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The Teleparallel Equivalent of General Relativity and the Gravitational Centre of Mass

José Wadih Maluf

Instituto de Física, Universidade de Brasília, C. P. 04385, 70.919-970 Brasília DF, Brazil; wadih@unb.br or jwmaluf@gmail.com

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Abstract: We present a brief review of the teleparallel equivalent of general relativity and analyse the expression for the centre of mass density of the gravitational field. This expression has not been sufficiently discussed in the literature. One motivation for the present analysis is the investigation of the localization of dark energy in the three-dimensional space, induced by a cosmological constant in a simple Schwarzschild-de Sitter space-time. We also investigate the gravitational centre of mass density in a particular model of dark matter, in the space-time of a point massive particle and in an arbitrary space-time with axial symmetry. The results are plausible, and lead to the notion of gravitational centre of mass (COM) distribution function.

Keywords: teleparallel gravity; gravitational centre of mass moment; dark energy; teleparallel equivalent of general relativity

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1. Introduction

The most popular and acceptable approach to the relativistic theory of gravitation is given by Einstein's theory of general relativity. However, nowadays there are several alternative formulations of theories for the gravitational field that attempt to explain the dark energy and dark matter problems, which do not find satisfactory explanations within the framework of Einstein's general relativity. Moreover, concepts such as energy, momentum, angular momentum and centre of mass of the gravitational field are usually defined only for asymptotically flat space-times, in the context of a 3+1 type formulation. The latter are definitions for the total quantities, and suffer from at least two restrictions: the definitions are valid only for asymptotically flat space-times, and there do not exist localized expressions for the densities of the energy-momentum and 4-angular momentum of the gravitational field. The ADM definition for the gravitational energy-momentum [1] is constructed out of the metric tensor, and by means of the metric tensor it is not possible to construct suitable scalar densities in the form of total divergences. The approach via pseudo-tensors is certainly not satisfactory. The notions of energy-momentum and angular momentum of the gravitational field have been extensively discussed in the literature, but not the concept of gravitational centre of mass.

The notion of centre of mass can be made clear in flat space-time. Any relativistic field theory in flat space-time is expected to be covariant under the inhomogeneous Lorentz transformations, or Poincaré transformations: the 4-rotations and space-time translations. The generators of these transformations satisfy an algebra, the algebra of the Poincaré group. The generators of the 4-rotations are composed by the generators of the ordinary 3 dimensional rotations, and by the generators of the boosts. The latter are related to the centre of mass moment of the field. Energy, momentum and angular momentum of the field constitute seven conserved integral quantities associated to the symmetries of the theory. The integrals are carried out over the whole three-dimensional space. The three other integral quantities are associated to the centre of mass of the field, which sometimes is also called the centre of energy [2].

In the notation of Ref. [2], the centre of mass integrals read

$$J^{0i} = tP^i - \int d^3x\, x^i T^{00} \,, \tag{1}$$

where 0 and i are time and space indices, P^i is the i-th component of the momentum of the field, and T^{00} is the energy component of the energy-momentum tensor of field. It is argued [2] that the J^{0i} components have no clear physical significance since J^{0i} can be made to vanish if the coordinate system is chosen to coincide with the "centre of energy" at $t = 0$. However, in the context of general relativity, Dixon [3–5] developed a procedure for describing the dynamics of extended bodies in an arbitrary gravitational field, and for this purpose a definition of the centre of mass of such bodies (considered as quasi-rigid bodies) was proposed. A general relation between the centre of mass 4-velocity and the energy-momentum of the body was obtained [6]. One is led to the concept of centre of mass world line, whose uniqueness depends on the strength of the gravitational field [6].

In the standard metric formulation of general relativity, the centre of mass moment for the gravitational field has been first considered by Regge and Teitelboim [7,8], and reconsidered by several other authors (see References [9–12] and references therein). The centre of mass integral was obtained in the context of the Hamiltonian formulation of general relativity. The idea was to require the variation of the total Hamiltonian to be well defined in an asymptotically flat space-time, where the standard asymptotic space-time translations and 4-rotations are considered as coordinate transformations at spacelike infinity. This requirement leads to the addition of boundary (surface) terms to the primary Hamiltonian, so that the latter has well defined functional derivatives, and therefore one may obtain the field equations in the Hamiltonian framework (Hamilton's equations) by means of a consistent procedure. In this way, one arrives at the total energy, momentum, angular momentum and centre of mass moment of the gravitational field, given by surface terms of the total Hamiltonian.

In this article we address the centre of mass moment of the gravitational field in the realm of the teleparallel equivalent of general relativity (TEGR), which is an alternative and mathematically consistent formulation of general relativity [13] (see also Reference [14], and chapters 5 and 6 of Reference [15] and references therein). The geometrical structure of the TEGR was already considered by Einstein [16,17] in his attempt to unify gravity and electromagnetism, and later on by Cho [18,19], Hayashi and Shirafuji [20,21], Hehl et al. [22], Nitsch [23], Schweizer et al. [24], Nester [25] and Wiesendanger [26]). In recent years the teleparallel geometrical structure has been used in modified theories of gravity, with the purpose of constructing cosmological models that provide a consistent explanation to the dark energy problem (see the review article [27] and references therein).

The TEGR is constructed out of the tetrad fields $e^a{}_\mu$, where $a = \{(0), (i)\}$ and $\mu = \{0, i\}$ are SO(3,1) and space-time indices, respectively. The extra six components of the tetrads (compared to the 10 components of the metric tensor) yield additional geometric structure, that allows to define field quantities that cannot be constructed in the ordinary metric formulation of the theory (such as non-trivial total divergences, for instance). The tetrad fields allow to use concepts and definitions of both Riemannian and Weitzenböck geometries.

