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Dark Matter as Gravitational Solitons in the Weak Field Limit

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Abstract: In this paper, we will describe the idea that dark matter partly consists of gravitational solitons (gravisolitons). The corresponding solution is valid for weak gravitational fields (weak field limit) with respect to a background metric. The stability of this soliton is connected with the existence of a special foliation and amazingly with the smoothness properties of spacetime. Gravisolitons have many properties of dark matter, such as no interaction with light but act on matter via gravitation. In this paper, we showed that the gravitational lensing effect of gravisolitons agreed with the lensing effect of usual matter. Furthermore, we obtained the same equation of state $w = 0$ as matter.

Keywords: dark matter; gravitational solitons; exotic smoothness of 4-manifolds

1. Introduction

In the 1930s, astronomers, among them the Swiss astrophysicist Fritz Zwicky [1,2], made observations which seem to imply that the orbits of stars in various galaxies did not have the expected relation between speed and distance from the galactic center. However, it took more than 30 years, until 1975, for careful studies of several different galaxies to confirm the results of Zwicky. Using new techniques, the astronomer Vera Rubin [3] showed that the speeds of stars in orbit in every galaxy tended to be independent of the distance from the center of the galaxy. However, this observation is in sharp contrast to the theory: If the visible mass of galaxies is responsible for star acceleration, then the speeds should vary as $\sqrt{1/r}$, with respect to the distance. With this contradiction in mind, astronomers and physicists quickly realized that almost all (90%) of the mass of galaxies resides in an invisible halo of unknown matter, called *dark matter*, sticking out into space for a distance around 10 times the visible galactic radius. Dark matter is matter that emits or reflects minimal to no light, but does have a gravitational influence. Evidence for dark matter is present in

- the motion of stars in galaxies,
- the orbits of galaxies in galaxy clusters,
- the temperature of intracluster gas in galaxy clusters and
- the gravitational lensing of distant galaxies.

The appearance of large-scale structure (e.g., the distribution of galaxies) is very hard to understand without Dark matter. This large scale structure is particularly encoded into the microwave background as measured by the satellites like Cosmic Background Explorer (COBE), Wilkinson Microwave Anisotropy Probe (WMAP) or PLANCK [4,5]. One way to accommodate this is to go to a dark matter model in which you have cold dark matter (CDM) to act as a seed for galaxy formation. But there are also results which contrast with the CDM hypothesis like the distribution of satellite galaxies around the Milky Way [6], the absence of dark matter in our sun system [7] and the existence of a vast polar structure of satellite galaxies around the Milky Way [8]. Dark matter can be divided into cold, warm, and hot categories, according to the free streaming length (FSL) indicating how far corresponding objects moved due to random motions in the early universe. Warm dark matter comprises particles with an FSL comparable to the size of a protogalaxy. Predictions based on warm dark matter are similar to those for cold dark matter on large scales, but with less small- scale density perturbations. A conjectured candidate is the sterile neutrino. The warm dark matter model is considered to be a better fit to observations. However, in both cases, there is currently no experimental detection of dark matter particles. Hot dark matter consists of particles, like the neutrino, whose FSL is much larger than the size of a protogalaxy. However, current limits on the neutrino masses by using cosmic microwave background data induces the result that the neutrinos cannot be dark matter. Here we will follow a different path. To explain all these properties, we need a kind of matter which cannot be compressed, couples to gravity only and is stable. In this paper we present such a substance: a gravitational soliton (in the weak field limit—it will be explained later). It is a special very stable structure of the spacetime appearing in two types, the gravisoliton $\mathcal{S}$ and the anti-gravisoliton $\mathcal{A}$. Furthermore, there is no coupling to light and therefore only small acoustic oscillations (caused by possible collisions of $\mathcal{SS}$ and $\mathcal{AA}$) are allowed.

Here we will consider gravisolitons as another component of dark matter. Loosely speaking, Gravisolitons are topological defects of the space like a density wave of space. They behave like matter but with different scattering properties. At first we will study the properties of the spacetime which is needed to generate gravitational solitons. Then we will connect the appearance of gravisolitons with a special codimension-1 foliation (also known as helical wobble, see [9]). In Section 4 we will discuss how a gravisoliton can be generated by matter. At the same time, this mechanism uncovers the topological reason for the appearance of gravisolitons: the underlying spacetime must admit an exotic smoothness structure. Dark matter was found by the gravitational lensing effect. Therefore, we will study the deflection of light rays by the gravisoliton in Section 5. Interestingly, we will find no difference to ordinary (baryonic) matter. Then we will discuss a coherent model of spacetime with all necessary properties. Here, we will see that all relevant spacetime with exotic smoothness structure are fulfilling these properties. Finally, in Section 7 we will also discuss the state equation for gravisolitons to be $w = 0$, i.e., no difference to usual matter. The discussion of strong field limit is reserved to a separate paper.

