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On the Hilbert Space in Quantum Gravity

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Abstract: This article deals with the fractional problem of Sturm–Liouville and the Hilbert space associated with the solutions of this differential equation. We apply a quantization procedure to Schwarzschild space–time and obtain a fractional differential equation. The Hilbert space for these solutions is established. We used equations arising from quantization for the FRW and Reissner–Nordstron metrics to build the respective Hilbert spaces.

Keywords: quantum gravity; teleparallelism; non-commutative gravity

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1. Introduction

The search for a quantum theory of gravitation has been attempted as a stepping stone towards a great unified field theory. Despite this, there is no consensus on how to achieve this goal. The two most widespread approaches are loop quantum gravity [1] and string theory [2]. The latter, despite the mathematical consistency and simple basic principles, does not present experimental verification. On the other hand, loop quantum gravity is based on quantization processes applied to the Hamiltonian formulation of general relativity. This generates a discretization in the space–time itself instead of one in the gravitation source. The lack of gravitational energy in this context leads to a difficult physical interpretation of the result.

In fact, theories, which have a well-defined energy-momentum tensor for the field in question, are less refractory to known quantization techniques. Among the best known of these techniques, we have the canonical quantization that requires the formulation of the field in the phase space [3]. None of these conditions are met by the standard approach to gravitation. On the other hand, an alternative theory equivalent to general relativity allows the conception of a very well-defined gravitational energy, the so-called teleparallel gravity [4,5]. Teleparallelism equivalent to general relativity (TEGR) is constructed out of the tetrad field and was introduced by Einstein himself as part of the same effort to find a unified theory. TEGR is not a priori formulated in the phase space, which makes canonical quantization difficult to implement. The Hamiltonian formulation can certainly overcome this difficulty [6], but we are interested in applying a quantization technique that acts on functions dependent only on coordinates. We refer to Weyl's quantization [7].

Weyl's quantization was introduced in the early days of quantum mechanics and has the property of transforming a coordinate function into an operator [8–10]. On the other hand, it shares with the canonical quantization the arbitrariness in the representation of the operators constructed from the coordinates. Given n variables denoted by $z_1, z_2, ..., z_n$, then the prescription

$$(z_1, z_2, ..., z_n) \rightarrow (\widehat{z}_1, \widehat{z}_2, ..., \widehat{z}_n)\,,$$

immediately quantizes a function f dependent on the classical variables. This quantization of the functions f is achieved by the following Weyl's map, $\mathcal{W} : f \rightarrow \widehat{f} = \mathcal{W}[f]$,

$$\mathcal{W}[f](z_1, z_2, ..., z_n) := \frac{1}{(2\pi)^n} \int d^n k d^n z f(z_1, z_2, ..., z_n) \exp\left(i \sum_{l=1}^{n} k_l (\widehat{z}_l - z_l) \right). \quad (1)$$

The operators constructed out of the classical variables obey the following relation

$$[\widehat{z}_i, \widehat{z}_j] = i\beta_{ij},$$

which means that the Weyl's quantization is essentially a non-commutative prescription. It is worth noting that such non-commutativity can be established for all coordinates one pair at a time, by choosing the beta components. On the other hand, non-commutativity involving the temporal coordinate is problematic, so we usually restrict ourselves to spatial coordinates or apply the technique to stationary systems. In addition, the very dependence of the function to be quantized influences this choice. Particularly when energy is used, its dependence on coordinates becomes the natural choice to use non-commutative coordinates. Weyl's quantization combined with TEGR creates a powerful approach to quantum gravity that has achieved consistent results such as the discretization of the charge-mass ratio [11], as well as the achievement of discrete levels of energy in a primordial universe [12]. On the other hand, improvements must be made such as the construction of the Hilbert space associated with the functions on which the resulting operators act. It is interesting to note that the three systems discussed in this article have different possible interpretations. Certainly until the advent of an experimental measurement, we may have a somewhat speculative interpretation. In Schwarzschild's case, a discrete mass spectrum points to a deeper quantum construction of matter, perhaps linked to a geometric structure of particles. Incidentally, such an idea applied to a charged black hole explains why the charge-mass ratio is discrete. It was even possible to associate the non-commutative parameter to the fundamental charge. As for the FLRW metric, its quantization provides a simple and ab initio explanation of the inflation process of the early universe.

