

Review

Higher Spin Fields in Hyperspace. A Review

Dmitri Sorokin ¹ and Mirian Tsulaia ^{2,*}

¹ Istituto Nazionale di Fisica Nucleare, Sezione di Padova, via F. Marzolo 8, 35131 Padova, Italy; dmitri.sorokin@pd.infn.it

² School of Physics M013, The University of Western Australia, 35 Stirling Highway, Crawley, Perth, WA 6009, Australia

* Correspondence: mirian.tsulaia@gmail.com

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Abstract: We give an introduction to the so-called tensorial, matrix or hyperspace approach to the description of massless higher-spin fields.

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1. Introduction

Every consistent theory of interacting higher spin fields necessarily includes an infinite number of such fields. For this reason, it is extremely important to develop a formalism which effectively includes an infinite number of fields into a simpler field-theoretical object. This formalism should yield correct field equations first of all at the free level and then be promoted to an interacting theory. An elegant geometrical approach to higher spin theories of this kind is known as the method of tensorial spaces. This approach was first suggested by Fronsdal [1]. Its explicit dynamical realization and further extensive developments have been carried out in [2–28].

In a certain sense, the method of tensorial spaces is reminiscent of the Kaluza–Klein theories. In such theories, one usually considers massless field equations in higher dimensions and then, assuming that the extra dimensions are periodic (compact), one obtains a theory in lower dimensions, which contains fields with growing masses. In the method of tensorial (super)spaces, one also considers theories in multi-dimensional space–times, but in this case the extra dimensions are introduced in such a way that they generate the fields with higher spins instead of the fields with increasing masses. A main advantage of the formulation of the higher spin theories on extended tensorial (super) spaces is that one can combine curvatures of an infinite number of bosonic and fermionic higher spin fields into a single “master” (or “hyper”) scalar and spinor field which propagate through the tensorial superspaces (also called hyperspaces). The field equations in the tensorial spaces are invariant under the action of $Sp(2n)$ group, whereas the dimensions of the corresponding tensorial spaces are equal to $\frac{n(n+1)}{2}$. The case of four space–time dimensions $D = 4$ is of particular interest since the approach of tensorial (super)spaces comprises all massless higher spin fields from zero to infinity. The free field equations are invariant under the $Sp(8)$ group, which contains a four dimensional conformal group $SO(2,4)$ as a subgroup. In fact, the entire structure of the $Sp(8)$ invariant formulation of the higher spin fields is a straightforward generalization of the conformally invariant formulation of the four-dimensional scalar and spinor fields. This allows one to use the experience and intuition gained from the usual conformal field theories for studying the dynamics of higher spin fields on flat and AdS backgrounds, and to construct their correlation functions.

Being intrinsically related to the unfolded formulation [29–33] of higher-spin field theory, the hyperspace approach provides an extra and potentially powerful tool for studying higher spin AdS/CFT correspondence (for reviews on higher-spin holography, see, e.g., [34,35]). The origin of higher-spin holographic duality can be traced back to the work of Flato and Fronsdal [36] who showed

that the tensor product of single-particle states of a 3D massless conformal scalar and spinor fields (singletons) produces the tower of all single-particle representations of 4D massless fields whose spectrum matches that of 4D higher spin gauge theories. The hyperspace formulation provides an explicit field theoretical realization of the Flato-Fronsdal theorem in which higher spin fields are embedded in a single scalar and spinor fields, though propagating in hyperspace. The relevance of the unfolded and hyperspace formulation to the origin of holography has been pointed out in [33]. In this interpretation, holographically dual theories share the same unfolded formulation in extended spaces which contains twistor-like variables and each of these theories corresponds to a different reduction, or “visualization”, of the same “master” theory.

In what follows, we will review main features and latest developments of the tensorial space approach, and associated generalized conformal theories. It is mainly based on Papers [3,8,10,13,23,24,27]. We hope that this will be a useful complement to a number of available reviews on the higher-spin gauge theories which reflect other aspects and different approaches to the subject

- Frame-like approach in higher-spin field theory [37–42].
- Metric-like approach [43–55].
- Review that address the both approaches [56].
- Higher-spin Holography [34,35,57,58].
- Reviews which contain both the metric-like approach and the hyperspace approach [59,60].
- A short review on the hyperspace approach [61].
- A short review that contains frame-like approach, hyperspaces and higher-spin holography [62].

The review is organized as follows. In Section 2 we introduce a general concept of flat hyperspaces. To this end we use somewhat heuristic argument, which includes a direct generalization of the famous twistor-like representation of a light-light momentum of a particle to higher dimensional tensorial spaces i.e., to hyperspaces. The basic fields in this set up are one bosonic and one fermionic hyperfield, which contain infinite sets of bosonic and fermionic field strengths of massless fields with spins ranging from zero to infinity. Physically interesting examples are hyperspaces associated with ordinary space–times of dimensions $D = 3, 4, 6$ and 10 . In what follows, we will always keep in mind these physical cases, though from the geometric perspective the tensorial spaces of any dimension have the same properties.

We demonstrate in detail that the solutions of wave equations in hyperspace are generating functionals for higher spin fields. These equations are nothing but a set of free conformal higher spin equations in $D = 3, 4, 6$ and 10 . The case of $D = 3$ describes only scalar and spinor fields, the case of $D = 4$ comprises the all massless bosonic and fermionic higher spin fields with spins from 0 to ∞ and the cases of $D = 6$ and $D = 10$ describe infinite sets of fields whose field strengths are self-dual multiforms. These fields carry unitary irreducible representations of the higher-dimensional conformal group and are sometimes called “spinning singletons” [63].

We then describe a generalized conformal group $Sp(2n)$ which contains a conventional conformal group $SO(2, D)$ as its subgroup (for $D = 3, 4, 6, 10$ and $n = 2, 4, 8, 16$, respectively) and show how the coordinates in hyperspace and the hyperfields transform under these generalized conformal transformations.

In Section 3, we consider an example of curved hyperspaces which are $Sp(n)$ group manifolds. An interesting property of these manifolds is that they are hyperspace generalizations of AdS_D spaces. Similarly to the AdS_D space which can be regarded as a coset space of the conformal group $SO(2, D)$, the $Sp(n)$ group manifold is a coset space of the generalized conformal group $Sp(2n)$. This results in the fact that the property of the conformal flatness of the AdS_D spaces (i.e., the existence of a basis in which the AdS metric is proportional to a flat metric) is also generalized to the case of hyperspaces. In particular, a metric on the $Sp(n)$ group manifold is flat up to a rotation of the $GL(n)$ group, the property that we call “ GL -flatness”.

In Section 4, we briefly discuss how the field equations given in the previous Sections can be obtained as a result of the quantization of (super)particle models on hyperspaces.

In Section 5, we derive the field equations on $Sp(n)$ group manifolds. We show that the field equations on flat hyperspaces and $Sp(n)$ group manifolds can be transformed into each other by performing a generalized conformal rescaling of the hyperfields. We discuss plane wave solutions on generalized AdS spaces and present a generalized conformal (i.e., $Sp(2n)$) transformations of the hyperfields on the $Sp(n)$ group manifolds. In all these considerations, the property of $GL(n)$ flatness plays a crucial role.

Section 6 describes a supersymmetric generalization of the construction considered in Section 2 and Section 7 deals with the supersymmetric generalization of the field theory on $Sp(n)$ introduced in Section 3. The generalization is straightforward but nontrivial. Instead of hyperspace, we consider hyper-superspaces and instead of hyperfields we consider hyper-superfields. The generalized superconformal symmetry is the $OSp(1|2n)$ supergroup and the generalized super- AdS spaces are $OSp(1|n)$ supergroup manifolds. We show that all the characteristic features of the hyperspaces and hyperfield equations are generalized to the supersymmetric case as well.

The direct analogy with usual D -dimensional CFTs suggests a possibility of considering generalized conformal field theories in hyperspaces. Sections 8 and 9 deal with such a theory which is based on the invariance of correlation functions under the generalized conformal group $Sp(2n)$. The technique used in these Sections is borrowed from usual D -dimensional CFTs and the correlation functions are obtained via solving the generalized Ward identities in (super) hyperspaces.

In Section 8, we derive $OSp(1|2n)$ invariant two-, three- and four-point functions for scalar super-hyperfields. The correlation functions for component fields can be obtained by simply expanding the results in series of the powers of Garssmann coordinates. Therefore, we shall not consider the derivation of $Sp(2n)$ invariant correlation functions for the component fields separately.

Finally, in Section 9, we introduce generalized conserved currents and generalized stress-tensors. Their explicit forms and the transformation rules under $Sp(2n)$ can be readily obtained from the free field equations and the transformation rules of the free hyperfields.

Further, we show how one can compute $Sp(2n)$ invariant correlation functions which involve the basic hyperfields together with higher rank tensors such as conserved currents and the generalized stress tensor. We show that the $Sp(2n)$ invariance itself does not impose any restriction on the generalized conformal dimensions of the basic hyperfields even if the conformal dimensions of the current and stress tensor remains canonical.

However, the further requirements of the conservation of the generalized current and generalized stress tensor fixes also the conformal dimensions of the basic hyperfields, implying that the generalized conformal theory will not allow for nontrivial interactions.

We briefly discuss possibilities of avoiding these restrictions by considering spontaneously broken $Sp(2n)$ symmetry or local $Sp(2n)$ invariance, which may lead to an interacting hyperfield theory.

Appendices contain some technical details such as conventions used in the review, a derivation of the field equations on $Sp(n)$ group manifolds and some useful identities.

2. Flat Hyperspace

Let us formulate the basic idea behind the introduction of tensorial space. We shall mainly concentrate on a tensorial extension of four-dimensional Minkowski space–time. A generalization to higher dimensional $D = 6$ and $D = 10$ spaces will be given later in this Section.

Consider a four dimensional massless scalar field. Its light-like momentum $p_m p^m = 0$, $m = 0, 1, 2, 3$ can be expressed via the Cartan–Penrose (twistor) representation as a bilinear combination of a commuting Weyl spinor λ_A and its complex conjugate $\bar{\lambda}_{\dot{A}}$ ($A, \dot{A} = 1, 2$)

$$p^m = \lambda^A (\sigma^m)_{A\dot{A}} \tilde{\lambda}^{\dot{A}}, \quad \text{or} \quad P_{A\dot{A}} = \lambda_A \bar{\lambda}_{\dot{A}}. \quad (1)$$

Obviously, since the spinors are commuting, one has $\lambda^A \lambda^B \varepsilon_{AB} \equiv \lambda^A \lambda_A = 0 = \bar{\lambda}^{\dot{A}} \bar{\lambda}_{\dot{A}}$ and therefore $P^{A\dot{A}} P_{A\dot{A}} = 0$, where the spinor indices are raised and lowered with the unit antisymmetric tensors ε^{AB} and ε_{AB} .

In order to generalize this construction to higher dimensions note that one can equivalently rewrite Equation (1) in terms of four-dimensional real Majorana spinors λ^α ($\alpha = 1, \dots, 4$)

$$p^m = \lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta. \tag{2}$$

Due to the Fierz identities

$$(\gamma^m)_{\alpha\beta} (\gamma_m)_{\gamma\delta} + (\gamma^m)_{\alpha\delta} (\gamma_m)_{\beta\gamma} + (\gamma^m)_{\alpha\gamma} (\gamma_m)_{\delta\beta} = 0 \tag{3}$$

satisfied by the Dirac matrices $(\gamma^m)_{\alpha\beta} = (\gamma^m)_{\beta\alpha}$ one has $p^m p_m = 0$. (The four-component spinor indices are raised and lowered by antisymmetric charge conjugation matrices $C^{\alpha\beta}$ and $C_{\alpha\beta}$ see the Appendix A.) Let us note that since identities similar to (3) hold also in $D = 3, 6$ and 10 , the Cartan–Penrose relation (2) is valid in these dimensions as well.

Let us continue with the four-dimensional case. The momentum $P_{A\dot{A}}$ is canonically conjugate to coordinates $x^{A\dot{A}}$. One can easily solve the quantum analogue of Equation (1)

$$\left(\frac{\partial}{\partial x^{A\dot{A}}} - i\lambda_A \bar{\lambda}_{\dot{A}} \right) \Phi(x, \lambda) = 0 \tag{4}$$

to obtain a plane wave solution for the massless scalar particle

$$\Phi(x, \lambda, \bar{\lambda}) = \phi(\lambda, \bar{\lambda}) e^{ix^{A\dot{A}} \lambda_A \bar{\lambda}_{\dot{A}}}, \tag{5}$$

or in terms of the Majorana spinors

$$\Phi(x, \lambda) = \phi(\lambda) e^{ix_m \lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta}, \tag{6}$$

with $\phi(\lambda)$ being an arbitrary spinor function.

Let us now consider the equation

$$P_{\alpha\beta} = \lambda_\alpha \lambda_\beta, \tag{7}$$

which looks like a straightforward generalization of (1) and see its implications. A space–time described by the coordinates $X^{\alpha\beta}$ (conjugate to $P_{\alpha\beta}$) is now ten-dimensional, since $X^{\alpha\beta}$ is a 4×4 symmetric matrix. A basis of symmetric matrices is formed by the four Dirac matrices $\gamma_{\alpha\beta}^m$ and their six antisymmetric products $\gamma_{\alpha\beta}^{mn} = -\gamma_{\alpha\beta}^{nm}$. In this basis, $X^{\alpha\beta}$ has the following expansion

$$X^{\alpha\beta} = \frac{1}{2} x^m (\gamma_m)^{\alpha\beta} + \frac{1}{4} y^{mn} (\gamma_{mn})^{\alpha\beta}. \tag{8}$$

The analogue of the wave Equation (4) is now

$$\left(\frac{\partial}{\partial X^{\alpha\beta}} - i\lambda_\alpha \lambda_\beta \right) \Phi(X, \lambda) = 0, \tag{9}$$

whose solution is

$$\Phi(X, \lambda) = e^{iX^{\alpha\beta} \lambda_\alpha \lambda_\beta} \phi(\lambda). \tag{10}$$

At this point, one might ask the question what is the meaning of Equation (9) and of the extra coordinates y^{mn} and λ^α ? As we shall see, the answer is that Equation (9) is nothing else but Vasiliev’s unfolded equations for free massless higher-spin fields in four-dimensional Minkowski space–time [29]. The wave function $\Phi(X, \lambda)$ depends on the coordinates x^m , y^{mn} and λ^α . While x^m parameterize the

conventional four-dimensional Minkowski space–time, the coordinates y^{mn} (and/or λ^α) are associated with integer and half-integer spin degrees of freedom of four-dimensional fields with spin values ranging from zero to infinity.

2.1. Higher Spin Content of the Tensorial Space Equations

To demonstrate the above statement let us first Fourier transform the wave function (10) into a conjugate representation with respect to the spinor variable λ_α considered in [4]

$$C(X, \mu) = \int d^4\lambda e^{-i\mu^\alpha \lambda_\alpha} \Phi(X, \lambda) = \int d^4\lambda e^{-i\mu^\alpha \lambda_\alpha + iX^{\alpha\beta} \lambda_\alpha \lambda_\beta} \phi(\lambda). \tag{11}$$

The function $C(X, \mu)$ obeys the equation

$$\left(\frac{\partial}{\partial X^{\alpha\beta}} - i \frac{\partial^2}{\partial \mu^\alpha \partial \mu^\beta} \right) C(X, \mu) = 0. \tag{12}$$

Let us expand the function $C(X, \mu)$ in series of the variables μ^α

$$C(X, \mu) = \sum_{n=0}^{\infty} C_{\alpha_1 \dots \alpha_n}(X) \mu^{\alpha_1} \dots \mu^{\alpha_n} = b(X) + f_\alpha(X) \mu^\alpha + \dots \tag{13}$$

and insert this expansion into the Equation (12). Then one finds that all the components of $C(X, \mu)$ proportional to the higher powers of μ^α are expressed in terms of two fields the scalar $b(X)$ and the spinor $f_\alpha(X)$. As a result of (13), these fields satisfy the relations [4]

$$\partial_{\alpha\beta} \partial_{\gamma\delta} b(X) - \partial_{\alpha\gamma} \partial_{\beta\delta} b(X) = 0, \tag{14}$$

$$\partial_{\alpha\beta} f_\gamma(X) - \partial_{\alpha\gamma} f_\beta(X) = 0. \tag{15}$$

The basic fields $b(X)$ and $f_\alpha(X)$ depend on x^m and y^{mn} . Let us now expand these fields in series of the tensorial coordinates y^{mn}

$$b(x, y) = \phi(x) + y^{m_1 n_1} F_{m_1 n_1}(x) + y^{m_1 n_1} y^{m_2 n_2} \hat{R}_{m_1 n_1, m_2 n_2}(x) + \sum_{s=3}^{\infty} y^{m_1 n_1} \dots y^{m_s n_s} \hat{R}_{m_1 n_1, \dots, m_s n_s}(x), \tag{16}$$

$$f^\alpha(x, y) = \psi^\alpha(x) + y^{m_1 n_1} \hat{\mathcal{R}}_{m_1 n_1}^\alpha(x) + \sum_{s=\frac{5}{2}}^{\infty} y^{m_1 n_1} \dots y^{m_{s-\frac{1}{2}} n_{s-\frac{1}{2}}} \hat{\mathcal{R}}_{m_1 n_1, \dots, m_{s-\frac{1}{2}} n_{s-\frac{1}{2}}}^\alpha(x). \tag{17}$$

Each four-dimensional component field in this expansion is antisymmetric under the permutation of the indices m_i and n_i and is symmetric with respect to the permutation of the pairs (m_i, n_i) with (m_j, n_j) . In order to answer the question about the physical meaning of these fields, let us first consider the scalar field Equation (14). Using the expression (8) for the tensorial coordinates and four-dimensional γ -matrix identities, one can decompose (14) as follows

$$\begin{aligned} \partial_p \partial^p b(x^l, y^{mn}) = 0, \quad \left(\partial_p \partial_q - 4 \partial_{pr} \partial^r_q \right) b(x^l, y^{mn}) = 0, \quad \epsilon^{pqrt} \partial_{pq} \partial_{rs} b(x^l, y^{mn}) = 0, \\ \epsilon^{pqrt} \partial_q \partial_{rt} b(x^l, y^{mn}) = 0, \quad \partial_q^p \partial_p b(x^l, y^{mn}) = 0. \end{aligned} \tag{18}$$

where $\partial_p = \frac{\partial}{\partial x^p}$ and $\partial_{pq} = \frac{\partial}{\partial y^{pq}}$. The meaning of Equations (18) is the following. The first equation is a Klein-Gordon equation. The second equation implies that the trace (with respect to the 4D Minkowski metric) of the tensor which comes with the s -th power of y^{mn} in the expansion (14) is expressed via the second derivative of the tensor which comes with the $(s - 2)$ -th power of y^{mn} . Therefore, traces are not independent degrees of freedom and the independent tensorial fields under consideration are effectively traceless. The third and fourth Equation in (18) imply that the tensor fields satisfy the

four-dimensional Bianchi identities, and the last equation implies that they are co-closed. These are equations for massless higher-spin fields written in terms of their curvatures $\hat{\mathcal{R}}_{m_1 n_1, \dots, m_{s-\frac{1}{2}} n_{s-\frac{1}{2}}}^\alpha(x)$. In four dimensions these equations are conformally invariant. Therefore one can conclude that in the expansion (16) the field $\phi(x)$ is a conformal scalar, $F_{mn}(x)$ is the field strength of spin-1 Maxwell field, the field $\hat{R}_{m_1 n_1, m_2 n_2}(x)$ is a linearized Riemann tensor for spin-2 graviton, etc.

The treatment of Equation (15) which describes half-integer higher-spin fields in terms of corresponding curvatures is completely analogous to the bosonic one (14). The independent equations for the conformal half-integer spin fields are

$$\gamma^p \partial_p f(x^l, y^{mn}) = 0, \tag{19}$$

$$(\partial_p - 2\gamma^r \partial_{pr}) f(x^l, y^{mn}) = 0 \tag{20}$$

From (19)–(20) one can derive the equation

$$\partial_{mn} f(x, y) = \frac{1}{2} \gamma_{[m} \partial_{n]} f(x, y) + \frac{1}{2} (\partial_{mn} + \frac{1}{2} \varepsilon_{mnpq} \partial^{pq} \gamma_5) f(x, y). \tag{21}$$

This equation describes the decomposition of the spinor-tensor $\partial_{mn} f$ into the part which contains the $D = 4$ space-time derivative of f and the “physical” part which is self-dual and gamma-traceless, i.e.,

$$\begin{aligned} \gamma^m (\partial_{mn} + \frac{1}{2} \varepsilon_{mnpq} \partial^{pq} \gamma_5) f(x, y) &= 0 \\ (\partial_{mn} + \frac{1}{2} \varepsilon_{mnpq} \partial^{pq} \gamma_5) f(x^l, y^{mn}) &= \frac{1}{2} \varepsilon_{mnr s} (\partial^{rs} + \frac{1}{2} \varepsilon^{rs pq} \partial^{pq} \gamma_5) f(x, y) \end{aligned} \tag{22}$$

Therefore, one can conclude that due to Equations (19) and (20) the field $\psi^\alpha(x)$ in the expansion (17) is a spin- $\frac{1}{2}$ field, the field $\hat{\mathcal{R}}_{m_1 n_1}^\alpha(x)$ corresponds to the field strength of the spin- $\frac{3}{2}$ Rarita–Schwinger field, while the other fields are the field strengths of the half-integer conformal higher-spin fields in $D = 4$.

Finally, let us define the hyperspaces associated with $D = 6$ and $D = 10$ space-time. The dynamics of the fields will be again determined by the equation (7) with the corresponding hyperspaces and the twistor-like variables λ_α defined as follows.

