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Curvature Invariants for Charged and Rotating Black Holes

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Abstract: Riemann curvature invariants are important in general relativity because they encode the geometrical properties of spacetime in a manifestly coordinate-invariant way. Fourteen such invariants are required to characterize four-dimensional spacetime in general, and Zakhary and McIntosh showed that as many as seventeen can be required in certain degenerate cases. We calculate explicit expressions for all seventeen of these Zakhary–McIntosh curvature invariants for the Kerr–Newman metric that describes spacetime around black holes of the most general kind (those with mass, charge, and spin), and confirm that they are related by eight algebraic conditions (dubbed syzygies by Zakhary and McIntosh), which serve as a useful check on our results. Plots of these invariants show richer structure than is suggested by traditional (coordinate-dependent) textbook depictions, and may repay further investigation.

Keywords: black holes; curvature invariants; general relativity

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1. Introduction

Quantities whose value is manifestly independent of coordinates are particularly useful in general relativity [1]. Riemann or curvature invariants, formed from the Riemann tensor and derivatives of the metric, are one example. In principle, one can construct fourteen such quantities in four-dimensional spacetime, since the Riemann curvature tensor has twenty independent components subject to six conditions on the metric [2]. These quantities have proved useful in, for example, classifying metrics [3,4] and deciding whether or not they are equivalent [5,6]. They have been applied to speed up the estimation of gravitational-wave signatures from black-hole collisions [7], to distinguish between "gravito-electrically" versus "gravito-magnetically dominated" regions of spacetime [8–10], to measure the mass and spin and locate the horizons of black holes [11–14], and to study perturbations of the Kerr metric [15], Lorentzian wormholes [16], and others [17].

More than fourteen independent curvature invariants may be required to describe certain degenerate cases in the presence of matter [18]. Zakhary and McIntosh (ZM) reviewed this problem and offered the first complete list of seventeen independent real curvature invariants for all possible metric types (6 Petrov types and 15 Segre types, or 90 types in all) [19]. However, the ZM invariants are defined in terms of spinorial quantities whose physical meaning can be obscure. Here, we apply the ZM formalism to the Kerr–Newman metric to obtain for the first time explicit algebraic expressions for all seventeen invariants for black holes of the most general kind (those with mass, charge, and spin). Our results will be of most astrophysical interest in the case of nonzero spin and zero charge (because all real black holes rotate, but few possess significant charge due to the preferential infall of opposite-charged matter).

2. Preliminaries

Our starting point is the line element for Kerr–Newman spacetime [20] expressed in Boyer–Lindquist coordinates $x^i = t, r, \theta, \phi$ [21–24]

$$ds^2 = -\frac{\Delta}{\rho^2}\left(c\,dt - a\sin^2\theta\,d\phi\right)^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)d\phi - ca\,dt\right]^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2\,, \tag{1}$$

where

$$\Delta \equiv r^2 - r_s r + a^2 + r_q^2 \quad , \quad \rho^2 \equiv r^2 + a^2\cos^2\theta\,. \tag{2}$$

Here, m is the mass of the black hole as measured at infinity, $r_s \equiv 2Gm/c^2$ is its Schwarzschild radius, $a \equiv J/(mc)$ and $r_q \equiv Gq^2/(4\pi\epsilon_0)$ where J and q are angular momentum and electric charge.

Switching to natural units where $c = G = 4\pi\epsilon_0 = 1$ and re-arranging terms, we re-express the metric in a form more amenable to symbolic computation [25]:

$$\begin{aligned}
g_{rr} &= \frac{r^2 + a^2\cos^2\theta}{r^2 - 2mr + a^2 + q^2}\,, \\
g_{\theta\theta} &= r^2 + a^2\cos^2\theta\,, \\
g_{\phi\phi} &= \left[r^2 + a^2 + \frac{a^2\left(2mr - q^2\right)\sin^2\theta}{r^2 + a^2\cos^2\theta}\right]\sin^2\theta\,, \\
g_{\phi t} &= g_{t\phi} = -\frac{a\left(2mr - q^2\right)\sin^2\theta}{r^2 + a^2\cos^2\theta}\,, \\
g_{tt} &= -\left(1 - \frac{2mr - q^2}{r^2 + a^2\cos^2\theta}\right)\,.
\end{aligned} \tag{3}$$

This metric reduces to the Kerr solution when $q = 0$, the Reissner–Nordström solution when $a = 0$, the Schwarzschild solution when $a = q = 0$, and Minkowski spacetime (in oblate spheroidal coordinates) when $m = 0$ [23,26]. The angular coordinates ϕ, θ are standard spherical polar angles (θ is the angular displacement from the black hole's spin axis) but the Boyer–Lindquist coordinate r is *not* conventional; it goes over to the usual radial distance far from the black hole, but goes to zero at the ring singularity and $-\infty$ at the center of the geometry.