This article is divided as follows. In Section 2, we introduce the basic ideas of TEGR, and we show the expression of gravitational energy that will be object of quantization. In Section 3, we approach the fractional problem of Sturm–Liouville and how to build the Hilbert space associated with the solutions of the differential equation. In Section 4, we obtain a fractional equation as a result of the quantization of Schwarzschild space–time. In Section 5, we show that the quantization of Reisner–Nordstrom space–time, obtained in a previous article, has a well-defined Hilbert space. In Section 6, we used the quantization of the FRW metric, already obtained in another article, to establish the associated Hilbert space. Finally, in Section 7 we present our final points. We adopt a unity system such that $G = c = 1$.

2. Teleparallelism Equivalent to General Relativity (TEGR)

TEGR is an alternative theory of gravitation whose dynamic variables are the tetrads. They also determine the state of the observer. That is, there is a single solution of the field equations for each reference system. This arbitrariness in the choice of the observer is a physical feature absent from the standard theory of gravitation, general relativity. In this sense, an expression of gravitational energy must be dependent on the reference system but invariant by coordinate transformations. The tetrad field relates two symmetries. On the one hand, the Greek indices denote coordinates transformation. On the other hand, the Latin indices denote Lorentz transformations.

Let's consider a manifold endowed with the following connection:

$$\Gamma_{\mu\lambda\nu} = e^a{}_\mu \partial_\lambda e_{a\nu} \,,$$

which is the Weitzenböck connection. It is curvature free, but has a torsion given by

$$T^a{}_{\lambda\nu} = \partial_\lambda e^a{}_\nu - \partial_\nu e^a{}_\lambda \,. \tag{2}$$

It is important to note that the tetrad is related to the metric tensor through $g_{\mu\nu} = e^a{}_\mu e_{a\nu}$. The metric tensor is the foundation of a Riemannian geometry in which Christoffel symbols, ${}^0\Gamma_{\mu\lambda\nu}$, are defined. The scalar curvature obtained from this connection is the invariant in the Hilbert–Einstein's action. Thus, a relationship between Christoffel symbols and Weitzenböck connection determines in itself an equivalence between TEGR and general relativity. Such a relationship is a mathematical identity given by

$$\Gamma_{\mu\lambda\nu} = {}^0\Gamma_{\mu\lambda\nu} + K_{\mu\lambda\nu} \,, \tag{3}$$

where $K_{\mu\lambda\nu}$, that is defined as

$$K_{\mu\lambda\nu} \;=\; \frac{1}{2}(T_{\lambda\mu\nu} + T_{\nu\lambda\mu} + T_{\mu\lambda\nu}) \,, \tag{4}$$

with $T_{\mu\lambda\nu} = e_{a\mu} T^a{}_{\lambda\nu}$, is the contortion tensor.

Due to the identity (3), the following equation holds

$$eR(e) \equiv -e(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^a T_a) + 2\partial_\mu(eT^\mu) \,. \tag{5}$$

It should be noted that the scalar curvature calculated out of the Weitzenböck connection vanishes identically. Hence, the Lagrangian density of TEGR may be written as

$$\begin{aligned} \mathfrak{L}(e_{a\mu}) \;&=\; -\kappa\, e\,(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^a T_a) - \mathfrak{L}_M \\ &\equiv\; -\kappa\, e \Sigma^{abc} T_{abc} - \mathfrak{L}_M \,, \end{aligned} \tag{6}$$

where $\kappa = 1/(16\pi)$, $\mathfrak{L}_M$ is the Lagrangian density of matter fields and Σ^{abc} is given by

$$\Sigma^{abc} = \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac}T^b - \eta^{ab}T^c) \,, \tag{7}$$

with $T^a = e^a{}_\mu T^\mu$. The Lagrangian density above yields the Einstein equation but doesn't share the same symmetries of the general relativity Lagrangian density because the total divergence in (5) was dropped out. Thus, the field equations read

$$\partial_\nu\Big(e\Sigma^{a\lambda\nu}\Big) = \frac{1}{4\kappa} e\, e^a{}_\mu (t^{\lambda\mu} + T^{\lambda\mu}) \,, \tag{8}$$

where

$$t^{\lambda\mu} = \kappa\Big[4\,\Sigma^{bc\lambda} T_{bc}{}^\mu - g^{\lambda\mu}\,\Sigma^{abc} T_{abc}\Big] \tag{9}$$

is the gravitational energy-momentum tensor. Such a tensor is conserved due to

$$\partial_\lambda \partial_\nu\Big(e\Sigma^{a\lambda\nu}\Big) \equiv 0 \,. \tag{10}$$

