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Lorentz Gauge and Coulomb Gauge for Tetrad Field of Gravity

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Abstract: In general relativity, an inertial frame can only be established in a small region of spacetime, and local inertial frames are mathematically represented by a tetrad field in gravity. The tetrad field is not unique due to the freedom to perform Lorentz transformations in local inertial frames, and there exists freedom to choose the local inertial frame at each spacetime point. The local Lorentz transformations are known as non-Abelian gauge transformations for the tetrad field, and to fix the gauge freedom corresponding to the Lorentz gauge $\partial^\mu \mathcal{A}_\mu = 0$ and the Coulomb gauge $\partial^i \mathcal{A}_i = 0$ in electrodynamics, the Lorentz gauge and Coulomb gauge for tetrad fields are proposed in the present work. Moreover, properties of the Lorentz gauge and the Coulomb gauge for tetrad fields are discussed to show their similarities to those in electromagnetic fields.

Keywords: general relativity; tetrad field; Lorentz gauge; Coulomb gauge

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1. Introduction

Spacetime is flat in special relativity, and a global inertial frame (GIF) can always be established. Any observer with a constant three-velocity in a GIF is an inertial observer, and the laws of physics look the same to all such observers. In general relativity, one main issue that no unique global inertial frame can be established, arises from flat-to-curved spacetime. However, in small regions of spacetime, the spacetime looks approximately flat, and freely falling observers who perform physics experiments in these small regions of spacetime around them would see that physics is approximately the same as that seen by an inertial observer in flat spacetime. Therefore, to retain the notation of inertial frames, one can define local inertial frames (LIFs) in the small enough regions of spacetime, but they cannot be globally extended throughout spacetime.

LIFs are described by the tetrad field [1,2], which is briefly discussed in the following. The freedom of the tetrad field due to the local Lorentz transformations (LLTs) provides different LIFs at the same spacetime point, and the choice of a LIF at the same spacetime point is, accordingly, quite arbitrary. In the view of gauge theory, Lorentz transformations play the role of gauge transformations. The redundant gauge freedoms compensate for elegant mathematical expressions but, meanwhile, result in an obstacle to identifying the real physical degrees of freedom. In electrodynamics, the vector potential $\mathcal{A}^\mu$ is used to describe the massless photon, where the Lorentz gauge $\partial^\mu \mathcal{A}_\mu = 0$ and Coulomb gauge $\partial^i \mathcal{A}_i = 0$ can both remove certain degrees of the gauge freedom. The Lorentz gauge, however, leaves some degrees of unphysical freedom, while the Coulomb gauge is sufficient to specify the two physical degrees of a photon's polarization. Similar to in electrodynamics, the Coulomb gauge for gravity in weak-field approximation was proposed and discussed in [3,4].

Due to the freedom of the local choice of inertial frame in curved spacetime, there is no way to determine the same direction of LIFs at two separated spacetime points. This is a dramatic problem; for instance, the spin entanglement state shared by two separated parties has an ambiguous meaning in curved spacetime. To overcome this issue, a gauge condition can be introduced to remove the degrees of freedom, as is usual in gauge theory, and an appropriate gauge condition leaving only physical freedom is further appreciated. In this paper, corresponding to those in electrodynamics, we investigate the Lorentz gauge and Coulomb gauge for tetrad fields, and the properties of the two gauge conditions are discussed.

2. Mathematical Representation of LIFs

In general relativity, spacetime is a four-dimensional Lorentzian manifold $\mathcal{M}$ with a metric tensor $g_{\mu\nu}(x)$, and the tangent space $T_x\mathcal{M}$ at some point $x \in \mathcal{M}$ is spanned by the natural basis $\{e_\mu = \partial/\partial x^\mu\}$. LIFs can be mathematically represented by a tetrad field ${e_\alpha}^\mu(x)$ [1,2], which gives a set of orthonormal basis vectors of a LIF, say

$$e_{\hat{\alpha}} = {e_\alpha}^\mu \partial_\mu \quad \{{e_\alpha}^\mu\} \in \mathrm{GL}(4, \mathbb{R}). \tag{1}$$