2. Spacetime Properties for the Generation of Gravitational Solitons

Dark matter has many properties known for matter itself, but it has also some other specific properties, like the very weak interaction to usual matter, which leads to the idea that dark matter consists only partly of (particle-like) matter (WIMPS etc.). Here we will follow this idea, i.e., we will consider dark matter as a special property of the space. It is just a trial which, nevertheless, is aiming toward deciding important question namely whether and how the idea can be realized by a spacetime itself. What properties of spacetime are required to this end? In the following sections we will describe these properties. First let us consider the following ansatz for the metric:

$$ds^2 = f(t,z)\left(dz^2 - dt^2\right) + g_{ab}(t,z)dx^a dx^b \tag{1}$$

for $a, b = 1, 2$ and with the non-diagonal 2-metric g_{ab}. The non-diagonality of g_{ab} is very important, it is responsible for the difference between non-linear solutions (for non-diagonal g_{ab}) and linear solutions (for diagonal g_{ab}). The linear case was first considered by Einstein and Rosen in 1937 (gravitational cylindrical waves). It is convenient to introduce the coordinates

$$\theta = \frac{1}{2}(z - t) \quad \zeta = \frac{1}{2}(z + t)$$

transforming $dz^2 - dt^2$ into $d\theta\, d\zeta$ and $\alpha = \sqrt{\det(g_{ab})}$ for the determinant. Then the Einstein equation $R_{ab} = 0$ is represented by the matrix equation

$$\frac{\partial}{\partial\theta}\left(\alpha\frac{\partial g}{\partial\zeta}\,g^{-1}\right) + \frac{\partial}{\partial\zeta}\left(\alpha\frac{\partial g}{\partial\theta}\,g^{-1}\right) = 0 \tag{2}$$

and we obtained the additional conditions from the equations $R_{00} = 0$, $R_{33} = 0$ and $R_{03} = 0$. For the following analysis, we will refer to the first two chapters of the book [10] where the solution is presented using the Inverse Scattering Method (ISM). The method itself is complicated and cannot be presented here completely. We will try to give some background here to make the paper as self-contained as possible. At first we must explain the wording *weak field limit*. A solution of the non-linear equation (2) can be obtained by the ISM as perturbation with respect to a background metric. Therefore, this solution is only valid for weakly gravitational fields. Then the solution is a small perturbation, i.e., we will only obtain a gravisoliton for weak gravitational fields. Or we will say that the gravisoliton is obtained in the weak field limit. The strong field limit will be considered in a separate paper.

Main idea relies on a proper introduction of a background metric. Let a particular solution g_0 serves as a background metric so that metric $g^{(1)}$ derived from the ISM method is a soliton solution with respect to g_0. g_0 can be chosen as a diagonal metric (for a reason discussed in Section 4)

$$g_0 = diag\left(\alpha e^{u_0}, \alpha e^{-u_0}\right)$$

where $\alpha = \sqrt{\det g_0}$ and

$$\frac{\partial^2\alpha}{\partial\theta\partial\zeta} = 0 \qquad \frac{\partial}{\partial\theta}\left(\alpha\frac{\partial u_0}{\partial\zeta}\right) + \frac{\partial}{\partial\zeta}\left(\alpha\frac{\partial u_0}{\partial\theta}\right) = 0$$

follows from (2). Now we will get a metric g from the background metric g_0 by using ISM. Main idea of the ISM is the introduction of two differential operators, the L-A or Lax pair, so that the eigenfunctions of these operators determine the new solution out from the old solution. In this case, we started with the constant diagonal (background) metric. Then the 1-soliton solution is given by

$$g^{(1)} = \frac{1}{\mu_1\cdot cosh(\rho_1)}\begin{pmatrix} (\mu_1^2 e^{\rho_1} + \alpha^2 e^{-\rho_1})e^{u_0} & \alpha^2 - \mu_1^2 \\ \alpha^2 - \mu_1^2 & (\mu_1^2 e^{\rho_1} + \alpha^2 e^{-\rho_1})e^{-u_0} \end{pmatrix}$$

where μ_1, ρ_1 are integration constants. It is interesting to note that the off-diagonal elements are the topological charge of the soliton. Unfortunately, from the physics point of view this solution is difficult to interpret. Therefore, we will study invariant reparameterizations to study the invariant properties of the solution. With the notation