In $D = 10$ the twistor-like variable λ_α is a 16-component Majorana–Weyl spinor. The gamma-matrices $\gamma_m^{\alpha\beta}$ and $\gamma_{m_1 \dots m_5}^{\alpha\beta}$ form a basis of the symmetric 16×16 matrices, so the $n = 16$ tensorial manifold is parameterized by the coordinates

$$\begin{aligned} X^{\alpha\beta} &= \frac{1}{16} \left(x^m \gamma_m^{\alpha\beta} + \frac{1}{2 \cdot 5!} y^{m_1 \dots m_5} \gamma_{m_1 \dots m_5}^{\alpha\beta} \right) = X^{\beta\alpha}, \\ (m &= 0, 1, \dots, 9; \quad \alpha, \beta = 1, 2, \dots, 16), \end{aligned} \tag{23}$$

where $x^m = X^{\alpha\beta} \gamma_{\alpha\beta}^m$ are associated with the coordinates of the $D = 10$ space-time, while the anti-self-dual coordinates

$$y^{m_1 \dots m_5} = X^{\alpha\beta} \gamma_{\alpha\beta}^{m_1 \dots m_5} = -\frac{1}{5!} \epsilon^{m_1 \dots m_5 n_1 \dots n_5} y_{n_1 \dots n_5},$$

describe spin degrees of freedom.

The corresponding field Equations are again (14) and (15) and the entire discussion repeats as in the case of $D = 4$. The crucial difference is that now the expansions (16) and (17) is performed in terms of the coordinates $y^{m_1 \dots m_5}$. As a result one obtains a description of conformal fields whose curvatures are self-dual with respect to each set of indexes $(m_i n_i p_i q_i r_i)$. These traceless rank 5s tensors $R_{[5]_1 \dots [5]_s}$ are automatically irreducible under $GL(10, \mathbb{R})$ due to the self-duality property, and are thus associated with the rectangular Young diagrams (s, s, s, s, s) which are made of five rows of

equal length s (“multi-five-forms”). The field equations, which are ten-dimensional analogues of the four-dimensional Equations (18), can be found in [13].

In $D = 6$ the commuting spinor λ_α is a symplectic Majorana–Weyl spinor. The spinor index can be decomposed as follows $\alpha = a \otimes i$ ($\alpha = 1, \dots, 8; a = 1, 2, 3, 4; i = 1, 2$). The tensorial space coordinates $X^{\alpha\beta} = X^{ai bj}$ are decomposed into

$$X^{ai bj} = \frac{1}{8} x^m \tilde{\gamma}_m^{ab} \epsilon^{ij} + \frac{1}{16 \cdot 3!} y_I^{mnp} \tilde{\gamma}_{mnp}^{ab} \tau_I^{ij}, \tag{24}$$

$$m, n, p = 0, \dots, 5; \quad a, b = 1, \dots, 4; \quad i, j = 1, 2; \quad I = 1, 2, 3$$

where $\epsilon^{12} = -\epsilon_{12} = 1$, and τ_I^{ij} ($I = 1, 2, 3$) provide a basis of 2×2 symmetric matrices, They are related to the usual $SU(2)$ -group Pauli matrices $\tau_{Iij} = \epsilon_{ijj'} \sigma_{Ii}^{j'}$. The matrices $\tilde{\gamma}_m^{ab}$ (where $\gamma_{ab}^m = 1/2 \epsilon_{abcd} \tilde{\gamma}^{m cd}$) form a complete basis of 4×4 antisymmetric matrices with upper (lower) indices transforming under an (anti)chiral fundamental representation of the non-compact group $SU^*(4) \sim Spin(1, 5)$. For the space of 4×4 symmetric matrices with upper (lower) indices, a basis is provided by the set of self-dual and anti-self-dual matrices $(\tilde{\gamma}^{mnp})^{ab}$ and γ_{ab}^{mnp} , respectively,

$$(\tilde{\gamma}^{mnp})^{ab} = \frac{1}{3!} \epsilon^{mnpqrs} \tilde{\gamma}_{qrs}^{ab}, \quad \gamma_{ab}^{mnp} = -\frac{1}{3!} \epsilon^{mnpqrs} (\gamma_{qrs})_{ab}. \tag{25}$$

The coordinates $x^m = x^{ai bj} \gamma_{ab}^m \epsilon_{ij}$ are associated with $D = 6$ space–time, while the self-dual coordinates

$$y_I^{mnp} = x^{ai bj} \gamma_{ab}^{mnp} \tau_{Iij} = -\frac{1}{3!} \epsilon^{mnpqrs} y_{qrs}^I, \tag{26}$$

describe spinning degrees of freedom.

The consideration proceeds as in the $D = 4$ and $D = 10$ case. Because of the form of the tensorial coordinates in (24) the six-dimensional analogue of the expansions (16) and (17) contains powers of y_i^{mnp} . Corresponding field strengths, which again describe conformal fields in six dimensions, are self-dual with respect to each set of the indexes $(m_i n_i p_i)$. In other words, one has an infinite number of conformally invariant (self-dual) “multi-3-form” higher-spin fields in the six-dimensional space–time which form the $(2[s] + 1)$ -dimensional representations of the group $SO(3)$.

In [9,16,21] Equation (12) has been generalized to include several commuting spinor variables $\mu^{p\alpha}$ ($p, q = 1, \dots, r$)

$$\left(\frac{\partial}{\partial X^{\alpha\beta}} \pm i \eta^{pq} \frac{\partial^2}{\partial \mu^{p\alpha} \partial \mu^{q\beta}} \right) C_{\pm}(X, \mu) = 0. \tag{27}$$

where $\eta^{pq} = \eta^{qp}$ is a nondegenerate metric. The value of r is called the “rank”. As we explained above, the free higher-spin fields in $D = 4$ are described by the rank-one equations in the ten-dimensional tensorial space. The higher-spin currents are fields of rank-two $r = 2$. These currents obey the equations with off-diagonal η^{pq} [19]. The currents $J(X, \mu^p)$ are bilinear in the higher-spin gauge fields \mathcal{C}_+ and \mathcal{C}_- , which obey the rank-one equation (27) $J = \mathcal{C}_+ \mathcal{C}_-$.

On the other hand, when considering rank-two equations the corresponding tensorial space can be embedded in the higher-dimensional tensorial space. From the discussion above, it follows that a natural candidate for such higher-dimensional space is the tensorial extension of $D = 6$ space–time. In this way one effectively linearizes the problem since the conformal currents in four dimensions are identified with the fields in $D = 6$ [21].

2.2. Four Dimensional Unfolded Higher-Spin Field Equations from the Hyperspace Field Equations

Let us rewrite, in the case of the $D = 4$ theory, the hyperspace relations in terms of the Weyl spinors. The momenta (7) take the form

$$P_{AB} = \lambda_A \lambda_B, \quad \bar{P}_{\dot{A}\dot{B}} = \bar{\lambda}_{\dot{A}} \bar{\lambda}_{\dot{B}}, \quad P_{A\dot{A}} = \lambda_A \bar{\lambda}_{\dot{A}}, \tag{28}$$

while Equation (7) splits into

$$\left(\sigma_{AB}^{mn} \frac{\partial}{\partial y^{mn}} + i \frac{\partial^2}{\partial \mu^A \partial \mu^B} \right) C(x, y, \mu) = 0, \tag{29}$$

$$\left(\bar{\sigma}_{\dot{A}\dot{B}}^{mn} \frac{\partial}{\partial y^{mn}} - i \frac{\partial^2}{\partial \bar{\mu}^{\dot{A}} \partial \bar{\mu}^{\dot{B}}} \right) C(x, y, \mu) = 0$$

and

$$\left(\sigma_{A\dot{A}}^m \frac{\partial}{\partial x^m} + i \frac{\partial^2}{\partial \mu^A \partial \bar{\mu}^{\dot{A}}} \right) C(x, y, \mu) = 0. \tag{30}$$

Equations (29) relate the dependence of $C(x, y, \mu)$ on the coordinates y^{mn} to its dependence on μ^α . Thus, using this relation, one can regard the wave function $C(x^m, \mu^\alpha) := C(X^{\alpha\beta}, \mu^\alpha)|_{y^{mn}=0}$ as the fundamental field.

The expansion of $C(x^m, \mu)$ in series of μ^A and $\bar{\mu}^{\dot{A}}$ is

$$C(x^p, \mu^A, \bar{\mu}^{\dot{A}}) = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} C_{A_1 \dots A_m, \dot{B}_1 \dots \dot{B}_n}(x^p) \mu^{A_1} \dots \mu^{A_m} \bar{\mu}^{\dot{B}_1} \dots \bar{\mu}^{\dot{B}_n}, \tag{31}$$

where the reality of the wave function implies $(C_{A_1 \dots A_m, \dot{B}_1 \dots \dot{B}_n})^* = C_{B_1 \dots B_n, \dot{A}_1 \dots \dot{A}_m}$, and by construction the spin-tensors are symmetric in the indices A_i and in \dot{B}_i .

The consistency of (30) implies the integrability conditions

$$\frac{\partial^2}{\partial \mu^{[A} \partial x^{B] \dot{B}}} C(x, \mu) = 0, \quad \frac{\partial^2}{\partial \bar{\mu}^{[\dot{A}} \partial x^{\dot{B}] B}} C(x, \mu) = 0. \tag{32}$$

We have thus obtained the equations of the Vasiliev’s unfolded formulation of free higher spin fields in terms of zero-forms. In this formulation the $C_{0,0}$ component (a physical scalar), $C_{A_1 \dots A_{2s},0}$ and $C_{0,\dot{A}_1, \dots, \dot{A}_{2s}}$ components of the expansion (31) correspond to the physical fields, while the other fields are auxiliary. The latter two fields are the self-dual and anti-self-dual components of the spin- s field strength. The nontrivial equations on the dynamical fields are [38] the Klein–Gordon equation for the spin zero scalar field $\partial^m \partial_m C_{0,0} = 0$ and the massless equations for spin $s > 0$ field strengths

$$\partial^{B\dot{B}} C_{BA_1 \dots A_{2s-1}}(x) = 0, \quad \partial^{B\dot{B}} C_{\dot{B}\dot{A}_1 \dots \dot{A}_{2s-1}}(x) = 0, \tag{33}$$

which follow from (32). All the components of $C(x^m, \mu^A, \bar{\mu}^{\dot{A}})$ that depend on both μ^A and $\bar{\mu}^{\dot{A}}$ are auxiliary fields expressed by (30) in terms of space–time derivatives of the dynamical fields contained in the analytic fields $C(x^m, \mu^A, 0)$ and $C(x^m, 0, \mu^{\dot{A}})$ and thus one arrives at the unfolded formulation of [38].

Let us summarize what we have considered by now. To describe the dynamics of higher-spin fields in four dimensions we have introduced extended ten-dimensional tensorial space, hyperspace, parameterized by the coordinates $X^{\alpha\beta}$ (8). The main object is a generating functional for higher-spin fields described by $C(X, \mu)$ or by $\Phi(X, \lambda)$. The generating functional depends on the tensorial coordinates $X^{\alpha\beta}$ and on the commuting spinors μ^α or λ^α . The dynamics is described by the field Equation (9) or (12). To obtain from these the higher-spin field equations in the ordinary space–time parameterized by the coordinates x^m one can use two options. In the first approach one gets rid of the tensorial coordinates y^{mn} and arrives at Vasiliev’s unfolded formulation in terms of the functional (31). Alternatively, one can first get rid of the commuting spinor variables and arrive at the equations for the bosonic (16) and fermionic (17) hyperfields. Both pictures provide the equations for the field strengths

of the higher-spin potentials, the difference being that these field strengths are realized either as tensors or spin-tensors.

2.3. Generalized Conformal Group $Sp(2n)$

Let us consider in more detail the symmetries of Equation (7) in which now the Greek indices α, β, \dots run from 1 to an arbitrary even integer $2n$. However, as we explained in the previous Section, the physically interesting cases are associated with $n = 2, 4, 8$, and 16 , which correspond to the number of space–time dimensions equal to $3, 4, 6$ and 10 , respectively.

It turns out that Equation (7) is invariant under the transformations of the $Sp(2n)$ group [5,8]

$$\delta\lambda_\alpha = g_\alpha^\beta \lambda_\beta - k_{\alpha\beta} X^{\beta\gamma} \lambda_\gamma, \tag{34}$$

$$\delta X^{\mu\nu} = a^{\mu\nu} + (X^{\mu\rho} g_\rho^\nu + X^{\nu\rho} g_\rho^\mu) - X^{\mu\rho} k_{\rho\lambda} X^{\lambda\nu}. \tag{35}$$

The constant parameters $a^{\alpha\beta} = a^{\beta\alpha}$, g_γ^α and $k_{\alpha\beta} = k_{\beta\alpha}$ correspond to the generators of generalized translations $P_{\alpha\beta}$, generalized Lorentz transformations and dilatations G_β^α (generated by the $GL(n)$ algebra) and generalized conformal boosts $K_{\alpha\beta}$. The differential operator representation of these generators have the form

$$P_{\mu\nu} = -i \frac{\partial}{\partial X^{\mu\nu}} \equiv -i \partial_{\mu\nu}, \tag{36}$$

$$G_\nu^\mu = -2i X^{\mu\rho} \partial_{\rho\nu} \tag{37}$$

and

$$K^{\mu\nu} = i X^{\mu\rho} X^{\nu\lambda} \partial_{\rho\lambda} \tag{38}$$

These symmetries are the hyperspace counterparts of the conventional Poincaré translations, Lorentz rotations, dilatations and conformal boosts of Minkowski space–time. The generalized Lorentz rotations are generated by the traceless operators $L_\mu^\nu = G_\mu^\nu - \frac{1}{n} \delta_\mu^\nu G_\lambda^\lambda$, forming the $SL(n)$ –algebra, whereas dilatations are generated by the trace of G_μ^ν . The generators (36), (37) and (38) form the $Sp(2n)$ algebra which plays the role of a generalized conformal symmetry in the hyperspace

$$\begin{aligned} [P_{\mu\nu}, P_{\rho\lambda}] &= 0, & [K^{\mu\nu}, K^{\rho\lambda}] &= 0, & [G_\nu^\mu, G_\lambda^\rho] &= i(\delta_\lambda^\mu G_\nu^\rho - \delta_\nu^\rho G_\lambda^\mu), \\ [P_{\mu\nu}, G_\lambda^\rho] &= -i(\delta_\mu^\rho P_{\nu\lambda} + \delta_\nu^\rho P_{\mu\lambda}), & [K^{\mu\nu}, G_\lambda^\rho] &= i(\delta_\lambda^\mu K^{\nu\rho} + \delta_\lambda^\nu K^{\mu\rho}), \\ [P_{\mu\nu}, K^{\lambda\rho}] &= \frac{i}{4}(\delta_\mu^\rho G_\nu^\lambda + \delta_\nu^\rho G_\mu^\lambda + \delta_\mu^\lambda G_\nu^\rho + \delta_\nu^\lambda G_\mu^\rho). \end{aligned} \tag{39}$$

From the structure of this algebra, one can see that the flat hyperspace \mathcal{M}_n can be realized as a coset manifold associated with the translations $P = \frac{Sp(2n)}{K \otimes GL(n)}$ where $K \otimes GL(n)$ is the semi–direct product of the Abelian group generated by the generalized conformal boosts $K_{\mu\nu}$ and the general linear group.

The generators of the translations, Lorentz rotations and conformal boosts of the conventional conformal group can be obtained from the $Sp(2n)$ generators as projections onto the x –space, for example $p_m = (\gamma_m)^{\mu\nu} P_{\mu\nu}$, etc.

Let us note that the $Sp(2n)$ algebra can be conveniently realized with the use of the twistor-like variables λ_α and their conjugate μ^α

$$[\mu^\alpha, \lambda_\beta] = \delta_\beta^\alpha. \tag{40}$$

In the twistor representation the generators of the $Sp(2n)$ group have the following form

$$P_{\alpha\beta} = \lambda_\alpha \lambda_\beta, \quad G_\alpha^\beta = \lambda_\alpha \mu^\beta, \quad K_{\alpha\beta} = \mu_\alpha \mu_\beta. \tag{41}$$

Equations (14) and (15) are invariant under the $Sp(2n)$ transformations (35), provided that the fields transform as follows

$$\delta b(X) = -(a^{\mu\nu}\partial_{\mu\nu} + \frac{1}{2}g_{\mu}^{\mu} + 2g_{\nu}^{\mu}X^{\nu\rho}\partial_{\mu\rho} - k_{\mu\nu}(\frac{1}{2}X^{\mu\nu} + X^{\mu\rho}X^{\nu\lambda}\partial_{\rho\lambda}))b(X), \tag{42}$$

$$\begin{aligned} \delta f_{\rho}(X) = & -(a^{\mu\nu}\partial_{\mu\nu} + \frac{1}{2}g_{\mu}^{\mu} + 2g_{\nu}^{\mu}X^{\nu\lambda}\partial_{\mu\lambda} - k_{\mu\nu}(\frac{1}{2}X^{\mu\nu} + X^{\mu\tau}X^{\nu\lambda}\partial_{\tau\lambda}))f_{\rho}(X) + \\ & -(g_{\rho}^{\nu} - k_{\lambda\rho}X^{\lambda\nu})f_{\nu}(X). \end{aligned} \tag{43}$$

Note that these variations contain the term $\frac{1}{2}(g_{\mu}^{\mu} - k_{\mu\nu}X^{\mu\nu})$, implying that the fields have the canonical conformal weight 1/2. A natural generalization of these transformations to fields of a generic conformal weight Δ is [4]

$$\delta b(X) = -(a^{\mu\nu}\partial_{\mu\nu} + \Delta(g_{\mu}^{\mu} - k_{\mu\nu}X^{\mu\nu}) + 2g_{\nu}^{\mu}X^{\nu\rho}\partial_{\mu\rho} - k_{\mu\nu}X^{\mu\rho}X^{\nu\lambda}\partial_{\rho\lambda})b(X), \tag{44}$$

$$\begin{aligned} \delta f_{\rho}(X) = & -(a^{\mu\nu}\partial_{\mu\nu} + \Delta(g_{\mu}^{\mu} - k_{\mu\nu}X^{\mu\nu}) + 2g_{\nu}^{\mu}X^{\nu\lambda}\partial_{\mu\lambda} - k_{\mu\nu}X^{\mu\tau}X^{\nu\lambda}\partial_{\tau\lambda})f_{\rho}(X) \\ & -(g_{\rho}^{\nu} - k_{\lambda\rho}X^{\lambda\nu})f_{\nu}(X). \end{aligned} \tag{45}$$

3. Hyperspace Extension of AdS Spaces

A hyperspace extension of AdS_D spaces is another coset of the $Sp(2n)$ group. Recall that the usual AdS_D space can be realized as the coset space (Here, K and \mathbb{D} denote the generalized conformal boosts and dilatation, respectively.) $\frac{SO(2,D)}{K \otimes (SO(1,D-1) \times \mathbb{D})}$ parameterized by the coset element $e^{\mathcal{P}_m x^m}$. The generators of the AdS_D boosts \mathcal{P}_m can be singled out from the generators of the four dimensional conformal group $SO(2, D)$ by taking a linear combination of the generators of the Poincaré translations P_m and conformal boosts K_m as $\mathcal{P}_m = P_m - \xi^2 K_m$, where ξ is the inverse of the AdS_D radius.

Analogously, for the case of the hyperspace extension of the AdS_D space let us consider the generators

$$\mathcal{P}_{\alpha\beta} = P_{\alpha\beta} - \frac{\xi^2}{16}K_{\alpha\beta}, \quad [\mathcal{P}, \mathcal{P}] \sim M, \quad [\mathcal{P}, M] \sim \mathcal{P}, \tag{46}$$

where $K_{\alpha\beta} = C_{\alpha\gamma}C_{\beta\delta}K^{\gamma\delta}$, $M_{\alpha\beta}$ stands for the symmetric part of the $GL(n)$ transformations $M_{\alpha\beta} = G_{(\alpha}^{\gamma}C_{\gamma\beta)} \equiv \frac{1}{2}(G_{\alpha}^{\gamma}C_{\gamma\beta} + G_{\beta}^{\gamma}C_{\gamma\alpha})$ and $C_{\alpha\beta} = -C_{\beta\alpha}$ is the $Sp(n)$ -invariant symplectic metric. One can see that the corresponding manifold is an $Sp(n)$ group manifold [8] which can be realized as a coset space $\frac{Sp(2n)}{K \otimes GL(n)}$ with the coset element $e^{(P - \frac{\xi^2}{16}K)_{\alpha\beta} X^{\alpha\beta}}$. Indeed, let us recall that $Sp(n)$ group is generated by $n \times n$ symmetric matrices $M_{\alpha\beta}$ which form the algebra

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -\frac{i\xi}{2} [C_{\gamma(\alpha}M_{\beta)\delta} + C_{\delta(\alpha}M_{\beta)\gamma}], \quad \alpha, \beta = 1, \dots, n. \tag{47}$$

As a group manifold, $Sp(n)$ is the coset $[Sp(n)_L \times Sp(n)_R]/Sp(n)$ which has the isometry group $Sp(n)_L \times Sp(n)_R$, the latter being the subgroup of $Sp(2n)$ generated by

$$M_{\alpha\beta}^L = P_{\alpha\beta} - \frac{\xi^2}{16}K_{\alpha\beta} - \frac{\xi}{4}M_{\alpha\beta} \quad M_{\alpha\beta}^R = P_{\alpha\beta} - \frac{\xi^2}{16}K_{\alpha\beta} + \frac{\xi}{4}M_{\alpha\beta}, \tag{48}$$

as one may see from the structure of the $Sp(2n)$ algebra (39). The generators $M_{\alpha\beta}$ form the diagonal $Sp(n)$ subalgebra of $Sp(n)_L \times Sp(n)_R$.

Let us note that, for the case of $n = 4$, i.e., for the case of four space–time dimensions, AdS_4 space is a coset subspace of $Sp(4) \sim SO(2, 3)$ of the maximal dimension. For $n > 4$, an AdS_D space is also a subspace of $Sp(n)$ manifold but is no longer the maximal coset of this group.