The variables $\{e_{\hat{\alpha}}\}$ are usually called *non-coordinate bases*, and the orthonormality is satisfied by

$$g(e_{\hat{\alpha}}, e_{\hat{\beta}}) = g_{\mu\nu}{e_\alpha}^\mu {e_\beta}^\nu = \eta_{\alpha\beta}, \tag{2}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric with signature $(-,+,+,+)$. (In this paper, for tetrad fields, Greek indices α, β, γ, δ, ... run over the four spacetime inertial coordinate labels; μ, ν, κ, λ, ... run over the four coordinate labels in general coordinates; Latin indices run from 1 to 3; and repeated indices are summed over.) As stressed before, the LIF at a point is not unique, and a Lorentz transformation $\Lambda(x)$ can give another tetrad field

$${e'_\alpha}^\mu = {e_\beta}^\mu {(\Lambda^{-1})^\beta}_\alpha, \tag{3}$$

and new orthogonal basis

$$e'_{\hat{\alpha}} = {e'_\alpha}^\mu \partial_\mu = e_{\hat{\beta}} {(\Lambda^{-1})^\beta}_\alpha. \tag{4}$$

It can be easily checked that the orthonormality of the basis is preserved by the Lorentz transformation, as required.

Actually, a non-coordinate basis at each spacetime point spans a vector space with the Minkowski metric $\eta_{\alpha\beta}$. In the language of differential geometry, the tetrad fields in the spacetime constitute a fiber bundle with the Minkowski space a fiber space at each spacetime point, and the Lorentz group $\mathrm{SO}(1,3)$ is the structure group.

3. Lorentz Connection One-Form

A connection ω_μ is required to define a covariant derivative in curved spacetime, such as the affine connection in the general coordinate, and the connection coefficients with respect to the non-coordinate bases $\{e_{\hat{\alpha}}\}$ are defined by

$$\mathcal{D}_\mu e_{\hat{\alpha}} = {\omega_\mu}^\beta{}_\alpha e_{\hat{\beta}}, \tag{5}$$

where $\mathcal{D}_\mu$ is the covariant derivative in LIF. The covariant derivative of a vector $V = V^\alpha e_{\hat{\alpha}}$ in LIF is $\mathcal{D}V = \mathcal{D}_\nu V^\alpha \mathrm{d}x^\nu \otimes e_{\hat{\alpha}}$, and according to Equation (5), the components can be easily derived:

$$\mathcal{D}_\mu V^\alpha = \partial_\mu V^\alpha + {\omega_\mu}^\alpha{}_\beta V^\beta. \tag{6}$$

Now, one can consider the covariant derivative of a vector $V = V^\mu \partial_\mu$ in the general coordinate

$$\nabla V \quad = \quad \nabla_\nu V^\mu \mathrm{d}x^\nu \otimes \partial_\mu,$$

where the components are

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda} V^\lambda, \tag{7}$$

with $\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}(\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\sigma\nu} - \partial_\sigma g_{\nu\lambda})$ the Levi–Civita connection coefficients, and one can have

$$\nabla_\nu V^\mu = e_\alpha{}^\mu \mathcal{D}_\nu V^\alpha. \tag{8}$$

Since the vector V can also be expressed as $V^\mu = e_\alpha{}^\mu V^\alpha$ in LIF, the covariant derivative can be rewritten as

$$\begin{aligned} \nabla_\nu V^\mu &= \nabla_\nu (e_\alpha{}^\mu V^\alpha) = \partial_\nu (e_\alpha{}^\mu V^\alpha) + \Gamma^\mu_{\nu\lambda}(e_\alpha{}^\lambda V^\alpha), \\ &= e_\alpha{}^\mu \left[\partial_\nu V^\alpha + (e^\alpha{}_\lambda \partial_\nu e_\beta{}^\lambda + e_\beta{}^\lambda e^\alpha{}_\sigma \Gamma^\sigma_{\nu\lambda}) V^\beta\right], \end{aligned} \tag{9}$$

where $e^\alpha{}_\mu$ is the inverse of $e_\alpha{}^\mu$. Therefore, one can find

$$\omega_\mu{}^\alpha{}_\beta = e^\alpha{}_\lambda \partial_\mu e_\beta{}^\lambda + e_\beta{}^\lambda e^\alpha{}_\sigma \Gamma^\sigma_{\mu\lambda}, \tag{10}$$

