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# Bekenstein Bound and Non-Commutative Canonical Variables

**Fabio Scardigli** [1,2]

[1] Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy; fabio@phys.ntu.edu.tw

[2] Department of Applied Mathematics, University of Waterloo, Waterloo, ON N2L 3G1, Canada

**Abstract:** A universal upper limit on the entropy contained in a localized quantum system of a given size and total energy is expressed by the so-called Bekenstein bound. In a previous paper [Buoninfante, L. et al. 2022], on the basis of general thermodynamic arguments, and in regimes where the equipartition theorem still holds, the Bekenstein bound has been proved practically equivalent to the Heisenberg uncertainty relation. The smooth transition between the Bekenstein bound and the holographic bound suggests a new pair of canonical non-commutative variables, which could be thought to hold in strong gravity regimes.

**Keywords:** entropy bounds; uncertainty principle; holographic bound

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## 1. Introduction

A universal upper bound on the entropy $S$ of a localized quantum system

$$S \leq \frac{2\pi k_B R E}{\hbar c}\,, \tag{1}$$

where $E$ is the total energy of the system and $R = \sqrt{A/4\pi}$ its size, with $A$ being the area of the minimal enclosing spherical surface, was proposed by Bekenstein [1] in 1981. This result arrived at the end of a whirlwind decade of investigations, which started with the puzzling proposal of Bekenstein himself about the entropy of a black hole [2–4], then the formulation of black-hole thermodynamics [5], and had its apex with Hawking's famous discovery of thermal radiation from black holes [6]. We note that, from the classical point of view, the entropy of a system is, in principle, unbounded from above. In fact, for $\hbar \to 0$, one obtains $S \leq \infty$. Therefore, the upper bound on the entropy has a clear quantum origin.

Even though Equation (1) was initially obtained via gedanken experiments involving black holes, namely, in strong-gravity regimes, such inequality does not display the Newton constant $G_N$. Therefore, the gravitational self—the interaction of the system—seems to be completely neglected. However, Schwarzschild black holes exactly saturate inequality (1), with an entropy given by $S = k_B A_H/(2\ell_p)^2$, where $A_H$ is the horizon area and $\ell_p = \sqrt{\hbar G_N/c^3}$ the Planck length.

Many arguments [7–11] have been made in favour of Equation (1), although also several counterexamples have been devised, thereby enriching a lively debate which is still ongoing [12–18]. During the years, several steps forward have, then, led to the formulation of the holographic principle [19–21], and the covariant [22] and causal [23] entropy bounds. In addition, in 2008, finally, Casini obtained a QFT rigorous proof of the Bekenstein bound in flat spacetime [24]. Of course, the ideas of Bekenstein have had a significant influence on (quantum) information theory, and the reader can easily check for examples in Refs. [25–27]. Various entropy bounds have also revealed many connections with cosmology [23,28–31], perturbative unitarity [32] and the Pauli principle [33].

In this paper, we are primarily interested in bringing to light the connections that such an entropy bound has with HUP. To this end, we show how general thermodynamic arguments allow to derive Bekenstein inequality (1) directly from HUP, and also how the reverse implication is true (at least for relativistic particles). The elementary and general nature of the present derivation helps to clarify why the Bekenstein bound has such a wide range of validity.

In the next section, we summarize the main lines of this derivation, following the footsteps of Ref. [34]. Successively, we will discuss the smooth transition between the Bekenstein bound and the Holographic bound, and the ensuing connection of the latter with non-commutative variables.

## 2. HUP and Bekenstein Bound

Consider a generic isolated quantum system localized inside a finite region of size $R$. From general thermodynamics, we know that the relation between the energy $E$, the entropy $S$, the volume $V$ of the system, and its temperature $T$ is given by

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_V . \tag{2}$$

Of course, relation (2) requires the differentiability of the function $S(E, V)$. In addition, we explicitly exclude (hypothetical) systems with a negative (unphysical) temperature. In what follows, we make two very general assumptions on the above isolated quantum system

(i) The system is in a regime where the equipartition theorem holds, namely, on average, the energy $\mu$ of each component of the system is approximately given by

$$\mu \simeq k_B T . \tag{3}$$

(ii) As our system is quantum, the momentum $p$ of each component should satisfy the de Broglie relation

$$p = \frac{\hbar}{\lambda} , \tag{4}$$

where $\lambda$ denotes the corresponding wavelength.