To calculate the curvature invariants for this metric, we used *Mathematica* [27] (For the convenience of readers, our code is included as Supplementary Material with this article). The Christoffel symbols and Riemann tensor are obtained as usual from $\Gamma^i{}_{jk} = \frac{1}{2}g^{il}\left(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}\right)$ and $R^i{}_{jkl} = \partial_k \Gamma^i{}_{jl} - \partial_l \Gamma^i{}_{jk} + \Gamma^o{}_{jl}\Gamma^i{}_{ok} - \Gamma^o{}_{jk}\Gamma^i{}_{ol}$, where ∂_k means $\partial/\partial x^k$ and repeated indices are summed over (Einstein summation convention). Contracting gives the Ricci tensor $R_{ij} = R^k{}_{ikj}$ and the Ricci (or curvature) scalar $R = g^{ij}R_{ij}$. Indices are lowered or raised with the metric, so that the Riemann tensor in fully covariant form, for example, is $R_{ijkl} = g_{io}R^o{}_{jkl}$.

We need the Weyl (or conformal curvature) tensor, defined in $n = 4$ dimensions by [2,23]

$$C_{ijkl} = R_{ijkl} + \frac{1}{6}R\left(g_{ik}g_{jl} - g_{il}g_{jk}\right) - \frac{1}{2}\left(g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik}\right)\,. \tag{4}$$

We also need the *dual* of the Weyl tensor, defined by [28]

$$C^*{}_{ijkl} = \frac{1}{2}E_{ijop}C^{op}{}_{kl}\,, \tag{5}$$

where E_{ijkl} is the Levi–Civita tensor, defined by [23]

$$E_{ijkl} = \frac{1}{\sqrt{|g|}}\tilde{\epsilon}^{ijkl} \,. \tag{6}$$

Here, g is the determinant of the metric so $\sqrt{|g|} = \left(r^2 + a^2\cos^2\theta\right)\sin\theta$, and $\tilde{\epsilon}^{ijkl} = -\tilde{\epsilon}_{ijkl}$ where $\tilde{\epsilon}_{ijkl}$ is the Levi–Civita symbol (or tensor density). We note from a computer algebra point of view that there should be no overlap between the names of variables used for physical quantities and those reserved for tensor indices (here i, j, k, l, o, p, u, v).

3. Results

With the above quantities in hand, we are in a position to obtain for the first time explicit expressions for all seventeen ZM invariants for the most general possible black holes using the metric in Equation (3). Their definitions fall into three groups:

- Weyl invariants:

$$\begin{aligned}
I_1 &= C_{ij}{}^{kl}C_{kl}{}^{ij} = C^{ijkl}C_{ijkl}\,,\\
I_2 &= -C_{ij}{}^{kl}C^{*}_{kl}{}^{ij} = -\tfrac{1}{2}E^{ij}{}_{kl}C^{klop}C_{ijop}\,,\\
I_3 &= C_{ij}{}^{kl}C_{kl}{}^{op}C_{op}{}^{ij} = C_{ijkl}C^{klop}C_{op}{}^{ij}\,,\\
I_4 &= -C_{ij}{}^{kl}C^{*}_{kl}{}^{op}C_{op}{}^{ij} = -\tfrac{1}{2}C_{ijkl}C^{klop}E_{op}{}^{uv}C_{uv}{}^{ij}\,.
\end{aligned} \tag{7}$$

- Ricci invariants:

$$\begin{aligned}
I_5 &= R = g_{ij}R^{ij}\,,\\
I_6 &= R_{ij}R^{ij} = R_{ij}g^{ki}g^{lj}R_{kl}\,,\\
I_7 &= R_i{}^jR_j{}^kR_k{}^i\,,\\
I_8 &= R_i{}^jR_j{}^kR_k{}^lR_l{}^i\,.
\end{aligned} \tag{8}$$