The definitions of the gravitational energy, momentum, angular momentum and centre of mass moment in the TEGR are not obtained according to the procedure described above, based on surface integrals of the total Hamiltonian. In the TEGR we first consider the Hamiltonian formulation of the theory [28,29]. The constraint equations of the theory (typically as $C = 0$) are equations that define the energy-momentum and the 4-angular momentum of the gravitational field [13] (i.e., $C = H - E = 0$). Moreover, the definitions of the energy-momentum and 4-angular momentum satisfy the algebra of the Poincaré group in the phase space of theory [13,30]. However, the energy-momentum definition, together with the gravitational energy-momentum tensor (but not the 4-angular momentum) may also be obtained directly from the Lagrangian field equations [13].

The gravitational centre of mass moment to be considered here yields the concept of gravitational centre of mass (COM) distribution function. One purpose of the present article is to show that a cosmological constant, which might be responsible for the dark energy, induces a very intense (divergent) gravitational COM distribution function in the vicinity of the cosmological horizon $r = R \simeq \sqrt{3/\Lambda}$ in a simple Schwarzschild-de Sitter space-time, in agreement with the hypothetical existence of dark energy. It seems that this result, obtained by means of tetrad fields, cannot be obtained in the context of the metric formulation of general relativity.

In Section 2 we present a brief review of the TEGR, emphasizing a recent simplified definition of the 4-angular momentum of the gravitational field, given by a total divergence. In Section 3 we investigate the gravitational COM distribution function of (i) the space-time of a massive particle in isotropic coordinates; (ii) the Schwarzschild-de Sitter space-time; (iii) a particular model of dark energy that arises from the non-local formulation of general relativity; and (iv) of an arbitrary space-time with axial symmetry. In the analysis of the first three cases above, which are spherically symmetric, we arrive at interesting results, that share similarities with the standard expressions in classical mechanics. For such space-times, the total centre of mass moment vanishes, as expected.

Notation: space-time indices $\mu, \nu, ...$ and SO(3,1) (Lorentz) indices $a, b, ...$ run from 0 to 3. The torsion tensor is given by $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$. The flat space-time metric tensor raises and lowers tetrad indices, and is fixed by $\eta_{ab} = e_{a\mu} e_{b\nu} g^{\mu\nu} = (-1, +1, +1, +1)$. The frame components are given by the inverse tetrads $\{e_a{}^\mu\}$. The determinant of the tetrad fields is written as $e = \det(e^a{}_\mu)$.

It is important to note that we assume that the space-time geometry is determined by the tetrad fields only, and thus the only possible non-trivial definition for the torsion tensor is given by $T^a{}_{\mu\nu}$. This tensor is related to the antisymmetric part of the Weitzenböck connection $\Gamma^\lambda_{\mu\nu} = e^{a\lambda}\partial_\mu e_{a\nu}$, which determines the Weitzenböck space-time and the distant parallelism of vector fields.

2. A Review of the Lagrangian and Hamiltonian Formulations of the TEGR

The TEGR is constructed out of the tetrad fields only. The first relevant consideration is an identity between the scalar curvature and an invariant combination of quadratic terms in the torsion tensor,

$$eR(e) \equiv -e\left(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^a T_a\right) + 2\partial_\mu(eT^\mu)\,, \tag{2}$$

where $T_a = T^b{}_{ba}$ and $T_{abc} = e_b{}^\mu e_c{}^\nu T_{a\mu\nu}$. The Lagrangian density for the gravitational field in the TEGR is given by [31]

$$\begin{aligned} L(e) &= -k\,e\left(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^a T_a\right) - \frac{1}{c}L_M \\ &\equiv -ke\Sigma^{abc}T_{abc} - \frac{1}{c}L_M\,, \end{aligned} \tag{3}$$

where $k = c^3/(16\pi G)$, L_M represents the Lagrangian density for the matter fields, and Σ^{abc} is defined by

$$\Sigma^{abc} = \frac{1}{4}\left(T^{abc} + T^{bac} - T^{cab}\right) + \frac{1}{2}\left(\eta^{ac}T^b - \eta^{ab}T^c\right)\,. \tag{4}$$

Thus, the Lagrangian density is geometrically equivalent to the scalar curvature density. The variation of $L(e)$ with respect to $e^{a\mu}$ yields the fields equations

$$e_{a\lambda}e_{b\mu}\partial_\nu(e\Sigma^{b\lambda\nu}) - e(\Sigma^{b\nu}{}_a T_{b\nu\mu} - \frac{1}{4}e_{a\mu}T_{bcd}\Sigma^{bcd}) = \frac{1}{4kc}eT_{a\mu}\,, \tag{5}$$

where $T_{a\mu}$ is defined by $\delta L_M/\delta e^{a\mu} = eT_{a\mu}$.

The field equations are equivalent to Einstein's equations. It is possible to verify by explicit calculations that the equations above can be rewritten as

$$\frac{1}{2}[R_{a\mu}(e) - \frac{1}{2}e_{a\mu}R(e)] = \frac{1}{4kc}\,T_{a\mu}\,, \tag{6}$$

Since the Lagrangian density (3) does not contain the total divergence that arises on the right hand side of Equation (2), it is not invariant under arbitrary local SO(3,1) transformations, but the field Equation (5) are covariant under such transformations.