This allows one to define the total energy-momentum vector as

$$P^a = \int_V d^3x\, e\, e^a{}_\mu (t^{0\mu} + T^{0\mu}) \,, \tag{11}$$

which can be expressed in view of the field equations as

$$P^a = 4k \int_V d^3x\, \partial_\nu\Big(e\,\Sigma^{a0\nu}\Big). \tag{12}$$

It should be noted that P^a is a vector under Lorentz transformations, but it is invariant under coordinate transformations, as expected for a energy-momentum vector.

3. The Fractional Sturm–Liouville Theory and Hilbert Space

In this section, we present the fractional theory of Sturm–Liouville. This subject is the background to analyze fractional ordinary differential equations (ODEs) by self-adjunct procedure and to establish a Hilbert space. The content of this section is based on references [13–18].

3.1. Preliminaries

In this subsection, we recall definitions of fractional integrals and derivatives. We focus on the Caputo's approach due to the smoothing of Riemann–Liouville regarding physical interpretations.

Definition 1. *Let's assume $\alpha > 0$, with $n-1 < \alpha < n$ and $n \in \mathbb{N}$, $[a,b] \subset \mathbb{R}$, in addition let f be a suitable real function. The Caputo fractional derivative is*

$$({}_CD^{\alpha}_{a+}f)(x) = (I^{n-\alpha}_{a+}D^n f)(x), \quad (x > a), \tag{13}$$

$$({}_CD^{\alpha}_{b-}f)(x) = (I^{n-\alpha}_{b-}D^n f)(x), \quad (x < b), \tag{14}$$

where

$$(I^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad (x > a), \tag{15}$$

$$(I^{\alpha}_{b-}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (x-t)^{\alpha-1} f(t)dt, \quad (x < b), \tag{16}$$

$\Gamma(\alpha)$ denotes the Euler gamma function, and D^n represents the usual derivative operator $D^n = \dfrac{\partial^n}{\partial x^n}$.

As an example, let's calculate the Caputo fractional derivative of function $f(x) = x^2$ of order $1/2$. First, we apply the differential operator $D = \dfrac{\partial}{\partial x}$ in $f(x) = x^2$, obtaining $D(x^2) = 2x$. Now, we calculate the integral

$$({}_CD^{1/2}_0 x^2) = \frac{1}{\Gamma(1/2)}\int_0^x (x-t)^{1/2} 2x dt.$$

Performing this integral and using $\Gamma(1/2) = \sqrt{\pi}$, we obtain

$$({}_CD^{1/2}_0 x^2) = \frac{2}{\sqrt{\pi}} x^{1/2}. \tag{17}$$

The Caputo fractional derivative satisfies several properties which can be find in references.

Next we present the definition of a relevant function to Caputo differential calculus which is called Mittag–Leffer function.

Definition 2. *The two parameters Mittag–Leffer function, $E_{a,b}(x)$, where $\Re\mathfrak{e}(a) > 0$, $\Re\mathfrak{e}(b) > 0$, is defined by*

$$E_{a,b}(x) = \sum_{k=o}^{\infty} \frac{x^k}{\Gamma(ak+b)}, \tag{18}$$

where we notice that $E_{1,1}(x) = e^x$ and $E_{a,1}(x) = E_a(x)$.

It is easy to show that the Caputo derivative of Mittag–Leffer function is given by

$$ {}_C D^{\gamma} E_{\alpha}(x^{\alpha}) = E_{\alpha}(x^{\alpha}). \tag{19}$$

The Mittag–Leffer function is a kind of generalization of exponential function to fractional calculus.

3.2. Fractional Sturm–Liouville Theory

In this section, we introduce fractional version to the Sturm–Liouville theory. For this proposal, let's begin with the following definition.