which are named the Lorentz (or spin) connection coefficients [1,2], and $\omega_\mu{}^{\alpha\beta} = -\omega_\mu{}^{\beta\alpha}$. Moreover, the non-coordinate bases $\{e_{\hat{\alpha}}\}$ satisfy the commutation relation $[e_{\hat{\alpha}}, e_{\hat{\beta}}] = f^\gamma{}_{\alpha\beta} e_{\hat{\gamma}}$, with $f^\gamma{}_{\alpha\beta}$ the coefficients of the anholonomy of tetrads [5], and the spin connection coefficients can also be expressed as [2,5]

$$\omega_\mu{}^\alpha{}_\beta = \frac{1}{2}\left(f_\beta{}^\alpha{}_\gamma + f_\gamma{}^\alpha{}_\beta - f^\alpha{}_{\beta\gamma}\right) e^\gamma{}_\mu. \tag{11}$$

Now, one can further define $\omega = \omega_\mu \mathrm{d}x^\mu$ as a Lie-algebra-valued connection one-form, and ω_μ take values in the Lie algebra $\mathfrak{so}(1,3)$ of the Lorentz group in the non-coordinate basis given by the tetrad. Denote the algebra generators of $\mathfrak{so}(1,3)$ by $J_{\alpha\beta}$, and then

$$\omega_\mu \equiv -\frac{\mathrm{i}}{2} \omega_\mu{}^{\alpha\beta} J_{\alpha\beta}. \tag{12}$$

In the non-coordinate basis, $J_{\alpha\beta}$ act as the vector representation of the Lorentz generators

$$(J_{\alpha\beta})^\gamma{}_\delta = \mathrm{i}\left(\eta_{\beta\delta}\eta^\gamma{}_\alpha - \eta_{\alpha\delta}\eta^\gamma{}_\beta\right), \tag{13}$$

which shows that ω_μ here are 4×4 matrices with the matrix elements $(\omega_\mu)^\alpha{}_\beta = \omega_\mu{}^\alpha{}_\beta$. Therefore, $\omega = \omega_\mu \mathrm{d}x^\mu$ is a matrix-valued one-form that can be defined by [2]

$$\omega^\alpha{}_\beta \equiv \omega_\mu{}^\alpha{}_\beta \mathrm{d}x^\mu. \tag{14}$$

It is easy to see that ω_μ transforms as a vector under a general coordinate transformation $x'^\mu = x'^\mu(x)$,

$$\omega'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} \omega_\nu(x). \tag{15}$$

Under an LLT, $e'_\alpha(x) = e_{\hat{\beta}}(\Lambda^{-1})^\beta{}_\alpha$; however, the Lorentz connection transforms inhomogeneously:

$$\omega'_\mu(x) = \Lambda(x)\omega_\mu(x)\Lambda^{-1}(x) + \Lambda(x)\partial_\mu \Lambda^{-1}(x). \tag{16}$$

For details of the tetrad field and Lorentz connection, see appendix J of [1].

4. Lorentz Gauge and Coulomb Gauge for Tetrad Fields

To obtain the gauge condition for a tetrad field, we first briefly review the Lorentz gauge in electrodynamics. Maxwell's theory of electromagnetism is described by the $\mathrm{U}(1)$ gauge group, and the connection one-form, physically representing the photon, is

$$\mathcal{A} = \mathcal{A}_\mu \mathrm{d}x^\mu, \tag{17}$$

where $\mathcal{A}_\mu$ is the vector potential in electrodynamics. Then, one may naively guess the Lorentz gauge for a tetrad field is $\partial^\mu \omega_\mu = 0$, corresponding to $\partial^\mu \mathcal{A}_\mu = 0$ in electrodynamics. However, this obviously does not work for a tetrad field. For one thing, the base manifold in Maxwell's theory is flat spacetime, while the one in a tetrad field is curved spacetime, and the ordinary derivative should be replaced by the covariant one. Additionally, the connection ω_μ is an element of Lie algebra $\mathfrak{so}(1,3)$, and the derivative could be further amended to be the gauge–covariant derivative in adjoint representation. To see how to construct the corresponding Lorentz gauge condition, we further see Maxwell's theory more mathematically in the following.

