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Scattering in Algebraic Approach to Quantum Theory—Associative Algebras

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Abstract: The definitions of scattering matrix and inclusive scattering matrix in the framework of formulation of quantum field theory in terms of associative algebras with involution are presented. The scattering matrix is expressed in terms of Green functions on shell (LSZ formula), and the inclusive scattering matrix is expressed in terms of generalized Green functions on shell. The expression for inclusive scattering matrix can be used also for quasi-particles (for elementary excitations of any translation-invariant stationary state, for example, for elementary excitations of equilibrium state). An interesting novelty is the consideration of associative algebras over real numbers.

Keywords: inclusive scattering matrix; generalized Green functions; associative algebras

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1. Introduction

The standard algebraic approach to quantum theory is based on consideration of associative algebra with involution *(algebra of observables). This algebra will be denoted by $\mathcal{A}$; we assume that it has a unit. Usually, it is assumed that $\mathcal{A}$ is an algebra over complex numbers (then the involution should be antilinear), but we consider also the case when $\mathcal{A}$ is an algebra over real numbers. The states are defined as positive linear functionals on $\mathcal{A}$ (one says that the functional ω is positive if $\omega(A^*A) \geq 0$ for $A \in \mathcal{A}$). The set of states is a convex cone denoted by $\mathcal{C}$. Proportional states are identified; hence, instead of the cone $\mathcal{C}$ one can consider the convex set $\mathcal{C}_0$ of states obeying $\omega(1) = 1$ (the set of normalized states).

Time translation (evolution operator) acts on $\mathcal{A}$ as involution preserving automorphism; the same is true for all other symmetries. To define particles and their scattering, we also need spatial translations; together, time and spatial translations span commutative group $\mathcal{T}$. The time translations are denoted by T_τ and spatial translations by $T_{\mathbf{x}}$ where $\mathbf{x} \in \mathbb{R}^d$. The induced transformations of the space of states are denoted by the same symbols. If $A \in \mathcal{A}$ we use the notation $A(\tau, \mathbf{x})$ for $T_\tau T_{\mathbf{x}} A$.[1]

Let us consider a stationary translation-invariant state ω and the pre Hilbert space $\mathcal{H}$ of the corresponding GNS (Gelfand–Naimark–Segal) representation of $\mathcal{A}$ (see, for example, [1]). Recall that in this representation there exists a cyclic vector Φ obeying $\omega(A) = \langle A\Phi, \Phi\rangle$. (One says that Φ is a cyclic vector if $\mathcal{H} = \mathcal{A}\Phi$). The translations descend to $\mathcal{H}$. In complex case,the infinitesimal translations (energy and momentum operators) are defined on a dense subset of Hilbert space $\bar{\mathcal{H}}$ (of the completion of $\mathcal{H}$). In a real case, they act on a dense subset of the complexification of $\bar{\mathcal{H}}$. (We use the same notation for translations in $\mathcal{H}$ as for automorphisms of $\mathcal{A}$. The element A of the algebra $\mathcal{A}$ and the corresponding operator in $\mathcal{H}$ are also denoted by the same symbol. Notice that representing A as an operator, we should consider translation of A as a conjugation with $T_\tau T_{\mathbf{x}}$, i.e., $A(\tau, \mathbf{x}) = T_\tau T_{\mathbf{x}} A T_{-\tau} T_{-\mathbf{x}}$).

Elements of $\mathcal{H}$ can be regarded as excitations of ω.

The *elementary space* $\mathfrak{h}$ is defined as a space of smooth fast decreasing functions on $\mathbb{R}^d \times \mathcal{I}$ (all of their derivatives should decrease faster than any power); it is equipped with L^2 metric. (Here $\mathcal{I}$ denotes a set consisting of m elements. We assume that $\mathcal{I}$ is finite. This means that there exists only a finite number of types of elementary particles. In principle, this number can be infinite; for example, in non-relativistic quantum mechanics we can have an infinite number of bound states. The considerations below can be easily generalized to the case of an infinite number of types of elementary particles.) We should consider real-valued functions if $\mathcal{A}$ is an algebra over $\mathbb{R}$ and complex-valued functions if $\mathcal{A}$ is an algebra over $\mathbb{C}$. It is convenient to consider the elements of $\mathfrak{h}$ as columns of m functions on $\mathbb{R}^d$. The spatial translations act naturally on this space; we assume that the time translations also act on $\mathfrak{h}$ and commute with spatial translations. In momentum representation, the spatial translation $T_{\mathbf{x}}$ is represented as multiplication by $e^{i\mathbf{xk}}$ and the time translation T_τ is represented as a multiplication by a matrix $e^{-i\tau E(\mathbf{k})}$. *We assume that* $E(\mathbf{k})$ *is a non-degenerate Hermitian matrix.*