- Mixed invariants:

$$\begin{aligned}
I_9 &= C_{ikl}{}^{j}R^{kl}R_j{}^i\,,\\
I_{10} &= -C^{*}_{ikl}{}^{j}R^{kl}R_j{}^i\,,\\
I_{11} &= R^{ij}R^{kl}(C_{oij}{}^{p}C_{pkl}{}^{o} - C^{*}_{oij}{}^{p}C^{*}_{pkl}{}^{o})\,,\\
I_{12} &= -R^{ij}R^{kl}(C^{*}_{oij}{}^{p}C_{pkl}{}^{o} + C_{olj}{}^{p}C^{*}_{pkl}{}^{o})\,,\\
I_{15} &= \tfrac{1}{16}R^{ij}R^{kl}(C_{oijp}C^{o}{}_{kl}{}^{p} + C^{*}_{oijp}C^{*o}{}_{kl}{}^{p})\,,\\
I_{16} &= -\tfrac{1}{32}R^{ij}R^{kl}(C_{ouvp}C^{o}{}_{ij}{}^{p}C^{u}{}_{kl}{}^{v} + C_{ouvp}C^{*o}{}_{ij}{}^{p}C^{u}{}_{kl}{}^{v}\\
&\quad - C^{*}_{ouvp}C^{*o}{}_{ij}{}^{p}C^{u}{}_{kl}{}^{v} + C^{*}_{ouvp}C^{o}{}_{ij}{}^{p}C^{*u}{}_{kl}{}^{v})\,,\\
I_{17} &= \tfrac{1}{32}R^{ij}R^{kl}(C^{*}_{ouvp}C^{o}{}_{ij}{}^{p}C^{u}{}_{kl}{}^{v} + C^{*}_{ouvp}C^{*o}{}_{ij}{}^{p}C^{*u}{}_{kl}{}^{v}\\
&\quad - C_{ouvp}C^{*o}{}_{ij}{}^{p}C^{u}{}_{kl}{}^{v} + C_{ouvp}C^{o}{}_{ij}{}^{p}C^{*u}{}_{kl}{}^{v})\,.
\end{aligned} \tag{9}$$

Note that I_{13} and I_{14} both vanish for the Kerr–Newman metric, which is of Petrov Type D and Segre type [(11)(1,1)] [19].

In principle, computer evaluation of Equations (7)–(9) is straightforward. In practice, the challenge is not so much computation as *simplification*: the resulting expressions may contain hundreds or even thousands of terms (for example, powers of $\cos\theta$, $\sin\theta$, $\cos 2\theta$, $\sin 2\theta$, etc.). These can be simplified in two ways. First, if working with the angular coordinate θ, one needs to specify its range (here, $0 \leq \theta \leq \pi$) and define a "trig simplifier" that converts any symbolic expression into a function of a single desired trigonometric function (here $\cos\theta$). Alternatively, one can re-express the metric in terms of a "rational polynomial" coordinate $\chi \equiv \cos\theta$ so that $\sin^2\theta = 1-\chi^2$, $d\theta = d\chi/\sqrt{1-\chi^2}$ and $\sqrt{|g|} = r^2 + a^2\chi^2$ [26]. The second method proves significantly faster in most cases. We obtain the following results (each organized in the most compact available form, or by powers of q and $a\cos\theta$ when no simpler form is found):