$$\begin{aligned} \frac{\partial}{\partial\zeta}\left(\ln\left(g_{11}\alpha^{-1}\right)\right) &= R_1\cos\left(\frac{\gamma+\omega}{2}\right) & \qquad g_{11}\alpha^{-1}\frac{\partial}{\partial\zeta}\left(g_{12}g_{11}^{-1}\right) &= R_1\sin\left(\frac{\gamma+\omega}{2}\right) \\ \frac{\partial}{\partial\theta}\left(\ln\left(g_{11}\alpha^{-1}\right)\right) &= R_2\cos\left(\frac{\gamma-\omega}{2}\right) & \qquad g_{11}\alpha^{-1}\frac{\partial}{\partial\theta}\left(g_{12}g_{11}^{-1}\right) &= R_2\sin\left(\frac{\gamma-\omega}{2}\right) \end{aligned} \tag{3}$$

we have a new parameterization so that R_1, R_2 and ω are an invariant parameterization (not depending on arbitrary linear transformations of the coordinates x^1, x^2) for the metric g (there is also a system of two equations to determine γ which depends only on R_1, R_2 and ω). Now we will make our main assumption:

$$\alpha = const. \longrightarrow R_1 = R_2 = const.$$

which will be motivated later. Then we obtain the following equation for the variable ω

$$\frac{\partial^2\omega}{\partial\zeta\partial\theta} = R_1 R_2 \sin\omega$$

which is the famous sine-Gordon equation. This equation admits soliton solutions of the form (now in coordinates (z,t))

$$\omega(z,t) = 4 \cdot \arctan\left(e^{mC(z-vt+z_o)}\right) \tag{4}$$

with velocity $v = -2\alpha\mu_1(\alpha^2+\mu_1^2)^{-1}$, $C = (1-v^2)^{-1}$ and mass $m = R_1 = R_2$ which are stable over the time. The constant μ_1 is an integration constant from the ISM. The solution depends on the sign of $\alpha^2 - \mu_1^2$ which is the topological charge of the soliton. The topological charge is defined as the difference $\omega(+\infty,t) - \omega(-\infty,t)$ in units of 2π. Usually, the solution (4) of the sine-Gordon equation describes a soliton where the sign of $\alpha^2-\mu_1^2$ (topological charge) determines whether it is a gravisoliton (positive sign) or anti-gravisoliton (negative sign). Or, the gravisoliton $\mathcal{S}$ has topological charge $+1$ whereas the anti-gravisoliton $\mathcal{A}$ has -1. The sine-Gordon soliton is a non-linear wave which can be described by a rotating vector. Then the sign of the topological charge gives the direction of the rotation (with respect to the propagation direction). The difference between $\mathcal{S}$ and $\mathcal{A}$ defines the behavior with respect to the scattering. The scattering of solitons like $\mathcal{SS}$ or $\mathcal{AA}$ shows an amazing behavior: there is a repulsive force so that these solitons bounce back by a collision. In contrast the soliton $\mathcal{SA}$ can be represented by a bound state (a so-called breather solution), so there is an attractive force in this system. By considering bound states of solitons (like $\mathcal{SA}$ but technically generated by a Bäcklund transformation) one obtains 2-gravisolitons which form only stable stationary solution (including an axisymmetry). These 2-solitons in a Minkowski background are equivalent to a Kerr-NUT or Schwarzschild solution [10]. Then one obtains an attraction of a test particle by a 2-gravisoliton (or 2-anti-gravisoliton). Summarizing this approach, we need to construct a spacetime

1. with metric (1): $ds^2 = f(t,z)\left(dz^2 - dt^2\right) + g_{ab}(t,z)dx^a dx^b$ where the 2-metric g_{ab} is parameterized by the $t,z-$coordinates only,
2. the 2-metric g_{ab} is non-diagonal, i.e., $g_{ab} \neq 0$ for $a \neq b$ and
3. $\det(g_{ab}) = const.$, i.e., the area of the surface defined by g_{ab} is fixed.

How to fulfill these conditions? At first one needs a spacetime which splits into a (non-trivial, curved) surface and a (conformally flat) $(1+1)-$dimensional subspace (first condition). The surface is locally generated by two independent vector fields (the dual to dx^1, dx^2) which are never orthogonal to each other (second condition). Furthermore this surface has a finite volume (third condition). The first and third conditions can be simply realized for instance by spaces like $\mathbb{R}^{1,1} \times S$ for every compact closed surface S. The problem is only with the second condition. In principle, we need a complicated codimension-1 foliation of a 3-dimensional manifold (formed by x^1, x^2, z) so that the three generating vector fields are never orthogonal to each other. In the next section we will construct this foliation for the 3-sphere but also for all other compact 3-manifolds.

3. A Codimension-1 Foliation Realizing the Conditions for Gravitational Solitons

In the previous section we discussed gravitational solitons, called gravisolitons, which are connected with the metric (1). Main characteristics of this metric is the appearance of a non-diagonal 2-metric g_{ab}. Then the non-diagonal term of the 2-metric can be interpreted as non-orthogonal vector fields in this space. With other words, we must construct a space or spatial subspace of the spacetime which is generated by non-orthogonal vector fields. A direct way would be the foliation of the space by surfaces described by the 2-metric g_{ab}. Does such foliation exist? And if it exists, what are conditions guaranteeing this which must be implied on the foliation and the spacetime?