To guarantee that time translations act in $\mathfrak{h}$, we assume that $E(\mathbf{k})$ is a smooth function of $\mathbf{k}$ and has at most polynomial growth. If $\mathfrak{h}$ consists of complex-valued functions, then diagonalizing the matrix $E(\mathbf{k})$ we can reduce the general case to the case when $m = 1$; this remark was used in [2]. (Notice, however, that the eigenvalues of $E(\mathbf{k})$ are not necessarily smooth functions of $\mathbf{k}$).

An *elementary excitation* of ω is defined as an isometric map $\Phi : \mathfrak{h} \to \mathcal{H} = \mathcal{A}\Phi$ commuting with translations.

If $\mathcal{A}$ is an algebra over $\mathbb{R}$, we assume that the elements of $\mathfrak{h}$ in momentum representation obey the reality condition $f^*(\mathbf{k}) = f(-\mathbf{k})$.

We show that imposing the condition of asymptotic commutativity we can define the scattering matrix and inclusive scattering matrix of elementary excitations of ω. To analyze the properties of scattering matrix, we assume that ω has a cluster property. Our results are based on ideas of [2]. (This paper was published as Section 13.3 of [1], an improved version of it was published recently in preprint form [2], v2). Notice that the results of [2] generalize the Haag–Ruelle theory dealing with a scattering matrix in Lorentz-invariant local quantum field theories. (See [3] for an exposition of the Haag–Ruelle theory closest to our approach and [4] or [1] for generalization of this theory to the case when Lorentz-invariance is not assumed).

If $\mathcal{A}$ is a C^*-algebra over complex numbers (as in [2]), one can identify (quasi-local) observables with self-adjoint elements of this algebra. For every normalized state ω, one defines a probability distribution of the observable a corresponding to a self-adjoint element A in such a way that for every continuous function f the expectation value of $f(a)$ is equal to $\omega(f(A))$. One can also consider global observables corresponding to infinitesimal automorphisms of $\mathcal{A}$, in particular, energy and momentum corresponding to time and spatial infinitesimal translations. We can talk about joint spectrum of energy and momentum operators in $\mathcal{H}$ (in the space of excitations of translation-invariant stationary state ω). We say that ω is a ground state if the energy operator in $\bar{\mathcal{H}}$ is positive definite.

For algebras over real numbers, one should consider skew-adjoint elements ($A = -A^*$) instead of self-adjoint elements (for algebras over complex numbers there exists an obvious one-to-one correspondence $A \to iA$ between skew- adjoint and self-adjoint elements). The definition of the probability distribution of physical quantity used in the case of complex numbers does not work; however, one can use the geometric approach to quantum theory to derive the probability distribution from decoherence [5].

Elementary excitations of ground state are called particles. Elementary excitations of arbitrary translation-invariant stationary state are called quasi-particles. Quasi-particles are in general unstable; this means that the conditions in the definition of elementary excitation are satisfied only approximately. For quasi-particles, the definition of scattering matrix does not work; however, the definition of inclusive scattering matrix still makes sense.

The conventional scattering matrix makes sense only if the theory has particle interpretation (see Section 2). It is proven that non-relativistic quantum mechanics has particle interpretation (see, for example, [6]); however, it is quite difficult to prove this fact in other theories. The situation with inclusive scattering matrix is much better; its existence can be proven by methods of the Haag–Ruelle theory and generalizations of this theory (see Section 4).