The equivalence between the TEGR and the standard metric formulation of general relativity is based on the equivalence of Equations (5) and (6). However, in the TEGR there are additional field quantities (like third order tensors) constructed by means of the tetrad fields, such as total divergences, for instance, that cannot be obtained in the standard metric formulation. These additional field quantities are covariant under the global Lorentz transformations, but not under local transformations. In the ordinary formulation of arbitrary field theories, energy, momentum, angular momentum and COM moment are frame dependent field quantities, that transform under the global SO(3,1) transformations. In particular, energy transforms as the zero component of the energy-momentum four-vector. This feature must hold also in the presence of the gravitational field. As an example, consider the total energy of a black hole, represented by the mass parameter m. As seen by a distant observer, the total energy of a static Schwarzschild black hole is given by $E = mc^2$. However, at great distances the black hole may be considered as a particle of mass m, and if it moves with constant velocity v, then its total energy as seen by the same distant observer is $E = \gamma mc^2$, where $\gamma = (1 - v^2/c^2)^{-1/2}$. Likewise, the gravitational momentum, angular momentum and the COM moment are also frame dependent field quantities in general, whose values are different for different frames and different observers. On physical grounds, energy, momentum, angular momentum and COM moment cannot be local Lorentz *invariant* field quantities, since these quantities depend on the frame, as we know from special relativity, which is the limit of the general theory of relativity when the gravitational field is weak or negligible.

After some rearrangements, Equation (5) may be written in the form [13]

$$\partial_\nu(e\Sigma^{a\mu\nu}) = \frac{1}{4k}ee^a{}_\nu(t^{\mu\nu} + \frac{1}{c}T^{\mu\nu})\,, \tag{7}$$

where

$$t^{\mu\nu} = k(4\Sigma^{bc\mu}T_{bc}{}^\nu - g^{\mu\nu}\Sigma^{bcd}T_{bcd})\,, \tag{8}$$

is interpreted as the gravitational energy-momentum tensor [13,32] and $T^{\mu\nu} = e_a{}^\mu T^{a\nu}$.

The Hamiltonian density of the TEGR is constructed as usual in the phase space of the theory. We first note that the Lagrangian density (3) does not depend on the time derivatives of e_{a0}. Therefore, the latter arise as Lagrange multipliers in the Hamiltonian density H. The momenta canonically conjugated to e_{a0} are denoted by Π^{a0}. The latter are primary constraints of the theory: $\Pi^{a0} \approx 0$. The momenta canonically conjugated to e_{ai} are given by $\Pi^{ai} = \delta L/\delta\dot{e}_{ai} = -4k\Sigma^{a0i}$. The Hamiltonian density is obtained by rewriting the Lagrangian density in the form $L = \Pi^{ai}\dot{e}_{ai} - H$, in terms of e_{ai}, Π^{ai} and Lagrange multipliers. After the Legendre transform is performed, we obtain the final form of the Hamiltonian density. It reads [29,30]

$$H(e,\Pi) = e_{a0}C^a + \lambda_{ab}\Gamma^{ab}. \tag{9}$$

where λ_{ab} are Lagrange multipliers. In the above equation we have omitted a surface term. $C^a = \delta H/\delta e_{a0}$ is a long expression of the field variables, and $\Gamma^{ab} = -\Gamma^{ba}$ are defined by

$$\Gamma^{ab} = 2\Pi^{[ab]} + 4ke(\Sigma^{a0b} - \Sigma^{b0a})\,. \tag{10}$$

After solving the field equations, the Lagrange multipliers are identified as $\lambda_{ab} = (1/4)$ $(T_{a0b} - T_{b0a} + e_a{}^0T_{00b} - e_b{}^0T_{00a})$. The constraints C^a may be written as

$$C^a = -\partial_i \Pi^{ai} - p^a = 0\,, \tag{11}$$

where p^a is an intricate expression of the field quantities.

The quantities C^a and Γ^{ab} are first class constraints. They satisfy an algebra similar to the algebra of the Poincaré group [29]. The integral form of the constraint equations $C^a = 0$ yields the gravitational energy-momentum P^a [13],

$$P^a = -\int_V d^3x\, \partial_i \Pi^{ai}\,, \tag{12}$$

where V is an arbitrary volume of the three-dimensional space and $\Pi^{ai} = -4k\Sigma^{a0i}$. In similarity to the definition above, the definition of the gravitational 4-angular momentum follows from the constraint equations $\Gamma^{ab} = 0$ [30]. However, it has been noted [33] that the second term on the right hand side of Equation (10) can be rewritten as a total divergence, so that the constraints Γ^{ab} become

$$\Gamma^{ab} = 2\Pi^{[ab]} - 2k\partial_i[e(e^{ai}e^{b0} - e^{bi}e^{a0})] = 0\,. \tag{13}$$

Therefore, the definition of the total 4-angular momentum of the gravitational field L^{ab} may be given by an integral of a total divergence, in similarity to Equation (12). We have

$$L^{ab} = -\int_V d^3x\, 2\Pi^{[ab]}\,, \tag{14}$$

where

$$2\Pi^{[ab]} = (\Pi^{ab} - \Pi^{ba}) = 2k\partial_i[e(e^{ai}e^{b0} - e^{bi}e^{a0})]\,. \tag{15}$$

It is easy to show [30] that expressions (12) and (14) satisfy the algebra of the Poincaré group in the phase space of the theory,

$$\begin{aligned} \{P^a, P^b\} &= 0\,, \\ \{P^a, L^{bc}\} &= \eta^{ab}P^c - \eta^{ac}P^b\,, \\ \{L^{ab}, L^{cd}\} &= \eta^{ad}L^{cb} + \eta^{bd}L^{ac} - \eta^{ac}L^{db} - \eta^{bc}L^{ad}\,. \end{aligned} \tag{16}$$

Therefore, from a physical point of view, the interpretation of the quantities P^a and L^{ab} is consistent.

Definitions (12) and (14) are invariant under coordinate transformations of the three-dimensional, under time reparametrizations, and under global SO(3,1) transformations. The gravitational energy is the zero component of the energy-momentum four vector P^a.

3. The Centre of Mass Moment

The gravitational centre of mass (COM) moment is given by the components

$$L^{(0)(i)} = -\int d^3x\, M^{(0)(i)}\,, \tag{17}$$

where

$$M^{(0)(i)} = 2\Pi^{[(0)(i)]} = 2k\partial_j[e(e^{(0)j}e^{(i)0} - e^{(i)j}e^{(0)0})]\,, \tag{18}$$

according to definition (15). The quantity $-M^{(0)(i)}$ is identified as the gravitational COM density. The evaluation of the expression above is very simple. One needs just to establish the suitable set of tetrad fields that define a frame in space-time.