With the connection one-form in Equation (17), the field strength for Maxwell's theory of electromagnetism can be introduced:

$$\begin{aligned} \mathcal{F} &\equiv \mathrm{d}\mathcal{A} = \frac{1}{2}\mathcal{F}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\ &= (\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu, \end{aligned} \tag{18}$$

and the Bianchi identity

$$\mathrm{d}\mathcal{F} = 0 \tag{19}$$

reduces to two of Maxwell's equations,

$$\boldsymbol{\partial} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0, \quad \boldsymbol{\partial} \cdot \boldsymbol{B} = 0. \tag{20}$$

For the other two of Maxwell's equations (with $\varepsilon_0 = \mu_0 = 1$),

$$\boldsymbol{\partial} \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{j}, \quad \boldsymbol{\partial} \cdot \boldsymbol{E} = \rho, \tag{21}$$

where ρ and $\boldsymbol{j}$ are the electric charge density and electric current density, respectively. To obtain the compact expressions corresponding to Equation (19), the adjoint of the exterior derivative (codifferential operator) in the four-dimensional Lorentzian manifold should first be introduced, say

$$\mathrm{d}^\dagger = *\mathrm{d}*, \tag{22}$$

where $*$ is the Hodge operator [2]. Therefore, $\mathrm{d}^\dagger \mathcal{F} = \partial^\mu \mathcal{F}_{\mu\nu}\mathrm{d}x^\nu$, and Equation (21) can be straightforwardly rewritten as

$$\mathrm{d}^\dagger \mathcal{F} = j, \tag{23}$$

with the one-form $j = -\rho \mathrm{d}t + \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{x}$. Furthermore, the Lorentz gauge $\partial^\mu \mathcal{A}_\mu = 0$ in electrodynamics can now be expressed as

$$\mathrm{d}^\dagger \mathcal{A} = 0 \tag{24}$$

and mathematically, the corresponding Lorentz gauge condition for the tetrad field is

$$\mathfrak{D}^\dagger \omega = 0, \tag{25}$$

where $\mathfrak{D} = \mathrm{d} + [\omega, \ ]$ is the gauge–covariant derivative in adjoint representation, and $\mathfrak{D}^\dagger = *\mathfrak{D}*$ is the codifferential operator for the four-dimensional Lorentzian manifold here. To make Equation (25) more explicit, we first come to the action of $\mathfrak{D}^\dagger$ on a Lie-algebra-valued one-form $\leftrightarrow = \varpi_\mu \mathrm{d}x^\mu$,

$$\begin{aligned}
\mathfrak{D}^\dagger \varpi &= *\mathfrak{D} * (\varpi_\mu \mathrm{d}x^\mu) \\
&= *\mathfrak{D}\left(\frac{\sqrt{g}}{3!}\varpi_\mu g^{\mu\lambda}\varepsilon_{\lambda\nu_2\nu_3\nu_4}\mathrm{d}x^{\nu_2}\wedge \mathrm{d}x^{\nu_3}\wedge \mathrm{d}x^{\nu_4}\right) \\
&= *\frac{1}{3!}\mathfrak{D}_\nu\left(\sqrt{g}g^{\mu\lambda}\varpi_\mu\right)\varepsilon_{\lambda\nu_2\nu_3\nu_4}\mathrm{d}x^{\nu}\wedge \mathrm{d}x^{\nu_2}\wedge \mathrm{d}x^{\nu_3}\wedge \mathrm{d}x^{\nu_4} \\
&= *\mathfrak{D}_\nu\left(\sqrt{g}g^{\mu\nu}\varpi_\mu\right)\mathrm{d}x^0\wedge \mathrm{d}x^1\wedge \mathrm{d}x^2\wedge \mathrm{d}x^3 \\
&= \mathfrak{D}_\nu\left(\sqrt{g}g^{\mu\nu}\varpi_\mu\right)\sqrt{g}\varepsilon^{0123} \\
&= -\frac{1}{\sqrt{g}}\mathfrak{D}_\nu\left(\sqrt{g}g^{\mu\nu}\varpi_\mu\right) \\
&= -\frac{1}{\sqrt{g}}\partial_\nu\left(\sqrt{g}g^{\mu\nu}\varpi_\mu\right) - g^{\mu\nu}[\omega_\nu, \varpi_\mu] \\
&= -g^{\mu\nu}\nabla_\mu\varpi_\nu - g^{\mu\nu}[\omega_\mu, \varpi_\nu]
\end{aligned} \tag{26}$$

where $g = |\det g_{\mu\nu}|$, $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}$ is the totally anti-symmetric Levi–Civita symbol, and $\varepsilon^{\mu_1\mu_2\mu_3\mu_4} = -g^{-1}\varepsilon_{\mu_1\mu_2\mu_3\mu_4}$ has been used. Now, according to the derivation in Equation (26), the Lorentz gauge in Equation (25) becomes

$$g^{\mu\nu}\nabla_\mu\omega_\nu = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\omega_\nu\right) = 0. \tag{27}$$