Notice that our considerations can be applied to the scattering of elementary excitations of any translation-invariant state. In particular, they can be applied to the scattering of elementary excitations of an equilibrium state. It is important to notice that our considerations can also be applied in non-equilibrium situation. The generalized Green functions we are using coincide with functions considered in Keldysh formalism of non-equilibrium statistical physics [7–9]. They also appear in the formalism of L-functionals that can be used to give a simple and transparent derivation of diagram technique for calculation of generalized Green functions in the framework of perturbation theory [1,10–12].

The ground state is not singled out in any way in our considerations.

The main goal of the present paper is to give an exposition of scattering theory in such a way that it can be easily compared with the definition of scattering in the geometric approach to quantum field theory [13] and in the approach based on Jordan algebras [14]. In particular, we analyze the notion of inclusive scattering matrix; this is important for comparison with the geometric approach where there exists a very natural generalization of this notion, but it seems that the conventional scattering matrix cannot be defined. The inclusive scattering matrix was defined in [10] in the formalism of L-functionals; the definition in the algebraic approach was given in [2], but in less general form than in the present paper and with different proofs. See also [1].

In the present paper we also consider the case when $\mathcal{A}$ is an algebra over $\mathbb{R}$; this is necessary for comparison with papers [13,14] where we consider both real and complex elementary spaces. There exists an opinion that complex numbers are important in the formulation of quantum mechanics. It is true that by imposing very natural axioms one can justify this opinion (see, for example, [15]). One of the goals of [5] and the present paper is to formulate axioms that allow us to avoid using complex numbers. Notice, however, that the case of algebras over real numbers is less natural; therefore, we do not discuss in detail some constructions in this case. The reader can skip everything related to algebras over real numbers.

2. Scattering Matrix

Let us consider an algebra with involution $\mathcal{A}$. We assume that spatial and time translations act as automorphisms of $\mathcal{A}$. We fix a translation-invariant stationary state ω; excitations of ω are defined as elements of pre-Hilbert space $\mathcal{H}$ obtained by GNS construction. The algebra $\mathcal{A}$ acts in $\mathcal{H}$. In what follows, we denote the operator in $\mathcal{H}$ corresponding to an element $A \in \mathcal{A}$ by the same letter. We assume that all operators we are dealing with are smooth. (We say that an operator corresponding to an element $B \in \mathcal{A}$ is smooth if $B = \int \alpha(\mathbf{x}, t) A(\mathbf{x}, t) d\mathbf{x} dt$, where $\alpha(\mathbf{x}, t) \in \mathcal{S}(\mathbb{R}^{d+1}, A \in \mathcal{A})$). Here, $\mathcal{S}$ stands for the space of smooth fast decreasing functions.

The element of $\mathcal{H}$ corresponding to ω is denoted by Φ. We define an elementary excitation as an isometric map of an elementary space $\mathfrak{h}$ into $\mathcal{H}$ commuting with spatial and time translations. (Recall that an elementary space consists of smooth fast decreasing functions depending on spatial variable $\mathbf{x}$ or on momentum variable $\mathbf{k}$ and on discrete variable taking m values. We simultaneously consider algebras over complex and real numbers; correspondingly, the functions considered below are complex-valued or real-valued).

Let us fix m elements $\phi_1, \cdots, \phi_m \in \mathfrak{h}$ and m operators $B_1, \cdots, B_m \in \mathcal{A}$ obeying $\Phi(\phi_i) = B_i \Phi$. The elements ϕ_i are columns of functions ϕ_i^α; together, they can be considered as a square matrix. *We assume that this matrix is invertible and commutes with the matrix $E(\mathbf{k})$ governing the time translation in $\mathfrak{h}$. If this condition is satisfied, we say that $B_1, \cdots, B_m$ are good operators.* We also require that $B_i^* \Phi = \Phi(\psi_i)$ for some $\psi_i \in \mathfrak{h}$.