Definition 3. *A Caputo fractional Sturm–Liouville problem is a fractional problem with boundary conditions in the form*

$$ -_C\, D^{\alpha}_{b-}({}_C u_C D^{\alpha}_{a+} y)(x) + v(x)y(x) = \lambda r(x)y(x), \tag{20}$$

$$ a < x < b, \qquad 1/2 < \alpha < 1 \tag{21}$$

$$ \alpha_1 y(a) + \alpha_2 I^{1-\alpha}_{b-}({}_C u_C D^{\alpha}_{a+} y)(x)|_{x=a} = 0, \tag{22}$$

$$ \beta_1 y(b) + \beta_2 I^{1-\alpha}_{b-}({}_C u_C D^{\alpha}_{a+} y)(x)|_{x=b} = 0, \tag{23}$$

where $L_a = -_C D^{\alpha}_{b-}({}_C u_C D^{\alpha}_{a+}) + v$ is a self-adjoint operator and the constants α_i, β_i satisfied in the boundary conditions verify $\alpha_1^2 + \alpha_2^2 \neq 0$, $\beta_1^2 + \beta_2^2 \neq 0$ and the functions u, v, r are continuous, such that $u > 0$ and $r > o$ in $x \in [a, b]$. The function r is called a weight function, and the values of λ for which there exist non-trivial solutions are called eigenvalues of the boundary value problem.

In this sense, the Caputo fractional Sturm–Liouville problem satisfies the following properties.

Caputo Fractional Sturm–Liouville problem properties:

1. All of the eigenvalues of the fractional Sturm–Liouville problem are real.
2. If y_1 and y_2 are two eigenvalues of the fractional Sturm–Liouville problem corresponding to eigenvalues λ_1 and λ_2, respectively, with $\lambda_1 \neq \lambda_2$, then

$$ \int_a^b r(x)y_1(x)y_2(x)dx = 0. \tag{24}$$

   That is, the eigenvalues corresponding to different eigenvalues have the property of orthogonality with respect to the weight function r.
3. For each eigenvalue, there is only one eigenfunction (except for multiples non zeros).
4. The eigenfunction corresponding to different eigenvalues are linearly independent.

If $y_n(x)$ are complex functions, orthogonality condition, Equation (24) becomes

$$ \int_a^b r(x)y_n^*(x)y_m(x)dx = 0; \qquad m \neq n, \tag{25}$$

where $y_n^*(x)$ is the conjugate complex of $y_n(x)$.

Due the hermiticity of operator $\mathcal{L}_{\dashv}$, i.e., $\overline{\mathcal{L}} = \mathcal{L}$, their eigenfunctions $y_n(x)$ form a complete set. This completeness means that any well-behaved function $f(x)$ can be approximated by a series

$$ f(x) = \sum_{n=0}^{\infty} a_n y_n(x), \tag{26}$$

where the coefficient a_n are given by

$$ a_n = \int_a^b p(x)f(x)y_m^*(x)dx. \tag{27}$$

Equation (26) is unique for a given set of $y_n(x)$. The functions $y_n(x)$ are a basis vector in an infinite dimensional Hilbert space $\mathcal{H}$. In this sense, we can define an Hilbert space $\mathcal{H}$ form the linear space with inner product defined as

$$\langle f(x), g(x)\rangle = \int_a^b r(x) f^*(x) g(x) dx. \tag{28}$$

Those functions $f(x)$ defined in $\mathcal{H}$ are integrable square functions, i.e.,

$$\langle f(x), f(x)\rangle = \int_a^b r(x) f^*(x) f(x) dx < \infty. \tag{29}$$

This framework of fractional Sturm–Liouville theory and Hilbert space will be useful in our discussion about quantum gravity in the next sections. It is worth mentioning that the fractional Sturm–Liouville problem reduces to the usual case when $\alpha = 1$.