Before one comes to the Coulomb gauge for the tetrad field, we first consider the corresponding expression to the Lorentz gauge in Equation (24) for the Coulomb gauge. Symbolically, the exterior derivative can be written as

$$\mathrm{d} \equiv \mathrm{d}t \wedge \frac{\partial}{\partial t} + \mathbf{d}, \tag{28}$$

where $\mathbf{d}$ is the spatial exterior derivative [6], and the partial differential operator acts only on the coefficients. Since

$$\mathrm{d}^\dagger \mathcal{A} = \partial^0 \mathcal{A}_0 + \mathbf{d}^\dagger \boldsymbol{\mathcal{A}}, \tag{29}$$

with $\boldsymbol{\mathcal{A}} = \mathcal{A}_i \mathrm{d}x^i$, $\mathbf{d}^\dagger = *\mathbf{d}*$, and $\mathbf{d}^\dagger \boldsymbol{\mathcal{A}} = \partial^i \mathcal{A}_i$, the Coulomb gauge in electrodynamics is

$$\mathbf{d}^\dagger \boldsymbol{\mathcal{A}} = 0. \tag{30}$$

According to the relationship between the Lorentz gauge $\partial^\mu \mathcal{A}_\mu = 0$ ($\mathrm{d}^\dagger \mathcal{A} = 0$) and the Coulomb gauge $\partial^i \mathcal{A}_i = 0$ ($\mathbf{d}^\dagger \boldsymbol{\mathcal{A}} = 0$) in electrodynamics, the Coulomb gauge for the tetrad field can be similarly generalized as

$$\boldsymbol{\mathfrak{D}}^\dagger \boldsymbol{\omega} = 0, \tag{31}$$

with $\boldsymbol{\omega} = \omega_i \mathrm{d}x^i$ and $\boldsymbol{\mathfrak{D}} = \mathbf{d} + [\boldsymbol{\omega}, \ ]$, and the gauge–covariant derivative in adjoint representation can be decomposed as

$$\mathfrak{D} = \mathrm{d}t \wedge \left(\frac{\partial}{\partial t} + [\omega_0, \ ] \right) + \boldsymbol{\mathfrak{D}}. \tag{32}$$

By some algebra, it is easy to obtain the explicit expression for the Coulomb gauge in Equation (31):

$$\frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \omega_j \right) = 0. \tag{33}$$

5. Remarks and Discussion

(i) Fermi–Walker transport is usually employed to define LIFs in curved spacetime, and the LIFs are established by parallel transport of the non-coordinate bases in curved spacetime [7]. The LIF of an observer in curved spacetime is then generally dependent on the path along which the non-coordinate bases are transported; for instance, the z-axes of LIFs at the same spacetime point can be different for observers who come along different paths. More seriously, a locally defined physical object, such as a localized qubit state, at a spacetime cannot be defined unambiguously, as the LIFs are dependent on the observer's path, and this seems to be "weird". However, if a gauge condition is carefully chosen and the gauge freedom of the tetrad fields is fixed, the LIF at each spacetime point can be unique up to a global gauge transformation. The LIFs are now independent of the observer's path, and one has a path-independent way of constructing "global" LIFs. Subsequently, the local physical object can be unambiguously defined with the "global" LIFs.

(ii) To determine the LIFs in the curved spacetime, the sixteen functions $e_\alpha{}^\mu(x)$ of tetrad field should be fixed. The orthonormality condition in Equation (2) represents ten constrains on the tetrad field, so there are still six degrees of freedom for the tetrad field, which are, exactly, the three degrees of pure boosts and three degrees of spatial rotations for the gauge transformation $\Lambda(x)$. As the Lorentz connection $\omega_\mu(x)$ is an antisymmetric matrix, both the Lorentz gauge and the Coulomb gauge in Equations (27) and (33), respectively, contain exactly six constrains and, roughly speaking, seem to fix the gauge transformation completely. However, more rigorously, under the gauge transformation in Equation (16), to preserve the Lorentz gauge condition in Equation (27), it is required that