Notice that

$$B_j(\tau, \mathbf{x})\Phi = \Phi(T_\tau T_\mathbf{x} \phi_j).$$

In momentum representation,

$$B_j(\tau, \mathbf{x})\Phi = \Phi((e^{-i\tau E(\mathbf{k})})^\alpha_\beta e^{i\mathbf{k}\mathbf{x}} \phi_j^\beta).$$

Let us consider a collection of smooth functions $f = (f^1(\mathbf{k}), \cdots, f^m(\mathbf{k}))$ decreasing faster than any power. (Using the notation $\mathcal{S}$ for the space of smooth fast decreasing functions, we can say that $f \in \mathcal{S}^m$). We define an operator $B(f,\tau)$ acting in $\overline{\mathcal{H}}$ (in the completion of $\mathcal{H}$) by the formula

$$B(f,\tau) = \int d\mathbf{x} \tilde{f}^j_\tau(\mathbf{x}) B_j(\tau,\mathbf{x}) \tag{1}$$

where $\tilde{f}^j_\tau(\mathbf{x})$ denotes the inverse Fourier transform with respect to $\mathbf{k}$ of the function $f^j_\tau(\mathbf{k}) = f^i(\mathbf{k})(e^{i\tau E(\mathbf{k})})^j_i$.

The operator $B(f,\tau)$ depends linearly of f; it specifies a generalized vector function of $\mathbf{k}, \tau$ that can be written in the form

$$B(\mathbf{k},\tau) = e^{i\tau E(\mathbf{k}))} \hat{B}(\tau,\mathbf{k}) \tag{2}$$

where $\hat{B}(\tau,\mathbf{k})$ is a Fourier transform with respect to $\mathbf{x}$ of $B_j(\tau,\mathbf{x})$ considered as a generalized vector function.

Let us prove that $B(f,\tau)\Phi$ does not depend on τ. Using (1), we obtain

$$B(f,\tau)\Phi = \int d\mathbf{x} \tilde{f}^i_\tau(\mathbf{x}) \Phi((e^{-i\tau E(\mathbf{k})})^\alpha_\beta e^{i\mathbf{kx}} \phi^\beta_i) =$$

$$\Phi(f^j(\mathbf{k})(e^{i\tau E(\mathbf{k})})^i_j (e^{-i\tau E(\mathbf{k})})^\alpha_\beta \phi^\beta_i) = \Phi(f^j \phi^\alpha_j). \tag{3}$$

(We have used the fact that the matrix $e^{i\tau E(\mathbf{k})}$ commutes with the matrix ϕ).

In what follows, we denote $f^j\phi^\alpha_j$ as $f\phi$ where f is considered as a column vector and ϕ as a square matrix.

Later, we will use this statement in the following form:

Lemma 1. $\dot{B}(f,\tau)\Phi = 0$.

Where dot denotes the derivative with respect to τ.

Definition 1. *Let us consider the function $f^j_\tau(\mathbf{x})$ corresponding to the collection $f = (f^1, \dots, f^m)$ of smooth fast decreasing functions. We say that a set $\tau K(f)$ is an essential support of the function $f^j_\tau(\mathbf{x})$ if for all n*

$$f^j_\tau(\mathbf{x}) < C_n(1 + |\mathbf{x}|^2 + \tau^2)^{-n}$$

where $\frac{\mathbf{x}}{\tau} \notin K$.

In the case when Fourier transforms $f^i(\mathbf{k})$ of functions $f^i(\mathbf{x})$ have compact support, one can assume that $K(f)$ is compact, but in general is not clear that one can find a compact set K obeying the conditions of this definition.

Let us impose the conditions of asymptotic commutativity on the operators $A_j \in \mathcal{A}$. This means that

$$||[A_j(\tau,\mathbf{x}), A_k]|| < \frac{C_n(\tau)}{1 + ||\mathbf{x}||^n}. \tag{4}$$

Here n is an arbitrary integer, and $C_n(\tau)$ is a polynomial. (The condition we have imposed can be weakened, see [2]).

Let us consider the vectors

$$\Psi(f_1,\tau_1,\dots,f_n,\tau_n) = B(f_1,\tau_1)\dots B(f_n,\tau_n)\Phi$$

where $f_i \in \mathcal{S}^m$ is a collection of m smooth fast decreasing functions on $\mathbb{R}^d$.

We say that $f_1,\dots,f_n$ are not overlapping if the sets $K_j = K(f_j)$ do not overlap (more precisely, we should assume that the distance between sets K_j and $K_{j'}$ is positive for $j \neq j'$).