4. Quantum Schwarzschild Equation

In this section, we study the eigenvalue equation related to the quantum Schwarzschild system. First, we consider the following metric,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{30}$$

where $f(r) = 1 - \frac{2M}{r}$ and $d\Omega$ is a solid angle element. M represent the mass of the system and r is the radial coordinate. From Equation (30), we obtain the density of energy

$$\mathfrak{H} = 4kr\sin\theta(1 - f^{1/2}). \tag{31}$$

It should be noted that the energy density is obtained from expression (11) or equivalently from (12), that is $\mathfrak{H} = \frac{\partial P^{(0)}}{\partial V}$. This function can be quantized by using the symmetrization rule and the operators

$$\widehat{\omega} = \omega, \quad \widehat{r} = i\beta\frac{\partial}{\partial\omega}, \tag{32}$$

where $\omega = \sin\theta$ and $\beta = \beta_{12}$ is the non-commutativity parameter. A particularly problematic feature of the energy density is its dependence on the radial coordinate with a fractional exponent. In previous articles [19], this difficulty was overcome through a power series expansion because the non-commutativity parameter must be very small. Here another possibility will be explored, namely the use of the fractional derivative that can be directly used in the energy operator. Then, using the condition $\frac{\beta}{M} \ll 1$ and the result $D^{1/2}(\omega) = \frac{2}{\sqrt{\pi}}\omega^{1/2}$, we obtain the following fractional differential equation,

$$\frac{\epsilon}{2k}\psi(\omega) = \left[i\beta + 2i\beta\omega\frac{\partial}{\partial\omega} - i^{3/2}\beta^{1/2}\sqrt{2M}\omega D^{1/2} - \frac{2i^{3/2}\beta^{1/2}\sqrt{2M}}{\sqrt{\pi}}\omega^{1/2}\right]\psi(\omega), \tag{33}$$

where ϵ is the eigenvalue of operator $\widehat{\mathfrak{H}}$. From now on, we represent the Caputo's fractional derivative ${}_CD^\alpha$ just by D^α. Equation (33) can be written as

$$c_1\omega D^1\psi(\omega) + c_2\omega D^{1/2}\psi(\omega) + c_3\omega^{1/2}\psi(\omega) + c_4\psi(\omega) = 0, \tag{34}$$

where $c_1 = 2i\beta$, $c_2 = -i^{3/2}\beta^{1/2}\sqrt{2M}$, $c_3 = -\frac{2i^{3/2}\beta^{1/2}\sqrt{2M}}{\sqrt{\pi}}$, $c_4 = i\beta - \epsilon$. If we multiply Equation (34) by integrator factor

$$\mu(\omega) = E_{1/2}(\frac{c_2}{c_1}\omega^{1/2}),$$

and use the following property of Mittag–Leffer function

$$D^{\alpha}[E_{\alpha}(x^{\alpha})] = E_{\alpha}(x^{\alpha}).$$

Equation (34) becomes

$$D^{1/2}\left[E_{1/2}(\frac{c_2}{c_1}\omega^{1/2})D^{1/2}\psi(\omega)\right] + \frac{c_3}{c_1\omega^{1/2}}E_{1/2}(\frac{c_2}{c_1}\omega^{1/2})\psi(\omega) + \frac{c_4}{c_1\omega}E_{1/2}(\frac{c_2}{c_1}\omega^{1/2})\psi(\omega) = 0. \quad (35)$$

Equation (35) is a Caputo's fractional Sturm–Liouville problem with weight function given by

$$g(\omega) = \frac{c_4}{c_1\omega}E_{1/2}(\frac{c_2}{c_1}\omega^{1/2}). \quad (36)$$

In this sense, the eigenvalues of this problem are real and the solutions $\psi_n(\omega)$ satisfy the following orthogonality condition,

$$\int_a^b \frac{c_4}{c_1\omega}E_{1/2}(\frac{c_2}{c_1}\omega^{1/2})\psi_n\omega\psi_m\omega d\omega = 0, \qquad m \neq n. \quad (37)$$

Then, we can define an Hilbert space in such functions where $\psi_n(\omega)$ are square integrable.

5. Quantum Reissner–Nordstrom System

In reference [11], the possibility of a charged particle being described by a quantized version of the Reissner–Nordstrom metric was explored. Here we deal with the Hilbert space as a theoretical advance of the quantum description. The metric is described by the following line element,

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\,,$$

with $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$, which leads to the following energy density

$$\mathfrak{H} = 4r\sin\theta\left[1 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{1/2}\right].$$