$$
\begin{aligned}
I_1 &= \frac{48}{(r^2+a^2\cos^2\theta)^6}\Big[m^2\left(r^2-a^2\cos^2\theta\right)\left(r^4-14r^2a^2\cos^2\theta+a^4\cos^4\theta\right) \\
&\quad -2mrq^2\left(r^4-10r^2a^2\cos^2\theta+5a^4\cos^4\theta\right) \\
&\quad +q^4\left(r^4-6r^2a^2\cos^2\theta+a^4\cos^4\theta\right)\Big]\,, \\
I_2 &= \frac{96a\cos\theta}{(r^2+a^2\cos^2\theta)^6}\Big[\left(3mr-2q^2\right)r-ma^2\cos^2\theta\Big] \\
&\quad \times\Big[mr\left(r^2-3a^2\cos^2\theta\right)-q^2\left(r^2-a^2\cos^2\theta\right)\Big]\,, \\
I_3 &= \frac{96}{(r^2+a^2\cos^2\theta)^9}\Big[\left(mr-q^2\right)r^2-\left(3mr-q^2\right)a^2\cos^2\theta\Big] \\
&\quad \times\Big[m^2\left(r^6-33r^4a^2\cos^2\theta+27r^2a^4\cos^4\theta-3a^6\cos^6\theta\right) \\
&\quad -mrq^2\left(2r^4-44r^2a^2\cos^2\theta+18a^4\cos^4\theta\right) \\
&\quad +q^4\left(r^4-14r^2a^2\cos^2\theta+a^4\cos^4\theta\right)\Big]\,, \\
I_4 &= \frac{96a\cos\theta}{(r^2+a^2\cos^2\theta)^9}\Big[\left(3mr-2q^2\right)r-ma^2\cos^2\theta\Big] \\
&\quad \times\Big[m^2\left(3r^6-27r^4a^2\cos^2\theta+33r^2a^4\cos^4\theta-a^6\cos^6\theta\right) \\
&\quad -2mrq^2\left(3r^4-18r^2a^2\cos^2\theta+11a^4\cos^4\theta\right) \\
&\quad +q^4\left(3r^4-10r^2a^2\cos^2\theta+3a^4\cos^4\theta\right)\Big]\,, \\
I_5 &= I_7 = 0\,, \\
I_6 &= \frac{4q^4}{(r^2+a^2\cos^2\theta)^4}\,, \\
I_8 &= \frac{4q^8}{(r^2+a^2\cos^2\theta)^8}\,, \\
I_9 &= \frac{-16q^4}{(r^2+a^2\cos^2\theta)^7}\Big[\left(mr-q^2\right)r^2-\left(3mr-q^2\right)a^2\cos^2\theta\Big]\,, \\
I_{10} &= \frac{-16q^4a\cos\theta}{(r^2+a^2\cos^2\theta)^7}\Big[\left(3mr-2q^2\right)r-ma^2\cos^2\theta\Big]\,, \\
I_{11} &= \frac{64q^4}{(r^2+a^2\cos^2\theta)^{10}}\Big[m^2\left(r^2-a^2\cos^2\theta\right)\left(r^4-14r^2a^2\cos^2\theta+a^4\cos^4\theta\right) \\
&\quad -2mrq^2\left(r^4-10r^2a^2\cos^2\theta+5a^4\cos^4\theta\right) \\
&\quad +q^4\left(r^4-6r^2a^2\cos^2\theta+a^4\cos^4\theta\right)\Big]\,, \\
I_{12} &= \frac{128q^4a\cos\theta}{(r^2+a^2\cos^2\theta)^{10}}\Big[\left(3mr-2q^2\right)r-ma^2\cos^2\theta\Big] \\
&\quad \times\Big[\left(mr-q^2\right)r^2-\left(3mr-q^2\right)a^2\cos^2\theta\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
I_{15} &= \frac{4q^4}{(r^2+a^2\cos^2\theta)^8}\left[\left(mr-q^2\right)^2+m^2a^2\cos^2\theta\right], \\
I_{16} &= \frac{8q^4}{(r^2+a^2\cos^2\theta)^{11}}\left[\left(mr-q^2\right)^2+m^2a^2\cos^2\theta\right] \\
&\quad \times\left[\left(mr-q^2\right)r^2-\left(3mr-q^2\right)a^2\cos^2\theta\right], \\
I_{17} &= \frac{8q^4 a\cos\theta}{(r^2+a^2\cos^2\theta)^{11}}\left[\left(3mr-2q^2\right)r-ma^2\cos^2\theta\right] \\
&\quad \times\left[\left(mr-q^2\right)^2+m^2a^2\cos^2\theta\right]. \qquad (10)
\end{aligned}
$$

These quantities represent a coordinate-invariant description of spacetime curvature around the most general possible black holes that is complete in the sense defined by Zakhary and McIntosh [19]. They do not seem to have been given before, apart from a derivation of the Kretschmann scalar [25] and a study of the Weyl invariants I_1 and I_2 [29]. Related expressions for an earlier set of invariants due to Carminati and McLenaghan [18] have been explored using *GRTensor* [30]. Other aspects of invariants have been discussed using specially designed software [31], and a general review of the application of symbolic computing to problems in general relativity was given by MacCallum [32].