Note that this second condition can only be defined for intrinsically quantum particles. In addition, as we know, Equation (4) is essentially equivalent to the Heisenberg relation between the momentum and position uncertainties.

It is also well-known that, although the equipartition theorem is a classical statement, it also holds true for a large majority of physical quantum systems in regimes close to classicality. In other words, Maxwell–Boltzmann statistics are a good approximation of quantum statistics in most of the systems in semiclassical regimes. For example, a gas of bosons at low frequencies or high temperatures is well-described by the standard Maxwell–Boltzmann statistics.

Now, from any quantum statistics distribution formula, we infer that the more a quantum system will be close to classicality, the more the energy $k_B T$ will satisfy the condition

$$k_B T \gtrsim \frac{\hbar c}{\lambda} = pc. \tag{5}$$

In addition, as our system is localized inside a volume of radius $R$, the inequality $\lambda \lesssim 2R$ must hold true, so that from Equations (2) and (5), we infer

$$\frac{\partial S}{\partial E} = \frac{1}{T} \lesssim \frac{k_B \lambda}{\hbar c} \lesssim \frac{2 k_B R}{\hbar c} , \tag{6}$$

where it is understood that the derivative is taken at constant volume.

Now, in general, $R$ and $E$ can be regarded as independent variables; therefore, we can easily integrate the above relation with the condition[1] $S(E = 0) = 0$, so obtaining

$$S \lesssim \frac{2\alpha k_B R E}{\hbar c}\,. \tag{7}$$

A "calibration factor" $\alpha$ has been inserted in order to account for the approximations so far performed. This factor cannot be exactly fixed by our thermodynamic argument, however, it was obtained in Ref. [34] by means of consistency arguments, and resulted as $\alpha = \pi$.

Remarkably, the above considerations and the connected bound (7) also remain valid when $R$ and $E$ are related via an equation of state. In fact, for a general and physically plausible radius–energy relation of the form $R = R(E)$, with $R(E)$ being a monotonically non-decreasing function of $E$, one can prove that the inequality (7) is still satisfied (see the Appendix for the proof).

We would like to stress that our result (7) has been derived on the basis of quite general hypotheses. In addition, we have not assumed any particular behavior of the entropy as a function of the energy and/or the number of the elementary constituents. Less complete attempts to trace the Bekenstein bound back to HUP can be found in Refs. [35,36].

It is important to also stress that the opposite implication, namely, a derivation of HUP from the Bekenstein bound, can be obtained, at least for relativistic particles [34], as also partially outlined in Ref. [37].

In fact, supposing that $R$ is the size of the wavepacket describing a particle of rest mass $m$, and supposing that such a particle is almost relativistic, i.e., $p \simeq E/c$, then the inequality (1) can be swiftly recast in the form

$$R\,p \simeq \frac{RE}{c} \geq \frac{\hbar}{2}\frac{S}{\pi k_B} \gtrsim \frac{\hbar}{2}\,, \tag{8}$$

which is valid for any system such that $S \gtrsim \mathcal{O}(k_B)$.[2] The calibration factor to be inserted again equals $\pi$, and, as before, cannot be determined with this heuristic approach.

By identifying the uncertainties in the position and in the momentum of our particle, respectively, as $\Delta x \simeq R$ and $\Delta p_x \simeq p$, (since the direction of motion of our particle is unknown a priori), then the Bekenstein inequality (8) can be immediately read as

$$\Delta x \Delta p_x \gtrsim \frac{\hbar}{2}\,, \tag{9}$$

which is the standard HUP for the particle in question. Together with the implication previously shown, the latter argument highlights a full physical consistency between the Bekenstein bound and Heisenberg uncertainty principle. Finally, it is interesting to note that this part of the argument suggests how to identify the canonical variables appearing in the Bekenstein inequality. This leads to the considerations of the next section.

## 3. Non-Commutative Variables

In the previous section, we proved a substantial equivalence between the Bekenstein bound and the Heisenberg uncertainty principle. Both principles are known to work well for systems in which gravity is negligible or weak. On the other hand, it is possible to relate the Bekenstein bound with the holographic bound via the inequality [30]

$$\begin{aligned} S_{BB} &\leq \frac{2\pi k_B E R}{\hbar c} = \frac{2\pi k_B G_N E R}{G_N \hbar c} \\ &= \frac{1}{4} k_B \frac{(4\pi R_g R)}{\ell_p^2} \leq \frac{1}{4} k_B \frac{(4\pi R^2)}{\ell_p^2} = S_{HB} \end{aligned} \tag{10}$$

where $R_g = 2G_N E/c^4$ is the gravitational radius associated with the energy $E$. The latter inequality holds because for a generic physical system $R_g \leq R$ holds always, namely, the gravitational radius lies well inside the physical size of the system for any non-collapsed object. The equality $R_g = R$ holds only if the object is inside its own gravitational radius, namely, it is itself a black hole. In this case, the two bounds coincide and are saturated.