First, we will give a flavor of the main idea without the technical details. Our task is now to construct a foliation of a 3-manifold into surfaces so that the three generating vector fields are never orthogonal to each other. This problem was solved by Thurston [9] who constructed a codimension-1 foliation of the 3- sphere which can be easily extended to all 3-manifolds. The details of the complicated construction are not important for the following discussion and can be found in [11]. Main idea is the usage of the group $PSL(2,\mathbb{R})$. Then the vector fields of the foliation are the left-invariant vector fields of the Lie algebra of this group. Furthermore, we will construct an invariant of the foliation (Godbillon–Vey invariant) which is directly connected to the non-orthogonal vector fields. Differently expressed: the vector fields are non- orthogonal if this invariant is non-zero. The reader not interested in the technical details of this construction can switch to the next section keeping in mind that there is such a foliation with the suitable properties.

A foliation (M,F) of a manifold M is an integrable Sub-bundle $F \subset TM$ of the tangent bundle TM. The leaves L of the foliation (M,F) are the maximal connected submanifolds $L \subset M$ with $T_x L = F_x$ for all $x \in L$. The number $\dim M - \dim L$ is the codimension of the foliation. A codimension-1 foliation on a 3-manifold M can be constructed by a smooth 1-form ω fulfilling the integrability condition $d\omega \wedge \omega = 0$. This construction can be found in [11]. In [12] the authors analyzed the local structure of the foliation: the three vector fields (forming the frame of the foliation) are not orthogonal to each other. This foliation is constructed by using the group $PSL(2,\mathbb{R})$, the isometry group of the hyperbolic plane $\mathbb{H}^2$. Let T,N,Z be the frame formed by three vector field dual to the one-forms ω,η,ξ as a basis for the tangent bundle TM (remember that the tangent bundle of every 3-manifold is trivial, see [13]). Now one considers the vector fields as left-invariant vector fields of the group $PSL(2,\mathbb{R})$ (forming the basis of the Lie algebra) fulfilling the commutator relations $[Z,N] = Z$, $[N,T] = T$ and $[Z,T] = N$. As shown in [12], the vector fields are not orthogonal to each other leading to a (local) non-zero torsion $\tau \neq 0$. For codimension-1 foliations of 3-manifolds, there exists the Godbillon–Vey invariant which we are going to describe now. Recall that codimension-1 foliation on M is defined by a $PSL(2,\mathbb{R})-$invariant integrable one-form ω ($d\omega \wedge \omega = 0$) and we define another one-form η by $d\omega = -\eta \wedge \omega$. Then the Godbillon–Vey class $gv = \eta \wedge d\eta$ is a closed form (by using the integrability) which is not exact. Furthermore, the integral

$$GV(M) = \int_M \eta \wedge d\eta \tag{5}$$

is a topological invariant of the foliation, known as Godbillon–Vey number $GV(M)$. From the physics point of view, it is the abelian Chern–Simons functional. The mathematical construction of the foliation can be found in [9,14] including a global calculation of the Godbillon–Vey invariant. However, here we will concentrate on the local expression for the Godbillon–Vey invariant. Let κ,τ be the curvature and torsion of a normal curve to the foliation, respectively. Furthermore, let T,N,Z be the frame formed by three vector field dual to the one-forms ω,η,ξ and let l_T be the second fundamental form of the leaf. Then the Godbillon–Vey class is locally given by

$$\eta \wedge d\eta = \kappa^2 \left(\tau + l_T(N,Z)\right) \omega \wedge \eta \wedge \xi \tag{6}$$

where $\tau \neq 0$ for $PSL(2,\mathbb{R})$ invariant foliations i.e., $[Z,N] = Z$, $[N,T] = T$ and $[Z,T] = N$. Therefore at least two vector fields are not orthogonal to each other and the corresponding metric g_{ab} is not diagonal. Finally, let M be a compact 3-manifold with codimension-1 foliation of non-zero Godbillon–Vey number $GV(M) \neq 0$. Then the spacetime $M \times \mathbb{R}$ (or $M \times [0,1]$ for finite time) has all properties to admit gravitational solitons (in the weak field limit).