$$g^{\mu\nu} \nabla_\mu (\Lambda \partial_\nu \Lambda^{-1}) + g^{\mu\nu} [\Lambda \omega_\mu \Lambda^{-1}, \Lambda \partial_\nu \Lambda^{-1}] = 0, \tag{34}$$

and obviously, the solution for the gauge transformation $\Lambda(x)$ satisfying the equation above is not unique, at least for some simple cases. Therefore, the Lorentz gauge for a tetrad field does not fix the gauge completely in general, and the same result holds for the Coulomb gauge. Nevertheless, the degrees of freedom unconstrained by the two gauges in a tetrad field should have different meanings, but it is not easy to completely clarify the differences due to complication from Abelian to non-Abelian gauge theory, and only brief discussions about the disparate characteristics of the two gauge conditions are provided next.

(iii) In a static case, the Lorentz gauge $\partial^\mu \mathcal{A}_\mu = 0$ coincides with the Coulomb gauge $\partial^i \mathcal{A}_i = 0$ in electrodynamics, but for a tetrad field in gravity, one does not have the same result. For spacetime with a static metric, the Lorentz gauge in Equation (27) reduces to $g^{i0} \nabla_i \omega_0 - g^{0\mu} \Gamma^\nu_{0\mu} \omega_\nu + g^{ij} \nabla_i \omega_j = 0$, and it is different from the Coulomb gauge in Equation (33) by a term $g^{i0} \nabla_i \omega_0 - g^{0\mu} \Gamma^\nu_{0\mu} \omega_\nu$. Moreover, the Lorentz gauge for a tetrad field is invariant under a general coordinate transformation in Equation (15), while the Coulomb gauge is not, and is only invariant under a spatial coordinate transformation. These show that the two gauge conditions undoubtedly have distinct properties for a tetrad field as well, and one may wonder which is the preferred option in relevant physics problems. Illuminated by the gauge theory of electromagnetism and the proposals in previous relevant works [3,4,8,9], it is reasonable to choose the Coulomb gauge as the physical gauge condition. Specifically, the non-coordinate basis $\{e_{\hat{\alpha}}\}$ obeys

$$\mathcal{D}_\mu \mathcal{D}_\nu e_{\hat{\alpha}} = \left[(\nabla_\mu \omega_\nu + \omega_\mu \omega_\nu)\right]^{\beta}{}_{\alpha} e_{\hat{\beta}}. \tag{35}$$

Denote $\boldsymbol{e} = (e_{\hat{0}}, e_{\hat{1}}, e_{\hat{2}}, e_{\hat{3}})$, and in the Lorentz gauge,

$$g^{\mu\nu}\mathcal{D}_\mu \mathcal{D}_\nu \boldsymbol{e} - \boldsymbol{e} g^{\mu\nu}\omega_\mu \omega_\nu = 0, \tag{36}$$

while in the Coulomb gauge, one has

$$g^{ij}\mathcal{D}_i \mathcal{D}_j \boldsymbol{e} - \boldsymbol{e} g^{ij}\omega_i \omega_j = 0. \tag{37}$$

Comparing Equation (37) with Equation (36) shows that for the Coulomb gauge, the non-coordinate basis $\{e_{\hat{\alpha}}\}$ is non-dynamic and does not instantaneously propagate in spacetime, which means the inertial effects from the freedom of the tetrad field are absent, leaving only gravitational effects on the LIFs. As a result, established by the tetrad field under the Coulomb gauge, the LIF stands a good chance to have advantages over the other ones, and this will be our further investigation elsewhere.

(iv) In gauge theory, the decomposition of gauge potential plays an important role in both theoretical physics and mathematics [10,11], and recently, a gauge decomposition approach was proposed to find a gauge–covariant description of the gluon spin and orbital angular momentum in [8], and it was further developed in [9]. It is a familiar practice to decompose the Lorentz connection into the physical part and the pure-gauge part. Mathematically, one can have the separation $\omega_\mu(x) = \hat{\omega}_\mu(x) + \bar{\omega}_\mu(x)$, with $\hat{\omega}_\mu(x)$ the physical part, while $\bar{\omega}_\mu(x)$ is the pure-gauge part, and under the gauge transformation in Equation (16),

$$\hat{\omega}'_\mu(x) = \Lambda(x)\hat{\omega}_\mu(x)\Lambda^{-1}(x), \tag{38}$$

$$\bar{\omega}'_\mu(x) = \Lambda(x)\bar{\omega}_\mu(x)\Lambda^{-1}(x) + \Lambda(x)\partial_\mu \Lambda^{-1}(x). \tag{39}$$