Due to its equivalence with HUP, we see that in the expression of the Bekenstein bound a pair of non-commuting canonical variables can be identified, namely, $(E/c, R)$, which for weak-gravity or "light" systems coincide with the usual non-commuting canonical pair $(p, R)$. The above sequence (10) of inequalities suggests that, as the energy of the system contained in the region $R$ grows (namely, $R_g$ grows), then the analytic expression of the Bekenstein bound naturally tends to become the analytic expression of the holographic bound. Therefore, we can reasonably wonder if the pair of non-commuting canonical variables $(p, R) \sim (E/c, R)$ naturally goes to identify another pair of non-commuting canonical variables, i.e., $(R, R)$ or $(t, R)$, this time more properly useful for the description of highly massive and gravitating systems. In other words, the expression of the holographic bound, $S_{HB} \sim R^2$, may suggest a different pair of dynamical non-commuting canonical variables $(R_i, R_k)$, to be used in strong-gravity or quantum-gravity situations (e.g., black holes), where holography becomes crucial.

For example, having in mind a Schwarzschild geometry, we can imagine that the correct spacelike local coordinates to be used on the surface of the event horizon are actually non-commutative variables, satisfying a commutation relation such as

$$[R\hat{\theta}, R\sin\theta\hat{\phi}] = R^2 \sin\theta \left[\frac{\hat{\theta}}{\sin\theta}, \hat{\phi}\right] = i\hbar\,, \tag{11}$$

which would be nothing else than a particular case of the general commutator between two non-commutative coordinates (see, e.g., Refs. [38,39])

$$[\hat{x}_j, \hat{x}_k] = i\hbar\Theta_{jk}\,. \tag{12}$$

Of course, the status of a proposal such as (11) can be qualified, at the moment, just as an educated guess, based solely on taking seriously the strong analogy between the variables acting, on one hand, in the Bekenstein bound and in HUP expression, and, on the other hand, the variables entering the analytic expression of the holographic bound.

We believe, however, that such a guess can have potentially rich consequences, worthy of being fully explored in future, more complete works.

## 4. Concluding Remarks

In this paper, we presented arguments in favour of a full consistency between the Heisenberg uncertainty principle and the Bekenstein bound on the entropy of a localized system with a given size and total energy, in particular in regimes where the equipartition theorem still holds. A comparison of such a result with the holographic bound has paved the way for an educated guess on the non-commuting nature of the canonical variables to be used in the description of systems where gravity is strong. Future works are being prepared to study the viability and coherence of this proposal.

*Appendix*

In this appendix, we show that our derivation of the inequality (7) holds for any monotonically non-decreasing function $R = R(E)$. Let $R(\varepsilon)$ and $g(R)$ be two positive, monotonically non-decreasing functions, respectively, of $\varepsilon$ (with $0 \leq \varepsilon \leq E$) and of $R$. By introducing the partial derivative $S'(\varepsilon) := \partial S/\partial\varepsilon$, the inequality (6) can be written in the compact form

$$S'(\varepsilon) \lesssim g(R(\varepsilon))\,. \tag{13}$$

We can now integrate the above inequality with the usual condition $S(\varepsilon = 0) = 0$ and obtain

$$S(E) = \int_0^E \mathrm{d}\varepsilon\, S'(\varepsilon) \lesssim \int_0^E \mathrm{d}\varepsilon\, g(R(\varepsilon)) \leq E\, g(R(E))\,, \tag{14}$$

where we used the fact that, also, $g(R(\varepsilon))$ is a monotonically non-decreasing function of $\varepsilon$ as it is a composition of two monotonically non-decreasing functions. Therefore, we proved that $S(E) \lesssim E\, g(R(E))$, which resumes the inequality (7).

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## Notes

[1] For completeness, we emphasize that the ansatz $S(E=0)=0$ contains the hidden assumption of a unique ground state.

[2] For instance, an electron can be in two possible states (spin up and spin down) and, therefore, its entropy is given by $S = k_B \log 2 \sim \mathcal{O}(k_B)$.

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