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Symplectic Structure of Intrinsic Time Gravity

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Abstract: The Poisson structure of intrinsic time gravity is analysed. With the starting point comprising a unimodular three-metric with traceless momentum, a trace-induced anomaly results upon quantization. This leads to a revision of the choice of momentum variable to the (mixed index) traceless momentric. This latter choice unitarily implements the fundamental commutation relations, which now take on the form of an affine algebra with SU(3) Lie algebra amongst the momentric variables. The resulting relations unitarily implement tracelessness upon quantization. The associated Poisson brackets and Hamiltonian dynamics are studied.

Keywords: intrinsic time; quantum gravity; canonical; quantization; symmetry

1. Introduction

A crucially important question in the quantization of gravity in 3+1 dimensions, as for any theory, is the choice of the fundamental dynamical variables of the classical theory, which upon quantization become promoted to quantum operators. In Loop Quantum Gravity (LQG) [1] the starting point for the classical theory are the Ashtekar variables, where a SU(2) gauge connection and a densitized triad form a canonically conjugate pair. This choice of variables turns the initial value constraints of GR from intractable non polynomial phase space functions, as they appear in the Arnowitt Deser Misner (ADM) theory [2], into polynomial form at the expense of an additional set of constraints related to the SU(2) gauge symmetry inherent in the theory. It is hoped that the polynomial form of the constraints in LQG make the constraints more tractable for quantization and the construction of a physical Hilbert space. The actual configuration variable in LQG which is subject to the quantization procedure is not the connection itself, but rather the holonomy of the connection, since the latter is well-defined in the quantum theory whereas the connection fails to exist [3]. Furthermore, the transformation properties of the holonomy more aptly are representative, at the kinematical level, of the symmetry properties of the theory [4]. Consequently, upon quantization in LQG all constraints and quantities must be rewritten in terms of the holonomies and the densitized triad, which themselves no longer form a canonical pair.

In LQG there exists only a manifold structure with no metric, and the metric is no longer fundamental, but becomes a derived quantity in terms of more fundamental variables. A main difficulty in LQG is the construction of a physical Hilbert space from solution of the Hamiltonian constraint. Whether one utilizes the self-dual version of the connection or its real counterparts as in the Barbero variables [5], the solution to the Hamiltonian constraint and its subsequent delineation of the physical Hilbert space, is a long and standing unresolved problem [2]. Consequently, the quantization of LQG remains complete only at the kinematical level (which is more suitably adapted to the fundamental variables), and the physical dynamics of gravity remain to be completely encoded within this procedure [4]. LQG can be contrasted with the standard ADM approach [4], wherein the fundamental variables are the spatial three metric and its conjugate momentum, constructed from the extrinsic curvature of the spatial slice of four-dimensional spacetime upon which the quantization must be performed. The corresponding initial value constraints are intractable due to various technical issues particularly related to ultraviolet divergences associated with operator products, which in LQG are absent. The choice of fundamental variables in the ADM approach poses the problem that as canonically conjugate variables, the momentum generates translations of the spatial three metric. Since the spectrum of both variables is the real line, then the positivity of the metric in the quantum theory cannot be guaranteed while having self-adjoint variables. Positivity of the spatial three metric is a crucially important condition that any quantum theory of gravity must satisfy, since spatial distances as measured by the theory must always be positive.

The theory of Intrinsic time quantum gravity (ITQG) [6] presented in this paper is driven by the motivation to solve all of the above difficulties. The choice of the configuration space variable in ITQG will be a unimodular spatial three metric metric and a momentum variable (ultimately known as a momentric) which generates dilations (more precisely SU(3) and SL(3,R) transformations of the metric). The importance of this choice of fundamental variables is that they will be self-adjoint in the quantum theory, while preserving the positivity and the unimodularity of the spatial three metric forming the configuration space variable. A common misconception of the price for such a result is that the variables cannot be canonically related, resulting in complications in their quantization. However, in the case of ITQG we will see that it is precisely their non canonical nature that makes them perfectly suited for quantization, and admits a group-theoretical interpretation as such, which resolves all of the aforementioned difficulties in the LQG and ADM approaches in one stroke.

In [6], a new formulation for quantization of the gravitational field in ITQG, is presented. The basic idea, as introduced in [7] and [8], is the concept of a new phase space for gravity which breaks the paradigm of four-dimensional spacetime covariance, shifting the emphasis to three dimensional spatial diffeomorphism invariance combined with a physical Hamiltonian which generates evolution with respect to intrinsic time. Through the constructive interference of wavefronts, classical spacetime emerges from the formalism, with direct correlation between intrinsic time intervals and proper time intervals of spacetime. In the present paper we will take a step back to analyse the motivations and canonical structure of ITQG, and then construct the fundamental variables and their commutation relations of the theory. These relations are noncanonical, which lead to the uncovering of an inherent $SU(3)$ structure for gravity. This presents certain advantages from the standpoint of quantization. The paper is thus structures as follows: Section 2 discusses the Poisson structure of the barred classical variables, Section 3 highlights the prelude to the quantum theory, Section 4 discusses the momentric operators and the SU(3) Lie algebra, Section 5 revisits the classical theory, and then lastly, Section 6 concludes the paper with some recommendations for similar future work in this direction.