4. How to Generate Gravitational Solitons

In the previous section we connected the gravitational soliton with a foliation of the 3-space. The result can be expressed by the simple statement: if the space admits a codimension-1 foliation with non-zero Godbillon–Vey invariant (5), locally given by (6), then, there is a gravitational soliton (gravisoliton) in space described by the metric (1). Therefore, if we can get a non-zero Godbillon invariant then a gravisoliton is generated. Starting point is a general but arbitrary foliation of the space into surfaces. In [15], we described a strong relation (using the work of [16]) between a surface and a spinor ψ. We will give a different meaning to the Godbillon–Vey invariant. In principle, this invariant is generated by a 3-form $\eta \wedge d\eta$ with the real 1-form η. Now we interpret η as a $U(1)$ gauge field A which is an imaginary valued 1-form i.e., $A = i\eta$. Then we couple this gauge field η to the spinor ψ. The Godbillon–Vey invariant is also known as abelian Chern–Simons form. Then the action of a spinor coupled to the gauge field given by the Chern–Simons form is a dynamical theory to generate a foliation of the space (see the Dirac–Chern–Simons functional below). The foliation is defined by the gradient flow. However, the gradient flow is equivalent to the Seiberg–Witten equations, a theory to detect different smoothness structures on 4-manifolds. A non-trivial solution of this gradient system is given by $\psi \neq 0, \eta \neq 0$, which is equivalent to the existence of a non-trivial Godbillon–Vey invariant which means that a gravisoliton is being generated. However, this solution gives also a non-trivial solution of the Seiberg–Witten equation for the 4-dimensional spacetime. This solution is a necessary condition for an exotic smoothness structure of the spacetime. A smoothness structure is the maximal smooth atlas of a manifold, i.e., the collection of smooth charts covering the manifold. In principle, the smoothness structure is the main topological information about the smooth manifold. Loosely speaking, all properties of differentiable quantities are determined by the smoothness structure, even one cannot define a differential equation on a manifold without fixing the smoothness structure. In dimension 4 there are manifolds which are homeomorphic to each other but not diffeomorphic, i.e., these manifolds differ by the smoothness structure. Main results of this section are the connection between the existence of gravisolitons in space and the smoothness structure of the spacetime.

Now we go into the details of this construction. In [15] we presented a geometrical/topological model for matter. In general, for a manifold M with boundary $\partial M = \Sigma$ one has the action (see [17])

$$S_{EH}(M) = \int_M R\sqrt{g}\, d^4x + \int_\Sigma H\,\sqrt{h}\, d^3x$$

where H is the mean curvature of the boundary with metric h. For the following discussion, we consider the boundary term

$$S_{EH}(\Sigma) = \int_\Sigma H\,\sqrt{h}\, d^3x \tag{7}$$

along the boundary Σ (a 3-manifold). Following [15], the action (7) over a 3-manifold Σ is equivalent to the Dirac action of a spinor over Σ. Interestingly, this relation is induced by a stronger relation between surfaces and spinors (see [18] for a complete discussion). Main result of [15] is the following relation between the corresponding Dirac operators

$$D^M\Phi = D^\Sigma\psi - H\psi \tag{8}$$

where D^Σ or D^M denote the Dirac operators on the 3-manifold Σ or 4-manifold M, respectively. Using the work of [16], the spinor ϕ directly defines the embedding (via an integral representation) of the 3-manifold. Then the restricted spinor $\Phi|_\Sigma = \phi$ is parallel transported along the normal vector and Φ is constant along the normal direction. However, then the spinor Φ must fulfill

$$D^M\Phi = 0 \tag{9}$$

i.e., Φ is a parallel spinor. Finally, we get

$$D^\Sigma\psi = H\psi \tag{10}$$

and

$$\int_\Sigma H\,\sqrt{h}\, d^3x = \int_\Sigma \bar{\psi}\, D^\Sigma\psi\,\sqrt{h} d^3x \tag{11}$$

Therefore, we get a direct relation between spinors and geometry. Now we will use this relation to get a connection to the foliation.

For every codimension-1 foliation, it is known: If the 1-form ω defines the leaf via the equation $\omega = const.$ then the dual of the 1-form η defines a vector normal to the leaf. However, then the spinor Φ (representing the leaf) is constant along the normal direction fulfilling the relation (9) i.e., Φ is a parallel spinor. Furthermore, in [15] we showed that a fermion is given by a knot complement admitting a hyperbolic structure. Then calculations in [19] implied that the particular knot is only important for the dynamical state (like the energy or momentum) but not for charges, flavors etc.

Now we will use this relation between spinors, surfaces and the foliation to reinterpret the Dirac equation $D^M\Phi = 0$ along the normal direction as covariant constant spinor

$$D^{\Sigma}_{\eta}\psi = 0$$

where the one-form η is interpreted as abelian gauge field (with structure group $U(1)$. By definition, the covariant constant 1-form ω

$$D_{\eta}\omega = d\omega + \eta \wedge \omega = 0$$

defines the foliation, because the integrability condition $\omega \wedge d\omega = 0$ is automatically fulfilled. Then we reinterpret the Godbillon–Vey invariant $gv = \eta \wedge d\eta$ of the foliation as abelian Chern–Simons form for the abelian gauge field η. Now we use the relation between the spinor and the surface (as given by $\omega = const.$) to define a similar relation as $D_{\eta}\omega = 0$: the spinor ψ is covariant constant with respect to η or