The equations to define the decomposition can be analogously constructed as

$$\frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\hat{\omega}_j\right) + g^{ij}[\bar{\omega}_i, \hat{\omega}_j] = 0, \tag{40}$$

$$\partial_\mu \bar{\omega}_\nu - \partial_\nu \bar{\omega}_\mu + [\bar{\omega}_\mu, \bar{\omega}_\nu] = 0. \tag{41}$$

(v) Parallel to Maxwell's Equations (20) and (21) in electrodynamics, one may wonder if there are analogical equations for tetrad fields in gravity. We first come to the field strength of a tetrad field:

$$\mathcal{R} \equiv \mathrm{d}\omega + \omega \wedge \omega = \frac{1}{2}\mathcal{R}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \tag{42}$$

where $\mathcal{R}_{\mu\nu}$ are the Lie-algebra-valued components of the field strength,

$$\mathcal{R}_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu], \tag{43}$$

and we can also have the curvature two-form $\mathcal{R}^\alpha{}_\beta = \frac{1}{2}R^\alpha{}_{\beta\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. With Equation (42), the Bianchi identity again gives $\mathfrak{D}\mathcal{R} = 0$, reducing to

$$\mathrm{d}\mathcal{R} + \omega \wedge \mathcal{R} - \mathcal{R} \wedge \omega = 0, \tag{44}$$

which is the analogical equation to Equation (20). It is a little more complicated to reach similar results for Equation (21), and before that, one may homoplastically have

$$\mathfrak{D}^{\dagger}\mathcal{R} = \left(\nabla^{\mu}\mathcal{R}_{\mu\nu} + [\omega^{\mu}, \mathcal{R}_{\mu\nu}]\right)\mathrm{d}x^{\nu}. \tag{45}$$

Further, consider Einstein's equation

$$R^{\alpha}{}_{\mu} = 8\pi G(T^{\alpha}{}_{\mu} - \frac{1}{2}Te^{\alpha}{}_{\mu}), \tag{46}$$

with $R^{\alpha}{}_{\mu} = R^{\alpha\beta}{}_{\mu\nu}e_{\beta}{}^{\nu}$ the Ricci tensor, $T = g^{\mu\nu}T_{\mu\nu}$ the trace of the energy–momentum tensor $T^{\mu}{}_{\nu}$, and G Newton's constant of gravitation, and one can obtain

$$\mathfrak{D}^{\dagger}\mathcal{R} = -4\pi G\mathrm{d}\mathcal{T}, \tag{47}$$

or, more explicitly,

$$\nabla^{\mu}\mathcal{R}_{\mu\nu} + [\omega^{\mu}, \mathcal{R}_{\mu\nu}] = -4\pi G\partial_{\nu}\mathcal{T}, \tag{48}$$

with the matrix elements $\mathcal{T}^{\alpha}_{\ \beta} = T\delta^{\alpha}{}_{\beta}$. This plays the role of the dynamic equation for the Lorentz connection $\omega_{\mu}(x)$ of a tetrad field. The influence of the Lorentz gauge in Equation (27) and the Coulomb gauge in Equation (33) on the dynamics of the Lorentz connection ω_{μ} is an interesting and important question, and it deserves systematical consideration in the future.

6. Conclusions and Summary

The gauge aspect of a tetrad field in gravity is investigated in this work, and the Lorentz gauge and the Coulomb gauge are proposed and discussed in the non-Abelian gauge theory of tetrad fields. Though the two gauge conditions are much more complicated than the corresponding ones in the U(1) gauge theory, some analogous features compared to electromagnetic fields are revealed in our discussions. The Coulomb gauge is more likely to be the physical gauge, and it is preferable for fixing the tetrad field. Completion of the gauge theory of a tetrad field in gravity is a sizable task that requires thorough consideration. Some related problems are still worth investigating. Applications of the gauge condition in physics are of particular interest and significance, and the details are for our further consideration.

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