The inverse tetrads $e_a{}^\mu$ are interpreted as a frame adapted to a particular class of observers in space-time. Let the curve $x^\mu(\tau)$ represent the timelike worldline C of an observer in space-time, where τ is the proper time of the observer. The velocity of the observer along C is given by $u^\mu = dx^\mu/d\tau$. A frame adapted to this observer is constructed by identifying the timelike component of the frame

$e_{(0)}{}^{\mu}$ with the velocity u^{μ} of the observer: $e_{(0)}{}^{\mu} = u^{\mu}(\tau)$. The three other components of the frame, $e_{(i)}{}^{\mu}$, are orthogonal to $e_{(0)}{}^{\mu}$, and may be oriented in the three-dimensional space according to the symmetry of the physical system. If the space-time has axial symmetry, for instance, then the $e_{(3)}{}^{\mu}$ components of the tetrad fields are chosen to be oriented, asymptotically, along the z axis of the coordinate system, i.e., $e_{(3)}{}^{\mu}(t, x, y, z) \simeq (0, 0, 0, 1)$ in the limit $r \to \infty$. A static observer in space-time is defined by the condition $u^{\mu} = (u^0, 0, 0, 0)$. Thus, a frame adapted to a static observer in space-time must satisfy the conditions $e_{(0)}{}^{i}(t, x^k) = (0, 0, 0)$.

An alternative way to characterise a frame in space-time is by means of the acceleration tensor ϕ_{ab} [34–38],

$$\phi_{ab} = \frac{1}{2}[T_{(0)ab} + T_{a(0)b} - T_{b(0)a}] \,. \tag{19}$$

This tensor is invariant under coordinate transformations and covariant under global SO(3,1) transformations, but not under local SO(3,1) transformations. It yields the inertial (i.e., the non-gravitational) accelerations that are necessary to impart to a frame in space-time in order to maintain the frame in a given inertial state. Three components of ϕ_{ab} yield the translational accelerations, and three other components yield the frequency of rotation of the frame. Altogether, these six components cancel the gravitational acceleration, so that the frame is kept in a particular inertial state.

In the following, we will evaluate the density of the centre of mass moment of four space-time configurations that exhibit spherical symmetry. In the four cases we will establish the frame of a static observer in space-time.

3.1. The Space-Time of a Massive Point Particle

The Schwarzschild solution in isotropic coordinates represents the space-time of a point massive particle [39,40]. It is obtained as an exact solution of Einstein's equations by writing the energy-momentum tensor in terms of a δ function of a point particle of mass M, with support at the origin of the coordinate system. The solution is described by the line element

$$ds^2 = -\alpha^2 c^2 dt^2 + \beta^2[dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)] \,, \tag{20}$$

where

$$\alpha^2 = \left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2 , \qquad \beta^2 = \left(1 + \frac{m}{2r}\right)^4 . \tag{21}$$

The parameter $m = GM/c^2$ represents the mass of the point particle that appears in the energy-momentum tensor. The line element above is clearly a solution of Equation (6), with the appropriate energy-momentum tensor $T_{a\mu}$ described in Reference [40].

By performing a coordinate transformation to (x, y, z) coordinates where

$$\begin{aligned} x &= r\sin\theta\cos\phi \,, \\ y &= r\sin\theta\sin\phi \,, \\ z &= r\cos\theta \,, \end{aligned} \tag{22}$$

the line element becomes

$$ds^2 = -\alpha^2 \, c^2 dt^2 + \beta^2(dx^2 + dy^2 + dz^2) \,. \tag{23}$$

The tetrad fields adapted to static observers is given by

$$e_{a\mu}(t, x, y, z) = \begin{pmatrix} -\alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix} . \tag{24}$$

Taking into account Equation (18), straightforward calculations yield $M^{(0)(1)} = 2k\,\partial_1\beta^2$, $M^{(0)(2)} = 2k\,\partial_2\beta^2$ and $M^{(0)(3)} = 2k\,\partial_3\beta^2$. It is easy to obtain

$$\begin{aligned} -M^{(0)(1)} &= d_g\, x\,, \\ -M^{(0)(2)} &= d_g\, y\,, \\ -M^{(0)(3)} &= d_g\, z\,. \end{aligned} \tag{25}$$

The quantity d_g is defined by

$$d_g = \frac{4\,k\,m}{r^3}\left(1+\frac{m}{2r}\right)^3 . \tag{26}$$

Therefore,

$$\begin{aligned} L^{(0)(1)} &= \int d^3x\, d_g x\,, \\ L^{(0)(2)} &= \int d^3x\, d_g y\,, \\ L^{(0)(3)} &= \int d^3x\, d_g z\,, \end{aligned} \tag{27}$$

where $d^3x = dx\,dy\,dz$ and $r^2 = x^2 + y^2 + z^2$. The expressions above remind the definition of centre of mass in classical mechanics. Given that $M^{(0)(i)} = 2k\,\partial_i\beta^2$, it is easy to see that all integrals given by Equation (17) vanish, namely, all components of the total centre of mass moment vanish. However, the field quantity (26) has the following properties:

$$\begin{aligned} r \to \infty &\quad : \quad d_g \to 0\,, \\ r \to 0 &\quad : \quad d_g \to \infty\,. \end{aligned} \tag{28}$$

Thus, d_g is more intense in the vicinity of the particle, and vanishes at spatial infinity. In view of Equations (27) and (28), d_g may be interpreted as the gravitational COM distribution function. It is clearly related to the intensity of the gravitational field. The analyses of the space-time configurations below support this interpretation, as we will see.