$$D^{\Sigma}_{\eta}\psi = 0\,.$$

To take the special properties of the foliation into account, we must consider a non-zero Godbillon–Vey invariant at the same time. That means that for a fixed foliation, the coupling between the abelian gauge field η and the spinor ψ to the Dirac–Chern–Simons action functional

$$S_{DCS} = \int\limits_{\Sigma} \left(\bar{\psi}\, D^{\Sigma}_{\eta}\psi\, \sqrt{h} d^3x + \eta \wedge d\eta \right)$$

on the 3-manifold is constant with the critical points at the solution

$$D^{\Sigma}_{\eta}\psi = 0 \quad d\eta = \tau(\psi,\psi)$$

where $\tau(\psi,\psi)$ is the unique quadratic form for the spinors locally given by $\bar{\psi}\gamma^{\mu}\psi$. If one has a spacetime $\Sigma \times I$, then we consider the solution which is translationally invariant. Alternatively, it is a spacetime with foliation induced by the foliation of Σ extended by a translation. Mathematically we must consider the gradient flow

$$\begin{aligned} \frac{d}{dt}\eta &= d\eta - \tau(\psi,\psi) \\ \frac{d}{dt}\psi &= D^{\Sigma}_{\eta}\psi \end{aligned}$$

From every solution of this gradient system, one can construct the corresponding foliation with Godbillon–Vey invariant. It is interesting to note that a solution with vanishing Godbillon–Vey invariant leads also to vanishing spinor. In [20,21] this system was shown to be equivalent to the Seiberg–Witten equation for $\Sigma \times I$ by using an appropriate choice of the $Spin_C$ structure . Then we have the result that a non-trivial foliation together with the existence of fermions induce a non-trivial solution of the gradient system which results in a non-trivial solution of the Seiberg–Witten equations. However, this non-trivial solution (i.e., $\psi \neq 0, \eta \neq 0$) is a necessary condition for the existence of an exotic smoothness structure (but not sufficient).

Here we get an unexpected relation between the existence of gravisolitons and topological (or better differential-topological) properties of the spacetime. This property of the spacetime is called exotic smoothness in mathematics (see the book [22] for an overview).

Again, exotic smoothness denotes a different (non-diffeomorphic) smooth atlas of a 4-manifold. Two 4-manifolds with different smoothness structures are topologically equivalent (homeomorphic) but smoothly different. From the physics point of view, two non-diffeomorphic spacetimes are different if they represent (physically) different systems.

Finally, we are concluding that the foliation above is connected with the smoothness properties of the spacetime. Furthermore, if one starts choosing an exotic smoothness structure instead of the standard one for the spacetime then, there gravisolitons must exist automatically. By using the relation between spinors and surfaces, there is a natural coupling between a spinor and the gravisoliton, or the gravisoliton is generated by a spinor (electrically charged). The description here is only the first sign of how the appearance of gravisolitons could be understood in terms of physical spinor fields. More work is needed to understand it completely.

5. Gravitational Lensing for Gravitational Solitons

In this section, we will investigate the gravitational lensing property of the gravitational solitons. To study the effect of these solitons on the propagation of light, we will introduce the optical metric and compare it with the metric of the gravitational soliton. Unexpectedly, we will obtain the same effect as for usual matter. Therefore, one cannot see a difference between the gravitational soliton and matter by using gravitational lensing. This section is important to understand that gravitational solitons cannot be distinguished from matter by gravitational lensing.

Dark matter was indirectly recognized by gravitational lensing. Therefore we are going to study the deflection of light induced by gravitational solitons. The idea is to introduce a metric, the optical metric, which describes this deflection of light.

The optical metric with the line element ds^2_{opt} in Schwarzschild coordinates (t, r, ϑ, ϕ) is usually given by

$$ds^2_{opt} = -exp\,(2A(r))\,dt^2 + exp\,(2B(r))\,dr^2 + r^2\left(d\vartheta^2 + sin^2\vartheta\,d\phi^2\right)$$

with some functions $A(r), B(r)$. To determine these two functions, one must put it in Einstein's field equations. Here one uses an energy-momentum tensor $T_{\mu\nu} = diag(\rho, p, p, p)$ in the local flat metric, where $\rho = \rho(r)$ denotes the density and $p = p(r)$ the pressure of the lens model. Then one obtains

$$exp\,(-2B(r)) = 1 - \frac{2\mu(r)}{r}$$

with

$$\mu(r) = 4\pi G \int\limits_0^r \rho(r')r'^2 dr'$$

where $\rho(r)$ denotes the energy density. However, the solution (4) implied that the dr^2 part of the corresponding gravitational soliton solution is in a good approximation constant, so that we get

$$exp\,(-2B(r)) = 1 - \frac{2\mu(r)}{r} = const.$$

However, then we will obtain

$$\mu(r) \sim r$$

and the energy density of a gravitational soliton is given by

$$\rho(r) \sim \frac{1}{r^2}$$

i.e., we obtain the same gravitational lensing effect as the singular isothermal sphere serving as a lensing model for galaxies. In [23], the deflection angle was determined from the above relation. Let us explain briefly how the deflection angle in the solitonic model is obtained. Let σ^2 be the amplitude of the soliton, then we set