3.1. GL-Flatness of Sp(n) Group Manifolds

Let us describe a property of GL-flatness of the Sp(n) group manifolds which is a generalization of the conformal flatness property of AdS_D spaces. By GL-flatness we mean that, in a local coordinate basis associated with X^{αβ}, the corresponding Sp(n) Cartan form Ω^{αβ} has the form

$$\Omega^{\alpha\beta} = dX^{\mu\nu} G_{\mu}^{\alpha}(X) G_{\nu}^{\beta}(X), \tag{49}$$

with the matrix G_{μ^α}(X) being

$$G_{\mu}^{\alpha}(X) = \delta_{\mu}^{\alpha} + \sum_{k=1}^{\infty} \left(-\frac{\xi}{4}\right)^k (X^k)_{\mu}^{\alpha}. \tag{50}$$

This expression implies that the Sp(n) Cartan form is obtained from the flat differential dX^{μν} by a specific GL(n) rotation of the latter.

This property can be demonstrated by showing that the Cartan forms (49) satisfy the Sp(n)-group Maurer–Cartan equations (see [8,23], for technical details)

$$d\Omega^{\alpha\beta} + \frac{\xi}{2} \Omega^{\alpha\gamma} \wedge \Omega_{\gamma}^{\beta} = 0. \tag{51}$$

The matrix G_{α^{-1μ}}(X) inverse to (50) depends linearly on X_{α^μ} and has a very simple form

$$G_{\alpha}^{-1\mu}(X) = \delta_{\alpha}^{\mu} + \frac{\xi}{4} X_{\alpha}^{\mu}. \tag{52}$$

Note that the possibility of representing the Cartan forms in the form (49) is a particular feature of the Sp(n) group manifold since, in general, it is not possible to decompose the components of the Cartan form into a “direct product” of components of some matrix G_{μ^α}.

3.2. An Explicit Form of the AdS₄ Metric

Let us now demonstrate that, for the case of n = 4 (D = 4), the pure x^m-dependent part of the matrix G_{μ^α}(X) indeed generates the metric on AdS₄ in a specific parameterization. To this end, we should evaluate the expression

$$\Omega^{\alpha\beta}(x^m) = \frac{1}{2} dx^m (\gamma_m)^{\delta\sigma} G_{\delta}^{\alpha} G_{\sigma}^{\beta} = \frac{1}{2} dx^m e_m^a (\gamma_a)^{\alpha\beta} + \frac{1}{4} dx^m \omega_m^{ab} (\gamma_{ab})^{\alpha\beta}, \tag{53}$$

where the dependence of the matrices X^{αβ} on the coordinates y^{mⁿ} (see Equation (8)) was discarded, i.e., X_{α^β} = ½ xⁿ (γ_n)_{α^β}. Denoting

$$x^2 = x^m x^n \eta_{mn}, \quad x_m = \eta_{mn} x^n \tag{54}$$

and, using the explicit form (50) of G_{μ^α}(X), one obtains

$$\Omega^{\alpha\beta}(x) = \frac{1}{2} \frac{dx^m}{[1 - (\frac{\xi}{8})^2 x^2]^2} \left[(\gamma_{\ell})^{\alpha\beta} \left([1 + (\frac{\xi}{8})^2 x^2] \delta_m^{\ell} - 2(\frac{\xi}{8})^2 \eta_{mn} x^n x^{\ell} \right) - \frac{\xi}{4} x^n (\gamma_{mn})^{\alpha\beta} \right]. \tag{55}$$

In this way, we obtain a four-dimensional space vierbein and spin-connection

$$e_m^a = \frac{1}{[1 - (\frac{\xi}{8})^2 x^2]^2} \left([1 + (\frac{\xi}{8})^2 x^2] \delta_m^a - 2(\frac{\xi}{8})^2 x^a x_m \right), \tag{56}$$

$$\omega_m^{ab} = \frac{-2\xi}{[1 - (\frac{\xi}{8})^2 x^2]^2} \delta_m^{[a} x^{b]} = -\frac{8(\frac{\xi}{8})}{(1 - (\frac{\xi}{8})^2 x^2)^2} (x^a \delta_m^b - x^b \delta_m^a). \tag{57}$$

The corresponding metric is

$$g_{mn} = \frac{1}{[1 - (\frac{\zeta}{8})^2 x^2]^4} \left([1 + (\frac{\zeta}{8})^2 x^2]^2 \eta_{mn} - 4(\frac{\zeta}{8})^2 x_m x_n \right), \tag{58}$$

It is well-known (see also Section 5.1) that the metric on AdS_D can be represented as an embedding in a flat $(D + 1)$ -dimensional space

$$ds^2 = \eta_{mn} dy^m dy^n - (dy^D)^2, \tag{59}$$

via the embedding constraint

$$\eta_{mn} y^m y^n - (y^D)^2 = -r^2. \tag{60}$$

Choosing the embedding coordinates for AdS_4 to be

$$y^m = \frac{1 + (\frac{\zeta}{8})^2 x^2}{[1 - (\frac{\zeta}{8})^2 x^2]^2} x^m, \quad y^4 = \sqrt{r^2 + x^2 \frac{1 + (\frac{\zeta}{8})^2 x^2}{[1 - (\frac{\zeta}{8})^2 x^2]^2}}, \tag{61}$$

one readily recovers the metric (58), with the parameter ζ being related to the AdS_4 radius r as follows

$$\zeta = \frac{2}{r}. \tag{62}$$

Finally, computing the Riemann tensor

$$R^{ab}{}_{mn} = -32(\frac{\zeta}{8})^2 \frac{1 + (\frac{\zeta}{8})^2 x^2}{[1 - (\frac{\zeta}{8})^2 x^2]^4} \left([1 + (\frac{\zeta}{8})^2 x^2] \delta_m^{[a} \delta_n^{b]} + 4(\frac{\zeta}{8})^2 x^{[a} \delta_{[m}^{b]} x_n] \right), \tag{63}$$

and the Ricci scalar

$$R = -192 \left(\frac{\zeta}{8} \right)^2 = -3\zeta^2, \tag{64}$$

one verifies that the metric (58) indeed corresponds to a space with constant negative curvature, i.e., the AdS_4 space.

4. Particles in Hyperspaces

In this Section, we would like to explain the physical meaning of the tensorial space coordinates as spin degrees of freedom from the perspective of the dynamics of a particle in hyperspace.

Historically, the first dynamical system in which the Fronsdal hyperspace proposal for higher-spin fields was realized explicitly was the twistor-like superparticle model of Bandos and Lukierski [2] which, for $D = 4$, possesses the generalized superconformal symmetry under $OSp(1|8)$. The original motivation behind this model was a geometric interpretation of commuting tensorial charges in an extended supersymmetry algebra. Its higher-spin content was found later in [3,64] where the quantum states of the superparticle were shown to form an infinite tower of massless higher-spin fields, and the relation of this model to the unfolded formulation was assumed. This relation was analyzed in detail in [4,5,8,10,13]. In addition to the relation to higher spins, the model of Bandos and Lukierski [2] has revealed other interesting features, such as the invariance under supersymmetry with tensorial charges (which are usually associated with brane solutions of Superstring and M-Theory). Moreover, it has provided the first example of a dynamical BPS system preserving more than half of the bulk supersymmetries. BPS states preserving $\frac{2n-1}{2n}$ supersymmetries

(with $n = 16$ for $D = 10, 11$) were then shown to be building blocks of any BPS states, and this led to a natural conjecture that they can be elementary constituents or “preons” of M–theory [65].

Let us consider the generic case of a particle moving in an $Sp(2n)$ -invariant hyperspace \mathcal{M} described by the action

$$S[X, \lambda] = \int E^{\alpha\beta}(X(\tau)) \lambda_\alpha(\tau) \lambda_\beta(\tau), \tag{65}$$

where $X^{\mu\nu}(\tau)$ are the hyperspace coordinates of the particle. The auxiliary commuting variables $\lambda_\alpha(\tau)$ ($\alpha = 1, \dots, n$) is a real spinor with respect to $Sp(n)$ and a vector with respect to $GL(n)$ (introduced in Section 2). Finally $E^{\alpha\beta}(X(\tau)) = E^{\beta\alpha}(X(\tau)) = dX^{\lambda\rho}(\tau) E_{\alpha\beta}^{\mu\nu}(X)$ is the pull-back on the particle worldline of the hyperspace vielbein. For flat hyperspace

$$E^{\alpha\beta}(X(\tau)) = d\tau \partial_\tau X^{\alpha\beta}(\tau) = dX^{\alpha\beta}(\tau), \tag{66}$$

and for the case of the $Sp(n)$ group manifold

$$E^{\alpha\beta}(X(\tau)) = \Omega^{\alpha\beta}(X), \tag{67}$$

where $\Omega^{\alpha\beta}$ is an $Sp(n)$ Cartan form. The latter can be taken in the GL -flat realization as in (49). The dynamics of particles on the $OSp(N|n)$ supergroup manifolds was considered for $N = 1$ in [8,10,66] and for generic values of N in [4,5], and, as we have already mentioned, the twistor-like superparticle in the $n = 32$ super-hyperspace was considered in [67] as a point-like model for BPS preons [65], the hypothetical $\frac{31}{32}$ -supersymmetric constituents of M-theory.

The action (65) is manifestly invariant under global $GL(n)$ transformations and *implicitly* invariant under global $Sp(2n)$ transformations, acting linearly on λ_ρ and non-linearly on $X^{\rho\nu}$. Thus, the model possesses the symmetry that Fronsdaal proposed as an underlying symmetry of higher-spin field theory in the case $n = 4, D = 4$ [1]. To make the $Sp(2n)$ invariance manifest, it is convenient to rewrite the action (65) in a twistor form (for simplicity we consider the flat case (66))

$$S[\lambda, \mu] = \int (d\mu^\alpha(\tau) \lambda_\alpha(\tau) - \mu^\alpha(\tau) d\lambda_\alpha(\tau)) = \int dZ^A Z_A, \tag{68}$$

where

$$\mu^\alpha = X^{\alpha\beta} \lambda_\beta, \tag{69}$$

and

$$Z_A = (\lambda_\alpha, \mu^\beta) \quad Z^A = C^{AB} Z_B = (\mu^\alpha, -\lambda_\beta), \quad A = 1, \dots, 2n, \tag{70}$$

form a linear representation of $Sp(2n)$

$$\delta Z_A = S_A^B Z_B, \quad S_A^B = \begin{pmatrix} g_\alpha^\beta & k_{\alpha\gamma} \\ a^{\delta\beta} & -(g_\gamma^\delta)^T \end{pmatrix}. \tag{71}$$

Hence, the bilinear form $dZ^A Z_A$ is manifestly $Sp(2n)$ invariant. Note that, as it follows from the action (68), the variables μ^α and λ_β are canonically conjugate coordinates and momenta of the particle. Upon quantization, they become the operators introduced in Section 2.3, Equation (40).

Using the relation (69) one can easily recover the $Sp(2n)$ transformation (35) of $X^{\alpha\beta}$.

Applying the Hamiltonian analysis to the particle model described by (65) and (66), one finds that the momentum conjugate to $X^{\alpha\beta}$ is related to the twistor-like variable λ_α via the constraint

$$P_{\alpha\beta} = \lambda_\alpha \lambda_\beta. \tag{72}$$

As we have already mentioned, this expression, e.g., in the case $n = 4$ for which $X^{\alpha\beta}$ is given in (8), is the direct analog and the generalization of the Cartan–Penrose (twistor) relation for the particle

momentum $P_m = \bar{\lambda} \gamma_m \lambda$. A difference is that in $D = 4$ the Penrose twistor relation is invariant under the phase transformation

$$\lambda_\alpha \rightarrow e^{i\varphi \gamma^5} \lambda_\alpha, \tag{73}$$

or in the two-component Weyl spinor notation $\lambda_A \rightarrow e^{i\varphi} \lambda_A$, while Equation (72) does not possess this symmetry. rather the symmetry of the model is $\mathbb{Z}_2 (\lambda_\alpha \rightarrow -\lambda_\alpha)$ subgroup of $U(1)$ and as a result in the model under consideration the phase component φ of λ_α is a dynamical degree of freedom. It turns out that upon quantization it is associated with the infinite number of massless quantum states (particles) with increasing spin (helicity). This is in contrast to the conventional twistor-like (super) particle models with a finite number of quantum states, considered e.g., in [68–79].

To understand the physical meaning of the phase φ , let us notice that Equation (72) is a constraint on possible values of the canonical momenta of the particle in the hyperspace. In the case $n = 4$ the Majorana spinor λ_α has four independent components. One of these components can be associated with the phase φ . The momentum $P_m = \bar{\lambda} \gamma_m \lambda$ of the particle along the four conventional Minkowski directions $x^m = \frac{1}{2} X^{\mu\nu} \gamma_{\mu\nu}^m$ of the hyperspace (8) is light-like. Therefore, P_m depends on three components of λ_α . It does not depend on the phase φ of λ_α , since it is invariant under the phase transformation (73). The momentum $P_{mn} = \bar{\lambda} \gamma_{mn} \lambda$ of the particle along the six additional tensorial directions $y^{mn} = \frac{1}{4} X^{\alpha\beta} \gamma_{\alpha\beta}^{mn}$ is not invariant under the phase transformations and, hence, depends on the four components of λ_α . However, we have already associated three of them with the light-like momentum P_m in $D = 4$. Therefore, the only independent $D = 4$ component of the momentum P_{mn} is associated with the $U(1)$ phase φ of λ_α , and as a result the motion of the particle along the six tensorial directions y^{mn} is highly constrained. This means that, effectively, the particle moves in the four-dimensional Minkowski space and along a single direction in the six additional dimensions whose coordinate is conjugate to the compact momentum–space direction parameterized by the periodic phase φ . As shown in [3,64], the coordinate conjugate to the compactified momentum φ takes, upon quantization, an infinite set of integer and half-integer values associated with the helicities of higher-spin fields. The half-integer and integer-spin states are distinguished by the discrete symmetry $\mathbb{Z}_2 (\lambda_\alpha \rightarrow -\lambda_\alpha)$.

The resulting infinite tower of discrete higher-spin states can be regarded [3,64] as an alternative to the Kaluza–Klein compactification mechanism akin to Fronsdal’s original proposal. In contrast to the conventional Kaluza–Klein theory, in the hyperspace particle model, the compactification occurs in momentum space and not in coordinate space. The phase φ in (73) can be regarded as a compactified component of the momentum (72), while the corresponding conjugate hyperspace coordinate is quantized and labels the discrete values of spin of fields in the effective conventional space–time.

As we have already seen by virtue of the Fierz identity (3) the twistor particle momentum is light-like ($P^m P_m = 0$) in $D = 3, 4, 6$ and 10. Therefore, in the hyperspaces corresponding to these space–time dimensions the first-quantized particles are massless [2,3,64]. Moreover, since the model is invariant under the generalized conformal group $Sp(2n)$, the quantum states of this particle in the hyperspaces containing the $D = 3, 4, 6$ and 10 Minkowski spaces as subspaces correspond to the conformal higher-spin fields introduced in Section 2.

Let us conclude this section with a brief comment on the model describing a particle propagating on the $Sp(n)$ group manifold. Its action has the form (65), with the corresponding Cartan form given by (67). The property of GL -flatness greatly simplifies the analysis of this case. Namely, since the Cartan forms of the $Sp(n)$ group manifold and the flat hyperspace are related as in Equation (49), one can simply reduce the classical $Sp(n)$ action to the flat one by redefining the spinor variables as follows $\lambda_\alpha \rightarrow G_\alpha^{-1\beta}(X)\lambda_\beta$. However, when quantizing this system we should work with variables that appropriately describe the geometry of the $Sp(n)$ background in which the particle propagates. Thus upon quantization one gets Equation (92) as explained in detail in [10].

5. Field Equations on $Sp(n)$ Group Manifold

5.1. Scalar Field on AdS_D . A Reminder

Before deriving the field equations of hyperfields on $Sp(n)$ group manifolds, let us recollect some well known facts about a scalar field propagating on AdS_D background. In the next subsection we will see that the form of the scalar field equation on $Sp(n)$ and its certain solutions are somewhat similar to those of the AdS scalar.

Conformally invariant scalar on AdS_4 is described by the field equation [80]

$$\left(D^m D_m + \frac{2}{r^2}\right) \phi(x) = 0, \tag{74}$$

here D_m is the usual covariant derivative on AdS_4 .

Equation (74) can be written in a so-called ambient space formalism. The ambient space is obtained by introducing one more time-like dimension and considering AdS_D as a hyperboloid in this higher dimensional space (for applications of this formalism to the description of higher-spin fields on AdS_D see for example [81–87])

$$\eta_{AB} y^A y^B = -r^2, \quad \eta_{AB} = \text{diag}(-1, 1, \dots, 1, -1), \quad A = 0, 1, \dots, D. \tag{75}$$

The AdS_D ambient-space generalization of (74) has the form

$$\left(\nabla^A \nabla_A + \frac{2(D-3)}{r^2}\right) \phi(y) = 0, \tag{76}$$

where

$$\nabla^A = \theta^{AB} \frac{\partial}{\partial y^B} \tag{77}$$

and

$$\theta^{AB} = \eta^{AB} + \frac{y^A y^B}{r^2} \tag{78}$$

is a projector, since in view of the relation (75) one has

$$\theta^{AB} \theta^{BC} = \theta^{AC}, \quad y^A \theta_A^B = 0, \quad y^A \nabla_A = 0, \quad \nabla^A y_A = D, \tag{79}$$

where the indexes A, B are raised and lowered with the metric η^{AB} and η_{AB} .

One also has the following identities

$$[\nabla_A, \nabla_B] = -y_A \nabla_B + y_B \nabla_A, \quad [\nabla^C \nabla_C, y^A] = 2\nabla^A + D y^A, \tag{80}$$

$$[\nabla^C \nabla_C, \nabla^A] = (2 - D) \nabla^A + 2y^A \nabla^D \nabla_D$$

where we have set $r^2 = 1$. The generators of the $SO(2, D - 1)$ group can be expressed as

$$M^{AB} = y^A \nabla^B - y^B \nabla^A. \tag{81}$$

One can check that the generators (81) can also be represented as

$$M_{AB} = y_A \partial_B - y_B \partial_A, \quad \partial_A = \frac{\partial}{\partial y^A}. \tag{82}$$

To form the $SO(2, D)$ conformal algebra we need extra generators. These generators are

$$M_{(D+1)A} = \partial_A + y_A y^B \partial_B + l y_A \tag{83}$$

Here l is the conformal weight of a field. For the scalar $l = 1$.

One can derive (83) as follows. Obviously (75) is invariant under the $SO(2, D - 1)$ rotations. In order to realize the conformal transformations in the ambient space one adds to it one more dimension i.e., considers $D + 2$ dimensional space, parameterized by the coordinates z^M , where $M = 0, 1, \dots, D + 1$. These coordinates are subject to the constraint

$$-(z^0)^2 + (z^1)^2 + (z^2)^2 + \dots + (z^{D-1})^2 - (z^D)^2 + (z^{D+1})^2 = z^M z^N g_{MN} = 0 \tag{84}$$

which is invariant under the group of rotations $SO(2, D)$ with the generators

$$M_{MN} = z_M \partial_N - z_N \partial_M. \tag{85}$$

One can solve the constraint (84) by introducing

$$y^A = r \frac{z^A}{z^{D+1}}, \tag{86}$$

satisfying Equation (75).

The generators M_{MN} (85) contain the generators M_{AB} of the AdS_D isometry group $SO(2, D - 1)$ and the generators $M_{(D+1),A}$ which extend the latter to the conformal group $SO(2, D)$ by taking the functions on the cone (84) to be homogeneous of degree $-l$

$$z^M \frac{\partial}{\partial z^M} f(z) = -l f(z). \tag{87}$$

In this way, one gets (83).

Then using the explicit realization of the generators (81), (83) as well as the commutation relations (80) between the operators it is straightforward to check invariance of the field Equation (76) under the conformal group $SO(2, D)$.

5.2. $Sp(n)$ Group-Manifold Equations

In the previous subsection we considered in detail a conformal scalar field on AdS_D . As we discussed in Section 3, the hyperspace generalization of AdS spaces are $Sp(n)$ group manifolds. We will now consider an $Sp(n)$ counterpart of the conformal scalar field Equation (74).