2. Poisson Structure of the Barred Classical Variables

Let $q_{ij}, \widetilde{\pi}^{ij}$ denote the spatial 3-metric and its conjugate momentum defined on a spatial slice Σ of a four dimensional spacetime of topology $M = \Sigma \times R$. In the ADM metric theory, the basic variables provide a canonical one form

$$\Theta_{ADM} = \int_{\Sigma} d^3x\, \widetilde{\pi}^{ij}(x)\delta q_{ij}(x). \tag{1}$$

Starting from this canonically conjugate pair, let us define as fundamental classical variables the following barred quantities $\overline{q}_{ij}$, a unimodular metric with $\det\overline{q}_{ij} = 1$, and a traceless momentum variable $\overline{\pi}^{ij}$ via the relations [7,8]

$$\overline{q}_{ij} = q^{-1/3}q_{ij};\ \ \overline{\pi}^{ij} = q^{1/3}\left(\widetilde{\pi}^{ij} - \frac{1}{3}q^{ij}\widetilde{\pi}\right), \tag{2}$$

where $\widetilde{\pi} = q_{ij}\widetilde{\pi}^{ij}$ with $\overline{q}_{ij}\overline{\pi}^{ij} = 0$. From Equation (2) we get the following cotangent space decomposition

$$\delta q_{ij} = q^{1/3}\left(\overline{q}_{ij}\delta\ln q^{1/3} + \delta\overline{q}_{ij}\right) \longrightarrow \delta\overline{q}_{ij} = \overline{P}^{kl}_{ij}\delta q_{kl} \tag{3}$$

where we have defined the traceless projector $\overline{P}^{ij}_{kl} = \frac{1}{2}(\delta^i_k\delta^j_l + \delta^j_k\delta^i_l) - \frac{1}{3}\overline{q}^{ij}\overline{q}_{kl}$, with $\overline{P}^{ij}_{kl}\overline{q}^{kl} = \overline{q}_{ij}\overline{P}^{ij}_{kl} = 0$. So we have $\overline{q}^{ij}\delta\overline{q}_{ij} = 0$, namely that the cotangent space elements $\delta\overline{q}^{ij}$ are traceless. The inverse relations

$$q_{ij} = q^{1/3}\overline{q}_{ij}; \;\; \widetilde{\pi}^{ij} = q^{-1/3}\big(\overline{\pi}^{ij} + \frac{1}{3}\overline{q}^{ij}\widetilde{\pi}\big) \tag{4}$$

take us from the barred back to the unbarred variables. Substitution of the left side of the arrow of Equation (3) into Equation (1) provides a clean separation of the barred gravitational degrees of freedom with canonical one-form [7]

$$\Theta = \int_\Sigma d^3x\widetilde{\pi}^{ij}\delta q_{ij} = \int_\Sigma d^3x\big(\widetilde{\pi}\delta\mathrm{ln}q^{1/3} + \overline{\pi}^{ij}\delta\overline{q}_{ij}\big), \tag{5}$$

where we have used $\overline{\pi}^{ij}\overline{q}_{ij} = \overline{q}^{ij}\delta\overline{q}_{ij} = 0$. Equation (5) yields a corresponding symplectic two-form

$$\Omega = \delta\Theta = \int_\Sigma d^3x\Big(\delta\widetilde{\pi}\wedge\delta\mathrm{ln}q^{1/3} + \delta\overline{\pi}^{ij}\wedge\delta\overline{q}_{ij}\Big). \tag{6}$$

While this may be the case, as we will see, the Poisson brackets which can arise from (6) are not unique, on account of subtleties due to the implementation of tracelessness of $\overline{\pi}^{ij}$.

A necessary condition for a consistent canonical quantization of the theory is that the correct Poisson brackets comprise the starting point at the classical level. So let us directly calculate via Equation (2) barred Poisson brackets with respect to the unbarred canonical structure, which is clearly known to be unambiguous. For the metric components we have $\{\overline{q}_{ij}(x), \overline{q}_{kl}(y)\} = 0$ which is encouraging, as the unbarred metric clearly is devoid of any momentum dependence. However, using the following relations

$$\frac{\delta q^{1/3}}{\delta q_{ij}} = \frac{1}{3}q^{1/3}q^{ij}; \;\; \frac{\delta q^{ij}}{\delta q_{mn}} = -q^{(im}q^{j)n}; \;\; \frac{\delta\overline{q}_{kl}}{\delta q_{ij}} = q^{-1/3}\overline{P}^{ij}_{kl}; \;\; \frac{\delta\overline{\pi}^{ij}}{\delta\widetilde{\pi}^{kl}} = q^{1/3}\overline{P}^{ij}_{kl}, \tag{7}$$

in conjunction with

$$\frac{\delta\overline{\pi}^{ij}}{\delta q_{kl}} = \frac{1}{3}\Big(q^{kl}\overline{\pi}^{ij} + q^{1/3}\big(q^{(ik}q^{j)l}q_{rs}\widetilde{\pi}^{rs} - q^{ij}\widetilde{\pi}^{kl}\big)\Big) = \frac{1}{3}q^{-1/3}\big(\overline{q}^{kl}\overline{\pi}^{ij} - \overline{q}^{ij}\overline{\pi}^{kl}\big) + \frac{1}{3}q^{1/3}q^{(ik}q^{j)l}\widetilde{\pi}, \tag{8}$$

we obtain the following Poisson bracket relations between barred metric and momentum

$$\{\overline{q}_{ij}(x), \overline{\pi}^{kl}(y)\} = \int_\Sigma d^3z\Big(\frac{\delta\overline{q}_{ij}(x)}{\delta q_{mn}(z)}\frac{\delta\overline{\pi}^{kl}(y)}{\delta\widetilde{\pi}^{mn}(z)} - \frac{\delta\overline{\pi}^{kl}(y)}{\delta q_{mn}(z)}\frac{\delta\overline{q}_{ij}(x)}{\delta\widetilde{\pi}^{mn}(z)}\Big) = \overline{P}^{kl}_{ij}\delta^{(3)}(x,y). \tag{9}$$