The thirteen nonzero invariants in Equation (10) are not all algebraically independent, as the choice of a specific metric uses up additional degrees of freedom. In fact, the Kerr–Newman metric has enough symmetries that only *five* of the ZM invariants (apart from the Ricci scalar $I_5 = R$) are actually independent [19]. A minimally independent set of ZM invariants for black holes with mass, charge, and spin can be formed by the Weyl invariants I_1, I_2, the Ricci invariant I_6, and the mixed invariants I_9, I_{10}. (This number drops to two in the case of uncharged—i.e., astrophysical—black holes described by the Kerr metric, for which I_1, I_2 alone are sufficient.) All five quantities are plotted in Figure 1.

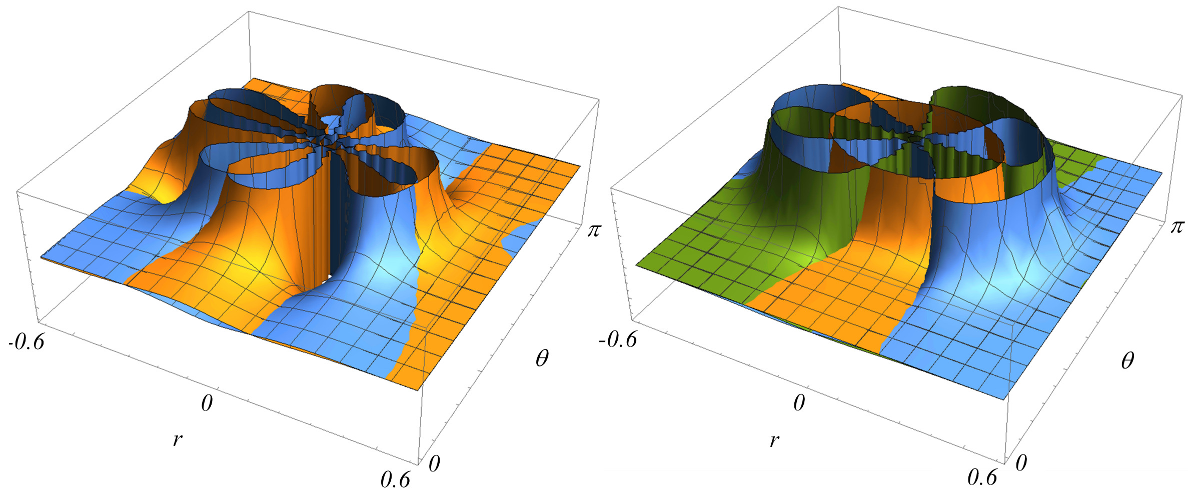

Figure 1. (**a**) The Weyl invariants I_1 (yellow) and I_2 (blue); and (**b**) the Ricci invariant I_6 (yellow) and mixed invariants I_9 (blue) and I_{10} (green), all plotted as a function the Boyer–Lindquist coordinates r and θ for a black hole of mass $m = 1$, angular momentum per unit mass $a = 0.6$ and charge $q = 0.8$.

The ZM invariants for Kerr–Newman black holes are therefore related by $13 - 5 = 8$ independent algebraic equations, which we obtain explicitly below. A subset of ZM invariants can also be related to the Kretschmann scalar $K \equiv R_{ijkl}R^{ijkl}$ [25]. In four dimensions, this relationship reads [8]

$$K = I_1 + 2I_6 - \tfrac{1}{3}I_5^2\,. \qquad (11)$$

This equation provides one check of our results.

More generally, proportionality relations between the curvature invariants have been dubbed "syzygies" by Zakhary and McIntosh, who defined the following needed quantities [19]:

$$\begin{aligned}
\mathbb{I} \equiv I_1 + iI_2 \quad &, \quad \mathbb{J} \equiv I_3 + iI_4 \,, \\
\mathbb{K} \equiv I_9 + iI_{10} \,, \quad &, \quad \mathbb{L} \equiv I_{11} + iI_{12} \,, \\
\mathbb{M}_1 \equiv I_{15} \,, \quad &, \quad \mathbb{M}_2 \equiv I_{16} + iI_{17} \,.
\end{aligned} \tag{12}$$

(Note that $\mathbb{K}$ here is not related to the Kretschmann invariant K.)

