity in all dimensions III. Quantum theory. *Class. Quantum Gravity* **2013**, *30*, 045003.

41.　Bodendorfer, N.; Thiemann, T.; Thurn, A. New variables for classical and quantum gravity in all dimensions IV. Matter Coupling. *Class. Quantum Gravity* **2013**, *30*, 045004.

42.　Zhang, X. Higher dimensional loop quantum cosmology. *Eur. Phys. J. C* **2016**, *76*, 395.

43. Taveras, V. Corrections to the Friedmann equations from loop quantum gravity for a universe with a free scalar field. *Phys. Rev. D* **2008**, *78*, 064072.
44. Ding, Y.; Ma, Y.; Yang, J. Effective scenario of loop quantum cosmology. *Phys. Rev. Lett.* **2009**, *102*, 051301.
45. Yang, J.; Ding, Y.; Ma, Y. Alternative quantization of the Hamiltonian in loop quantum cosmology. *Phys. Lett. B* **2009**, *682*, 1–7.
46. Bojowald, M.; Brizuela, D.; Hernandez, H.H.; Koop, M.J.; Morales-Tecotl, H.A. High-order quantum back-reaction and quantum cosmology with a positive cosmological constant. *Phys. Rev. D* **2011**, *84*, 043514.
47. Ashtekar, A.; Campiglia, M.; Henderson, A. Loop quantum cosmology and spin foams. *Phys. Lett. B* **2009**, *681*, 347.
48. Ashtekar, A.; Campiglia, M.; Henderson, A. Casting loop quantum cosmology in the spin foam paradigm. *Class. Quantum Gravity* **2010**, *27*, 135020.
49. Ashtekar, A.; Campiglia, M.; Henderson, A. Path integrals and the WKB approximation in loop quantum cosmolog. *Phys. Rev. D* **2010**, *82*, 124043.
50. Qin, L.; Huang, H.; Ma, Y. Path integral and effective Hamiltonian in loop quantum cosmology. *Gen. Relativ. Gravit.* **2013**, *45*, 1191–1210.
51. Qin, L.; Deng, G.; Ma, Y. Path Integrals and Alternative Effective Dynamics in Loop Quantum Cosmology. *Commun. Theor. Phys.* **2012**, *57*, 326–332.
52. Qin, L.; Ma, Y. Coherent state functional integrals in quantum cosmology. *Phys. Rev. D* **2012**, *85*, 063515.
53. Qin, L.; Ma, Y. Coherent state functional integral in loop quantum cosmology: Alternative dynamics. *Mod. Phys. Lett.* **2012**, *27*, 1250078.
54. Singh, P. Are loop quantum cosmos never singular? *Class. Quantum Gravity* **2009**, *26*, 125005.
55. Gomar, L.; Martin-Benito, M.; Marugan, G. Quantum corrections to the Mukhanov-Sasaki equations. *Phys. Rev. D.* **2016**, *93*, 104025.
56. Cai, Y.; Wilson-Ewing, E. A ΛCDM bounce scenario. *J. Cosmol. Astropart. Phys.* **2015**, *2015*, 006.
57. Agullo, I.; Ashtekar, A.; Nelson, W. A quantum gravity extension of the inflationary scenario. *Phys. Rev. Lett.* **2012**, *109*, 251301.
58. Amorós, J.; Haro, J.; Odintsov, S.D. $R + \alpha R^2$ loop quantum cosmology. *Phys. Rev. D* **2014**, *89*, 104010.
59. Bonga, B.; Gupt, B. Phenomenological investigation of a quantum gravity extension of inflation with the Starobinsky potential. *Phys. Rev. D* **2016**, *93*, 063513.
60. Zhang, X. Immirzi parameter and quasinormal modes in four and higher spacetime dimensions. *Front. Phys.* **2016**, *11*, 110401.
61. Alesci, E.; Cianfrani, F. A new perspective on cosmology in Loop Quantum Gravity. *Europhys. Lett.* **2013**, *104*, 10001.
62. Alesci, E.; Cianfrani, F. Quantum-reduced loop gravity: Cosmology. *Phys. Rev. D* **2013**, *87*, 083521.
63. Diener, P.; Gupt, B.; Singh, P. Chimera: A hybrid approach to numerical loop quantum cosmology. *Class. Quantum Gravity* **2014**, *31*, 025013.
64. Diener, P.; Gupt, B.; Singh, P. Numerical simulations of a loop quantum cosmos: robustness of the quantum bounce and the validity of effective dynamics. *Class. Quantum Gravity* **2014**, *31*, 105015.