Let us start with an $Sp(n)$ analogue of Equation (9). To this end one should replace the flat derivative $\partial_{\alpha\beta}$ with the covariant derivative on $Sp(n)$ group manifold. The covariant derivatives $\nabla_{\alpha\beta}$ satisfy the $Sp(n)$ algebra

$$[\nabla_{\alpha\beta}, \nabla_{\gamma\delta}] = \frac{\xi}{2} (C_{\alpha(\gamma} \nabla_{\delta)\beta} + C_{\beta(\gamma} \nabla_{\delta)\alpha}). \tag{88}$$

Due to the GL -flatness these covariant derivatives have a simple form

$$\nabla_{\alpha\beta} = G_{\alpha}^{-1\mu}(X) G_{\beta}^{-1\nu}(X) \partial_{\mu\nu}, \tag{89}$$

where $G_{\alpha}^{-1\mu}(X)$ was defined in (52). Further, one should replace the spinor product $\lambda_{\alpha} \lambda_{\beta}$ in (8) with an expression which like the covariant derivatives $\nabla_{\alpha\beta}$ also satisfies the $Sp(n)$ algebra. This can be done by introducing new variables

$$\tilde{Y}_{\alpha} \equiv \lambda_{\alpha} + \frac{i\xi}{8} \frac{\partial}{\partial \lambda^{\alpha}} \tag{90}$$

Obviously, the spinorial variables Y_{α} do not commute among each other

$$[\tilde{Y}_{\alpha}, \tilde{Y}_{\beta}] = \frac{i\xi}{4} C_{\alpha\beta}. \tag{91}$$

Using the covariant derivatives $\nabla_{\alpha\beta}$ and the variables Y_α . one can write an $Sp(n)$ analogue of Equation (9) as

$$\left[\nabla_{\alpha\beta} - \frac{i}{2} (\tilde{Y}_\alpha \tilde{Y}_\beta + \tilde{Y}_\beta \tilde{Y}_\alpha) \right] \Phi(X, \lambda) = 0. \tag{92}$$

Similarly, one finds an $Sp(n)$ version of Equation (12)

$$\left[\nabla_{\alpha\beta} - \frac{i}{2} (Y_\alpha Y_\beta + Y_\beta Y_\alpha) \right] C(X, \mu) = 0, \quad Y_\alpha \equiv \frac{\xi}{8} \mu_\alpha + i \frac{\partial}{\partial \mu^\alpha}. \tag{93}$$

To obtain the equations for component fields one should expand, e.g. the functional $C(X, \mu)$ in power of μ^α

$$C(X, \mu) = \sum_{n=0}^{\infty} C_{\alpha_1 \dots \alpha_n}(X) \mu^{\alpha_1} \dots \mu^{\alpha_n} = B(X) + F_\alpha(X) \mu^\alpha + \dots \tag{94}$$

Plugging this expansion into (92) one can show that similarly to the case of the flat hyperspace only zeroth and the first components in the expansion in terms of the variables μ^α are independent fields whereas the other fields are expressed in terms of derivatives of the independent ones. The independent hyperfields $B(X)$ and $F_\alpha(X)$ satisfy Equations [10]

$$\begin{aligned} \nabla_{\alpha[\beta} \nabla_{\gamma]\delta} B(X) &= \frac{\xi}{16} \left(C_{\alpha[\beta} \nabla_{\gamma]\delta} - C_{\delta[\gamma} \nabla_{\beta]\alpha} + 2C_{\beta\gamma} \nabla_{\alpha\delta} \right) B(X) \\ &+ \frac{\xi^2}{64} \left(2C_{\alpha\delta} C_{\beta\gamma} - C_{\alpha[\beta} C_{\gamma]\delta} \right) B(X), \end{aligned} \tag{95}$$

$$\nabla_{\alpha[\beta} F_{\gamma]}(X) = -\frac{\xi}{4} \left(C_{\alpha[\gamma} F_{\beta]}(X) + 2C_{\beta\gamma} F_\alpha(X) \right). \tag{96}$$

The derivation of these equations which are $Sp(n)$ versions of Equations (14) and (15) is straightforward and is given in the Appendix B.

Note that if one introduce the covariant derivatives $D_{\alpha\beta}$ acting on the spinors as follows (see [23] for more details)

$$D_{\alpha\beta} F_\gamma(X) = \nabla_{\alpha\beta} F_\gamma(X) + \frac{\xi}{4} C_{\gamma(\alpha} F_{\beta)}(X) \tag{97}$$

the form of Equations (95) and (96) simplifies to

$$D_{\alpha[\beta} D_{\gamma]\delta} B(X) = \frac{\xi^2}{8^2} \left(2C_{\alpha\delta} C_{\beta\gamma} - C_{\alpha[\beta} C_{\gamma]\delta} \right) B(X), \tag{98}$$

$$D_{\alpha[\beta} F_{\gamma]}(X) = 0. \tag{99}$$

We see that Equation (98) reminds that of the AdS scalar field (74), especially when we contract its indices.

5.2.1. Connection between the Fields in Flat Hyperspaces and $Sp(n)$ Group Manifolds

One can check [23] using the equations

$$\partial_{\mu\nu} G^{-1\alpha\beta}(X) = \frac{\xi}{8} (\delta_\mu^\alpha \delta_\nu^\beta + \delta_\mu^\beta \delta_\nu^\alpha), \tag{100}$$

and

$$\partial_{\mu\nu} (\det G(X))^k = \frac{\xi k}{8} (\det G(X))^k (G_{\mu\nu}(X) + G_{\nu\mu}(X)), \tag{101}$$

that the fields $B(X)$ and $F_\alpha(X)$ satisfying Equations (95) and (96) are related to the fields $b(X)$ and $f_\mu(X)$ satisfying the flat hyperspace Equations (14) and (15) as follows

$$B(X) = (\det G(X))^{-\frac{1}{2}} b(X), \tag{102}$$

$$F_\alpha(X) = (\det G(X))^{-\frac{1}{2}} G_\alpha^{-1\mu}(X) f_\mu(X). \tag{103}$$

These relations are similar to the relations between the conformally invariant scalar and spinor equations in the conventional flat and *AdS* spaces and reduce to them in the case of $n = 2, D = 3$.

5.2.2. Plane Wave Solutions

Equations (92) and (93) can be solved to obtain “plane-wave” solutions. Let us consider the case of the *Sp*(4) group manifold. One can check that Equations (92) and (93) have the following solutions

$$\Phi(X, \lambda) = \int d^4\mu \sqrt{\det G^{-1}(X)} e^{iX^{\alpha\beta}(\lambda_\alpha + \frac{\xi}{8}\mu_\alpha)(\lambda_\beta + \frac{\xi}{8}\mu_\beta) + i\lambda_\alpha\mu^\alpha} \varphi(\mu), \tag{104}$$

$$C(X, \mu) = \int d^4\lambda \sqrt{\det G^{-1}(X)} e^{iX^{\alpha\beta}(\lambda_\alpha + \frac{\xi}{8}\mu_\alpha)(\lambda_\beta + \frac{\xi}{8}\mu_\beta) - i\lambda_\alpha\mu^\alpha} \varphi(\lambda). \tag{105}$$

These solutions describe plane-wave-like fields in the *GL*-flat parameterization of the metric [10]. They can be compared with the plane-wave solutions for the higher-spin curvatures on *AdS*₄ given in [8,88]. The latter can be found by solving the *AdS*₄ deformation of the field Equations (33)

$$D_{M\dot{M}}C_{A_1,\dots,A_{n+2s},\dot{A},\dots,\dot{A}_n}(x) = e_{MM}^{A\dot{A}}C_{A_1,\dots,A_{n+2s},A,\dot{A},\dots,\dot{A}_n}(x) - n(n+2s)e_{MM,\{A\dot{A}}C_{A_2,\dots,A_{n+2s},\dot{A},\dots,\dot{A}_n}(x) \tag{106}$$

where $D_{M\dot{M}}$ is a covariant derivative on *AdS*₄ and $e_{MM}^{A\dot{A}}$ are the corresponding vierbeins in the Weyl spinor representation. The physical higher-spin curvatures satisfy the equations

$$e_{A\dot{A}}^{M\dot{M}}D_{MM}C^{A_1,\dots,A_{2s}}(x) = 0 \tag{107}$$

whereas the auxiliary fields are expressed via derivatives of the physical fields with the help of Equation (106). Choosing the *AdS*₄ metric in the conformally flat form

$$e_{MM}^{A\dot{A}} = e^{\frac{\rho(x)}{2}} \delta_M^A \delta_{\dot{M}}^{\dot{A}} \quad \rho(x) = \ln \frac{4}{(1 - (\frac{x}{r})^2)^2} \tag{108}$$

one can find the plane wave solutions of Equation (107)

$$C_{A_1,\dots,A_{2s}}(x) = \frac{\partial}{\partial \mu^{A_1}} \dots \frac{\partial}{\partial \mu^{A_{2s}}} C(x, \mu, \bar{\mu})|_{\mu=\bar{\mu}=0} \tag{109}$$

with

$$C(x, \mu, \bar{\mu}) = \int d^2\lambda d^2\bar{\lambda} \Phi(\lambda, \bar{\lambda}) \cdot \exp \left(i(\mu_A \bar{\mu}_{\dot{A}} + \lambda_A \bar{\lambda}_{\dot{A}}) x^{A\dot{A}} - \frac{\rho(x)}{2} + \left(1 - (\frac{x}{r})^2 \right)^{\frac{1}{2}} (\mu^A \lambda_A + \bar{\mu}^{\dot{A}} \bar{\lambda}_{\dot{A}}) \right). \tag{110}$$

Comparing (110) with (105), one can see that the latter is a direct generalization of the *AdS*₄ plane-wave solution to the case of the *Sp*(4) group manifold.

As a simplest example of this construction let us note that the conformal scalar on *AdS*₄ discussed in Section 5.1 admits a plane-wave solution [8] of the form

$$\phi(x) = \int d^2\lambda d^2\bar{\lambda} e^{ix^{A\dot{A}}\lambda_A \bar{\lambda}_{\dot{A}} - \frac{1}{2}\rho(x)} \phi_0(\lambda, \bar{\lambda}) \tag{111}$$

which can be checked substituting the expression (111) into the field Equation (74).

5.3. $Sp(2n)$ Transformations of the Fields

Using the relation between the fields of weight $\Delta = \frac{1}{2}$ on flat hyperspace and on $Sp(n)$ group manifold (102) we have the following relation between the $Sp(2n)$ transformations of the weight- $\frac{1}{2}$ fields on $Sp(n)$ and in flat hyperspace

$$\delta B(X) = (\det G(X))^{-\frac{1}{2}} \delta b(X), \tag{112}$$

$$\delta F_\alpha(X) = (\det G(X))^{-\frac{1}{2}} G_\alpha^{-1\mu}(X) \delta f_\mu(X). \tag{113}$$

Note that in the above expressions the matrix $G_\alpha^\mu(X)$ is not varied since it is form-invariant, i.e., $G(X')$ has the same form as $G(X)$.

Then, the $Sp(n)$ -variations of $B(X)$ and $F_\alpha(X)$ have the following form [23]

$$\delta B(X) = -(a^{\alpha\beta} \mathcal{D}_{\alpha\beta} + \frac{1}{2}(g_\alpha^\alpha - k_{\alpha\beta} X^{\alpha\beta}) + 2g_\beta^\alpha X^{\beta\gamma} \mathcal{D}_{\alpha\gamma} - k_{\alpha\beta} X^{\alpha\gamma} X^{\beta\delta} \mathcal{D}_{\gamma\delta}) B(X), \tag{114}$$

$$\delta F_\sigma(X) = -(a^{\alpha\beta} \mathcal{D}_{\alpha\beta} + \frac{1}{2}(g_\alpha^\alpha - k_{\alpha\beta} X^{\alpha\beta}) + 2g_\beta^\alpha X^{\beta\gamma} \mathcal{D}_{\alpha\gamma} - k_{\alpha\beta} X^{\alpha\gamma} X^{\beta\delta} \mathcal{D}_{\gamma\delta}) F_\sigma(X) - (g_\sigma^\beta - k_{\sigma\alpha} X^{\alpha\beta}) F_\beta(X),$$

where the derivative $\mathcal{D}_{\alpha\beta}$ is defined as

$$\mathcal{D}_{\alpha\beta} = \partial_{\alpha\beta} + \frac{\tilde{\zeta}}{16} (G_{\alpha\beta}(X) + G_{\beta\alpha}(X)). \tag{115}$$

Using

$$\partial_{\mu\nu} G_\rho^\sigma(X) = \frac{\tilde{\zeta}}{8} (G_{\rho\mu}(X) G_\nu^\sigma(X) + G_{\rho\nu}(X) G_\mu^\sigma(X)), \tag{116}$$

one can check that these derivatives commute with each other $[\mathcal{D}_{\alpha\beta}, \mathcal{D}_{\gamma\delta}] = 0$ just as in the flat case.

Let us note that the relation between the flat and $Sp(n)$ hyperfields of an arbitrary weight Δ and the form of the corresponding $Sp(2n)$ transformations require additional study since for this one should know the form of $Sp(2n)$ -invariant equations satisfied by these fields, which is still an open problem.

6. Supersymmetry

In this Section, we present a supersymmetric generalization of the $Sp(2n)$ invariant systems. We will mainly follow [24].

6.1. Flat Hyper-Superspace and Its Symmetries

The concept of hyperspaces, hyperfields and of the corresponding field equations can be generalized to construct supersymmetric $OSp(1|2n)$ invariant systems and the corresponding infinite-dimensional higher-spin supermultiplets. In this section we shall describe this generalization in detail.

The flat hyper-superspace (see e.g., [3,4,12]) is parameterized by $\frac{n(n+1)}{2}$ bosonic matrix coordinates $X^{\mu\nu} = X^{\nu\mu}$ and n real Grassmann-odd ‘‘spinor’’ coordinates θ^μ ($\mu = 1, \dots, n$). The supersymmetry variation

$$\delta\theta^\mu = \epsilon^\mu, \quad \delta X^{\mu\nu} = -i\epsilon^{(\mu}\theta^{\nu)}, \tag{117}$$

leaves invariant the Volkov-Akulov-type one-form

$$\Pi^{\mu\nu} = dX^{\mu\nu} + i\theta^{(\mu}d\theta^{\nu)}. \tag{118}$$

The supersymmetry transformations form a generalized super-translation algebra

$$\{Q_\mu, Q_\nu\} = 2P_{\mu\nu}, \quad [Q_\mu, P_{\nu\rho}] = 0, \quad [P_{\mu\nu}, P_{\rho\lambda}] = 0, \quad (119)$$

with $P_{\mu\nu}$ generating translations along $X^{\mu\nu}$.

The realization of $P_{\mu\nu}$ and Q_μ as differential operators is given by

$$P_{\mu\nu} = -i \frac{\partial}{\partial X^{\mu\nu}} \equiv -i\partial_{\mu\nu}, \quad Q_\mu = \partial_\mu - i\theta^\nu \partial_{\nu\mu}, \quad \partial_\mu \equiv \frac{\partial}{\partial \theta^\mu}, \quad (120)$$

The algebra (119) is invariant under rigid $GL(n)$ transformations

$$Q'_\mu = g_\mu{}^\nu Q_\nu, \quad P'_{\mu\nu} = g_\mu{}^\rho g_\nu{}^\lambda P_{\rho\lambda}, \quad (121)$$

generated by

$$G_\mu{}^\nu = -2i(X^{\nu\rho} + \frac{i}{2}\theta^\nu\theta^\rho)\partial_{\rho\mu} - i\theta^\nu Q_\mu, \quad (122)$$

which act on $P_{\mu\nu}$ and Q_μ as follows

$$[P_{\mu\nu}, G_\lambda{}^\rho] = -i(\delta_\mu^\rho P_{\nu\lambda} + \delta_\nu^\rho P_{\mu\lambda}), \quad [Q_\mu, G_\nu{}^\rho] = -i\delta_\mu^\rho Q_\nu, \quad (123)$$

and close into the $gl(n)$ algebra as in (39)

$$[G_\nu{}^\mu, G_\lambda{}^\rho] = i(\delta_\lambda^\mu G_\nu{}^\rho - \delta_\nu^\rho G_\lambda{}^\mu). \quad (124)$$

The algebra (119), (123) and (124) is the hyperspace counterpart of the conventional super-Poincaré algebra enlarged by dilatations. That this is so can be most easily seen by taking $n = 2$ (i.e., $\mu = 1, 2$), in which case this algebra is recognized as the $D = 3$ super-Poincaré algebra with $G_\mu{}^\nu - \frac{1}{2}\delta_\mu^\nu G_\rho{}^\rho = M_m(\gamma^m)_\mu{}^\nu$ ($m = 0, 1, 2$) generating the $SL(2, R) \sim SO(1, 2)$ Lorentz rotations and $\mathbb{D} = \frac{1}{2}G_\rho{}^\rho$ being the dilatation generator. Note that the factor $\frac{1}{2}$ in the definition of the dilatation generator is required in order to have the canonical scaling of the momentum generator $P_{\mu\nu}$ with weight 1 and the supercharge Q_μ with weight $\frac{1}{2}$, as follows from Equation (123).

This algebra may be further extended to the $OSp(1|2n)$ algebra, generating generalized superconformal transformations of the flat hyper-superspace, by adding the additional set of supersymmetry generators

$$S^\mu = -(X^{\mu\nu} + \frac{i}{2}\theta^\mu\theta^\nu)Q_\nu, \quad (125)$$

and the generalized conformal boosts

$$K^{\mu\nu} = i(X^{\mu\rho} + \frac{i}{2}\theta^\mu\theta^\rho)(X^{\nu\lambda} + \frac{i}{2}\theta^\nu\theta^\lambda)\partial_{\rho\lambda} - i\theta^{(\mu}S^{\nu)}. \quad (126)$$

The generators S^μ and $K^{\mu\nu}$ form a superalgebra similar to (119)

$$\{S^\mu, S^\nu\} = -2K^{\mu\nu}, \quad [S^\mu, K^{\nu\rho}] = 0, \quad [K^{\mu\nu}, K^{\rho\lambda}] = 0, \quad (127)$$

while the non-zero (anti)commutators of S^μ and $K^{\mu\nu}$ with Q_μ , $P_{\mu\nu}$ and $G_\mu{}^\nu$ read

$$\begin{aligned} \{Q_\mu, S^\nu\} &= -G_\mu{}^\nu, \quad [S^\mu, P_{\nu\rho}] = i\delta_{(\nu}^\mu Q_{\rho)}, \\ [Q_\mu, K^{\nu\rho}] &= -i\delta_\mu^{(\nu} S^{\rho)}, \quad [S^\mu, G_\nu{}^\rho] = i\delta_\nu^\mu S^\rho. \end{aligned} \quad (128)$$

Let us note that in the case $n = 4$, in which the physical space-time is four-dimensional the generalized superconformal group $OSp(1|8)$ contains the $D = 4$ conformal symmetry group $SO(2, 4) \sim SU(2, 2)$ as a subgroup, but not the superconformal group $SU(2, 2|1)$. The reason being that,

although $OSp(1|8)$ and $SU(2,2|1)$ contain the same number of (eight) generators, the anticommutators of the former close on the generators of the whole $Sp(8)$, while those of the latter only close on an $U(2,2)$ subgroup of $Sp(8)$, and the same supersymmetry generators cannot satisfy the different anti-commutation relations simultaneously. In fact, the minimal OSp -supergroup containing $SU(2,2|1)$ as a subgroup is $OSp(2|8)$.

6.2. Scalar Superfields and Their $OSp(1|2n)$ -Invariant Equations of Motion

Let us now consider a superfield $\Phi(X, \theta)$ transforming as a scalar under the super-translations (120)

$$\delta\Phi(X, \theta) = -(\epsilon^\alpha Q_\alpha + ia^{\mu\nu} P_{\mu\nu}) \Phi(X, \theta). \tag{129}$$

To construct equations of motion for $\Phi(X, \theta)$ which are invariant under (129) and comprise the equations of motion of an infinite tower of integer and half-integer higher-spin fields with respect to conventional space-time, we introduce the spinorial covariant derivatives

$$D_\mu = \partial_\mu + i\theta^\nu \partial_{\nu\mu}, \quad \{D_\mu, D_\nu\} = 2i\partial_{\mu\nu}, \tag{130}$$

which (anti)commute with Q_μ and $P_{\mu\nu}$.

The Φ -superfield equations then take the form [12]

$$D_{[\mu} D_{\nu]} \Phi(X, \theta) = 0, \tag{131}$$

As was shown in [12], these superfield equations imply that all the components of $\Phi(X, \theta)$ except for the first and the second one in the θ^μ -expansion of $\Phi(X, \theta)$ should vanish

$$\Phi(X, \theta) = b(X) + i\theta^\mu f_\mu(X) + i\theta^\mu \theta^\nu A_{\mu\nu}(X) + \dots, \tag{132}$$

(i.e., $A_{\mu_1 \dots \mu_k} = 0$ for $k > 1$) while the scalar and spinor fields $b(X)$ and $f_\mu(X)$ satisfy Equations (14) and (15).

The superfield Equations (131) are invariant under the generalized superconformal $OSp(1|2n)$ symmetry, provided that $\Phi(X, \theta)$ transforms as a scalar superfield with the “canonical” generalized scaling weight $\frac{1}{2}$, i.e.,

$$\begin{aligned} \delta\Phi(X, \theta) = & -(\epsilon^\mu Q_\mu + \xi_\mu S^\mu + ia^{\mu\nu} P_{\mu\nu} + ik_{\mu\nu} K^{\mu\nu} + ig_\mu{}^\nu G_\nu{}^\mu) \Phi(X, \theta) \\ & -\frac{1}{2} \left(g_\mu{}^\mu - k_{\mu\nu}(X^{\mu\nu} + \frac{i}{2}\theta^\mu \theta^\nu) + \xi_\mu \theta^\mu \right) \Phi(X, \theta), \end{aligned} \tag{133}$$

where the factor $\frac{1}{2}$ in the second line is the generalized conformal weight and $\epsilon^\mu, \xi_\mu, a^{\mu\nu}, k_{\mu\nu}$ and $g_\mu{}^\nu$ are the rigid parameters of the $OSp(1|2n)$ transformations.