3.2. The Schwarzschild-de Sitter Space-Time

The line element of the Schwarzschild-de Sitter space-time is given by

$$ds^2 = -\alpha^2\,dt^2 + \frac{1}{\alpha^2}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\,, \tag{29}$$

where

$$\alpha^2 = 1 - \frac{2m}{r} - \frac{r^2}{R^2}\,, \tag{30}$$

$R = \sqrt{3/\Lambda}$ and Λ is the cosmological constant. Here we are considering the speed of light $c = 1$. The Schwarzschild-de Sitter space-time has been considered in the TEGR in Reference [41]. The set of tetrad fields adapted to stationary observers in space-time is given by

$$e_{a\mu} = \begin{pmatrix} -\alpha & 0 & 0 & 0 \\ 0 & \alpha^{-1}\sin\theta\,\cos\phi & r\cos\theta\,\cos\phi & -r\sin\theta\,\sin\phi \\ 0 & \alpha^{-1}\sin\theta\,\sin\phi & r\cos\theta\,\sin\phi & r\sin\theta\,\cos\phi \\ 0 & \alpha^{-1}\cos\theta & -r\sin\theta & 0 \end{pmatrix} . \tag{31}$$

After long but simple calculations we find that the components of Equation (18) read

$$
\begin{aligned}
-M^{(0)(1)} &= 4k\sin\theta\left(\frac{1}{\alpha}-1\right) r\sin\theta\cos\phi\,,\\
-M^{(0)(2)} &= 4k\sin\theta\left(\frac{1}{\alpha}-1\right) r\sin\theta\sin\phi\,,\\
-M^{(0)(3)} &= 4k\sin\theta\left(\frac{1}{\alpha}-1\right) r\cos\theta\,.
\end{aligned}
\tag{32}
$$

We identify $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ as usual, and write Equation (17) as

$$
\begin{aligned}
L^{(0)(1)} &= \int d^3x\, 4k\sin\theta\left(\frac{1}{\alpha}-1\right) x\,,\\
L^{(0)(2)} &= \int d^3x\, 4k\sin\theta\left(\frac{1}{\alpha}-1\right) y\,,\\
L^{(0)(3)} &= \int d^3x\, 4k\sin\theta\left(\frac{1}{\alpha}-1\right) z\,,
\end{aligned}
\tag{33}
$$

where $d^3x = dr\, d\theta\, d\phi$. Integration in the angular variables implies the vanishing of the three integrals $L^{(0)(i)}$, i.e., the total centre of mass vanishes, as expected. The equations above may be written exactly as Equation (27) provided we identify

$$
d_g = 4k\,\sin\theta\left(\frac{1}{\alpha}-1\right) \equiv 4k\,\sin\theta\, f(r)\,. \tag{34}
$$

The analysis of the expression above leads to interesting results. Let r_1 and r_2 denote the two horizons of the Schwarzschild-de Sitter space-time, $\alpha(r_1) = 0$ and $\alpha(r_2) = 0$, so that $r_1 < r_2$. The radius r_1 is close to the Schwarzschild radius, $r_1 \approx \frac{2m}{r}$, and $r_2 \approx R$. We have

$$
\begin{aligned}
r \to r_1 \quad &: \quad f(r) \to \infty\,,\\
r \to r_2 \quad &: \quad f(r) \to \infty\,.
\end{aligned}
\tag{35}
$$

The function $f(r)$ is defined by Equation (34). The minimum of $f(r)$ is given by

$$
\frac{df}{dr} = -\frac{1}{\alpha^2}\frac{d\alpha}{dr} = 0\,,
$$

and takes place at $r_{min} = (mR^2)^{1/3}$. Thus, d_g is intense close to both r_1 and r_2, i.e., close to the Schwarzschild and cosmological horizons.

The radial position r_{min} is related to the inertial accelerations of an observer. In order to understand this feature, we evaluate the translational (non-gravitational) accelerations of a frame given by Equation (19). We find

$$
\begin{aligned}
\phi_{(0)(1)} &= \frac{d\alpha}{dr}\sin\theta\cos\phi\,,\\
\phi_{(0)(2)} &= \frac{d\alpha}{dr}\sin\theta\sin\phi\,,\\
\phi_{(0)(3)} &= \frac{d\alpha}{dr}\cos\theta\,.
\end{aligned}
\tag{36}
$$

We define the inertial acceleration vector Φ as

$$
\Phi(r) = (\phi_{(0)(1)}, \phi_{(0)(2)}, \phi_{(0)(3)}) \equiv \phi(r)\hat{r} = \frac{d\alpha}{dr}\hat{r}\,, \tag{37}
$$

where $\hat{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Since

$$\frac{d\alpha}{dr} = \frac{1}{\alpha}\left(\frac{m}{r^2} - \frac{r}{R^2}\right),$$

we see that

$$r_1 < r < r_{min}: \quad \frac{d\alpha}{dr} > 0 \quad \rightarrow \quad \phi(r) > 0\,,$$
$$r_{min} < r < r_2: \quad \frac{d\alpha}{dr} < 0 \quad \rightarrow \quad \phi(r) < 0\,. \tag{38}$$

Thus, given that the inertial acceleration $\phi(r) > 0$ is repulsive in the region $r_1 < r < r_{min}$, the gravitational acceleration is attractive in this interval, as expected. By means of a similar argument, we see that the gravitational acceleration is repulsive in the region $r_{min} < r < r_2$, as expected.

In view of the analysis above, we may interpret d_g given by Equation (34) as the gravitational COM distribution function, in similarity to Equation (26), and therefore one may understand the gravitational repulsion as attraction to a region of intense gravitational COM distribution function, which, in the present case, is the region in the vicinity of the cosmological horizon. If dark energy is indeed related to the existence of a cosmological constant, then it is natural that it is concentrated close to the radius $r_2 \approx R = \sqrt{3/\Lambda}$ in the context of a simple Schwarzschild-de Sitter model.