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}$$

implying $\mu(r) = 2\sigma^2 r$. This is known as the solution of the Tolman– Oppenheimer–Volkoff equation in the nonrelativistic limit (i.e., first order in σ^2). The Gaussian curvature vanishes for $r > 0$ in this limit, so that the equatorial plane in the optical metric is a cone with a singular vertex at $r = 0$. The deficit angle of the conical optical metric is the deflection angle δ. Thus, for our model the deflection angle is given by

$$\delta = 4\pi\sigma^2$$

in agreement with the gravitational lensing of galaxies. It is interesting to note that this lensing is similar to the lensing of a cosmic string. Finally, gravitational solitons have the same effect as usual matter and gravitational lensing cannot distinguish between the gravitational soliton and matter.

6. A Coherent Model for a Spacetime with These Properties

In the Section 4, we found an unexpected relation between gravisolitons and (differential-)topological properties of the spacetime. In particular, the existence of gravisolitons is connected with a class of complicated foliations (with non-trivial Godbillon–Vey invariant) leading to a coupling of the gravisoliton to a spinor. Then the last relation (spinor-gravisoliton coupling) can be fulfilled for a class of spacetimes, known as spacetime with exotic smoothness structure. In this section, we will construct a suitable spacetime and study its properties. Finally, we will see that for this class of spacetimes we obtain the remaining conditions which are needed to construct the gravisoliton (i.e., $\alpha = \det(g) = const.$ and the non-diagonality of the 2-metric).

In a series of papers we have described the class of spacetimes with exotic smoothness structure. The influence of the smoothness structure on the topological properties of the spacetime is tremendous. Let us consider the spacetime given by the smooth manifold $S^3 \times \mathbb{R}$. In the standard smoothness structure on $S^3 \times \mathbb{R}$ one has a global foliation of the spacetime into slides $S^3 \times \{t\}$ for every $t \in \mathbb{R}$ (or by copies of S^3). In contrast to that, the 4-spacetime $S^3 \times \mathbb{R}$ with exotic smoothness structure is given by a sequence $\ldots \to \Sigma_1 \to \Sigma_2 \to \ldots$ of topological transitions where all Σ_i have the same homology like $S^2 \times S^1$. In the limit of infinite transitions, one gets a manifold which is topologically equivalent to the 3-sphere. We remark that this is a general rule for all exotic versions of spacetimes $N \times \mathbb{R}$ (with N a compact 3-manifold). To understand the general structure of the Σ_n, we must understand the general decomposition of 3-manifolds into pieces.

Every 3-manifold can be decomposed into prime 3-manifolds according to

$$K_1 \# \cdots \# K_{n_1} \#_{n_2} S^1 \times S^2 \#_{n_3} S^3 / \Gamma$$

where Γ is a finite subgroup of $SO(4)$. Every (irreducible) 3-manifold K_i can be also split into a hyperbolic and a graph manifold. As shown in [18] and further extended in [15,19], the 3-manifolds K_i are the matter part. It consists of hyperbolic 3-manifolds as fermions and graph manifolds (torus bundles) for the interaction (gauge bosons). The further details of this model are not important here, but we want to call the reader's attention to the sum operation #. By definition we have for the connecting sum $K_1 \# K_2$ the expression

$$K_1 \# K_2 = \left(K_1 \setminus D^3\right) \cup_{S^2} \left(S^2 \times [0,1]\right) \cup_{S^2} \left(K_2 \setminus D^3\right)$$

with the tube $S^2 \times [0,1]$ to identify $K_1 \setminus D^3$ and $K_2 \setminus D^3$ along its common boundary S^2. Therefore we obtained an additional manifold $S^2 \times [0,1]$, called sphere bundle, between the matter

K_i. The corresponding spacetime version of these sphere bundles admit the topology $S^2 \times [0,1]^2$. Obviously, we have the following properties:

1. $S^2 \times [0,1]^2$ admits a Lorentz metric with coordinates (x^1, x^2) for the sphere S^2 and (z,t) for $[0,1]^2$ and
2. the line element is given by

$$ds^2 = f(t,z)\left(dz^2 - dt^2\right) + g_{ab}(t,z)dx^a dx^b$$

in agreement with (1).