Scalar superfields with anomalous generalized conformal dimension Δ transform under $OSp(1|2n)$ as

$$\begin{aligned} \delta\Phi(X, \theta) = & -(\epsilon^\mu Q_\mu + \xi_\mu S^\mu + ia^{\mu\nu} P_{\mu\nu} + ik_{\mu\nu} K^{\mu\nu} + ig_\mu{}^\nu G_\nu{}^\mu) \Phi(X, \theta) \\ & -\Delta \left(g_\mu{}^\mu - k_{\mu\nu}(X^{\mu\nu} + \frac{i}{2}\theta^\mu \theta^\nu) + \xi_\mu \theta^\mu \right) \Phi(X, \theta). \end{aligned} \tag{134}$$

It is instructive to demonstrate how the generalized conformal dimension Δ , which is defined to be the same for all values of n in $OSp(1|2n)$, is related to the conventional conformal weight of scalar superfields in various space-time dimensions. As we have already mentioned in Section 6.1, the dilatation operator should be identified with $\mathbb{D} = \frac{1}{2}G_\mu{}^\mu$. Therefore, considering a $GL(n)$ transformation (134) with the parameter $g_\mu{}^\nu$

$$\delta\Phi(X, \theta) = -ig_\mu{}^\nu G_\nu{}^\mu \Phi(X, \theta),$$

the part of the transformation corresponding to the dilatation reads

$$\delta_{\mathbb{D}}\Phi(X, \theta) = -\frac{i}{n}g_{\mu}^{\mu} G_{\nu}^{\nu}\Phi(X, \theta) = -\frac{2i}{n}g_{\mu}^{\mu}\mathbb{D}\Phi(X, \theta) = -i\tilde{g}\mathbb{D}\Phi(X, \theta), \tag{135}$$

where $\tilde{g} = \frac{2}{n}g_{\mu}^{\mu}$ is the genuine dilatation parameter. From (134), it then follows that the conventional conformal weight Δ_D of the scalar superfield is related to the generalized one Δ via

$$\Delta_{\mathbb{D}} = \frac{n}{2}\Delta, \quad D = \frac{n}{2} + 2. \tag{136}$$

In the $n = 2$ case corresponding to the $\mathcal{N} = 1, D = 3$ scalar superfield theory the two definitions of the conformal dimension coincide, whereas in the case $n = 4$ describing conformal higher-spin fields in $D = 4$ one finds $\Delta_4 = 2\Delta$. Relation (136) indeed provides the correct conformal dimensions of scalar superfields (and consequently of their components) in the corresponding space–time dimensions. For instance, when $\Delta = \frac{1}{2}$, in $D = 3$ one finds $\frac{1}{2}$ as the canonical conformal dimension of the scalar superfield, while in the cases $D = 4$ and $D = 6$ ($n = 8$) it is found to be equal to one and two, respectively. For convenience, we shall henceforth associate the scaling properties of the fields to the universal D - and n -independent generalized conformal weight Δ .

6.3. Infinite-Dimensional Higher-Spin Representation of $\mathcal{N} = 1, D = 4$ Supersymmetry

Using the example of $n = 4$ ($D = 4$) we will now show that in four space–time dimensions, the fields of integer and half-integer spin $s = 0, \frac{1}{2}, 1, \dots, \infty$ encoded in $b(X)$ and $f_{\mu}(X)$ (see Section 2.1) form an irreducible infinite-dimensional supermultiplet with respect to the supersymmetry transformations generated by the *generalized* super-Poincaré algebra (119). The hyperfields $b(X)$ and $f_{\mu}(X)$, satisfying (14) and (15), transform under the supertranslations (129) as follows

$$\delta b(X) = -i\epsilon^{\mu} f_{\mu}(X), \quad \delta f_{\mu}(X) = -\epsilon^{\nu} \partial_{\nu\mu} b(X). \tag{137}$$

and their expansion in terms of the y^{mn} coordinates is given in (16) and (17).

The fact that the higher–spin fields should form an infinite-dimensional representation of the generalized $\mathcal{N} = 1, D = 4$ supersymmetry (119) is prompted by the observation that the spectrum of bosonic fields contains a single real scalar field $\phi(x)$, which alone cannot have a fermionic superpartner, while each field with $s > 0$ has two helicities $\pm s$. Indeed, from (137), we obtain an infinite entangled chain of supersymmetry transformations for the $D = 4$ fields

$$\begin{aligned} \delta\phi(x) &= -i\epsilon^{\mu} \psi_{\mu}(x), \\ \delta\psi_{\mu}(x) &= \epsilon^{\nu} (\gamma_{\nu\mu}^m \partial_m \phi(x) + \gamma_{\nu\mu}^{mn} F_{mn}(x)), \\ \delta F_{mn}(x) &= -i\epsilon^{\mu} \left(R_{\mu mn}(x) - \frac{1}{2} \partial_{[m} (\gamma_{n]} \psi)_{\mu}(x) \right), \\ \delta R_{\mu mn}(x) &= \frac{1}{2} \partial_{[m} (\gamma_{n]} \delta\psi(x))_{\mu} - \frac{1}{2} \epsilon^{\nu} \gamma_{\nu\mu}^p \partial_p F_{mn}(x) \\ &\quad - \epsilon^{\nu} \gamma_{\nu\mu}^{pq} \left(R_{pq, mn}(x) - \frac{1}{2} \partial_q \eta_{p[m} \partial_{n]} \phi(x) \right), \end{aligned} \tag{138}$$

and so on.

The algebraic reason behind the appearance of the infinite-dimensional supermultiplet of the $D = 4$ higher–spin fields is related to the following fact. In the $n = 4, D = 4$ case the superalgebra (119) takes the following form

$$\{Q_{\mu}, Q_{\nu}\} = (\gamma^m)_{\mu\nu} P_m + (\gamma^{mn})_{\mu\nu} Z_{mn}, \tag{139}$$

where P_m is the momentum along the four-dimensional space–time and $Z_{mn} = -Z_{nm}$ are the tensorial charges associated with the momenta along the extra coordinates y^{mn} .

On the other hand, the conventional $N = 1, D = 4$ super-Poincaré algebra is

$$\{Q_\mu, Q_\nu\} = (\gamma^m)_{\mu\nu} P_m. \tag{140}$$

Though both algebras have the same number of the supercharges Q_μ , their anti-commutator closes on different sets of bosonic generators. Thus, the super-Poincaré algebra (140) is not a subalgebra of (139). Hence the representations of (139) do not split into (finite-dimensional) representations of the standard super-Poincaré algebra. In this sense the supersymmetric higher-spin systems under consideration differ from most of supersymmetric models of finite-dimensional super-Poincaré or AdS higher-spin supermultiplets considered in the literature (see e.g., [40,46,89–112]).

7. Hyperspace Extension of Supersymmetric AdS Spaces

In Section 3 we have seen that the hyperspace extension of AdS spaces are $Sp(n)$ group manifolds. In this section we consider their minimal supersymmetric extension, namely $OSp(1|n)$ supergroup manifolds.

The $OSp(1|n)$ superalgebra is formed by n anti-commuting supercharges Q_α and $\frac{n(n+1)}{2}$ generators $M_{\alpha\beta} = M_{\beta\alpha}$ of $Sp(n)$

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= 2M_{\alpha\beta}, & [Q_\alpha, M_{\beta\gamma}] &= \frac{i\tilde{\zeta}}{2} C_{\alpha(\beta} Q_{\gamma)}, \\ [M_{\alpha\beta}, M_{\gamma\delta}] &= -\frac{i\tilde{\zeta}}{2} (C_{\gamma(\alpha} M_{\beta)\delta} + C_{\delta(\alpha} M_{\beta)\gamma}), \end{aligned} \tag{141}$$

The $OSp(1|n)$ algebra (141) is recognized as a subalgebra of $OSp(1|2n)$ (see the Section 6.1) with the identifications

$$Q_\alpha = (Q_\alpha + \frac{\tilde{\zeta}}{4} S_\alpha), \quad M_{\alpha\beta} = P_{\alpha\beta} - \frac{\tilde{\zeta}^2}{16} K_{\alpha\beta} - \frac{\tilde{\zeta}}{4} G_{(\alpha\beta)}. \tag{142}$$

The $OSp(1|n)$ manifold is parameterized by the coordinates $(X^{\mu\nu}, \theta^\mu)$ and its geometry is described by the Cartan forms

$$\Omega = \mathcal{O}^{-1} d\mathcal{O}(X, \theta) = -i\Omega^{\alpha\beta} M_{\alpha\beta} + iE^\alpha Q_\alpha, \tag{143}$$

where $\mathcal{O}(X, \theta)$ is an $OSp(1|n)$ supergroup element. The Cartan forms satisfy the Maurer–Cartan equations associated with the $OSp(1|n)$ superalgebra (141)

$$d\Omega^{\alpha\beta} + \frac{\tilde{\zeta}}{2} \Omega^{\alpha\gamma} \wedge \Omega_\gamma{}^\beta = -iE^\alpha \wedge E^\beta, \quad dE^\alpha + \frac{\tilde{\zeta}}{2} E^\gamma \wedge \Omega_\gamma{}^\alpha = 0, \tag{144}$$

with the external differential acting from the right.

7.1. GL Flatness of $OSp(1|n)$ Group Manifolds

There is a supersymmetric generalization of the $GL(n)$ flatness property of $Sp(n)$ group manifolds to the case of $OSp(1|n)$ supergroup manifolds [8]. In particular, the Maurer–Cartan Equations (144) are solved by the following forms

$$\Omega^{\alpha\beta} = dX^{\mu\nu} G_\mu{}^\alpha G_\nu{}^\beta(X) + \frac{i}{2} (\Theta^\alpha \mathcal{D}\Theta^\beta + \Theta^\beta \mathcal{D}\Theta^\alpha) = \Pi^{\mu\nu} \mathcal{G}_\mu{}^\alpha \mathcal{G}_\nu{}^\beta(X, \Theta), \tag{145}$$

$$E^\alpha = P(\Theta^2) \mathcal{D}\Theta^\alpha - \Theta^\alpha \mathcal{D}P(\Theta^2) \tag{146}$$

where Θ is related to θ as follows

$$\theta^\alpha = \Theta^\beta G_\beta{}^{-1\alpha} P^{-1}(\Theta^2), \quad \Theta^2 = \Theta^\alpha \Theta_\alpha, \quad P^2(\Theta^2) = 1 + \frac{i\tilde{\zeta}}{8} \Theta^2, \tag{147}$$

while the covariant derivative

$$\mathcal{D}\Theta^\alpha = d\Theta^\alpha + \frac{\xi}{4}\Theta^\beta\omega_\beta^\alpha(X), \tag{148}$$

contains the Cartan form of the $Sp(n)$ group manifold

$$\omega^{\alpha\beta}(X) = dX^{\mu\nu}G_\mu^\alpha(X)G_\nu^\beta(X), \tag{149}$$

and

$$\mathcal{G}_\alpha^\beta(X, \Theta) = G_\alpha^\beta(X) - \frac{i\xi}{8}(\Theta_\alpha - 2G_\alpha^\gamma(X)\Theta_\gamma)\Theta^\beta, \tag{150}$$

where $G_\alpha^\beta(X)$ is given in (50). The inverse matrix of (150) is

$$\begin{aligned} \mathcal{G}_\alpha^{-1\beta}(X, \Theta) &= G_\alpha^{-1\beta}(X) - \frac{i\xi}{8}(\Theta^\delta G_{\delta\alpha}^{-1}(X))(\Theta^\delta G_\delta^{-1\beta}(X))P^{-2}(\Theta^2) \\ &= G_\alpha^{-1\beta}(X) - \frac{i\xi}{8}\theta_\alpha\theta^\beta = \delta_\alpha^\beta + \frac{\xi}{4}(X_\alpha^\beta - \frac{i}{2}\theta_\alpha\theta^\beta) \end{aligned} \tag{151}$$

with $G_\alpha^{-1\beta}(X)$ given in (52).

7.2. Field Equations on $OSp(1|n)$ Supergroup Manifold

The scalar superfield equation on $OSp(1|n)$ has the form [12]

$$\left(\nabla_{[\alpha}\nabla_{\beta]} - \frac{i\xi}{8}C_{\alpha\beta}\right)\Phi_{OSp}(X, \theta) = 0, \tag{152}$$

where the Grassmann-odd covariant derivatives ∇_α and their bosonic counterparts $\nabla_{\alpha\beta}$ satisfy the $OSp(1|n)$ superalgebra similar to (141), namely

$$\{\nabla_\alpha, \nabla_\beta\} = 2i\nabla_{\alpha\beta} \tag{153}$$

$$[\nabla_\gamma, \nabla_{\alpha\beta}] = \frac{\xi}{2}C_{\gamma(\alpha}\nabla_{\beta)}, \tag{154}$$

$$[\nabla_{\alpha\beta}, \nabla_{\gamma\delta}] = \frac{\xi}{2}(C_{\alpha(\gamma}\nabla_{\delta)\beta} + C_{\beta(\gamma}\nabla_{\delta)\alpha}). \tag{155}$$

while the $OSp(1|n)$ covariant derivatives are obtained from the flat superspace ones by the following GL transformations

$$\begin{aligned} \nabla_\alpha &= \mathcal{G}_\alpha^{-1\mu}(X, \Theta)D_\mu, \\ \nabla_{\alpha\beta} &= \mathcal{G}_\alpha^{-1\mu}(X, \Theta)\mathcal{G}_\beta^{-1\nu}(X, \Theta)\left(\partial_{\mu\nu} + 2iD_{(\mu}\ln\left((\det G(X))^{\frac{1}{2}}P^{-1}(\Theta^2)\right)\right)D_{\nu)}. \end{aligned} \tag{156}$$

Connection between Superfields on Flat Hyper-Superspace and on $OSp(1|n)$ Supergroup Manifolds

Using the relations given in Appendix C one can show that the superfield $\Phi_{OSp}(X, \theta)$ satisfying (152) is related to the superfield $\Phi(X, \theta)$ satisfying the flat superspace Equation (131) by the super-Weyl transformation

$$\begin{aligned} \Phi_{OSp(1|n)}(X, \theta) &= (\det \mathcal{G}(X, \Theta))^{-\frac{1}{2}}\Phi_{flat}(X, \theta) \\ &= (\det G(X))^{-\frac{1}{2}}P(\Theta^2)\Phi_{flat}(X, \theta), \end{aligned} \tag{157}$$

Substituting (132) into (157) and using the definition (147), together with the fact that on the mass shell all higher components in (132) vanish, we find

$$\begin{aligned} \Phi_{OSp(n)}(X, \theta) &= (\det G(X))^{-\frac{1}{2}}b(X) \\ &+ \Theta^\alpha(\det G(X))^{-\frac{1}{2}}G_\alpha^{-1\mu}(X)f_\mu(X) + O(\Theta^2, b(X)), \end{aligned} \tag{158}$$

where the first two terms are the fields

$$B(X) = (\det G(X))^{-\frac{1}{2}} b(X), \quad F_\alpha(X) = (\det G(X))^{-\frac{1}{2}} G_\alpha^{-1\mu}(X) f_\mu(X) \tag{159}$$

propagating on the $Sp(n)$ group manifold, and $O(\Theta^2, b(X))$ stands for higher order terms in Θ^2 which only depend on $b(X)$. The fields (159) satisfy the equations of motion on $Sp(n)$ group manifolds (95)–(96). Note that in these equations the covariant derivatives are restricted to the bosonic group manifold $Sp(n)$, i.e., $\nabla_{\alpha\beta} = G_\alpha^{-1\mu}(X) G_\beta^{-1\nu}(X) \partial_{\mu\nu}$.

7.3. $OSp(1|2n)$ Transformations of Superfields

Since the flat superspace field equation is invariant under the generalized superconformal $OSp(1|2n)$ transformations (133), the above relation leads us to conclude that also the $OSp(1|n)$ superspace Equations (152) are invariant under the $OSp(1|2n)$ transformations, under which the superfield $\Phi_{OSp}(X, \theta)$ varies as follows

$$\begin{aligned} \delta\Phi_{OSp}(X, \theta) = & -(\epsilon^\mu Q_\mu + \zeta_\mu S^\mu + ia^{\mu\nu} P_{\mu\nu} + ik_{\mu\nu} K^{\mu\nu} + ig_\mu{}^\nu G_\nu{}^\mu) \Phi_{OSp}(X, \theta) \\ & -\frac{1}{2} \left(g_{\mu\nu} X^{\mu\nu} + \frac{i}{2} \theta^\mu \theta^\nu \right) + \zeta_\mu \theta^\mu \Phi_{OSp}(X, \theta). \end{aligned} \tag{160}$$

Here,

$$P_{\mu\nu} = -iD_{\mu\nu} = -i(\partial_{\mu\nu} + \frac{\zeta}{8} G_{(\mu\nu)}(X, \Theta)), \tag{161}$$

and

$$Q_\mu = Q_\mu - \frac{i\zeta}{8} \Theta_\mu P(\Theta). \tag{162}$$

Using the relations given in the Appendix C one may check that the operators (161) and (162) obey the flat hyperspace supersymmetry algebra

$$[P_{\mu\nu}, P_{\rho\sigma}] = 0, \quad \{Q_\mu, Q_\nu\} = -2P_{\mu\nu}, \quad [P_{\mu\nu}, Q_\rho] = 0. \tag{163}$$

The other generators of the $OSp(1|2n)$ are

$$S^\mu = -(X^{\mu\nu} + \frac{i}{2} \theta^\mu \theta^\nu) Q_\nu, \quad G_\mu{}^\nu = -2i(X^{\nu\rho} + \frac{i}{2} \theta^\nu \theta^\rho) D_{\rho\mu} - i\theta^\nu Q_\mu, \tag{164}$$

and

$$K^{\mu\nu} = i(X^{\mu\rho} + \frac{i}{2} \theta^\mu \theta^\rho)(X^{\nu\lambda} + \frac{i}{2} \theta^\nu \theta^\lambda) D_{\rho\lambda} - i\theta^{(\mu} S^{\nu)}. \tag{165}$$

Taking into account the commutation relations (163) we see that the operators $Q_\mu, S^\mu, P_{\mu\nu}, G_\mu{}^\nu$ and $K^{\mu\nu}$ obey the same $OSp(1|2n)$ algebra as the operators $Q_\mu, S^\mu, P_{\mu\nu}, G_\mu{}^\nu$ and $K^{\mu\nu}$ considered in the Section 6.1.

8. Generalized CFT. Part I. Correlation Functions in $OSp(1|2n)$ -Invariant Models

In the previous sections, we have described the generalized conformal group $Sp(2n)$ and generalized conformal supergroup $OSp(1|2n)$. We introduced the fundamental fields and superfields and showed how they transform under generalized conformal transformations.

In this Section we shall construct two-, three- and four-point correlation functions of these fields, by requiring the $Sp(2n)$ symmetry of the correlators, i.e., by solving the corresponding Ward identities. In other words we will follow the conventional approach adopted in multidimensional CFTs (see e.g., [113]). In particular, we will consider $OSp(1|2n)$ invariant correlation functions from which the $Sp(2n)$ invariant correlation functions can be recovered as components of the expansions

of the former in series of the Grassman coordinates θ^μ . $Sp(2n)$ -invariant correlation functions in the tensorial spaces have been studied in [11,23,24,27] and in the unfolded formulation in [114].

8.1. Two-Point Functions

Let us denote the two-point correlation function by

$$W(Z_1, Z_2) = \langle \Phi(X_1, \theta_1) \Phi(X_2, \theta_2) \rangle. \tag{166}$$

The invariance under supersymmetry transformation generated by the operators Q , Equation (120), requires that

$$\epsilon^\mu \left(\frac{\partial}{\partial \theta_1^\mu} - i\theta_1^\nu \frac{\partial}{\partial X_1^{\mu\nu}} + \frac{\partial}{\partial \theta_2^\mu} - i\theta_2^\nu \frac{\partial}{\partial X_2^{\mu\nu}} \right) W(Z_1, Z_2) = 0, \tag{167}$$

which implies

$$\langle \Phi(X_1, \theta_1) \Phi(X_2, \theta_2) \rangle = W(\det|Z_{12}|), \tag{168}$$

where

$$Z_{12}^{\mu\nu} = X_1^{\mu\nu} - X_2^{\mu\nu} - \frac{i}{2} \theta_1^\mu \theta_2^\nu - \frac{i}{2} \theta_1^\nu \theta_2^\mu \tag{169}$$

is the interval between two points in hyper-superspace which is invariant under the rigid supersymmetry transformations (117).

We next require the invariance of the correlator under the S -supersymmetry (125)

$$\begin{aligned} & \xi_\mu \left[(X_1^{\mu\nu} + \frac{i}{2} \theta_1^\mu \theta_1^\nu) \left(\frac{\partial}{\partial \theta_1^\nu} - i\theta_1^\rho \frac{\partial}{\partial X_1^{\nu\rho}} \right) + (X_2^{\mu\nu} + \frac{i}{2} \theta_2^\mu \theta_2^\nu) \left(\frac{\partial}{\partial \theta_2^\nu} - i\theta_2^\rho \frac{\partial}{\partial X_2^{\nu\rho}} \right) \right] \cdot \\ & W(\det|Z_{12}|) \\ & + \xi_\mu \left(\frac{i}{2} \theta_1^\mu + \frac{i}{2} \theta_2^\mu \right) W(\det|Z_{12}|) = 0, \end{aligned} \tag{170}$$

which is solved by

$$W(\det|Z_{12}|) = c_2(\det|Z_{12}|)^{-\frac{1}{2}} \Rightarrow \langle \Phi(X_1, \theta_1) \Phi(X_2, \theta_2) \rangle = c_2(\det|Z_{12}|)^{-\frac{1}{2}}. \tag{171}$$

The two-point function (171) reproduces the correlators of the component bosonic and fermionic hyperfields $b(X)$ and $f_\mu(X)$ after the expansion of the former in powers of the Grassmann coordinates $\theta_1^{(\mu} \theta_2^{\nu)}$. Since on the mass shell the superfield (132) has only two non-zero components, all terms in the θ -expansion of the two-point function (171), starting from the ones quadratic in $\theta_1^{(\mu} \theta_2^{\nu)}$, should vanish. This is indeed the case, as a consequence of the field equations.