The function d_g plays the role of a gravitational COM density. However, mathematically it is not a density. The integrands in Equations (27) and (33) are in fact densities, but not d_g alone. In Newtonian mechanics, d_g in Equations (27) and (33) plays the role of mass density.

3.3. Dark Matter Simulated by Non-Local Gravity

A non-local formulation of general relativity, based on a geometrical framework similar to the one established by Equations (2)–(4) has been developed by Hehl, Mashhoon and collaborators [42–44]. One interesting consequence of this development is an extension of Newtonian gravity that may play a relevant role in the dynamics of galaxies, and might provide an explanation that is expected to come from dark matter models of gravity. We restrict the considerations to a simplified space-time with spherical symmetry, so that Equations (29), (31), (33) and (34) remain valid.

The Newtonian approximation is established by

$$\alpha^2 = -g_{00} \simeq 1 + \frac{2\Phi_g}{c^2}\,, \tag{39}$$

where Φ_g is the Newtonian potential, and $2\Phi_g/c^2 << 1$.

It follows that

$$f(r) = \frac{1}{\alpha} - 1 \simeq -\frac{\Phi_g}{c^2}\,. \tag{40}$$

The Newtonian potential that arises in the non-local formulation of gravity is given by [43,44]

$$\Phi_g \simeq -\frac{GM}{r} + \frac{GM}{\lambda}\ln\left(\frac{r}{\lambda}\right), \tag{41}$$

where λ is a constant length, and is taken to be $\lambda \approx 1kpc = 3260$ light-years. Consequently, the influence of the second term on the right hand side of Equation (41) in the solar system is negligible. Therefore, we find

$$f(r) = \frac{1}{\alpha} - 1 \simeq \frac{m}{r} - \frac{m}{\lambda}\ln\left(\frac{r}{\lambda}\right). \tag{42}$$

In the expression above, $m = GM/c^2$. For values of r within a galaxy, $r < \lambda$ and thus $-(m/\lambda)\ln(r/\lambda)$ is positive, and decreases as $1/r$ with increasing values of r, a result that shows that the gravitational field is sufficiently intense at the borders of a galaxy to explain the rotation curves of spiral galaxies. The function $d_g = 4k\,\sin\theta f(r)$ may again be understood as the gravitational COM distribution function of the spherically symmetric space-time.

3.4. Arbitrary Space-Time with Axial Symmetry

The analysis of a space-time that is not spherically symmetric allows to obtain the generalization of Equations (27) and (33). Let us consider an arbitrary space-time with axial symmetry. It is described by following line element,

$$ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2 + 2g_{03}d\phi\, dt\,, \tag{43}$$

where all metric components depend on r and θ, but not on ϕ : $g_{\mu\nu} = g_{\mu\nu}(r,\theta)$. The determinant $e = \sqrt{-g}$ is $e = [g_{11}g_{22}\,\delta]^{1/2}$, where

$$\delta = g_{03}g_{03} - g_{00}g_{33}\,.$$

The inverse metric components are $g^{00} = -g_{33}/\delta$, $g^{03} = g_{03}/\delta$ and $g^{33} = -g_{00}/\delta$.

The set of tetrad fields in spherical coordinates that is adapted to static observers in space-time is given by

$$e_{a\mu} = \begin{pmatrix} -A & 0 & 0 & -C \\ 0 & \sqrt{g_{11}}\,\sin\theta\cos\phi & \sqrt{g_{22}}\cos\theta\cos\phi & -D\,r\sin\theta\sin\phi \\ 0 & \sqrt{g_{11}}\,\sin\theta\sin\phi & \sqrt{g_{22}}\cos\theta\sin\phi & D\,r\sin\theta\cos\phi \\ 0 & \sqrt{g_{11}}\,\cos\theta & -\sqrt{g_{22}}\sin\theta & 0 \end{pmatrix}. \tag{44}$$

The functions A, C and D are defined such that Equation (44) yields (43). They read

$$\begin{aligned} A(r,\theta) &= (-g_{00})^{1/2}\,, \\ C(r,\theta) &= -\frac{g_{03}}{(-g_{00})^{1/2}}\,, \\ D(r,\theta) &= \frac{1}{(r\sin\theta)}\left[\frac{\delta}{(-g_{00})}\right]^{1/2}. \end{aligned} \tag{45}$$

After simple calculations, we find that Equations (17) and (18) yield

$$\begin{aligned} L^{(0)(1)} &= \int d^3x\, d_{g1}\,(r\sin\theta\cos\phi)\,, \\ L^{(0)(2)} &= \int d^3x\, d_{g2}\,(r\sin\theta\cos\phi)\,, \\ L^{(0)(3)} &= \int d^3x\, d_{g3}\,(r\cos\theta)\,, \end{aligned} \tag{46}$$

where now we have

$$\begin{aligned} d_{g1} = d_{g2} &= 2k\Big\{-\frac{1}{r}\partial_1\left[\frac{g_{22}\delta}{(-g_{00})}\right]^{1/2} - \frac{1}{r\sin\theta}\,\partial_2\left[\left(\frac{g_{11}\delta}{(-g_{00})}\right)^{1/2}\cos\theta\right] \\ &\quad +\frac{1}{r\sin\theta}(g_{11}g_{22})^{1/2}\Big\}\,, \\ d_{g3} &= 2k\Big\{-\frac{1}{r}\partial_1\left[\frac{g_{22}\delta}{(-g_{00})}\right]^{1/2} + \frac{1}{r\cos\theta}\,\partial_2\left[\left(\frac{g_{11}\delta}{(-g_{00})}\right)^{1/2}\sin\theta\right]\Big\}. \end{aligned} \tag{47}$$

In the flat space-time, the quantities above vanish. It is not difficult to see that if the metric tensor components above represent the exterior gravitational field of a typical rotating source, the expressions above are not divergent. Note that $L^{(0)(1)}$ and $L^{(0)(2)}$ vanish due to integration in ϕ, as a consequence of the axial symmetry, but $L^{(0)(3)}$ is non-vanishing in general.