Lemma 2. *If $f_1, \ldots, f_n$ do not overlap the vectors*

$$\Psi(f_1, \tau_1, \ldots, f_n, \tau_n) = B(f_1, \tau_1) \ldots B(f_n, \tau_n)\Phi$$

have a limit in $\bar{\mathcal{H}}$ as τ_j tend to $-\infty$; this limit will be denoted by

$$\Psi(f_1, \ldots, f_n | -\infty)$$

The set spanned by such limits will be denoted by $\mathcal{D}_-$.

Let us sketch the proof of this lemma for the case when $\tau_1 = \ldots = \tau_n = \tau$. It is sufficient to check that $\int_{-\infty}^{0} ||\dot{\Psi}(\tau)||d\tau$ is finite. (Here, $\Psi(\tau)$ stands for $\Psi(f_1, \tau, \ldots, f_n, \tau)$, and $\dot{\Psi}$ denotes the derivative with respect to τ). The derivative $\dot{\Psi}$ is a sum of n terms; every term contains n factors, and one of these factors is a derivative. We can interchange the factors because the commutators can be neglected as $\tau \to \infty$; this follows from the condition that the functions f_j do not overlap and from the fact that for non-overlapping families, essential supports of functions $f^i_\tau(\mathbf{x}, j)$ and $f^i_\tau(\mathbf{x}, j')$ are far away for large τ.

We use this remark to shift the factor with the derivative to the right. It remains to apply Lemma 1.

Let us define the *in*-operators a^+_{in} by the formula

$$a^+_{in}(f\phi) = a^+_{in}(f^j\phi^\alpha_j) = \lim_{\tau \to -\infty} B(f, \tau). \tag{5}$$

Lemma 2 gives conditions on f that guarantee the existence of this limit as a strong limit on the set $\mathcal{D}_-$.

Let us introduce the asymptotic bosonic Fock space $\mathcal{H}_{as}$ as a Fock representation of canonical commutation relations

$$[b(\rho), b(\rho')] = [b^+(\rho), b^+(\rho')] = 0, [b(\rho), b^+(\rho')] = \langle \rho, \rho' \rangle$$

where $\rho, \rho' \in \mathfrak{h}$.

We define the Møller matrix S_- as a map $\mathcal{H}_{as} \to \overline{\mathcal{H}}$ obeying

$$a^+_{in}(\rho)S_- = S_- b^+(\rho), S_-|0\rangle = |0\rangle$$

where $|0\rangle$ stands for the Fock vacuum.

Notice that spatial and time translations act naturally in $\mathcal{H}_{as}$. The Møller matrix commutes with these translations.

Operators $a_{in}(\rho)$ are defined on the image of S_- by the formula

$$a_{in}(\rho)S_- = S_- b(\rho).$$

They are Hermitian conjugate to $a^+_{in}(\rho)$.

One can give a direct definition of the Møller matrix by the formula

$$S_- b^+(f^k_1\phi^\alpha_k) \cdots b^+(f^k_n\phi^\alpha_k)|0\rangle = \Psi(f_1, \ldots, f_n | -\infty)$$

or equivalently

$$S_-(b^+(g_1) \cdots b^+(g_n)|0\rangle) = \lim_{\tau \to -\infty} B(g_1\phi^{-1}, \tau) \cdots B(g_n\phi^{-1}, \tau)\Phi.$$

If $g_1, \cdots, g_n$ do not overlap the vector

$$B(g_1\phi^{-1}, \tau) \cdots B(g_n\phi^{-1}, \tau)\Phi \tag{6}$$

describes n distant particles as $\tau \to -\infty$.

It is convenient to require a strong cluster property (see Section 2 of [2]) to analyze the Møller matrix. (This condition can be weakened).

Let us make the following

Assumption 1. *The subset of the space $\mathcal{S}^{mn}$ spanned by non-overlapping families $(f_1, \dots, f_n)$ contains an open dense subset of $\mathcal{S}^{mn}$.*

This assumption is not restrictive; see the discussion in Section 4.2 of [2].

Using the above assumption and cluster property, one can prove the theorem below.