To see this, let us recall that in the separated points the two-point function of the bosonic hyperfield of weight $\frac{1}{2}$ satisfies the free field equation. Therefore for $X_{\alpha\beta}^1 \neq X_{\alpha\beta}^2$ one has (when the two points coincide, one can define an analog of the Dirac delta-function in the tensorial spaces, see [5] for the relevant discussion)

$$(\partial_{\mu\nu}^1 \partial_{\rho\sigma}^1 - \partial_{\mu\rho}^1 \partial_{\nu\sigma}^1) \langle b(X_1) b(X_2) \rangle = (\partial_{\mu\nu}^1 \partial_{\rho\sigma}^1 - \partial_{\mu\rho}^1 \partial_{\nu\sigma}^1) (\det|X_{12}|)^{-\frac{1}{2}} = 0. \tag{172}$$

Similarly, for $X_{\alpha\beta}^1 \neq X_{\alpha\beta}^2$ the fermionic two-point function satisfies the free field equation for the fermionic hyperfield. Written in terms of the superfields, these equations are encoded in the superfield equation (for $Z_{12} \neq 0$)

$$(D_\mu^1 D_\nu^1 - D_\nu^1 D_\mu^1) \langle \Phi(X_1, \theta_1) \Phi(X_2, \theta_2) \rangle = (D_\mu^1 D_\nu^1 - D_\nu^1 D_\mu^1) (\det|Z_{12}|)^{-\frac{1}{2}} = 0. \tag{173}$$

Expanding the two-point function $(\det|Z_{12}|)^{-\frac{1}{2}}$ in powers of the Grassmann variables

$$\begin{aligned}
 (\det|Z_{12}|)^{-\frac{1}{2}} &= (\det|X_{12}|)^{-\frac{1}{2}} \\
 &- i\partial_{\alpha\beta}(\det|X_{12}|)^{-\frac{1}{2}}\theta_1^{(\alpha}\theta_2^{\beta)} - \frac{1}{2}\partial_{\gamma\delta}\partial_{\alpha\beta}(\det|X_{12}|)^{-\frac{1}{2}}\theta_1^{(\alpha}\theta_2^{\beta)}\theta_1^{(\gamma}\theta_2^{\delta)} + \dots,
 \end{aligned}
 \tag{174}$$

one may see that the terms in the expansion starting from $(\theta_1^{(\mu}\theta_2^{\nu)})^2$ vanish due to the free field Equation (172). From Equations (171) and (174) and from the explicit form of the superfield (132), one may immediately reproduce the correlation functions for the component fields [11]

$$\langle b(X_1)b(X_2) \rangle = c_2(\det|X_{12}|)^{-\frac{1}{2}}, \tag{175}$$

$$\langle f_\mu(X_1)f_\nu(X_2) \rangle = \frac{ic_2}{2}(X_{12})_{\mu\nu}^{-1}(\det|X_{12}|)^{-\frac{1}{2}}. \tag{176}$$

The two-point functions on the $OSp(1|n)$ manifold may now be obtained from (171) via the rescaling (157), which relates the superfields in flat superspace and on the $OSp(1|n)$ group manifold

$$\begin{aligned}
 \langle \Phi_{OSp}(X_1, \theta_1)\Phi_{OSp}(X_2, \theta_2) \rangle &= \\
 (\det G(X_1))^{-\frac{1}{2}}P(\Theta_1^2)(\det G(X_2))^{-\frac{1}{2}}P(\Theta_2^2)\langle \Phi(X_1, \theta_1)\Phi(X_2, \theta_2) \rangle.
 \end{aligned}
 \tag{177}$$

Finally, as in the $D = 3$ case, one may derive the superconformally invariant two-point function for superfields carrying an arbitrary generalized conformal weight Δ , which on flat hyper superspace has the form

$$\langle \Phi^{\Delta_1}(X_1, \theta_1)\Phi^{\Delta_2}(X_2, \theta_2) \rangle = c_2(\det|Z_{12}|)^{-\Delta}, \quad \Delta_1 = \Delta_2 = \Delta. \tag{178}$$

8.2. Three-Point Functions

The three-point functions for the superfields with arbitrary generalized conformal dimensions Δ_i , ($i = 1, 2, 3$)

$$W(Z_1, Z_2, Z_3) = \langle \Phi(X_1, \theta_1)\Phi(X_2, \theta_2)\Phi(X_3, \theta_3) \rangle, \tag{179}$$

may be computed in a way similar to the two-point functions using the superconformal Ward identities. The invariance under Q -supersymmetry implies that they depend on the superinvariant intervals Z_{ij} , i.e.,

$$\langle \Phi(X_1, \theta_1)\Phi(X_2, \theta_2)\Phi(X_3, \theta_3) \rangle = W(Z_{12}, Z_{23}, Z_{31}), \tag{180}$$

where

$$Z_{ij}^{\mu\nu} = X_i^{\mu\nu} - X_j^{\mu\nu} - \frac{i}{2}(\theta_i^\mu\theta_j^\nu + \theta_i^\nu\theta_j^\mu), \quad i, j = 1, 2, 3. \tag{181}$$

Invariance under S -supersymmetry then fixes the form of the function W to be

$$\begin{aligned}
 &\langle \Phi(X_1, \theta_1)\Phi(X_2, \theta_2)\Phi(X_3, \theta_3) \rangle \\
 &= c_3(\det Z_{12})^{-\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)}(\det Z_{23})^{-\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1)}(\det Z_{31})^{-\frac{1}{2}(\Delta_3+\Delta_1-\Delta_2)}.
 \end{aligned}
 \tag{182}$$

Let us note that the three-point function is not annihilated by the operator entering the free equations of motion (131) for generic values of the generalized conformal dimensions, including the case in which the values of all the generalized conformal dimensions are canonical

$$\begin{aligned}
 &(D_\mu^1 D_\nu^1 - D_\nu^1 D_\mu^1)\langle \Phi(X_1, \theta_1), \Phi(X_2, \theta_2), \Phi(X_2, \theta_2) \rangle \\
 &= c_3(D_\mu^1 D_\nu^1 - D_\nu^1 D_\mu^1) \left((\det|Z_{12}|)^{-\frac{1}{4}}(\det|Z_{23}|)^{-\frac{1}{4}}(\det|Z_{31}|)^{-\frac{1}{4}} \right) \neq 0.
 \end{aligned}
 \tag{183}$$

Again, the three-point functions on the supergroup manifold $OSp(1|n)$ can be obtained via the Weyl rescaling (157), as in the case of the two-point functions (177)

$$\begin{aligned} & \langle \Phi_{OSp}(X_1, \theta_1) \Phi_{OSp}(X_2, \theta_2) \Phi_{OSp}(X_3, \theta_3) \rangle \\ &= (\det G(X_1))^{-\frac{1}{2}} P(\Theta_1^2) (\det G(X_2))^{-\frac{1}{2}} P(\Theta_2^2) (\det G(X_3))^{-\frac{1}{2}} P(\Theta_3^2) \cdot \\ & \langle \Phi(X_1, \theta_1) \Phi(X_2, \theta_2) \Phi(X_3, \theta_3) \rangle. \end{aligned} \tag{184}$$

8.3. Four-Point Functions

Finally, let us consider, first in flat hyper superspace, the correlation function of four real scalar superfields with arbitrary generalized conformal dimensions, Δ_i ($i = 1, 2, 3, 4$)

$$W(Z_1, Z_2, Z_3) = \langle \Phi(X_1, \theta_1) \Phi(X_2, \theta_2) \Phi(X_3, \theta_3) \Phi(X_4, \theta_4) \rangle. \tag{185}$$

Invariance under Q -supersymmetry again implies that the correlation function depends only on the superinvariant intervals $Z_{ij}^{\mu\nu}$ (181). Following the analogy with conventional conformal field theory we find

$$W(X_1, X_2, X_3, X_4) = c_4 \prod_{ij, i < j} \frac{1}{(\det |Z_{ij}|)^{k_{ij}}} \tilde{W}(z, z'), \tag{186}$$

with W being an arbitrary function of the cross-ratios

$$z = \det \left(\frac{|Z_{12}| |Z_{34}|}{|Z_{13}| |Z_{24}|} \right), \quad z' = \det \left(\frac{|Z_{12}| |Z_{34}|}{|Z_{23}| |Z_{14}|} \right), \tag{187}$$

subject to the crossing symmetry constraints

$$\tilde{W}(z, z') = \tilde{W} \left(\frac{1}{z}, \frac{z'}{z} \right) = \tilde{W} \left(\frac{z}{z'}, \frac{1}{z'} \right). \tag{188}$$

Furthermore, the k_{ij} 's are constrained by the invariance of the four-point function under the S -supersymmetry to satisfy

$$\sum_{j \neq i} k_{ij} = \Delta_i. \tag{189}$$

Similar to the case of two- and three-point functions, the four-point function of the scalar superfields on $OSp(1|n)$ can be obtained from (186) via the Weyl re-scaling (157).

8.4. An Example. $\mathcal{N} = 1$ $D = 3$ Superconformal Models

As we mentioned earlier, the case of $D = 3$ is the simplest example of “hyperspace” which in this case coincides with the three-dimensional space time itself, and the fundamental fields are just the scalar $b(x)$ and the two-component spinor $f_\alpha(x)$. All known results for three-dimensional (super)conformal theories are reproduced from the above generic formulas restricted to the case of $n = 2$ and $D = 3$, as we will show on the example of $\mathcal{N} = 1$ $D = 3$ superconformal two- and three-point functions.

The superconformally invariant two- and three-point correlation functions of the $\mathcal{N} = 1, D = 3$ scalar supermultiplet model have been constructed in [115].

Let us use the spinor-tensor representation for the description of the three-dimensional space-time coordinates

$$x^{\alpha\beta} = x^{\beta\alpha} = x^m (\gamma_m)^{\alpha\beta}, \tag{190}$$

where now $\alpha, \beta = 1, 2$ are $D = 3$ spinorial indices and $m = 0, 1, 2$ is the vectorial one. Since (190) provides a representation of the symmetric 2×2 matrices $x^{\alpha\beta}$, no extra coordinates, like y^m , are present and, hence, no higher-spin fields.

The inverse matrix of (190), $x_{\alpha\beta}^{-1}$

$$x^{\alpha\beta} x_{\beta\gamma}^{-1} = \delta_\alpha^\gamma, \tag{191}$$

takes the simple form

$$x_{\alpha\beta}^{-1} = -\frac{1}{x^m x_m} x^n (\gamma_n)_{\alpha\beta} = -\frac{1}{x^2} x_{\alpha\beta}. \tag{192}$$

We may now consider a real scalar superfield in $D = 3$

$$\Phi(x, \theta) = \phi(x) + i\theta^\alpha f_\alpha(x) + \theta^\alpha \theta_\alpha F(x), \tag{193}$$

with $\phi(x)$ being a physical scalar, $f_\alpha(x)$ a physical fermion and $F(x)$ an auxiliary field.

If (193) satisfies the free equation of motion (131), which in the $D = 3$ case reduces to

$$D^\alpha D_\alpha \Phi(x, \theta) = 0. \tag{194}$$

This equation implies that on the mass shell the auxiliary field $F(x)$ vanishes, the scalar field $\phi(x)$ satisfies the massless Klein–Gordon equation and $f_\alpha(x)$ satisfies the massless Dirac equation. The field Equation (194) is superconformally invariant if the superfield $\Phi(x, \theta)$ has the canonical conformal weight $\Delta = \frac{1}{2}$.

Let us consider a superconformal transformation of (193). The Poincaré supersymmetry transformations of Φ are

$$\delta\Phi(x, \theta) = \epsilon^\alpha \left(\frac{\partial}{\partial\theta^\alpha} - i\theta^\beta \frac{\partial}{\partial x^{\alpha\beta}} \right) \Phi(x, \theta) = \epsilon^\alpha Q_\alpha \Phi(x, \theta). \tag{195}$$

They encode the supersymmetry transformations of the component fields

$$\delta\phi(x) = i\epsilon^\alpha f_\alpha(x), \tag{196}$$

$$\delta f_\alpha(x) = -2i\epsilon_\alpha F(x) - \epsilon^\beta \partial_{\alpha\beta} \phi(x), \tag{197}$$

$$\delta F(x) = \frac{1}{2} \epsilon^\alpha \partial_{\alpha\beta} f^\beta(x), \tag{198}$$

where we have made use of the identity

$$\theta^\alpha \theta^\beta = \frac{1}{2} C^{\alpha\beta} (\theta^\gamma \theta_\gamma). \tag{199}$$

Under conformal supersymmetry, $\Phi(x, \theta)$ transforms as follows

$$\delta\Phi(x, \theta) = \zeta_\alpha (x^{\alpha\beta} + \frac{i}{2} \theta^\alpha \theta^\beta) Q_\beta \Phi(x, \theta) - i(\zeta_\alpha \theta^\alpha) \Delta \Phi(x, \theta), \tag{200}$$

where Δ is the conformal weight of the superfield. The superconformal transformations of the component fields are

$$\delta\phi(x) = i\zeta_\alpha x^{\alpha\beta} f_\beta(x), \tag{201}$$

$$\delta f_\alpha(x) = -2i\zeta_\beta x^\beta_\alpha F(x) + \zeta_\beta x^{\beta\gamma} \partial_{\gamma\alpha} \phi(x) + \zeta_\alpha \Delta \phi(x), \tag{202}$$

$$\delta F(x) = \frac{1}{2} \zeta_\alpha x^{\alpha\beta} \partial_{\beta\gamma} f^\gamma(x) - \frac{1}{2} \zeta_\alpha \left(\frac{1}{2} - \Delta \right) f^\alpha(x). \tag{203}$$

The conformal weights of ϕ , f_α and F are Δ , $\Delta + \frac{1}{2}$ and $\Delta + 1$, respectively.

As we have already seen, the two-point function for a superfield of an arbitrary noncanonical dimension has the form (178). Expanding the expression on the right hand side of (178) in powers of θ , we obtain

$$(\det|z_{12}|)^{-\Delta} = (\det|x_{12}|)^{-\Delta} - i\partial_{\alpha\beta} (\det|x_{12}|)^{-\Delta} \theta_1^{(\alpha} \theta_2^{\beta)} - \frac{1}{2} \partial_{\gamma\delta} \partial_{\alpha\beta} (\det|x_{12}|)^{-\Delta} \theta_1^{(\alpha} \theta_2^{\beta)} \theta_1^{(\gamma} \theta_2^{\delta)}. \tag{204}$$

Using the identities

$$\partial_{\alpha\beta}(\det|x|)^{-\Delta} = -\Delta x_{\alpha\beta}^{-1} \det|x|^{-\Delta}, \tag{205}$$

and

$$\partial_{\alpha\beta}\partial_{\gamma\delta}(\det|x|)^{-\Delta} = \Delta \left(\Delta x_{\alpha\beta}^{-1}x_{\gamma\delta}^{-1} + \frac{1}{2}x_{\alpha\gamma}^{-1}x_{\beta\delta}^{-1} + \frac{1}{2}x_{\beta\gamma}^{-1}x_{\alpha\delta}^{-1} \right) (\det|x|)^{-\Delta}, \tag{206}$$

one may rewrite the expression (204) as

$$(\det|z_{12}|)^{-\Delta} = (\det|x_{12}|)^{-\Delta} \left(1 - i\Delta \frac{x_{12}^m(\gamma_m)_{\alpha\beta}}{x_{12}^2} \theta_1^\alpha \theta_2^\beta - \frac{(2\Delta-1)\Delta}{4} \frac{1}{x_{12}^2} \theta_1^2 \theta_2^2 \right). \tag{207}$$

Thus, from Equation (204) or (207), one may immediately read off the expressions for the correlation functions of the component fields of the superfield (193)

$$\langle \phi(x_1)\phi(x_2) \rangle = c_2(\det|x_{12}|)^{-\frac{1}{2}}, \tag{208}$$

$$\langle f_\alpha(x_1)f_\beta(x_2) \rangle = -ic_2\partial_{\alpha\beta}(\det|x_{12}|)^{-\frac{1}{2}}, \tag{209}$$

$$\langle \phi(x_1)f_\alpha(x_2) \rangle = 0, \quad \langle F(x_1)\phi(x_2) \rangle = 0, \quad \langle F(x_1)f_\alpha(x_2) \rangle = 0, \tag{210}$$

$$\langle F(x_1)F(x_2) \rangle = -\frac{c_2}{8}\partial^{\alpha\beta}\partial_{\alpha\beta}(\det|x|)^{-\Delta}. \tag{211}$$

Let us note that when the superfield $\Phi(x, \theta)$ has the canonical conformal dimension $\Delta = \frac{1}{2}$, due to the identity

$$C^{\alpha\gamma}C^{\beta\delta}\partial_{\alpha\beta}^1\partial_{\gamma\delta}^1(\det|x_{12}|)^{-\frac{1}{2}} = -\frac{1}{2}\eta^{mn}\frac{\partial}{\partial x_1^m}\frac{\partial}{\partial x_1^n}(\det|x_{12}|)^{-\frac{1}{2}}, \tag{212}$$

the last term in (204) is proportional to the δ -function if one moves to the Euclidean signature. Then, one has for the two-point function for the auxiliary field

$$\langle F(x_1)F(x_2) \rangle = -\frac{\pi}{4}c_2\delta^{(3)}(x_1 - x_2). \tag{213}$$

Note that the correlation functions of the auxiliary field F with the physical fields and with itself (for $x_1^m \neq x_2^m$) vanish.

On the other hand, if the conformal weight of the superfield (193) is anomalous, i.e., $\Delta \neq \frac{1}{2}$, the correlators of the auxiliary field with the physical ones still vanish (in agreement with the fact that their conformal weights are different), but the $\langle FF \rangle$ correlator is

$$\begin{aligned} \langle F(x_1)F(x_2) \rangle &= -c_2\frac{(2\Delta-1)\Delta}{4}\frac{1}{x_{12}^2}(\det|x_{12}|)^{-\Delta} \\ &= -c_2\frac{(2\Delta-1)\Delta}{4}(\det|x_{12}|)^{-\Delta-1}. \end{aligned} \tag{214}$$

This situation may correspond to an interacting quantum $\mathcal{N} = 1$ superconformal field theory [116], where the auxiliary field is non-zero, and fields acquire anomalous dimensions due to quantum corrections.

The consideration of three-point functions is analogous. Using the expression for the three-point function (182) and expanding it in series of the θ_i^h variables, we get for the component fields whose labels of scaling dimension we skip for simplicity

$$\langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle = c_3(\det|x_{12}|)^{-k_1}(\det|x_{23}|)^{-k_2}(\det|x_{31}|)^{-k_3}, \tag{215}$$

$$\begin{aligned} &\langle f_\alpha(x_1)f_\beta(x_2)\phi(x_3) \rangle \\ &= -ic_3\frac{k_1x_{12}^m(\gamma_m)_{\alpha\beta}}{x_{12}^2}(\det|x_{12}|)^{-k_1}(\det|x_{23}|)^{-k_2}(\det|x_{31}|)^{-k_3} \\ &= -ic_3k_1x_{12}^m(\gamma_m)_{\alpha\beta}(\det|x_{12}|)^{-k_1-1}(\det|x_{23}|)^{-k_2}(\det|x_{31}|)^{-k_3}, \end{aligned} \tag{216}$$

$$\begin{aligned} &\langle f_\alpha(x_1)F(x_2)f_\beta(x_3) \rangle \\ &= c_3 \frac{k_1 k_2}{2x_{12}^2 x_{23}^2} (\gamma_m)_\alpha^\delta (\gamma_n)_{\delta\beta} (x_{12}^m) (x_{23}^n) (\det|x_{12}|)^{-k_1} (\det|x_{23}|)^{-k_2} (\det|x_{31}|)^{-k_3} \end{aligned} \tag{217}$$

$$= c_3 \frac{k_1 k_2}{2} (\gamma_m)_\alpha^\delta (\gamma_n)_{\delta\beta} (x_{12}^m) (x_{23}^n) (\det|x_{12}|)^{-k_1-1} (\det|x_{23}|)^{-k_2-1} (\det|x_{31}|)^{-k_3},$$

$$\langle F(x_1)F(x_2)\phi(x_3) \rangle = -\frac{c_3}{8} \partial^m \partial_m ((\det|x_{12}|)^{-k_1} (\det|x_{23}|)^{-k_2} (\det|x_{31}|)^{-k_3}). \tag{218}$$

The remaining three-point functions containing an odd number of fermions, as well as the correlator $\langle F\phi\phi \rangle$, vanish. Note that, dimensional arguments would allow for a non-zero $\langle F\phi\phi \rangle$ correlator, but supersymmetry forces it to vanish. The correlator $\langle F(x_1)F(x_2)F(x_3) \rangle$ is zero as well, since it is proportional to $(\gamma_m \gamma_n \gamma_p) x_{12}^m x_{23}^n x_{31}^p = 2i\epsilon_{mnp} x_{12}^m x_{23}^n x_{31}^p = 0$.

Moreover, from the above expressions we see that superconformal symmetry does not fix the values of the scaling dimensions Δ_i . This indicates that quantum operators may acquire anomalous dimensions and the quantum $\mathcal{N} = 1, D = 3$ superconformal theory of scalar superfields can be non-trivial, in agreement e.g., with the results of [116].

If the value of Δ were restricted by superconformal symmetry to its canonical value and no anomalous dimensions were allowed (for all the operators which are not protected by supersymmetry) one would conclude that the conformal fixed point is that of the free theory. This is the case, for instance, for the $\mathcal{N} = 1, D = 4$ Wess-Zumino model in which the chirality of $\mathcal{N} = 1$ matter multiplets and their three-point functions restricts the scaling dimensions of the chiral scalar supermultiplets to be canonical. This implies that in the conformal fixed point the coupling constant is zero, i.e., the theory is free [117,118].

9. Generalized CFT. Part II

In this Section, we shall continue our consideration of the generalized CFT based on the symmetries of the generalized conformal group $Sp(2n)$. We shall mainly follow [27].

9.1. Conserved Currents

In Section 2, we introduced the bosonic and fermionic fields in hyperspace which play the role of the scalar and fermionic fields in ordinary conformal field theory. In order to continue the analogy with CFTs let us consider the fields $b_\Delta^A(X)$ and $f_{\mu\Delta}^A(X)$ where now $A = 1, \dots, N$ is an index of an internal $O(N)$ group (not to be confused with the Weyl spinor indices of the previous Sections) and Δ are corresponding generalized conformal weights.

The two point functions of these fields are similar to those obtained in the previous section, with an obvious generalization including the “color” indexes

$$\langle b_{\Delta_1}^A(X_1), b_{\Delta_2}^B(X_2) \rangle = c_{bb} (\det|X_{12}|)^{-\Delta} \delta^{AB}, \tag{219}$$

$$\langle f_{\alpha(\Delta_1)}^A(X_1), f_{\beta(\Delta_2)}^B(X_2) \rangle = c_{ff} (\det|X_{12}|)^{-\Delta} (X_{12})_{\alpha\beta}^{-1} \delta^{AB}, \tag{220}$$

where $\Delta_1 = \Delta_2 = \Delta$, and $(X_{12})_{\alpha\beta} = (X_1)_{\alpha\beta} - (X_2)_{\alpha\beta}$.