Thus, by using the following quantization rule, $\sin\theta \to \hat{\sin}\,\theta = \omega$ and $r \to \hat{r} = \beta\frac{\partial}{\partial\omega}$, we have the equation

$$\left\{-\frac{4\beta^2\omega}{2Q}\frac{\partial^2}{\partial\omega^2} + 4\beta\left[\omega\left(1 + \frac{M}{Q}\right) - \frac{\beta}{2Q}\right]\frac{\partial}{\partial\omega} + \left[2\beta\left(1 + \frac{M}{Q}\right) - \epsilon - 4Q\omega\right]\right\}\psi(\omega) = 0. \quad (38)$$

Equation (38) can be written as

$$\left[b_1\omega\frac{\partial^2}{\partial\omega^2} + (b_2\omega - b_3)\frac{\partial}{\partial\omega} + (b_4 + b_5\omega)\right]\psi(\omega) = 0, \quad (39)$$

where $b_1 = -\frac{4\beta^2}{2Q}$, $b_2 = 4\beta\left(1 + \frac{M}{Q}\right)$, $b_3 = \frac{2\beta^2}{Q}$, $b_4 = 2\beta\left(1 + \frac{M}{Q}\right) - \epsilon$ and $b_5 = -4Q$.

If we multiply Equation (39) by the integrator factor $\mu(\omega) = \frac{b_1}{b_3} e^{\frac{b_3}{b_1}\omega}$, we obtain

$$\frac{\partial}{\partial\omega}\left(\frac{b_1}{b_2} e^{\frac{b_3}{b_1}\omega} \frac{\partial\psi(\omega)}{\partial\omega}\right) + \frac{1}{b_3\omega} e^{\frac{b_3}{b_1}\omega}(b_4 + b_5\omega)\psi(\omega) = 0. \tag{40}$$

Equation (40) is the adjoint form of Equation (39). In this way, Equation (40) represents a usual Sturm–Liouville problem with weight function given by

$$g(\omega) = \frac{b_4}{b_3\omega} e^{\frac{b_3}{b_1}\omega}. \tag{41}$$

In this case, the eigenvalues of operator $\widehat{\mathfrak{H}}$ are real and quantized. The functions $\psi_n(\omega)$ are orthogonal with respect to weight function r, that is

$$\int_0^\infty \frac{b_4}{b_3\omega} e^{\frac{b_3}{b_1}\omega} \psi_n(\omega)\psi_m(\omega) d\omega = 0, \qquad n \neq m. \tag{42}$$

Then, the set $\{\psi_n(\omega)\}$ is complete and an general state $f(\omega)$ can be expressed in terms of this basis,

$$f(\omega) = \sum_{n=0}^{\infty} c_n \psi_n(\omega), \tag{43}$$

with

$$c_n = \int_0^\infty \frac{b_4}{b_3\omega} e^{\frac{b_3}{b_1}\omega} f(\omega)\psi_n(\omega) d\omega. \tag{44}$$

This shows that we have a Hilbert space $\mathcal{H}$ with square integrable functions. Due to $\widehat{\mathfrak{H}}$ being self-adjointed, their eigenvalues are real. In this way, ϵ_n can be represent a measure of a physical quantity.

6. Quantum FLRW System

In reference [12], the quantization of a homogeneous and isotropic universe is explored. The FLRW metric is

$$ds^2 = -dt^2 + a^2\left[\frac{dr^2}{1 - k\,r^2} + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right].$$

Then the energy density is

$$k\,\mathfrak{H} = \frac{a^2(3\,\dot{a}^2 - k)\,\sqrt{\dot{a}^2 + k\,(1 - a^2)}}{4\,(\dot{a}^2 + k)} - \frac{a\,(3\,\dot{a}^2 + k)}{4\,\sqrt{k}} \arctan\left[\frac{\sqrt{k}\,a}{\sqrt{\dot{a}^2 + k\,(1 - a^2)}}\right].$$

It should be noted that we can choose any representation for the operators in the quantization process. Here we use $a \to \hat{a} = a$ and $\dot{a} \to \hat{\dot{a}} = \beta\frac{\partial}{\partial a}$. This leads to the equation

$$\left[15a^2 + \frac{66}{8} + \left(6a^2 + \frac{76}{8}\right)a\frac{d}{da} + \left(\frac{a^2}{2} + \frac{31}{8}\right)a^2\frac{d^2}{da^2}\right]\psi(a) = \epsilon\psi(a)\,, \tag{45}$$

where $\epsilon = \frac{k^{3/2}E}{\beta^2}$ and E is the observable energy. Equation (45) can be written as