The next two properties of the spacetime are directly related to the exotic smoothness structure on it. However, from the point of view of geometry, exotic smoothness is connected with hyperbolic geometry (see for instance [24,25] and the result stating that 4-manifolds with positive scalar curvature in the standard smoothness structure fail to be paired with hyperbolic geometry). In [26] we analyzed this situation and explained the reason: exotic smoothness structures enforce the appearance of saddle points (in the Morse-theoretic sense, i.e., non-canceling 2-/3-handle pairs) carrying a hyperbolic geometry. The corresponding analysis was extended in [27,28]. This fact has two important consequences: hyperbolic 3- and 4-manifolds (with finite volume, i.e., the neighborhood of the saddle points) have the property called *Mostow rigidity*. As shown by Mostow [29] and extended by Prasad [30], every hyperbolic $n-$manifold $n > 2$ (with finite volume) has the following property: *Every diffeomorphism (especially every conformal transformation) of a hyperbolic $n-$manifold (with finite volume) for $n > 2$ is induced by an isometry.* Therefore, one cannot scale a hyperbolic 3- and 4-manifold. Then the volume $vol(\,)$ and the curvature (both combined into the Chern–Simons invariant) are topological invariants. Secondly, the foliation of this hyperbolic 4-manifold part admits a non-trivial Chern–Simons invariant and also a non-trivial Godbillon–Vey invariant (foliations of type III) [31]. Both consequences imply the following properties:

1. The background for the sphere bundle has a constant scale-invariant volume (by Mostow rigidity) from the matter, i.e., $\det(g) = const$.
2. The metric g is non-diagonal (otherwise the Godbillon–Vey invariant of the foliation vanishes, see [32] and Sections 3 and 4).

All four properties together are the prerequisites to obtain the model of the gravisoliton as dark matter.

7. The Equation of State for Gravisolitons

In the astrophysical context, the scaling behavior of the energy density $\rho(a)$ depends on the scale factor $a(t)$. Now we consider the fluid equation

$$\dot{\rho} + \frac{3\dot{a}}{a}(\rho + p) = 0$$

and the equation of state $p = w \cdot \rho$. Usually, one assumes a constant w so that one gets

$$\rho \sim a^{-3(1+w)}$$

For gravisolitons, the energy E is constant. In the previous section we discussed that the volume of the sphere S^2 in the sphere bundle $S^2 \times [0,1]$ (representing the gravisoliton) is fixed. Furthermore, we know from exotic smoothness structure that this sphere bundle is the equator region of a 3-sphere which carries a hyperbolic structure. By Mostow rigidity (see Section 6), the volume of $S^2 \times [0,1]$ is also fixed, i.e., gravisolitons do not scale with the expansion of the universe. Then we obtain

$$\rho_{soliton} = \sum_i^N \frac{\lambda E_i}{V}$$

where N is the number of gravisolitons in our cosmic horizon, λ is the linear density of gravisolitons, and E_i is the energy of each gravisoliton. Then we can get the dependence of ρ on the scale a

$$\rho_{soliton}(a) \sim \frac{1}{a^3} = a^{-3}$$

and then

$$3 = 3(1 + w_{soliton}) \Longrightarrow w_{soliton} = 0$$

Then we have the following consequences:

(i) The state equation of the gravisoliton (as dark matter) is $p = 0$ or $w = 0$.
(ii) Gravisolitons interact only via gravity with matter by an attractive force.
(iii) Gravisolitons have two possible scattering behaviors: two gravisolitons scatter so that it looks repulsively. In contrast, gravi- and anti-gravisolitons scatter so that it looks attractively.

In [33], the state equation for dark matter was investigated by using the gravitational lensing. For two clusters, the Coma and the CL0024 cluster (see Figure 2 in [33]), the variation of the equation of state parameter w over the radius of the cluster was studied. In contrast to the expected value at $w = 0$, the both curves are located at the value $w = -1/3$ but the error bars are large enough to get $w = 0$ also. Newer measurements [34] got the value $w = 0$. In [35] the equation of state for dark matter was studied. In particular, the inverse cosmic volume law for Dark Matter was tested by allowing its equation of state to vary independently for different times values in the cosmic history (i.e., eight redshift bins in the range $z = 10^5$ and $z = 0$). Again, the equation of state $w = 0$ was also confirmed during the cosmic evolution. According to consequence (i) above, our results agree with these measurements. Therefore, the detection of gravisolitons is rather complicate.

However, what is different for gravisolitons? From the mathematics point of view, the corresponding foliation leading to gravisolitons (see Section 3) admits torsion. Here we expect that the polarization properties of the light after the gravitational lensing must be different. We conjecture that the light must be circularly polarized. However, what about the strong field limit? One of the main results is the close relation between the smoothness properties and the existence of gravisolitons. As discussed in our previous work, the space admits a fractal structure. Gravisolitons will follow this structure and the distribution of dark matter looks like a fractal which will be shown in a forthcoming paper.

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