Having introduced global $O(N)$ symmetry, one can construct bosonic and fermionic bilinears

$$J_{\mu\nu}^{AB}(X) = b^A(X) \partial_{\mu\nu} b^B(X) - b^B(X) \partial_{\mu\nu} b^A(X), \tag{221}$$

$$J_{\mu\nu}^{AB}(X) = f_\mu^A(X) f_\nu^B(X) + f_\nu^A(X) f_\mu^B(X). \tag{222}$$

These bilinears correspond to conserved $O(N)$ currents. Indeed one can check that the currents (221) and (222) satisfy the generalized conservation conditions (first introduced in [6])

$$\partial_{\mu\nu} J_{\alpha\beta}^{AB}(X) - \partial_{\mu\alpha} J_{\nu\beta}^{AB}(X) - \partial_{\beta\nu} J_{\alpha\mu}^{AB}(X) + \partial_{\beta\alpha} J_{\nu\mu}^{AB}(X) = 0 \tag{223}$$

provided that the fields $b^A(X)$ and $f_\mu^A(X)$ satisfy the free equations of motion (14) and (15).

Knowing the $Sp(2n)$ transformations (42) and (43) of the fields $b^A(X)$ and $f_\mu^A(X)$ and using Equations (221) and (222), one can derive the $Sp(2n)$ transformations of the conserved currents

$$\delta_a J_{\mu\nu}^{AB}(X) = -a^{\alpha\beta} \partial_{\alpha\beta} J_{\mu\nu}^{AB}(X) \tag{224}$$

$$\delta_g J_{\mu\nu}^{AB}(X) = -\left(g_\alpha^\alpha + 2g_\alpha^\beta X^{\alpha\gamma} \partial_{\beta\gamma}\right) J_{\mu\nu}^{AB}(X) - 2g_{(\mu}^\rho J_{\rho\nu)}^{AB}(X) \tag{225}$$

$$\delta_k J_{\mu\nu}^{AB}(X) = (k_{\alpha\beta} X^{\alpha\beta} + k_{\alpha\beta} X^{\alpha\gamma} X^{\beta\delta} \partial_{\gamma\delta}) J_{\mu\nu}^{AB}(X) + 2k_{(\mu\alpha} X^{\alpha\beta} J_{\beta\nu)}^{AB}(X) \tag{226}$$

From this transformation laws i.e, from the coefficients in front of the terms g_α^α and $k_{\alpha\beta} X^{\alpha\beta}$ one can conclude that the generalized conformal dimension Δ_J of the currents (221) and (222) is equal to 1. The same conclusion can be reached from the fact that (221) and (222) correspond to free currents and the generalized conformal dimension of the fields $b(X)$ and $f_\mu(X)$ is equal to $\frac{1}{2}$. Using the general expression (136), one can see that the generalized conformal dimension is related to the usual scaling dimension as follows. Recall (see Section 2.3) that $SL(n)$ subalgebra of $GL(n)$ algebra is parameterized by $l_\mu^\nu = g_\mu^\nu - \frac{1}{n} \delta_\mu^\nu g_\rho^\rho$. Let us rewrite Equation (225) as

$$\delta_g J_{\mu\nu}^{AB}(X) = -\left(\frac{n+2}{n} g_\alpha^\alpha + 2g_\alpha^\beta X^{\alpha\gamma} \partial_{\beta\gamma}\right) J_{\mu\nu}^{AB}(X) - 2l_{(\mu}^\rho J_{\rho\nu)}^{AB}(X) \tag{227}$$

and define a weight Δ_1 as follows

$$\Delta_1 = 1 + \frac{2}{n}. \tag{228}$$

Then using the relations (136) one can see that

$$\Delta_{\mathbb{D},1} = D - 1 \tag{229}$$

which is the canonical conformal weight of a spin-1 field.

9.2. Stress Tensor

Since we are considering a generalized CFT it is natural to define a generalized stress tensor, which contains a usual CFT stress tensor when projected to the x -subspace. Taking

$$\tilde{T}_{\mu\nu,\rho\sigma}(X) = (\partial_{\mu\nu} b(X))(\partial_{\rho\sigma} b(X)) - \frac{1}{3} b(X)(\partial_{\mu\nu} \partial_{\rho\sigma} b(X)) \tag{230}$$

and

$$\tilde{T}_{\mu\nu,\rho\sigma}(X) = f_\rho(X) \partial_{\mu\nu} f_\sigma(X) \tag{231}$$

we define the generalized stress tensor as a symmetrized combination

$$T_{\mu\nu,\rho\sigma}(X) = \tilde{T}_{\mu\nu,\rho\sigma}(X) + \tilde{T}_{\mu\rho,\nu\sigma}(X) + \tilde{T}_{\mu\sigma,\nu\rho}(X) \tag{232}$$

The reason of taking the expression (232) as a definition for the generalized stress tensor instead of (230) and (231) is that (232) transforms properly under the $Sp(2n)$ transformations

$$\delta_a T_{\mu\nu\rho\sigma}(X) = -a^{\alpha\beta} \partial_{\alpha\beta} T_{\mu\nu,\rho\sigma}(X), \tag{233}$$

$$\delta_g T_{\mu\nu\rho\sigma}(X) = -(g_\alpha^\alpha + 2g_\alpha^\beta X^{\alpha\gamma} \partial_{\beta\gamma}) T_{\mu\nu\rho\sigma}(X) - g_\mu^\alpha T_{\alpha\nu\rho\sigma}(X) - \dots - g_\sigma^\alpha T_{\mu\nu\rho\alpha}(X), \tag{234}$$

$$\delta_k T_{\mu\nu\rho\sigma}(X) = (k_{\alpha\beta} X^{\alpha\beta} + k_{\alpha\beta} X^{\alpha\gamma} X^{\beta\delta} \partial_{\gamma\delta}) T_{\mu\nu\rho\sigma}(X) + k_{\mu\alpha} X^{\alpha\beta} T_{\beta\nu\rho\sigma}(X) + \dots + k_{\sigma\alpha} X^{\alpha\beta} T_{\mu\nu\rho\beta}(X). \tag{235}$$

The transformations above are again derived using the transformations for the free fields (42) and (43) and the explicit form of the stress energy tensor (232). Again, using (136), one can see that the generalized conformal dimension of the stress tensor is $\Delta_T = 1$, whereas the conformal dimension Δ_2 (analogous to the expression (228) for $s = 1$ current) is

$$\Delta_2 = 1 + \frac{4}{n} \tag{236}$$

and the canonical spin-2 field weight is

$$\Delta_{D,2} = D$$

in compliance with the general formula $\Delta_{D,s} = D + s - 2$.

Like the conserved current $J_{\mu\nu}^{AB}$, the stress energy tensor satisfies the generalized conservation conditions

$$\partial_{\mu\nu} T_{\alpha\beta\gamma\delta}(X) - \partial_{\mu\alpha} T_{\nu\beta\gamma\delta}(X) - \partial_{\beta\nu} T_{\alpha\mu\gamma\delta}(X) + \partial_{\beta\alpha} T_{\nu\mu\gamma\delta}(X) = 0 \tag{237}$$

provided the fields satisfy the free equations of motion (14) and (15).

9.3. Higher Spin Conserved Currents

By analogy with $J_{\alpha\beta}(X)$ and $T_{\alpha\beta\gamma\delta}(X)$ one can introduce [6] higher-spin conserved currents $T_{\alpha_1\dots\alpha_{2s}}(X)$ ($2s = 1, 2, 3, \dots$) which transform under $Sp(2n)$ as follows

$$\delta_a T_{\alpha_1\dots\alpha_{2s}}(X) = -a^{\mu\nu} \partial_{\mu\nu} T_{\alpha_1\dots\alpha_{2s}}(X), \tag{238}$$

$$\begin{aligned} \delta_g T_{\alpha_1\dots\alpha_{2s}}(X) = & -(\Delta_s g_{\mu}^{\mu} + 2g_{\nu}^{\mu} X^{\nu\rho} \partial_{\mu\rho}) T_{\alpha_1\dots\alpha_{2s}}(X) \\ & - 2sl_{(\alpha_1}^{\mu} T_{\alpha_2\dots\alpha_{2s})\mu}(X), \end{aligned} \tag{239}$$

$$\begin{aligned} \delta_k T_{\alpha_1\dots\alpha_{2s}}(X) = & (k_{\mu\nu} X^{\mu\nu} + k_{\mu\nu} X^{\mu\rho} X^{\nu\lambda} \partial_{\rho\lambda}) T_{\alpha_1\dots\alpha_{2s}}(X) \\ & + 4k_{\mu(\alpha_1} X^{\mu\nu} T_{\alpha_2\dots\alpha_{2s})\nu}(X), \end{aligned} \tag{240}$$

where

$$\Delta_s = 1 + \frac{2s}{n}. \tag{241}$$

Again, using the relations (136), one can see that

$$\Delta_{\mathbb{D},s} = D + s - 2 \tag{242}$$

which is a conventional expression for a canonical conformal weight for a field with spin s .

The higher spin currents obey $Sp(2n)$ conservation conditions [6]

$$\partial_{\mu\nu} T_{\alpha\beta\gamma(2s-2)}(X) - \partial_{\mu\alpha} T_{\nu\beta\gamma(2s-2)}(X) - \partial_{\beta\nu} T_{\alpha\mu\gamma(2s-2)}(X) + \partial_{\alpha\beta} T_{\mu\nu\gamma(2s-2)}(X) = 0. \tag{243}$$

9.4. Two-Point Correlation Functions of the Currents

We have already considered two-point functions for scalar and spinorial hyperfields (219) and (220). Using these expressions as well as the expressions for the generalized conserved currents (221) and (222), it is straightforward to compute the two-point functions of two currents

$$\langle J_{\alpha\beta}^{AB}(X_1), J_{\mu\nu}^{CD}(X_2) \rangle = C_{JJ} (\det |X_{12}|)^{-1} (P_{12})_{\alpha\beta,\mu\nu} (\delta^{AC} \delta^{BD} - \delta^{AD} \delta^{BC}). \tag{244}$$

Here, we introduced an $Sp(2n)$ -invariant tensor structure (When checking the invariance under the generalized conformal boosts notice that the first pair of the indices of $(P_{12})_{\alpha\beta,\gamma\delta}$ gets rotated with the matrix $k_{\alpha\sigma} X_1^{\sigma\delta}$ and the second pair gets rotated with $k_{\mu\sigma} X_2^{\sigma\delta}$) (which we call P -structure)

$$(P_{ab})_{\alpha\beta,\mu\nu} = (X_{ab}^{-1})_{\mu\alpha} (X_{ab}^{-1})_{\nu\beta} + (X_{ab}^{-1})_{\nu\alpha} (X_{ab}^{-1})_{\mu\beta} \tag{245}$$

$a, b = 1, 2$ and $a \neq b$. which will be one of the building blocks for higher point correlation functions as well.

One more building block for the correlation functions is $(X_{12})_{\alpha\beta}^{-1}$ which is $Sp(2n)$ invariant when considered as a bilocal tensor

$$\begin{aligned} \delta_{tot}(X_{12}^{-1})_{\alpha\beta} &= -(X_{12}^{-1})_{\alpha\gamma}(\delta X_1 - \delta X_2)^{\gamma\delta}(X_{12}^{-1})_{\delta\beta} \\ &+ 2g_{(\alpha}{}^\gamma(X_{12}^{-1})_{\beta)\gamma}k_{\alpha\gamma}X_1^{\gamma\delta}(X_{12}^{-1})_{\delta\beta} - (X_{12}^{-1})_{\alpha\delta}X_2^{\delta\gamma}k_{\gamma\beta} = 0. \end{aligned}$$

Similarly, for the two stress tensors one finds

$$\langle T_{\alpha\beta\gamma\delta}(X_1), T_{\mu\nu\rho\sigma}(X_2) \rangle = C_{TT} \frac{1}{\det |X_{12}|} ((P_{12})_{\alpha\beta,\mu\nu}(P_{12})_{\gamma\delta,\rho\sigma} + \text{symm.}), \tag{246}$$

where the total symmetrization of the both sets of indices $(\alpha\beta\gamma\delta)$ and $(\mu\nu\rho\sigma)$ is assumed.

It is instructive to recall the similar expressions for two-point functions in the usual CFT

$$\langle T_{\mu_1, \dots, \mu_n}^{(l)}(x_1), T_{\nu_1, \dots, \nu_n}^{(l)}(x_2) \rangle = c_{TT} \frac{g_{\mu_1\nu_1}(x_{12}) \dots g_{\mu_n\nu_n}(x_{12})}{(x_{12})^l} - \text{traces} \tag{247}$$

with

$$g_{\mu\nu} = \delta_{\mu\nu} - \frac{x_\mu x_\nu}{x^2}. \tag{248}$$

Obviously, the $Sp(2n)$ -invariant structure $(P_{12})_{\alpha\beta,\gamma\delta}$ is a generalization of $g_{\mu\nu}$. Notice also that the expressions for two-point functions (244)–(246) can be obtained from solving generalized Ward identities, as it has been done for the case of scalar and spinor hyperfields. The generalized Ward identity for an n -point function

$$\langle \Phi_{\alpha_1 \dots \alpha_{r_1}}^{\Delta(1)}(X_1) \dots \Phi_{\beta_1 \dots \beta_{r_k}}^{\Delta(k)}(X_k) \rangle \equiv G_{\alpha_1 \dots \alpha_{r_1} \dots \beta_1 \dots \beta_{r_k}}(X_1, \dots, X_k). \tag{249}$$

is as follows

$$\begin{aligned} &\sum_{i=1}^k \left[\Delta_i (g_{\mu}{}^\mu - k_{\mu\nu} X_i^{\mu\nu}) + \delta X_i^{\mu\nu} \frac{\partial}{\partial X_i^{\mu\nu}} \right] G_{\alpha_1 \dots \alpha_{r_1} \dots \beta_1 \dots \beta_{r_k}}(X_1, \dots, X_k) \\ &+ \sum_{j=1}^1 (g_{\alpha_j}{}^{\mu_j} - k_{\alpha_j\nu} X_1^{\nu\mu_j}) G_{\mu_1 \dots \mu_j \dots \mu_{r_1} \dots \beta_1 \dots \beta_{r_k}}(X_1, \dots, X_k) + \dots \\ &+ \sum_{j=1}^{r_k} (g_{\beta_j}{}^{\mu_j} - k_{\beta_j\nu} X_k^{\nu\mu_j}) G_{\alpha_1 \dots \alpha_{r_k} \dots \mu_1 \dots \mu_j \dots \mu_{r_k}}(X_1, \dots, X_k) = 0, \end{aligned} \tag{250}$$

It is straightforward to check that the two-point functions solve Equations (250).

9.5. Three Point Functions: *bbb* and *ffb*

Three-point functions for three scalars and for two fermions and a scalar (computed firstly in [11]) have been given in Section 8.2 in the supersymmetric form and as a particular example for $D = 3$ were given in Section 8.4. The only difference with the case without supersymmetry is that the overall constants in front of the non-supersymmetric ones are independent of each other

$$\langle b_{\Delta_1}(X_1)b_{\Delta_2}(X_2)b_{\Delta_3}(X_3) \rangle = C_{bbb} (\det |X_{12}|)^{-k_3} (\det |X_{23}|)^{-k_1} (\det |X_{13}|)^{-k_2}, \tag{251}$$

$$\langle f_\alpha(X_1)f_\beta(X_2)b(X_3) \rangle = c_{ffb} (X_{12}^{-1})_{\alpha\beta} (\det |X_{12}|)^{-k_3} (\det |X_{23}|)^{-k_1} (\det |X_{13}|)^{-k_2}. \tag{252}$$

$$k_a = \frac{1}{2}(\Delta^{(a+1)} + \Delta^{(a+2)} - \Delta^{(a)}), \quad \text{cycl.} \quad (a = 1, 2, 3). \tag{253}$$

9.6. Three-Point Functions with J and T

Now, we would like to consider three-point functions which include the generalized conserved current $J_{\alpha\beta}^{AB}(X)$ and generalized stress tensor $T_{\alpha\beta\gamma\delta}(X)$. These can give us an answer whether an interacting generalized conformal field theory based on $Sp(2n)$ symmetry exists. As we shall see below, the answer to this question is negative.

Our strategy is as follows. As we have seen the generalized conformal weights of $J_{\alpha\beta}^{AB}(X)$ and $T_{\alpha\beta\gamma\delta}(X)$ are equal to one, $\Delta_J = \Delta_T = 1$. If we assume that the corresponding symmetries are not broken by interactions, then the values of Δ_J and Δ_T will remain the same. Therefore, we would like to construct $Sp(2n)$ -invariant three- and higher-order correlation functions which include $J_{\alpha\beta}^{AB}(X)$, $T_{\alpha\beta\gamma\delta}(X)$ and other operators \mathcal{O} and see if the conservation conditions (223) and (237) along with $Sp(2n)$ invariance allow for the operators \mathcal{O} to have anomalous dimensions. We will find that this is unfortunately not the case for $n > 2$.

First let us introduce one more $Sp(2n)$ -invariant tensor structure (which we call Q-structure)

$$(Q_{ab}^c)_{\alpha\beta} = (X_{ac}^{-1})_{\alpha\beta} - (X_{bc}^{-1})_{\alpha\beta}, \quad a, b, c = 1, 2, 3 \tag{254}$$

This structure, along with (245) and

$$(p_{ab})_{\alpha\beta} = (X_a^{\alpha\beta} - X_b^{\alpha\beta})^{-1}, \quad a, b = 1, 2, \quad a \neq b. \tag{255}$$

is a building block for all the $Sp(2n)$ -invariant correlation functions. In other words, the most general multi-point function can be written as a sum over all possible polynomials of a required rank of the three structures $p_{ab} = X_{ab}^{-1}$, P_{ab} and Q_{ab}^c times a pre-factor

$$\langle \Phi \dots \Phi \rangle = G(p_{ab}, P_{ab}, Q_{ab}^c | X_{ab}). \tag{256}$$

Following this prescription one can immediately write the simplest three-point function of two scalars (with generalized conformal dimensions $\Delta_1 = \Delta_2 = \Delta$) and a conserved current (with $\Delta_J = 1$)

$$\begin{aligned} \langle b_{\Delta_1}(X_1) b_{\Delta_2}(X_2) J_{\alpha\beta}(X_3) \rangle &= \\ &= C_{bbJ} (\det |X_{12}|)^{-k_3} (\det |X_{13}|)^{-k_2} (\det |X_{23}|)^{-k_1} (Q_{12}^3)_{\alpha\beta}, \end{aligned} \tag{257}$$

and a three-point function of the two scalars (with $\Delta_1 = \Delta_2 = \Delta$) and the stress tensor (with $\Delta_T = 1$)

$$\begin{aligned} \langle b(X_1) b(X_2) T_{\alpha\beta\gamma\delta}(X_3) \rangle &= C_{bbT} (\det |X_{12}|)^{-k_3} (\det |X_{13}|)^{-k_2} \times \\ &(\det |X_{23}|)^{-k_1} ((Q_{12}^3)_{\alpha\beta} (Q_{12}^3)_{\gamma\delta} + (Q_{12}^3)_{\alpha\gamma} (Q_{12}^3)_{\beta\delta} + (Q_{12}^3)_{\alpha\delta} (Q_{12}^3)_{\beta\gamma}), \end{aligned} \tag{258}$$

where k_a are restricted according to (253). One can see that $Sp(2n)$ invariance alone does not impose any requirement on the generalized conformal dimension Δ of the scalar field.

The next step is to require the conservation of the current $J_{\alpha\beta}^{AB}(X)$ and the stress tensor $T_{\alpha\beta\gamma\delta}(X)$ according to Equations (223) and (237). This implies

$$k_1 = k_2 = \frac{1}{2}, \quad \text{and any } k_3. \tag{259}$$

Therefore, in this case, no restriction on generalized conformal dimension of the scalar field appears i.e., anomalous dimension and therefore interactions are allowed. At this, the current and the stress tensor remain conserved, and their dimensions remain canonical $\Delta_J = \Delta_T = 1$.

The next nontrivial example is a three point-function of two conserved currents and one scalar operator $\mathcal{O}(X)$ of dimension Δ . From the $Sp(2n)$ -invariance condition we have

$$\langle J_{\mu\nu}(X_1)\mathcal{O}(X_2)J_{\alpha\beta}(X_3)\rangle = (\det|X_{12}|)^{-\frac{\Delta}{2}}(\det|X_{13}|)^{-\frac{2-\Delta}{2}}(\det|X_{23}|)^{-\frac{\Delta}{2}} \times (\mathcal{A}[(Q_{12}^3)_{\alpha\beta}(Q_{23}^1)_{\mu\nu}] + \mathcal{B}(P_{13})_{\mu\nu,\alpha\beta}) \tag{260}$$

where \mathcal{A} and \mathcal{B} are some constants. Again, one can see that $Sp(2n)$ symmetry alone does not impose any restriction on the generalized conformal dimension of $\mathcal{O}(X)$.

However, imposing the current conservation condition (223), one gets

$$\mathcal{A} = \mathcal{B}, \quad \text{and} \quad \Delta = 1, \tag{261}$$

that is the dimension of the operator $\mathcal{O}(X)$ is fixed (Since the canonical dimension of the field $b(X)$ is equal to $\frac{1}{2}$ it is natural to assume that the operator $\mathcal{O}(X)$ is a composite one $\mathcal{O}(X) = b^2(X)$.) by the current conservation condition. Let us note that from the point of view of the x -space the current $J_{\alpha\beta}^{AB}(X)$ contains higher spin currents as a result of its expansion in series of y coordinates. Therefore, this result is in accordance with the theorem of [119] stating that the conformal field theories which contain conserved higher-spin currents should be free.