$$\left[-\epsilon + h_3(a) + h_2(a)a\frac{d}{da} + h_1(a)a^2\frac{d^2}{da^2}\right]\psi(a) = 0, \tag{46}$$

where $h_1 = \frac{1}{2}\left(a^2 + \frac{31}{4}\right)a^2$, $h_2(a) = 2\left(3a^2 + \frac{19}{4}\right)a$, $h_3(a) = 15a^2 + \frac{66}{8}$. Multiplying Equation (46) by integrator factor

$$\mu(a) = \frac{2\left(a^2 + \frac{31}{4}\right)^{127/31}}{a^{45/31}}, \tag{47}$$

we obtain

$$\frac{d}{da}\left(\frac{\left(a^2 + \frac{31}{4}\right)^{158/31}}{a^{76/31}}\frac{d\psi(a)}{da}\right) + h_3(a)\mu(a)\psi(a) - \epsilon\mu(a)\psi(a) = 0. \tag{48}$$

Equation (48) is the self-adjoint of Equation (47). Equation (48) represents a usual Sturm–Liouville problem with weight function given by

$$r(a) = \frac{2\left(a^2 + \frac{31}{4}\right)^{127/31}}{a^{45/31}}. \tag{49}$$

In this case, the eigenvalues of operator $\widehat{\mathfrak{H}}$ are real and quantized. The functions $\psi_n(a)$ are orthogonal with respect to weight function r, that is

$$\int_0^b \frac{2\left(a^2 + \frac{31}{4}\right)^{127/31}}{a^{45/31}}\psi_n(a)\psi_m(a)da = 0, \qquad n \neq m. \tag{50}$$

Then, the set $\{\psi_n(a)\}$ is complete and an general state $f(a)$ can be expressed in terms of this basis,

$$f(a) = \sum_{n=0}^{\infty} c_n\psi_n(a), \tag{51}$$

with

$$c_n = \int_0^{\infty} \frac{2\left(a^2 + \frac{31}{4}\right)^{127/31}}{a^{45/31}} f(a)\psi_n(a)da. \tag{52}$$

This shows that we have a Hilbert space $\mathcal{H}$ with square integrable functions. Because $\widehat{\mathfrak{H}}$ is self-adjointed, their eigenvalues are real. In this way, ϵ_n can represent a measure of a physical quantity.

7. Conclusions

In this article, we apply a quantization procedure to Schwarzschild space–time and obtain a differential equation in terms of the Caputo fractional derivative. With this, we were able to establish Hilbert's space for this configuration. In previous articles, quantization procedures were applied to Reissner–Nordstron space–time, as well as to the FRW metric. The first describes a charged black hole, and the second describes an isotropic and homogeneous Universe. Thus, we show how the respective functions of Hilbert space obey certain orthogonal conditions. It is interesting to note that once Hilbert's space has been defined, the eigenvalues of the equations resulting from the quantization process have real values and represent experimentally verifiable quantities. It is worth clarifying that the quantization process is done in the space of one of the coordinates, that is, one of the operators resulting from the process is always multiplicative. The passage of the gravitational energy function to the quantum operator can be problematic due to the semi-integer powers of the coordinates on which the function depends. This is circumvented with the use of the fractional derivative of Caputo. In particular, the conditions which establish a Hilbert space for the FLRW metric strengthen the interpretation of a multiverse. Each discrete function may describe a specific universe. The orthogonality relationship guarantees the physical independence of each solution. This establishes a very well-defined procedure for the construction of the respective Hilbert space and therefore for a promising quantum gravitational theory. The quantization procedure used here depends only on the existence of a gravitational energy, although we defend the concept of energy obtained in the scope of the TEGR; the method extends to any definition of energy in the literature. In particular, a very well-accepted approach to quantum gravity, the loop quantum gravity, could benefit from this quantization process. As a future perspective we hope to understand the limitations or advantages of Weyl quantization in the 3 + 1 decomposition of the Hamiltonian formulation of gravitation. The greatest difficulty would be in the choice of operators since they are defined from an anti-commutation relation of two (or more) coordinates. In ADM decomposition, the dynamic field is generic. This problem is analogous to the Schroedinger equation in which we need to define the potential so that the equation can be solved; otherwise we have a generic Hamiltonian.

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