Finally, we obtain the following relation amongst the barred momentum components

$$\{\overline{\pi}^{ij}(x), \overline{\pi}^{kl}(y)\} = \int_\Sigma d^3z\Big(\frac{\delta\overline{\pi}^{ij}(x)}{\delta q_{mn}(z)}\frac{\delta\overline{\pi}^{kl}(y)}{\delta\widetilde{\pi}^{mn}(z)} - \frac{\delta\overline{\pi}^{kl}(y)}{\delta q_{mn}(z)}\frac{\delta\overline{\pi}^{ij}(x)}{\delta\widetilde{\pi}^{mn}(z)}\Big) = \frac{1}{3}\big(\overline{q}^{kl}\overline{\pi}^{ij} - \overline{q}^{ij}\overline{\pi}^{kl}\big)\delta^{(3)}(x,y). \tag{10}$$

The Poisson brackets between barred variables are noncanonical. But we will show that they yield the same barred contribution as the symplectic two form (6) which can be seen as follows. From the calculated Poisson brackets the following Poisson matrix can be constructed

$$P^{IJ} = \begin{pmatrix} \{\overline{q}_{ij}(x), \overline{q}_{kl}(y)\} & \{\overline{q}_{ij}(x), \overline{\pi}^{kl}(y)\} \\ \{\overline{\pi}^{kl}(y), \overline{q}_{ij}(x)\} & \{\overline{\pi}^{ij}(x), \overline{\pi}^{kl}(y)\} \end{pmatrix} = \begin{pmatrix} 0 & \overline{P}^{kl}_{ij} \\ -\overline{P}^{kl}_{ij} & \frac{1}{3}\big(\overline{q}^{kl}\overline{\pi}^{ij} - \overline{q}^{ij}\overline{\pi}^{kl}\big) \end{pmatrix}\delta^{(3)}(x,y). \tag{11}$$

In Poisson geometry, a two form $\Omega = \frac{1}{2}\Omega_{IJ}\delta q^I\wedge\delta q^J$ on the phase space $q^I \equiv \overline{q}_{ij}, \overline{\pi}^{ij}$ can be constructed whose components are the inverse of the Poisson matrix. If Ω is closed ($\delta\Omega = 0$) and nondegenerate, then it is said to be a symplectic two form. Making the identifications $\{\overline{q},\overline{\pi}\} \sim \beta$ and $\{\overline{\pi},\overline{\pi}\} \sim \alpha$, then the inverse of the Poisson matrix for the barred variables is of the form

$$P^{-1} = \begin{pmatrix} 0 & \beta \\ -\beta & \alpha \end{pmatrix}^{-1} = \begin{pmatrix} \beta^{-1}\alpha\beta^{-1} & -\beta^{-1} \\ \beta^{-1} & 0 \end{pmatrix}, \tag{12}$$

which does not exist since the projector $\overline{P}^{kl}_{ij}$ is uninvertible. This suggests, naively, that the symplectic structure associated with the above Poisson brackets does not exist.

One method of quantization of a theory is to promote Poisson brackets directly into quantum commutators. The Poisson brackets for a generic theory can be read off directly from its symplectic two form, and which in turn is defined from the Poisson matrix by constructing the inverse of the latter. We would like to construct the symplectic two form for ITQG by inverting the Poisson matrix P^{IJ} constructed in equation Equation (11). The Poisson matrix in its present form is uninvertible since it consists of projectors P^{kl}_{ij} in its block off-diagonal positions denoted by the symbol β. In the process of inversion of P^{IJ}, as shown above with P^{-1}, it is necessary to have β^{-1}. But β^{-1} does not exist on account of the fact that projectors are not invertible, which suggests, naively, that ITQG does not have a well-defined symplectic structure.

To get around this technical difficulty we will add a trace part to the Poisson matrix, parametrized by a parameter γ which we will ultimately remove after all calculations have been performed. While this distorts the theory of ITQG to a new theory parametrized by γ, it renders the resulting Poisson matrix invertible to allow progress to the corresponding symplectic two form, parametrized by γ, since the previously offending terms β now become β_γ, which as in Equation (12) are now invertible. Thus we have

$$\beta_\gamma \equiv (P_\gamma)^{ij}_{kl} = P^{ij}_{kl} + \gamma\overline{q}^{ij}\overline{q}_{kl} \longrightarrow \beta^{-1}_\gamma = P^{kl}_{mn} + \frac{1}{9\gamma}\overline{q}^{kl}\overline{q}_{mn}. \tag{13}$$

So now, we can invert the resulting object, and we have that

$$\beta^{-1}_\gamma\alpha\beta^{-1}_\gamma = \frac{1}{3}\big(\overline{P}^{mn}_{kl} + \frac{1}{9\gamma}\overline{q}^{mn}\overline{q}_{kl}\big)\big(\overline{q}^{kl}\overline{\pi}^{ij} - \overline{q}^{ij}\overline{\pi}^{kl}\big)\big(\overline{P}^{rs}_{ij} + \frac{1}{9\gamma}\overline{q}^{rs}\overline{q}_{ij}\big) = -\frac{1}{9\gamma}\big(\overline{\pi}^{mn}\overline{q}^{rs} - \overline{q}^{mn}\overline{\pi}^{rs}\big), \tag{14}$$

where we have used $\overline{P}^{ij}_{kl}\overline{q}_{ij} = \overline{q}^{kl}\overline{\pi}_{kl} = 0$ and $\overline{P}^{ij}_{kl}\overline{\pi}^{kl} = \overline{\pi}^{ij}$, which assumes that $\overline{\pi}^{ij}$ is traceless. So the inverse of the Poisson matrix parametrized by γ is given by

$$P^{-1} = \begin{pmatrix} -\frac{1}{9\gamma}\big(\overline{q}^{kl}\overline{\pi}^{ij} - \overline{q}^{ij}\overline{\pi}^{kl}\big) & -\big(\overline{P}^{kl}_{mn} + \frac{1}{9\gamma}\overline{q}^{kl}\overline{q}_{mn}\big) \\ \overline{P}^{ij}_{rs} + \frac{1}{9\gamma}\overline{q}^{ij}\overline{q}_{rs} & 0 \end{pmatrix}\delta^{(3)}(x,y),$$