Let us note, however, that in the simplest case of $n = 2$, i.e., $D = 3$ CFTs with the $Sp(4)$ conformal group the two conditions (261) are reduced to one (see [27] for technical details)

$$\mathcal{A}(D - 1 - \Delta) - \mathcal{B}\Delta = 0. \tag{262}$$

This means that the conformal dimension Δ of the operator $\mathcal{O}(X)$ remains undetermined, and hence this analysis does not ban the existence of interacting $D = 3$ CFTs, as is well known.

9.7. General Case

Let us now discuss the general structure of the three-point correlators of conserved currents which are symmetric tensors of rank $r = 2s$ with s being an integer “spin”. To this end, it is convenient to hide the tensor indices away by contracting them with auxiliary variables λ_a^α , where a refers to the point of the operator insertion:

$$(p_{ab})_{\alpha\beta} \Rightarrow p_{ab} = (X_{ab}^{-1})_{\alpha\beta} \lambda_a^\alpha \lambda_b^\beta \quad \text{no summation over } a, b. \tag{263}$$

$$(P_{bc})_{\alpha\beta,\gamma\delta} \Rightarrow P_{ab} = 2p_{ab}p_{ba} = (P_{ab})_{\alpha\beta,\gamma\delta} \lambda_a^\alpha \lambda_a^\beta \lambda_b^\gamma \lambda_b^\delta \quad \text{no summation over } a, b, \tag{264}$$

$$(Q_{bc}^a)_{\alpha\beta} \Rightarrow Q_{bc}^a = (Q_{bc}^a)_{\alpha\beta} \lambda_a^\alpha \lambda_a^\beta \quad \text{no summation over } a. \tag{265}$$

For instance, the correlator of two scalar operators \mathcal{O} of the same dimension Δ with a conserved current of an integer spin- s obeying (243) is

$$\langle \mathcal{O}(X_1)\mathcal{O}(X_2)J_s(X_3)\rangle = C(\det|X_{12}|)^{-\frac{2-\Delta}{2}}(\det|X_{13}|)^{-\frac{1}{2}}(\det|X_{23}|)^{-\frac{1}{2}}(Q_{12}^3)^s. \tag{266}$$

The current conservation condition leads to the same result as for the case of $s = 1, 2$, i.e., $k_1 = k_2 = \frac{1}{2}$, which means that the dimensions of the scalar operators are arbitrary.

However, if we consider a three-point function of a scalar operator and two conserved currents

$$J_s(X) = J_{\alpha_1 \dots \alpha_{2s}}(X) \lambda^{\alpha_1} \dots \lambda^{\alpha_{2s}} \tag{267}$$

of ranks $2s_1$ and $2s_2$ with $s \geq 1$, we will again find that, up to an overall factor, all the free parameters in the correlator are fixed. For example,

$$\langle J_3(X_1)J_1(X_2)O(X_3) \rangle = C \frac{(Q_{23}^1)^3 Q_{13}^2 - 3(Q_{23}^1)^2 P_{12}}{(\det |X_{12}| \det |X_{13}| \det |X_{23}|)^{1/2}}. \tag{268}$$

From the discussion above, one can conclude that in order to describe the $Sp(2n)$ -invariant three-point functions, we can borrow the generating functions of 3-point correlators of free symmetric higher-spin fields in conventional conformal theories [114,120–123] simply because the $Sp(2n)$ group contains the corresponding conformal group $SO(2, D)$ as a subgroup, or, in other words, the correlators in the free CFTs can be covariantly embedded into the $Sp(2n)$ invariant correlators. For example, a generating function of the three-point functions of currents built out of free scalars $b(X)$ is

$$\langle J(X_1)J(X_2)J(X_3) \rangle = \frac{\cos(p_{12}) \cos(p_{13}) \cos(p_{23}) \exp\left(\frac{1}{2}[Q_{23}^1 + Q_{13}^2 + Q_{12}^3]\right)}{(\det |X_{12}| \det |X_{23}| \det |X_{13}|)^{1/2}}. \tag{269}$$

It contains the operators $J_s(X)$, $s = 0, 1, 2, \dots$ and the correlator $\langle J_{s_1} J_{s_2} J_{s_3} \rangle$ is obtained as the coefficient in front of $(\lambda_1)^{2s_1} (\lambda_2)^{2s_2} (\lambda_3)^{2s_3}$.

The generating function obtained from the currents built out of the free fermions $f_\alpha(X)$ is

$$\langle J(X_1)J(X_2)J(X_3) \rangle = \frac{\sin(p_{12}) \sin(p_{13}) \sin(p_{23}) \exp\left(\frac{1}{2}[Q_{23}^1 + Q_{13}^2 + Q_{12}^3]\right)}{(\det |X_{12}| \det |X_{23}| \det |X_{13}|)^{1/2}}. \tag{270}$$

The generating function of multi-point correlators can be found in [114,122–125].

The above expressions deal with the bosonic symmetric tensor currents of even rank. The generating function which produces three-point correlators involving two fermionic currents of odd ranks is similar, see e.g., [119].

As a further development of this subject, it would be of interest to carry out the study of other aspects of the $Sp(2n)$ -invariant higher-spin systems, in particular, to explore their links to recent results on conformal higher-spin theories in AdS_D backgrounds (see e.g., [126–130]) and to $Sp(2n)$ -invariant unfolded higher-spin structures discussed in [131].

9.8. Breaking $Sp(2n)$ Symmetry

As it follows from the discussion above, to have an interacting generalized conformal field theory based on $Sp(2n)$ symmetry, one has to break this symmetry down to a subgroup. Obviously, to still use $Sp(2n)$ symmetry as a symmetry of the theory, it should be broken spontaneously rather than explicitly. On the other hand, the question whether a symmetry is broken spontaneously or explicitly could be simpler to address if one had the corresponding Lagrangian, which would produce the field Equations (14) and (15) (and/or their possible nonlinear or massive deformations). Unfortunately, such a Lagrangian is still lacking.

In this respect, let us mention that the issue of breaking $Sp(8)$ symmetry via current interactions in the unfolded formulation has been addressed in [26]. In particular, analyzing the system of equations

$$DC(x, \mu, \bar{\mu}) = F(\omega, J(x, \mu, \bar{\mu})), \quad D_2 J(x, \mu, \bar{\mu}) = 0, \tag{271}$$

where $D = d + \omega$ is a spin connection, J is a current which is bilinear in the higher-spin functional C and D_2 is the corresponding kinetic operator (see the discussion around the Equation (27)), the authors showed that the $Sp(8)$ symmetry is broken to the four-dimensional conformal group $SO(2, 4)$.

In the hyperspace framework, one may try to approach this problem as follows. First, one should construct a nonlinear deformation of Equations (14) and (15)

$$\partial_{\alpha\beta}\partial_{\gamma\delta} b(X) - \partial_{\alpha\gamma}\partial_{\beta\delta} b(X) = F_b(b, f, A), \tag{272}$$

$$\partial_{\alpha\beta}f_{\gamma}(X) - \partial_{\alpha\gamma}f_{\beta}(X) = F_f(b, f, A). \tag{273}$$

with some unknown functions $F_b(b, f, A)$ and $F_f(b, f, A)$. It is natural to expect that these functions depend also on higher-spin potentials A , in addition to the higher-spin curvatures contained in the hyperfields $b(X)$ and $f_{\mu}(X)$. Note that, in the unfolded description of the $Sp(8)$ -invariant system, higher-spin gauge potentials were introduced, at the linearized level, in [16]. As a necessary step forward, one should understand whether and how the Equations (272) may result from a (non-linear) generalization of the construction of [16].

The right hand sides of the Equation (272) should be chosen under the requirement that the analysis of the Equations (272) and (273), similar to the one carried out for the free equations in Section 2.1 leads to a physically meaningful nonlinear equations in the x -space. This is an interesting open problem for a future study.

10. Conclusions

The idea to formulate higher-spin theories in an extended (super) space, where extra coordinates generate higher spins (by analogy with the Kaluza–Klein theories where compact extra dimensions generate “higher masses”) seems to be very attractive, especially taking into account a level of complexity of higher-spin theories formulated in an ordinary space–time.

The underlying symmetry of this formulation is the $Sp(2n)$ group, which contains the corresponding D -dimensional conformal group as a subgroup. This allows one to borrow, for the analysis of the $Sp(2n)$ -invariant systems, an intuition and techniques from conventional Conformal Field Theories.

To summarize, the reviewed approach generalizes familiar concepts to higher-dimensional tensorial spaces and the correspondence looks schematically as follows

- Space–time coordinates x^m are extended to tensorial coordinates $X^{\alpha\beta}$.
- Cartan–Penrose relation $P_{AA} = \lambda_A \bar{\lambda}_{\dot{A}}$ gets extended to the hyperspace twistor-like relation $P_{\alpha\beta} = \lambda_{\alpha} \lambda_{\beta}$ which determines free dynamics of fields in the tensorial space with the momentum $P_{\alpha\beta}$ conjugate to $X^{\alpha\beta}$.
- AdS_D space is extended to the $Sp(n)$ group manifold.
- Conformal scalar $\phi(x)$ and conformal spinor $\psi_{\mu}(x)$ become the “hyperscalar” $b(X)$ and the “hyperspinor” $f_{\mu}(X)$.
- D -dimensional conformal group $SO(2, D)$ is extended to the $Sp(2n)$ group which underlies the Generalized Conformal Field Theory of the fields $b(X)$ and $f_{\mu}(X)$.

We have shown that the hyperspace approach describes (in $D = 3, 4, 6$ and 10) free dynamics of an infinite set of massless conformal higher-spin fields in an elegant compact form. An important and non-trivial problem is to find a non-linear generalization of this formulation which would correspond to an interacting higher-spin theory. This problem has been addressed by several authors. As we have seen, it is related to the necessity to break the $Sp(2n)$ symmetry in an appropriate way. Attempts to construct such a generalization in the framework of hyperspace supergravity and a non-linear realization of the $OSp(1|8)$ supergroup were undertaken, respectively, in [12,14]. Obstacles encountered in these papers may be related to the fact that their constructions utilized only higher-spin field strengths but did not include couplings to higher-spin gauge potentials, while the consistent formulation of nonlinear equations of massless higher-spin fields contains both [37–39]. Therefore, to successfully address the problem of interactions it is important to incorporate higher-spin potentials in the hyperspace approach, e.g., by further elaborating on the construction of [16].

Another issue, which can be related to the previous one, is a question of consistent breaking $Sp(2n)$ symmetry. The manifestation of this breaking was observed e.g. in higher-spin current interactions [26]. As we have seen in Section 9, when considering generalized CFT based on global $Sp(2n)$ invariance (see [27]), the requirement of generalized current conservation turns out to be too strong to allow for the basic hyperfields to have anomalous conformal dimensions and again points at the necessity to (spontaneously) break $Sp(2n)$ invariance.

Theories with spontaneously broken $Sp(2n)$ symmetry might be also useful for studying massive higher-spin fields in hyperspaces. A consideration of theories with local $Sp(n)$ invariance i.e., some sort of generalized gravity is yet another interesting and widely unexplored area.

Finally, let us mention that field Equations (14) and (15) for the fields in hyperspaces remind (a part of) weak section conditions of exceptional field theories (see [132] for a review and references). This similarity can be relevant for higher-spin extensions of these theories, provided the section conditions can be properly relaxed (see e.g., [133,134] for a discussion of this point). It would be interesting to further elaborate on this issue, as a connection to the E_{11} framework [18].

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Appendix A. Conventions

The γ -matrices satisfy the following anti-commutation relations

$$(\gamma^m)^\alpha{}_\delta (\gamma^n)^\delta{}_\beta + (\gamma^n)^\alpha{}_\delta (\gamma^m)^\delta{}_\beta = 2\eta^{mn} \delta^\alpha{}_\beta, \tag{A1}$$

where m, n and other Latin letters are space-time vector indices, and α, β and other Greek letters label spinorial indices. Throughout the paper “(,)” denotes symmetrization and “[,]” denotes antisymmetrization with weight one. The symplectic matrix $C^{\alpha\beta} = -C^{\beta\alpha}$ is used to relate upper and lower spinorial indexes as follows

$$\mu^\alpha = C^{\alpha\beta} \mu_\beta, \quad \mu_\alpha = -C_{\alpha\beta} \mu^\beta, \quad C^{\alpha\gamma} C_{\gamma\beta} = -\delta^\alpha{}_\beta. \tag{A2}$$

The differentiation by hypercoordinates $X^{\alpha\beta}$ is as follows

$$\frac{dX^{\alpha\beta}}{dX^{\gamma\delta}} \equiv \partial_{\alpha\beta} X^{\gamma\delta} = \frac{1}{2} (\delta^\alpha{}_\gamma \delta^\beta{}_\delta + \delta^\beta{}_\gamma \delta^\alpha{}_\delta), \tag{A3}$$

$$\partial_{\mu\nu} X_{\alpha\beta}^{-1} = -\frac{1}{2} (X_{\mu\alpha}^{-1} X_{\nu\beta}^{-1} + X_{\mu\beta}^{-1} X_{\nu\alpha}^{-1}) \tag{A4}$$

and

$$\partial_{\mu\nu} (\det X) = X_{\mu\nu}^{-1} (\det X) \tag{A5}$$

where

$$X_{\mu\nu}^{-1} X^{\nu\alpha} = \delta_\mu^\alpha. \tag{A6}$$

Let us note that the product of an even number of $X^{\alpha\beta}$ matrices is antisymmetric in spinorial indexes, whereas the product of an odd number of $X^{\alpha\beta}$ is a symmetric matrix. For example,

$$X^{\alpha\gamma} X_\gamma{}^\beta = -X^{\beta\gamma} X_\gamma{}^\alpha, \quad X^\alpha{}_\gamma X^\gamma{}_\delta X^{\delta\beta} = +X^\beta{}_\delta X^\delta{}_\gamma X^{\gamma\alpha}, \quad \text{etc.} \tag{A7}$$

Appendix B. Derivation of the field Equations on $Sp(n)$

Let us evaluate the operator $Y_{(\alpha} Y_{\beta)}$ in (93):

$$\frac{1}{2}(Y_{\alpha} Y_{\beta} + Y_{\beta} Y_{\alpha}) \equiv Y_{(\alpha} Y_{\beta)} = \left(\frac{\xi}{8}\right)^2 \mu_{\alpha} \mu_{\beta} + \frac{i\xi}{8} \left(\mu_{\alpha} \frac{\partial}{\partial \mu^{\beta}} + \mu_{\beta} \frac{\partial}{\partial \mu^{\alpha}} \right) - \frac{\partial}{\partial \mu^{\alpha}} \frac{\partial}{\partial \mu^{\beta}} . \tag{A8}$$

Appendix B.1. Fermionic Equation

Consider Equation (93). Substituting into it the expansion (94) one gets for the term linear in μ^{α}

$$\nabla_{\alpha\beta} F_{\gamma}(X) \mu^{\gamma} + \frac{\xi}{8} (C_{\gamma\alpha} F_{\beta}(X) + C_{\gamma\beta} F_{\alpha}(X)) \mu^{\gamma} = 0 \tag{A9}$$

The second term comes from $-\frac{i}{2}(Y_{\alpha} Y_{\beta} + Y_{\alpha} Y_{\beta})$ acting on $F_{\gamma} \mu^{\gamma}$. From this equation, one gets (96).

Appendix B.2. Bosonic Equation

Equation (93) to the zeroth order in μ^{α} becomes:

$$\nabla_{\alpha\beta} B(X) = iY_{(\alpha} Y_{\beta)} \cdot \frac{1}{2} B_{\gamma\delta}(X) \mu^{\gamma} \mu^{\delta} . \tag{A10}$$

Obviously, only the double μ -derivative in $Y_{(\alpha} Y_{\beta)}$ will contribute to this order. Thus, we have:

$$\nabla_{\alpha\beta} B(X) = -i \frac{\partial}{\partial \mu^{\alpha}} \frac{\partial}{\partial \mu^{\beta}} \cdot \frac{1}{2} B_{(\gamma\delta)}(X) \mu^{\gamma} \mu^{\delta} \tag{A11}$$

Therefore,

$$\nabla_{\alpha\beta} B(X) = -i B_{(\alpha\beta)}(X) , \tag{A12}$$

Which indicates that all the higher order components in the expansion (94) are expressed in terms of $B(X)$ and $F_{\alpha}(X)$.

To zeroth order in μ^{α} , we compute:

$$(\nabla_{\alpha\beta} - iY_{(\alpha} Y_{\beta)})(\nabla_{\gamma\delta} - iY_{(\gamma} Y_{\delta)}) \left[B(X) + \frac{1}{2} B_{\rho\sigma}(X) \mu^{\rho} \mu^{\sigma} + \frac{1}{4!} B_{\rho\sigma\tau\lambda}(X) \mu^{\rho} \mu^{\sigma} \mu^{\tau} \mu^{\lambda} + \dots \right] = 0 . \tag{A13}$$

$$\begin{aligned} 0 = & \nabla_{\alpha\beta} \nabla_{\gamma\delta} B(X) + (C_{\alpha\gamma} C_{\beta\delta} + C_{\beta\gamma} C_{\alpha\delta}) B(X) + \left(\frac{\xi}{8}\right)^2 B_{(\alpha\beta\gamma\delta)}(X) \\ & + i \left(\frac{\xi}{8}\right) \left[C_{\alpha\gamma} B_{(\beta\delta)}(X) + C_{\alpha\delta} B_{(\beta\gamma)}(X) + C_{\beta\gamma} B_{(\alpha\delta)}(X) + C_{\beta\delta} B_{(\alpha\gamma)}(X) \right] \\ & + i \left[\nabla_{\gamma\delta} B_{(\alpha\beta)}(X) + \nabla_{\alpha\beta} B_{(\gamma\delta)}(X) \right] . \end{aligned} \tag{A14}$$

Now, using (A12), this becomes:

$$\begin{aligned} 0 = & \nabla_{\alpha\beta} \nabla_{\gamma\delta} B(X) + \left(\frac{\xi}{8}\right)^2 (C_{\alpha\gamma} C_{\beta\delta} + C_{\beta\gamma} C_{\alpha\delta}) B(X) + B_{(\alpha\beta\gamma\delta)}(X) \\ & - \frac{\xi}{8} \left[C_{\alpha\gamma} \nabla_{\beta\delta} + C_{\alpha\delta} \nabla_{\beta\gamma} + C_{\beta\gamma} \nabla_{\alpha\delta} + C_{\beta\delta} \nabla_{\alpha\gamma} \right] B(X) \\ & - \left[\nabla_{\gamma\delta} \nabla_{\alpha\beta} + \nabla_{\alpha\beta} \nabla_{\gamma\delta} \right] B(X) . \end{aligned} \tag{A15}$$

Using the algebra (155) for the covariant derivatives $\nabla_{\alpha\beta}$, we can write:

$$\nabla_{\gamma\delta} \nabla_{\alpha\beta} B(X) = \left(\frac{\xi}{8}\right)^2 (C_{\alpha\gamma} C_{\beta\delta} + C_{\beta\gamma} C_{\alpha\delta}) B(X) + B_{(\alpha\beta\gamma\delta)}(X) - \frac{1}{2} [\nabla_{\alpha\beta}, \nabla_{\gamma\delta}] B(X) . \tag{A16}$$

From this equation, we obtain the bosonic Equation (95). Let us note that exchange of indexes as $\alpha \leftrightarrow \gamma$ and $\beta \leftrightarrow \delta$:

$$\nabla_{\alpha\beta}\nabla_{\gamma\delta}B(X) = (\frac{\xi}{8})^2(C_{\alpha\gamma}C_{\beta\delta} + C_{\beta\gamma}C_{\alpha\delta})B(X) + B_{(\alpha\beta\gamma\delta)}(X) + \frac{1}{2}[\nabla_{\alpha\beta}, \nabla_{\gamma\delta}]B(X) . \tag{A17}$$

and subtraction of (A16) and (A17) leads to an identity.

Appendix C. Some Identities for Supercoordinates on $OSp(1|n)$ Group Manifold

The supercoordinates on $OSp(1|n)$ group manifold obey some useful relations in particular

$$\theta^\alpha \mathcal{G}_\alpha^\beta = \Theta^\beta P(\Theta^2), \quad \theta^\alpha = \Theta^\beta \mathcal{G}_\beta^{-1\alpha} P(\Theta^2), \tag{A18}$$

$$Q_\beta \Theta^\alpha = P^{-1}(\Theta^2) \left(G_\beta^\alpha + \frac{i\xi}{8} \Theta_\beta \Theta^\alpha + \frac{i\xi}{8} G_\beta^\sigma \Theta_\sigma \Theta^\alpha + \left(\frac{i\xi}{8} \right)^2 \Theta^2 \Theta_\beta \Theta^\alpha \right), \tag{A19}$$

$$(Q_\beta \Theta^\alpha) \Theta_\alpha = P(\Theta^2) \left(G_\beta^\sigma + \frac{i\xi}{8} \Theta_\beta \Theta^\sigma \right) \Theta_\sigma, \tag{A20}$$

$$\partial_{\alpha\beta} \Theta^\gamma = \frac{\xi}{4} \Theta_{(\alpha} G_{\beta)}^\delta (\delta_\delta^\gamma + \frac{i\xi}{8} \Theta_\delta \Theta^\gamma), \tag{A21}$$

$$D_\beta \mathcal{G}_\alpha^\gamma = \frac{i\xi}{4} P(\Theta^2) (\Theta_\alpha - 2G_{\alpha\rho} \Theta_\rho) \mathcal{G}_\beta^\gamma \tag{A22}$$

$$\partial_{\alpha\beta} \mathcal{G}_\gamma^\delta = \frac{\xi}{4} \mathcal{G}_{\gamma(\alpha} \mathcal{G}_{\beta)}^\delta, \tag{A23}$$

and

$$Q_\alpha \mathcal{G}_{\mu\nu} = -\frac{i\xi}{4} P(\Theta^2) \Theta_\nu \mathcal{G}_{\mu\alpha}. \tag{A24}$$

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