Using Equation (10), we are able to confirm the existence of syzygies similar to those proposed by Zakhary and McIntosh for metrics of Petrov Type D. However, our proportionality constants differ somewhat from theirs. Specifically, we find for the Kerr–Newman metric that

$$\begin{aligned}
\mathbb{I}^3 &= 12\mathbb{J}^2 \,, \\
I_6^2 &= 4I_8 \,, \\
3\mathbb{L}^2 &= \mathbb{I}\mathbb{K}^2 \,, \\
16I_6\mathbb{M}_1 &= \mathbb{K}\bar{\mathbb{K}} \,, \\
3072I_6^2\mathbb{M}_2^2 &= \mathbb{I}\mathbb{K}^2\bar{\mathbb{K}}^2 \,,
\end{aligned} \tag{13}$$

where an overbar denotes the complex conjugate, and where

$$\begin{aligned}
\mathbb{I} &= \frac{48(m(r+ia\cos\theta)-q^2)^2}{(r+ia\cos\theta)^2(r-ia\cos\theta)^6} \,, \\
\mathbb{J} &= \frac{96(m(r+ia\cos\theta)-q^2)^3}{(r+ia\cos\theta)^3(r-ia\cos\theta)^9} \,, \\
\mathbb{K} &= \frac{16q^4(q^2-m(r+ia\cos\theta))}{(r+ia\cos\theta)^5(r-ia\cos\theta)^7} \,, \\
\mathbb{L} &= \frac{64q^4(q^2-m(r+ia\cos\theta)^2}{(r+ia\cos\theta)^6(r-ia\cos\theta)^{10}} \,, \\
\mathbb{M}_1 &= \frac{4q^4((mr-q^2)^2+m^2a^2\cos^2\theta)}{(r^2+a^2\cos^2\theta)^8} \,, \\
\mathbb{M}_2 &= \frac{8q^4(mr-q^2-ima\cos\theta)(m(r+ia\cos\theta)-q^2)^2}{(r+ia\cos\theta)^9(r-ia\cos\theta)^{11}} \,.
\end{aligned} \tag{14}$$

We stress that the syzygies in Equation (13) provide a rigorous cross-check of the expressions we have obtained in Equation (10). That these equations provide exactly eight constraints can be seen by expanding and equating real and imaginary parts of Equation (13), whereupon

$$\begin{aligned}
I_1(I_1^2-3I_2^2) &= 12(I_3^2-I_4^2) \,, \\
I_2(3I_1^2-I_2^2) &= 24I_3I_4 \,, \\
I_6^2 &= 4I_8 \,, \\
3(I_{11}^2-I_{12}^2) &= I_1(I_9^2-I_{10}^2)-2I_2I_9I_{10} \,, \\
6I_{11}I_{12} &= I_2(I_9^2-I_{10}^2)+2I_1I_9I_{10} \,, \\
16I_6I_{15} &= I_9^2+I_{10}^2 \,, \\
3072I_6^2(I_{16}^2-I_{17}^2) &= I_1(I_9^2+I_{10}^2)^2 \,, \\
6144I_6^2I_{16}I_{17} &= I_2(I_9^2+I_{10}^2)^2 \,.
\end{aligned} \tag{15}$$

These equations may be readily used to check our main results in Equation (10).

4. Discussion

The plots in Figure 1 show that the geometry of spacetime inside charged and rotating black holes is far from simple. Curvature is particularly extreme near the ring singularity at $r = 0, \theta = \pi/2$, and is negative over significant regions of the phase space, as discussed by several workers [29,33]. This is interesting since the Weyl tensor encodes the degrees of freedom corresponding to a free gravitational field [34]. The fluctuations themselves have been attributed to conflicting contributions from the gravito-electric and gravito-magnetic components of this field, the latter generated by the black hole's rotation [8,10,35].

Plots in the full r, θ phase space are rich in information, but can be hard to reconcile intuitively with more conventional representations of the interior structure of rotating black holes such as that shown in Figure 2a [22]. Here, the axis of rotation is marked with a vertical line. The ring is the singularity. The inner shell (located at $r = m + \sqrt{m^2 - a^2}$) is the horizon, while the outer shell is the static limit. This diagram, by contrast with the plots in Figure 1, suggests that spacetime curvature inside the black hole is constant and positive. Both inferences are incorrect.