and the associated two form Ω inherits the γ dependence

$$\begin{aligned}\Omega^\gamma &= \frac{1}{2}\Omega^\gamma_{IJ}\delta q^I\delta q^J \\ &= \int_\Sigma d^3x\Big[-\frac{1}{18\gamma}\big(\overline{q}^{kl}\overline{\pi}^{ij} - \overline{q}^{ij}\overline{\pi}^{kl}\big)\delta\overline{q}_{ij}\wedge\delta\overline{q}_{kl} + \big(P^{ij}_{kl} + \frac{1}{9\gamma}\overline{q}^{ij}\overline{q}_{kl}\big)\delta\overline{q}_{ij}\wedge\delta\overline{\pi}_{kl} + \frac{1}{2}(0)_{ijkl}\delta\overline{\pi}^{ij}\wedge\delta\overline{\pi}^{kl}\Big].\end{aligned} \tag{15}$$

But $\overline{q}^{ij}\delta\overline{q}_{ij} = 0$, causing the $\delta\overline{q}\wedge\delta\overline{q}$ term and the γ contribution to the $\delta\overline{q}\wedge\delta\overline{\pi}$ term of (15) vanish. The quantity $(0)_{ijkl}$ in Equation (15) is basically to highlight the fact that that term, while zero is nontrivially so. Rather than omit this term, we wanted to highlight the fact that it is a tensorial quantity forming the coefficient of the $\delta\overline{\pi}\wedge\delta\overline{\pi}$ two form. This facilitates the keeping track for the reader of each individual term, of which there should be of the type including $\delta\overline{q}\wedge\delta\overline{q}$ and $\delta\overline{q}\wedge\delta\overline{\pi}$. There is no

$\delta\overline{\pi}\wedge\delta\overline{\pi}$ term since $\{\overline{q}_{ij},\overline{q}_{kl}\}=0$. All explicit γ dependence in the symplectic form has disappeared, so the $\gamma\rightarrow 0$ limit can be safely taken, yielding

$$\begin{aligned}\lim_{\gamma\rightarrow 0}\Omega^{\gamma} &= \int_{\Sigma} d^3x P^{ij}_{kl}\delta\overline{q}_{ij}\wedge\delta\overline{\pi}^{kl}\\ &= \int_{\Sigma} d^3x\delta\overline{q}_{ij}\wedge\delta\overline{\pi}^{ij}-\frac{1}{3}\int_{\Sigma} d^3x(\overline{q}^{ij}\delta\overline{q}_{ij})\wedge(\overline{q}_{kl}\delta\overline{\pi}^{kl})=\int_{\Sigma} d^3x\delta\overline{q}_{ij}\wedge\delta\overline{\pi}^{ij}.\end{aligned} \tag{16}$$

The vanishing of the $\frac{1}{3}$ term is due to $\overline{q}^{ij}\delta\overline{q}_{ij}=0$ or alternatively by the Leibniz rule for the momentum term

$$(\overline{q}^{ij}\delta\overline{q}_{ij})\wedge(\overline{q}_{kl}\delta\overline{\pi}^{kl})=\delta\overline{q}_{ij}\wedge\delta(\overline{q}^{ij}\overline{q}^{kl}\overline{\pi}_{kl})-\delta q_{ij}\wedge\delta q^{ij}(\overline{q}_{kl}\overline{\pi}^{kl})-(\overline{q}^{ij}\delta\overline{q}_{ij})\wedge(\overline{\pi}^{kl}\delta\overline{q}_{kl})=0 \tag{17}$$

due additionally to $\overline{q}_{ij}\overline{\pi}^{ij}=0$. This implies that the tracelessness of $\overline{\pi}^{ij}$ must be conjugate to the fact that infinitesimal variations in $\overline{q}_{ij}$ are traceless. Hence (16) is the same as the barred contribution to (6), with the difference that the tracelessness of $\overline{\pi}^{ij}$ has been implicitly enforced due to a unimodular metric. This calculation demonstrates that extreme care must be exercised when extracting Poisson brackets from a symplectic two form, particular when the index structure of the fundamental variables has implicit symmetries. The requirement to implement the noncanonical Poisson brackets at the quantum level will pose nontrivial issues, which we will address in the next few sections. Let us display, for completeness, the fundamental Poisson brackets for the barred phase space

$$\{\overline{q}_{ij}(x),\overline{q}_{kl}(y)\}=0;\;\;\{\overline{q}_{ij}(x),\overline{\pi}^{kl}(y)\}=\overline{P}^{ij}_{kl}(x,y);\;\;\{\overline{\pi}^{ij}(x),\overline{\pi}^{kl}(y)\}=\tfrac{1}{3}(\overline{q}^{kl}\overline{\pi}^{ij}-\overline{q}^{ij}\overline{\pi}^{kl})\delta^{(3)}(x,y). \tag{18}$$

The basic Poisson brackets are noncanonical, which can be seen as the price to be paid for choosing $\overline{\pi}^{ij}$ to be traceless at the classical level, or alternatively, the price for choosing unimodular metric variables.