In the equations above we obtain $d_{g1} = d_{g2}$ because of the axial symmetry of the space-time. We see that, in general, we may have three different COM distribution functions, one for each direction in the three-dimensional space, in contrast to the situation in classical mechanics, where there is a single mass density in the definition of centre of mass.

4. Conclusions

In this article we have investigated the definition of centre of mass of the gravitational field, in the realm of the teleparallel equivalent of general relativity. The analysis of the gravitational centre of mass density leads to the concept of COM distribution function. We may understand the latter as a quantity that provides a description of the intensity of the gravitational field in space-time. The emergence of this quantity justifies the analysis of the centre of mass density of arbitrary configurations of the gravitational field, including gravitational wave configurations. We have applied this definition to the space-time endowed with a positive cosmological constant. At the speculative level, dark energy might be a consequence of the existence of a positive cosmological constant that induces a strong gravitational acceleration very far from our present location in the universe. In the simple model established by the Schwarzschild-de Sitter space-time, dark energy is roughly located in the region beyond $r = r_{min} = (mR^2)^{1/3}$, according to Equation (38).

The centre of mass moment naturally arises in the Hamiltonian formulation of the teleparallel equivalent of general relativity, and its definition is obtained from the primary constraints of the theory—Equation (13). It is given by Equations (17) and (18). The analysis led us to interpret the quantity d_g in the integrand of Equations (27), (33) and (46) as the gravitational COM distribution function. Although d_g plays the role of a density, mathematically it is not a density. It vanishes when the gravitational field is turned off. The expressions of $L^{(0)(i)}$ given by Equations (27) and (33) do remind the standard expression of centre of mass in classical mechanics. The distribution function d_g in the three-dimensional space is related to the intensity of the gravitational field. In the space-time of a point massive particle, d_g is intense (and in fact diverges) in the vicinity of the particle, and in the Schwarzschild-de Sitter space-time d_g is positive definite and diverges at both the Schwarzschild and cosmological horizons, which are precisely the regions where the gravitational field is more intense.

In relativistic field theory or in the Newtonian approximation of general relativity, energy, momentum and angular momentum are frame dependent field quantities, and so they are, in general, in the present context. In particular, the gravitational COM moment is evaluated in the frame adapted to an arbitrary observer in space-time. The gravitational centre of mass given by Equations (17) and (18) is invariant under coordinate transformations of the three-dimensional space, and under time reparametrizations. It transforms covariantly under global SO(3,1) transformations, provided the tetrad fields transform as $\tilde{e}^a{}_\mu = \Lambda^a{}_b e^b{}_\mu$, where $\Lambda^a{}_b$ are matrices of the SO(3,1) group. However, definition (17) is not covariant under local SO(3,1) transformations. In relativistic field theory, the COM definition is also not covariant under local SO(3,1) transformations.

We conclude that repulsion, in the Schwarzschild-de Sitter space-time, is in fact attraction to a region of intense gravitational COM distribution function. We have seen that in the region $r < r_{min} = (mR^2)^{1/3}$ the gravitational acceleration is attractive, and is repulsive in the dark energy region $r > r_{min}$. We expect the present analysis to be useful in the investigation of realistic cosmological models endowed with a positive cosmological constant.