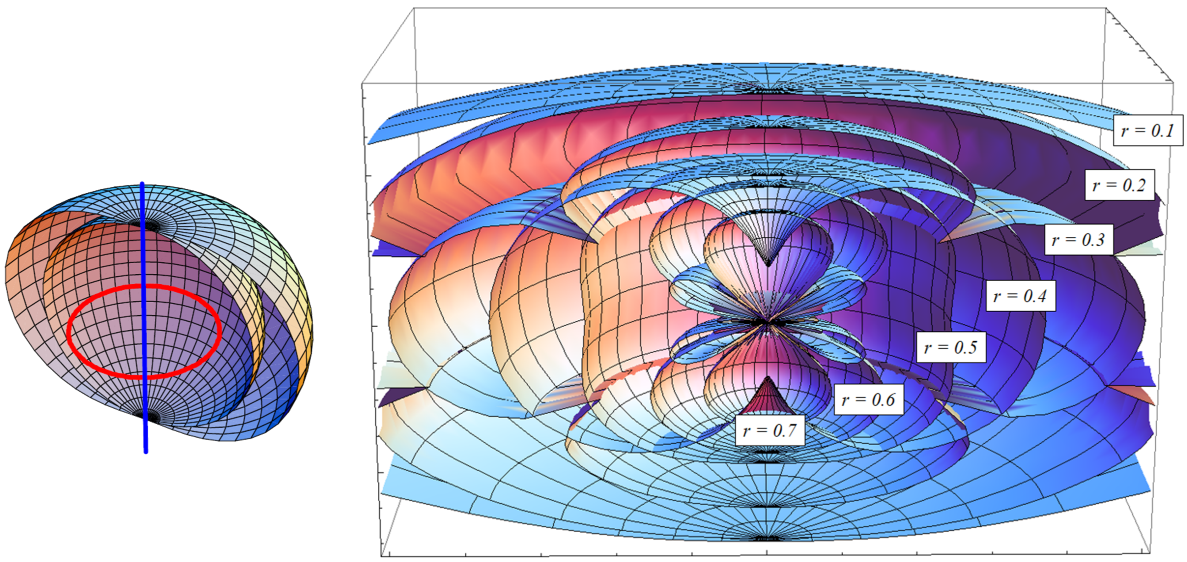

Figure 2. (**a**) A typical textbook representation of the interior structure of a rotating black hole. We argue that such coordinate-dependent depictions can be misleading, and should be supplemented by illustrations involving invariants. (**b**) The magnitude of the Weyl invariant I_1 in ordinary spherical coordinates (ϕ, θ), plotted for several values of the Boyer–Lindquist radial coordinate r.

In Figure 2b, we "unpack" one of the ZM invariants, I_1, and plot the logarithm of its magnitude in standard spherical polar coordinates θ, ϕ for several values of r. This figure is meant to be illustrative, as a contrast with Figure 2a, and does not cover the entire phase space (for example, only positive values of r are shown). Nevertheless, we feel it has considerable pedagogical value. The contours plotted here are surfaces of constant r. Thus, one can see at a glance, for example, that I_1 becomes extremely large near the equatorial plane close to the ring singularity (i.e., for small $r = 0.1$ or 0.2). As one gets farther from the singularity ($r = 0.3, 0.4, \ldots$) the magnitude of I_1 is no longer so large near the equatorial plane, but its shape as a function of polar angle becomes increasingly complex. This behavior is in stark contrast to that suggested by the traditional textbook representation in Figure 2a. Both kinds of figures have their uses, but the one on the right conveys a truth that the one on the left cannot. The traditional view misleads because it is not based on invariant quantities; it is simply a figure drawn in a particular coordinate system (Boyer–Lindquist). It is wrong in precisely the same way that the size of Greenland is wrong on a Mercator projection map of the world. Just as it is impossible to draw a map of the Earth realistically on a flat sheet of paper, it is impossible to portray the interior of a black hole realistically—unless what is shown is invariant.

Black holes are fascinating objects, not just for their inherent geometrical properties but because they reverse the usual course of scientific discovery. Usually, we observe first, and use mathematics later on to organize and explain what we have observed. However, the interior of the black hole is, by definition, the one place we will never be able to observe. Here, we can explore only through mathematics. Visualization, and hopefully deeper physical understanding, will follow later. The expressions obtained here for the ZM invariants are perhaps a first step in this process.

Supplementary Materials: The following are available online at http://www.mdpi.com/2218-1997/6/2/22/s1.

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