The original motivation was to obtain a symplectic form parametrized by γ and then to take the limit as γ approaches zero. But as one can see from the above that the wedge products in the resulting symplectic two form have coefficients proportional to γ^{-1}, which in the limit as γ approaches zero would be ill-defined. However, note form the arguments provided from Equation (15) through to Equation (18), that the individual wedge products of the fundamental variables all vanish on account of the unimodularity of the configuration space variable $\bar{q}_{ij}$ and the tracelessness of the momentum $\bar{pi}^{ij}$. Hence the proper procedure is to leave γ arbitrary in the symplectic two form, which is immaterial since all terms which depend on γ automatically vanish. The result is that the symplectic two form reduces to $\delta\bar{q}\wedge\delta\bar{\pi}$ form as in Equation (17), whence γ is conspicuously absent. So the justification that the parametrization of the Poisson matrix by the parameter does not affect the results of the symplectic two form is that for all nonzero γ, we can transition from the Poisson matrix to the symplectic two form by inversion as per the standard procedure, yielding a symplectic two form which is independent of the parameter γ. It is the unique choice of unimodular and traceless variables, which makes this the case, which admits a complete quantization of these variables.

3. A Prelude into the Quantum Theory

Having determined the Poisson brackets for the barred phase space, the next step is to implement them at the quantum level. In proceeding to the quantum theory according to the Heisenberg–Dirac prescription, we must promote all classical variables A,B to operators $\widehat{A},\widehat{B}$ and all Poisson brackets to commutators $\{A,B\}\rightarrow\frac{1}{(i\hbar)}[\widehat{A},\widehat{B}]$. So the fundamental Poisson brackets (18) yield the following equal-time commutation relations

$$[\overline{q}_{ij}(x,t),\overline{q}_{kl}(y,t)]=0;[\overline{q}_{ij}(x,t),\widehat{\overline{\pi}}^{kl}(y,t)]=i\hbar\overline{P}^{ij}_{kl}(x,y);\;\;[\widehat{\overline{\pi}}^{ij}(x,t),\widehat{\overline{\pi}}^{kl}(y,t)]=\tfrac{i\hbar}{3}(\overline{q}^{kl}\widehat{\overline{\pi}}^{ij}-\overline{q}^{ij}\widehat{\overline{\pi}}^{kl})\delta^{(3)}(x,y), \tag{19}$$

where we have chosen an operator ordering with the momenta to the right. Since the momentum components fail to commute, then we are restricted to wavefunctionals $\psi[\overline{q}]$ in the metric representation. A representation of the classically traceless momentum as a vector field

$$\widehat{\overline{\pi}}^{ij}(x)\psi[\overline{q}] \longrightarrow \frac{\hbar}{i}\Big[\overline{P}^{ij}_{kl}\frac{\delta}{\delta\overline{q}_{kl}} + \frac{1}{3}\big(\overline{q}^{ij}\overline{\pi}^{kl} - \overline{q}^{kl}\overline{\pi}^{ij}\big)\frac{\delta}{\delta\overline{\pi}^{kl}}\Big]\psi[\overline{q}] = \frac{\hbar}{i}\overline{P}^{ij}_{kl}(x)\frac{\delta\psi[\overline{q}]}{\delta\overline{q}_{kl}(x)} \tag{20}$$

correctly reproduces the commutation relations (19) (The term of (20) from the $\overline{\pi}^{ij}, \overline{\pi}^{kl}$ commutation relation does not contribute for wavefunctionals $\psi[\overline{q}]$ polarized in the metric representation.). However, Equation (20) does not constitute a self-adjoint operator since

$$\frac{\hbar}{i}\frac{\delta}{\delta\overline{q}_{kl}(x)}\overline{P}^{ij}_{kl}(x) = \frac{\hbar}{i}\overline{P}^{ij}_{kl}(x)\frac{\delta}{\delta\overline{q}_{kl}(x)} - \frac{2\hbar}{3i}\overline{q}^{ij}\delta^{(3)}(0). \tag{21}$$

So $\overline{q}_{ij}\widehat{\overline{\pi}}^{ij} = 0 \neq \widehat{\overline{\pi}}^{ij}\overline{q}_{ij}$, namely that the momentum in (20) is left-traceless, but is not right-traceless. A self- adjoint operator can be constructed by averaging the left-traceless and right-traceless versions $\frac{1}{2}\big(\frac{\delta}{\delta\overline{q}_{ij}}\overline{P}^{kl}_{ij} + \overline{P}^{kl}_{ij}\frac{\delta}{\delta\overline{q}_{ij}}\big)$. However, the resulting operator, while self-adjoint, is neither traceless from the left nor from the right. So it appears that tracelessness is a property which is nontrivial to enforce at the quantum level in the $\bar{q}_{ij}, \overline{\pi}^{ij}$ variables.

The quantity $\delta^{(3)}(0)$ in Equation (21) is an ultraviolet singularity in field theory, which results from evaluating the commutation relations at the same spatial point. It is a formal expression more rigorously defined by a limiting procedure in the coincidence limit of the arguments x and y. It is necessary to perform the commutation relations at the same spatial point in order to reorder the fundamental operators in Equations (20) and (21), which are defined at the same spatial point, which is necessary in order to evaluate self-adjointness. This operator ordering induced ambiguity, parametrized by $\delta^{(3)}(0)$, highlights that the variables in their present form, while solving the aforementioned problem of the symplectic structure, are still not ideally suited for quantization. This will ultimately lead us to the choice of the momentric $\bar{\pi}^i_j$, in lieu of the momentum variable $\bar{\pi}^{ij}$, which being self adjoint as we will demonstrate in the remainder of this paper, will eliminate the presence of any such $\delta^{(3)}(0)$ divergences in the quantum theory.