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References

1. Arnowitt, R.; Deser, S.; Misner, C.W. *Gravitation: An Introduction to Current Research*; Witten, L., Ed.; Wiley: New York, NY, USA, 1962.
2. Weinberg, S. *Gravitation and Cosmology*; Wiley: New York, NY, USA, 1972.
3. Dixon, W.G. Dynamics of Extended Bodies in General Relativity. I. Momentum and Angular Momentum. *Proc. R. Soc. Lond. A* **1970**, *314*, 499–527.
4. Dixon, W.G. Dynamics of Extended Bodies in General Relativity. II. Moments of the Charge-Current Vector. *Proc. R. Soc. Lond. A* **1970**, *319*, 509–547.
5. Dixon, W.G. The definition of multipole moments for extended bodies. *Gen. Relativ. Gravit.* **1973**, *4*, 199–209.
6. Ehlers, J.; Rudolph, E. Dynamics of extended bodies in general relativity center-of-mass description and quasirigidity. *Gen. Relativ. Gravit.* **1977**, *8*, 197–217.
7. Regge, T.; Teitelboim, C. Role of surface integrals in the Hamiltonian formulation of general relativity. *Ann. Phys.* **1974**, *88*, 286–318.
8. Hanson, A.; Regge, T.; Teitelboim, C. *Constrained hamiltonian systems*; Accademia Nazionale dei Lincei: Roma, Italy, 1976.
9. Beign, R.; ÓMurchadha, N. The Poincaré group as the symmetry group of canonical general relativity. *Ann. Phys.* **1987**, *174*, 463–498.
10. Baskaran, D.; Lau, S.R.; Petrov, A.N. Center of mass integral in canonical general relativity. *Ann. Phys.* **2003**, *307*, 90–131.
11. Nester, J.M.; Ho, F.H.; Chen, C.M. Quasilocal Center-of-Mass for Teleparallel Gravity. 2004, arXiv:gr-qc/0403101.
12. Nester, J.M.; Meng, F.F.; Chen, C.M. Quasilocal Center-of-Mass. *J. Korean Phys. Soc.* **2004**, *45*, S22–S25.
13. Maluf, J.W. The teleparallel equivalent of general relativity. *Ann. Phys.* **2013**, *525*, 339–357.
14. Aldrovandi, R.; Pereira, J.G. *Teleparallel Gravity: An Introduction*; Springer: Heidelberg, Germany, 2013.
15. Blagojevic, M.; Hehl, F.W. *Gauge Theories of Gravitation*; Imperial College: London, UK, 2013.
16. Einstein, A. Riemannsche Geometrie unter Aufrechterhaltung des Begriffes des Fernparallelismus (Riemannian Geometry with Maintaining the Notion of Distant Parallelism). In *Sitzungsberichte der Preussischen Akademie der Wissenshcaften*; Verlag der Akademie der Wissenschaften: Berlin, Germany, 1928; pp. 217–221.
17. Einstein, A. Unified Field Theory based on Riemannian Metrics and Distant Parallelism. *Math. Ann.* **1930**, *102*, 685–697.
18. Cho, Y.M. Einstein Lagrangian as the translational Yang-Mills Lagrangian. *Phys. Rev. D* **1976**, *14*, 2521–2525;
19. Cho, Y.M. Gauge theory of Poincaré symmetry. *Phys. Revs. D* **1976**, *14*, 3335–3341.
20. Hayashi, K.; Shirafuji, T. New general relativity. *Phys. Rev. D* **1979**, *19*, 3524–3554.
21. Hayashi, K.; Shirafuji, T. Addendum to "New general relativity". *Phys. Rev. D* **1981**, *24*, 3312–3315.
22. Hehl, F.W. Four lectures on poincaré Gauge field theory. In *Cosmology and Gravitation: Spin, Torsion, Rotation and Supergravity*; Bergmann, P.G., de Sabbata, V., Eds.; Plenum Press: New York, NY, USA, 1980.
23. Nitsch, J. The macroscopic limit of the poincaré gauge field theory on gravitation. In *Cosmology and Gravitation: Spin, Torsion, Rotation and Supergravity*; Bergmann, P.G., de Sabbata, V., Eds.; Plenum Press: New York, NY, USA, 1980.
24. Schweizer, M.; Straumann, N.; Wipf, A. Post-Newtonian Generation of Gravitational Waves in a Theory with Torsion. *Gen. Relativ. Gravit.* **1980**, *12*, 951–961.
25. Nester, J.M. Positive energy via the teleparallel Hamiltonian. *Int. J. Mod. Phys. A* **1989**, *4*, 1755–1772.
26. Wiesendanger, C. Translational gauge invariance and classical gravitodynamics. *Class. Quantum Gravity* **1995**, *12*, 585–603.
27. Cai, Y.-F.; Capozziello, S.; de Laurentis, M.; Saridakis, E.N. $f(T)$ Teleparallel Gravity and Cosmology. 2015, arXiv:1511.07586.
28. Maluf, J.W.; da Rocha-Neto, J.F. Hamiltonian formulation of general relativity in the teleparallel geometry. *Phys. Rev D* **2001**, *64*, 084014.
29. Da Rocha-Neto, J.F.; Maluf, J.W.; Ulhoa, S.C. Hamiltonian formulation of unimodular gravity in the teleparallel geometry. *Phys. Rev. D* **2010**, *82*, 124035.

30. Maluf, J.W.; Ulhoa, S.C.; Faria, F.F.; da Rocha-Neto, J.F. The angular momentum of the gravitational field and the Poincaré group. *Class. Quantum Gravity* **2006**, *23*, 6245–6256.
31. Maluf, J.W. Hamiltonian formulation of the teleparallel description of general relativity. *J. Math. Phys.* **1994**, *35*, 335–343.
32. Maluf, J.W. The gravitational energy-momentum tensor and the gravitational pressure. *Ann. Phys. (Berlin)* **2005**, *14*, 723–732.
33. Da Rocha-Neto, J.F.; Maluf, J.W. The angular momentum of plane-fronted gravitational waves in the teleparallel equivalent of general relativity. *Gen. Relativ. Gravit.* **2014**, *46*, 1667.
34. Mashhoon, B.; Muench, U. Length measurement in accelerated systems. *Ann. Phys. (Berlin)* **2002**, *11*, 532–547.
35. Mashhoon, B. Vacuum electrodynamics of accelerated systems: Nonlocal Maxwell's equations. *Ann. Phys. (Berlin)* **2003**, *12*, 586–598.
36. Maluf, J.W.; Faria, F.F.; Ulhoa, S.C. On reference frames in spacetime and gravitational energy in freely falling frames. *Class. Quantum Gravity* **2007**, *24*, 2743–2754.
37. Maluf, J.W.; Faria, F.F. On the construction of Fermi-Walker transported frames. *Ann. Phys. (Berlin)* **2008**, *17*, 326–335.
38. Maluf, J.W. Repulsive gravity near naked singularities and point massive particles. *Gen. Relativ. Gravit.* **2014**, *46*, 1734.
39. Parker, E.P. Distributional geometry. *J. Math. Phys.* **1979**, *20*, 1423–1426.
40. Katanaev, M.O. Point massive particle in General Relativity. *Gen. Relativ. Gravit.* **2013**, *45*, 1861–1875.
41. Ulhoa, S.C.; da Rocha-Neto, J.F.; Maluf, J.W. The gravitational energy problem for cosmological models in teleparallel gravity. *Int. J. Mod. Phys. D* **2010**, *19*, 1925–1935.
42. Hehl, F.W.; Mashhoon, B. Formal framework for a nonlocal generalization of Einstein's theory of gravitation. *Phys. Rev. D* **2009**, *79*, 064028.
43. Blome, H.J.; Chicone, C.; Hehl, F.W.; Mashhoon, B. Nonlocal modification of Newtonian gravity. *Phys. Rev. D* **2010**, *81*, 065020 .
44. Hehl, F.W.; Mashhoon, B. Nonlocal gravity simulates dark matter. *Phys. Lett. B* **2009**, *673*, 279–282.