4. Momentric Operators and the SU(3) Lie Algebra

Let us define a mixed-index version of the momentum, namely the momentric variables $\overline{P}^i_j = \overline{q}_{jm}\overline{\pi}^{im}$. We first compute the commutator of $\overline{P}^i_j$ with the barred metric. This is given by

$$[\overline{P}^i_j(x), \overline{q}_{kl}(y)] = [\overline{q}_{jm}(x)\overline{\pi}^{mi}(x), \overline{q}_{kl}(y)] = \overline{q}_{jm}(x)[\overline{\pi}^{mi}(x), \overline{q}_{kl}(y)] = -i\hbar\overline{q}_{jm}\overline{P}^{mi}_{kl}\delta^{(3)}(x,y) \equiv \tfrac{\hbar}{i}\overline{E}^i_{j(kl)}\delta^{(3)}(x,y) \tag{22}$$

where we have used (19), with the "superspace vielbein" defined as $\overline{E}^i_{j(kl)} = \frac{1}{2}\big(\delta^i_k\overline{q}_{jl} + \delta^i_l\overline{q}_{jk}\big) - \frac{1}{3}\delta^i_j\overline{q}_{kl}$. So we will rather adopt the pair $\overline{q}_{ij}, \overline{P}^i_j$ as the fundamental variables, and recompute the fundamental relations (19) with respect to them.

For the commutators amongst the momentric components themselves the following identity involving commutation relations regarding generic operators $\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}$ will be useful

$$[\widehat{A}\widehat{B}, \widehat{C}\widehat{D}] = \widehat{A}[\widehat{B}, \widehat{C}]\widehat{D} + \widehat{C}[\widehat{A}, \widehat{D}]\widehat{B} + [\widehat{A}, \widehat{C}]\widehat{B}\widehat{D} + \widehat{C}\widehat{A}[\widehat{B}, \widehat{D}]. \tag{23}$$

Note that the proper operator ordering has been preserved in (23). So we have the following, suppressing the x-y dependence in the intermediate steps and suppressing the hats to avoid cluttering up the notation,

$$\begin{aligned}[\overline{P}^i_j(x),\overline{P}^k_l(y)] &= [\overline{q}_{jm}(x)\overline{\pi}^{im}(x),\overline{q}_{ln}(y)\overline{\pi}^{kn}(y)] \\ &= \overline{q}_{jm}[\overline{\pi}^{im},\overline{q}_{ln}]\overline{\pi}^{kn} + \overline{q}_{ln}[\overline{q}_{jm},\overline{\pi}^{kn}]\overline{\pi}^{im} + [\overline{q}_{jm},\overline{q}_{ln}]\overline{\pi}^{im}\overline{\pi}^{kn} + \overline{q}_{ln}\overline{q}_{jm}[\overline{\pi}^{im},\overline{\pi}^{kn}] \\ &= \frac{\hbar}{i}\Big[\overline{q}_{jm}\overline{P}^{im}_{ln}\overline{\pi}^{kn} - \overline{q}_{ln}\overline{P}^{kn}_{jm}\overline{\pi}^{im} + 0 + \frac{1}{3}\overline{q}_{ln}\overline{q}_{jm}(\overline{q}^{kn}\overline{\pi}^{im} - \overline{q}^{im}\overline{\pi}^{kn})\Big]\delta^{(3)}(x,y).\end{aligned} \tag{24}$$

In the third line of Equation (24) we have used the fundamental equal time commutation relations (19). For completeness, let us display some of the intermediate steps from Equation (24). For the first term on the right hand side we have

$$\overline{q}_{jm}\overline{P}^{im}_{ln}\overline{\pi}^{kn} = \overline{q}_{jm}\Big(\frac{1}{2}(\delta^i_l\delta^m_n + \delta^i_n\delta^m_l) - \frac{1}{3}\overline{q}^{im}\overline{q}_{ln}\Big)\overline{\pi}^{kn} = \frac{1}{2}(\delta^i_l\overline{P}^k_j + \overline{q}_{jl}\overline{\pi}^{ki}) - \frac{1}{3}\delta^i_j\overline{P}^k_l. \tag{25}$$

For the middle term we have

$$\overline{q}_{ln}\overline{P}^{kn}_{jm}\overline{\pi}^{im} = \overline{q}_{ln}\Big(\frac{1}{2}(\delta^k_j\delta^n_m + \delta^k_m\delta^n_j) - \frac{1}{3}\overline{q}^{kn}\overline{q}_{jm}\Big)\overline{\pi}^{im} = \frac{1}{2}(\delta^k_j\overline{P}^i_l + \overline{q}_{lj}\overline{\pi}^{ik}) - \frac{1}{3}\delta^k_l\overline{P}^i_j. \tag{26}$$

For the last term on the right hand side of (24) we have

$$\frac{1}{3}\overline{q}_{ln}\overline{q}_{jm}(\overline{q}^{kn}\overline{\pi}^{im} - \overline{q}^{im}\overline{\pi}^{kn}) = \frac{1}{3}(\delta^k_l\overline{P}^i_j - \delta^i_j\overline{P}^k_l). \tag{27}$$

Substitution of Equations (25)–(27) into Equation (24) yields the result that

$$[\overline{P}^i_j(x),\overline{P}^k_l(y)] = \frac{\hbar}{i}\Big[\frac{1}{2}(\delta^i_l\overline{P}^k_j - \delta^k_j\overline{P}^i_l) + \frac{2}{3}(\delta^k_l\overline{P}^i_j - \delta^i_j\overline{P}^k_l)\Big]\delta^{(3)}(x,y). \tag{28}$$

Note that the algebra closes (if not for the precise cancellation of terms of the form $\overline{q}_{jl}\overline{\pi}^{ki}$, this would not be the case). While the algebra (28) closes on the momentric variables $\overline{P}^i_j$, it does not enforce the vanishing of the trace $\overline{P} = \delta^j_i\overline{P}^i_j$. This can be seen by contraction of (28) with δ^i_j, wherein

$$[\overline{P}(x),\overline{P}^i_j(y)] = -\frac{2\hbar}{i}(\overline{P}^i_j - \frac{1}{3}\delta^i_j\overline{P})\delta^{(3)}(x,y) \equiv 2i\hbar\overline{\pi}^i_j\delta^{(3)}(x,y), \tag{29}$$

where $\overline{\pi}^i_j$ denotes the traceless part of the momentric. Note that $\overline{P} = 0$ in Equation (29) leads to a contradiction, whereas the relation (22) implies $[\overline{P},\overline{q}_{ij}] = 0$ due to tracelessless of $\overline{E}^i_{j(kl)}$.

Still, it is interesting in Equation (29) that the commutator of $\overline{P}^i_j$ with its trace yields it traceless part $\overline{\pi}^i_j$. So let us evaluate the commutation relations involving the traceless part (suppressing the coordinate dependence for simplicity)

$$\begin{aligned}[\overline{\pi}^i_j(x),\overline{\pi}^k_l(y)] &= [\overline{P}^i_j - \frac{1}{3}\delta^i_j\overline{P},\overline{P}^k_l - \frac{1}{3}\delta^k_l\overline{P}] = [\overline{P}^i_j,\overline{P}^k_l] - \frac{1}{3}\delta^k_l[\overline{P}^i_j,\overline{P}] - \frac{1}{3}\delta^i_j[\overline{P},\overline{P}^k_l] + \frac{1}{9}\delta^i_j\delta^k_l[\overline{P},\overline{P}] \\ &= \frac{i\hbar}{2}(\delta^i_l\overline{P}^k_j - \delta^k_j\overline{P}^i_l)\delta^{(3)}(x,y)\end{aligned} \tag{30}$$

where we have used (28) and (29). We can now make the substitution $\overline{P}^i_j = \overline{\pi}^i_j + \frac{1}{3}\delta^i_j\overline{P}$, and the trace part cancels out to yield $[\overline{\pi}^i_j, \overline{\pi}^k_l] = \frac{i\hbar}{2}\left(\delta^i_l\overline{\pi}^k_j - \delta^k_j\overline{\pi}^i_l\right)$. The final result of our commutation relations (19), in terms of the traceless momentric variables $\overline{\pi}^i_j$ is given by

$$[\overline{q}_{ij}(x), \overline{q}_{kl}(y)] = 0; \;\; [\widehat{\overline{\pi}}^i_j(x), \overline{q}_{kl}(y)] = \frac{\hbar}{i}\overline{E}^i_{j(kl)}\delta^{(3)}(x,y); \;\; [\widehat{\overline{\pi}}^i_j(x), \widehat{\overline{\pi}}^k_l(y)] = \frac{i\hbar}{2}(\delta^i_l\overline{\pi}^k_j - \delta^k_j\overline{\pi}^i_l)\delta^{(3)}(x,y). \tag{31}$$

Note that Equation (31) implies a representation of the momentric as a vector field

$$\widehat{\overline{P}}^i_j = \frac{\hbar}{i}\frac{\delta}{\delta\overline{q}_{kl}}\overline{E}^i_{j(kl)} = \frac{\hbar}{i}\overline{E}^i_{j(kl)}\frac{\delta}{\delta\overline{q}_{kl}} + \frac{\hbar}{i}\left[\frac{\delta}{\delta\overline{q}_{kl}}, \overline{E}^i_{j(kl)}\right] = \frac{\hbar}{i}\overline{E}^i_{j(kl)}\frac{\delta}{\delta\overline{q}_{kl}}, \tag{32}$$

which is both self-adjoint and left-right traceless, implements the commutation relations, and is traceless in the sense that $\delta^j_i\widehat{\overline{\pi}}^i_j = \widehat{\overline{\pi}}^i_j\delta^j_i = 0$. There are a few things to note regarding (31). First, upon contraction with δ^j_i, yields consistently that the trace $\delta^j_i\overline{\pi}^i_j = 0$ vanishes as well as its comutator with all quantities. Secondly, the traceless momentric variables by themselves form a $SU(3)$ current algebra, and also generate an affine algebra with the metric, which unlike (19) preserves the positivity of the metric $\overline{q}_{ij}$. Thus, the fundamental variables $\overline{q}_{ij}, \overline{\pi}^i_j$ will be the prime choice for the quantum theory which, at the kinematical level, will involve constructing unitary, irreducible representations of the $SU(3)$ Lie algebra. Also of note is that the the object $\Delta = \overline{\pi}^j_i\overline{\pi}^i_j$ encodes to the quadratic Casimir of $SU(3)$, which by definition must commute with all traceless momentric components $[\Delta, \overline{\pi}^i_j] = 0$.

The Gell–Mann matrices satisfy the relations

$$[\lambda_A, \lambda_B]^i_j = if^C_{AB}(\lambda_C)^i_j; \;\; \{\lambda_A, \lambda_B\}^i_j = d_{ABC}(\lambda_C)^i_j \tag{33}$$

with totally antisymmetric structure constants f_{ABC}, and totally symmetric d_{ABC}. We will exploit the aforementioned index structure by projection of the momentric onto the Gell–Mann matrices

$$T^A = (\lambda^A)^j_i\overline{\pi}^i_j \longrightarrow \overline{\pi}^i_j = 2T^A(\lambda_A)^i_j, \tag{34}$$

where we have used the $SU(3)$ completeness relation $(\lambda^A)^i_j(\lambda^A)^k_l = \frac{1}{2}\left(\delta^k_j\delta^i_l - \frac{1}{3}\delta^i_j\delta^k_l\right)$. The $SU(3)$ Lie algebra is of rank 2, and therefore has two Casimir operators, $C^{(2)}$ and $C^{(3)}$ given by

$$C^{(2)} = (\lambda_A)^j_i(\lambda_A)^i_j = T^AT^A; \;\; C^{(3)} = d_{ABC}(\lambda_A)^i_j(\lambda_B)^j_k(\lambda_C)^k_i = \epsilon^{ijk}\epsilon_{mnl}\overline{\pi}^m_i\overline{\pi}^n_j\overline{\pi}^l_k \propto 6\det\overline{\pi}^i_j. \tag{35}$$

Note for $C^{(3)}$ that the pair of epsilon symbols is totally symmetric under interchange of any index pair $(i,m), (j,n), (k,l)$, which is consistent with the total symmetry of d_{ABC}.

5. The Classical Theory, Revisited

Having determined the ideal variables for quantization as the unimodular- traceless momentric pair $\overline{q}_{ij}, \overline{\pi}^i_j$, we will now re-evaluate the Poisson brackets of the theory. This provides a basis for correlation of quantum predictions to the classical dynamics. First, the fundamental Poisson brackets are given by

$$\{\overline{q}_{ij}(x), \overline{q}_{kl}(y)\} = 0, \{\overline{q}_{ij}(x), \overline{\pi}^k_l(y)\} = \overline{E}^i_{j(kl)}\delta^{(3)}(x,y); \;\; \{\overline{\pi}^i_j(x), \overline{\pi}^k_l(y)\} = \frac{1}{2}(\delta^i_l\overline{\pi}^k_j - \delta^k_j\overline{\pi}^i_l)\delta^{(3)}(x,y). \tag{36}$$

So the Poisson brackets between phase space functions A and B is given by

$$\begin{aligned}\{A,B\} &= \int_\Sigma d^3x \int_\Sigma d^3y \Big[\frac{\delta A}{\delta \overline{q}_{ij}(x)}\{\overline{q}_{ij}(x),\overline{q}_{kl}(y)\}\frac{\delta B}{\delta \overline{q}_{kl}(y)} + \frac{\delta A}{\delta \overline{q}_{ij}(x)}\{\overline{q}_{ij}(x),\overline{\pi}^k_l(y)\}\frac{\delta B}{\delta \overline{\pi}^k_l(y)} \\ &+ \frac{\delta A}{\delta \overline{\pi}^i_j(x)}\{\overline{\pi}^i_j(x),\overline{q}_{kl}(y)\}\frac{\delta B}{\delta \overline{q}_{kl}(y)} + \frac{\delta A}{\delta \overline{\pi}^i_j(x)}\{\overline{\pi}^i_j(x),\overline{\pi}^k_l(y)\}\frac{\delta B}{\delta \overline{\pi}^k_l(y)}\Big] \\ &= \int_\Sigma d^3z \Big[\overline{E}^k_{j(ij)}\Big(\frac{\delta A}{\delta \overline{q}_{ij}}\frac{\delta B}{\delta \overline{\pi}^k_l} - \frac{\delta B}{\delta \overline{q}_{ij}}\frac{\delta A}{\delta \overline{\pi}^k_l}\Big) + \frac{\delta A}{\delta \overline{\pi}^i_j}\overline{\pi}^i_l\frac{\delta B}{\delta \overline{\pi}^j_l} - \frac{\delta A}{\delta \overline{\pi}^i_j}\overline{\pi}^k_j\frac{\delta B}{\delta \overline{\pi}^k_i}\Big].\end{aligned} \tag{37}$$

In General relativity, we will be interested in the evolution of the basic variables with respect to T, gauge-invariant part of intrinsic time $\ln q^{1/3}$, under the action of a physical Hamiltonian

$$H_{Phys} = \int_\Sigma d^3x \bar{H}(x) = \int_\Sigma d^3x \sqrt{\overline{\pi}^j_i \overline{\pi}^i_j + \mathcal{V}[q_{ij}]}, \tag{38}$$

where $\mathcal{V}$ is a potential term which depends on the metric. The Hamilton's equations for the basic variables with respect to the Poisson brackets (37) are given by

$$\begin{aligned}\frac{\delta \overline{q}_{ij}(x)}{\delta T} &= \{\overline{q}_{ij}(x), H_{Phys}\} = \frac{1}{\bar{H}}\overline{E}^k_{l(ij)}\overline{\pi}^l_k; \\ \frac{\delta \overline{\pi}^i_j(x)}{\delta T} &= \{\overline{\pi}^i_j(x), H_{Phys}\} = \frac{1}{\bar{H}}\Big[\frac{1}{2}\overline{E}^i_{j(kl)}\frac{\delta \mathcal{V}}{\delta \overline{q}_{kl}} + \overline{\pi}^i_l\overline{\pi}^l_j - \overline{\pi}^k_j\overline{\pi}^i_k\Big] = \frac{1}{2\bar{H}}\overline{E}^i_{j(kl)}\frac{\delta \mathcal{V}}{\delta \overline{q}_{kl}}.\end{aligned} \tag{39}$$

As a quick consistency check, contraction of the first equation of (39) with $\overline{q}^{ij}$ and contraction of the second equation with δ^j_i shows that if $\overline{q}_{ij}$ is unimodular and $\overline{\pi}^i_j$ is traceless at time T_0, then these properties will be preserved under evolution in intrinsic time by the Hamilton's equations.

6. Conclusions

The consistent quantization of 3+1 gravity is one of the biggest unsolved problems in theoretical physics spanning the past 100 years of approaches which, while leading to insights into certain often complementary aspects of the problem, have so far not provided a complete solution due to various technical and conceptual difficulties and issues. The novelty of the author's approach is the claim that with ITQG, one has a complete and consistent quantization of gravity which provides a possible resolution to the long-standing problem, while solving the difficulties inherent in all of the approaches so far, in one stroke.

For future work, we aim to follow the work of this paper with a similar work by focusing on some 2+1 aspect of ITQG, with the aim of studying the thermodynamic aspects of the BTZ black hole. Also, looking at the initial wave function, one difference from the case of 3+1 gravity seems to be the observation that there is no Cotton-York tensor in two spatial dimensions. So we should expect just a Ricci curvature-squared higher derivative rendition of the theory. This then will help us to be able to exploit the SU(2) structure of the theory, which will go a long way towards learning about the physical Hilbert space.

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