Theorem 1. *The Møller matrix S_- is a well-defined isometric operator.*

Notice first of all that it is not clear from our definitions that the *in*-operators and Møller matrices are well-defined (they can depend on the choice of operators B_j). In other words, the operator S_- a priori can be multivalued. However, we can prove that this operator is isometric and use the fact that an isometric operator is necessarily single-valued.

To prove that the operator S_- preserves the inner product, we express the inner product

$$\langle B(f_1, \tau_1) \dots B(f_n, \tau_n)\Phi, B'(f'_1, \tau'_1) \dots B'(f'_{n'}, \tau'_{n'})\Phi\rangle$$

in terms of truncated correlation functions. We assume that both $(f_1, \dots, f_n)$ and $(f'_1, \dots, f'_{n'})$ do not overlap. Then, it follows from Definition 1 and the cluster property that only two-point correlations $\langle B'^*(f', \tau')B(f, \tau)\Phi, \Phi\rangle = \langle B(f, \tau)\Phi, B'(f', \tau')\Phi\rangle$ contribute in the limit $\tau_j, \tau'_{j'} \to -\infty$. Calculating the two-point correlation functions by means of (9), we see that S_- is an isometry. We assumed that the vectors corresponding to families of non-overlapping functions span a dense subset of $\mathcal{H}_{as}$,;hence, S_- can be extended to an isometric embedding of the space $\overline{\mathcal{H}_{as}}$ into $\overline{\mathcal{H}}$.

Taking $\tau \to +\infty$ instead of $\tau \to -\infty$, we obtain the definition of the Møller matrix S_+ and of *out*-operators $a^+_{out}(\rho), a_{out}(\rho)$. If Møller matrices are surjective operators, we can define the scattering matrix (S-matrix) by the formula $S = S_+^{-1}S_-$. In this case, one says that *the theory has particle interpretation.*[2]

In other words, we can say that the theory has particle interpretation if for a dense subset of $\mathcal{H}$, the time evolution can be represented as a linear combination of vectors

$$e^{-iH\tau}\Psi(f_1, \dots, f_n| - \infty) = \Psi(T_\tau f_1, \dots, T_\tau f_n| - \infty)$$

for $\tau < 0$ and of vectors

$$e^{-iH\tau}\Psi(f_1, \dots, f_n| + \infty) = \Psi(T_\tau f_1, \dots, T_\tau f_n| + \infty)$$

fot $\tau > 0$.

This means that generically both for $\tau \to -\infty$ and for $\tau \to +\infty$ we obtain a collection of distant particles (the wave functions $T_\tau f_k$ have distant essential supports if functions f_k do not overlap). Without assumption that the theory has particle interpretation, we can define the scattering matrix by the formula $S = S_+^* S_-$; however, this definition gives a unitary operator only in the case when the image of S_- coincides with the image of S_+.

Until now, all of our considerations were applicable both to algebras over real numbers and algebras over complex numbers. In what follows, we restrict ourselves to algebras over complex numbers. Notice, however, that we can apply the considerations below to algebras over real numbers complexifying the elementary space $\mathfrak{h}$ and the pre Hilbert space $\mathcal{H}$. The proofs do not require any serious modifications.

Let us diagonalize the matrix $E(\mathbf{k})$; corresponding eigenvalues are denoted by $\epsilon_j(\mathbf{k})$, and eigenvectors are denoted $\rho_j^\alpha(\mathbf{k})$. (We assume that these eigenvectors constitute an orthonormal system). In momentum representation, generalized eigenvectors vectors of time and spatial translations are $\rho_j^\alpha(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}_0)$. We consider *in*- and *out*-operators corresponding to these eigenvectors as generalized functions of $\mathbf{k}$; they are denoted

$$a_{in}(\mathbf{k},j), a_{in}^+(\mathbf{k},j), a_{out}(\mathbf{k},j), a_{out}^+(\mathbf{k},j).$$

For example, $\int d\mathbf{k}\sum_i f^i(\mathbf{k})a_{out}^+(\mathbf{k},i) = a_{out}^+(f\rho)$ where $f\rho = f^i(\mathbf{k})\rho_i^\alpha(\mathbf{k}))$. (Here, ρ is considered as a square matrix). These operators can be interpreted as annihilation and creation *in*- and *out*- operators of particles with momentum $\mathbf{k}$. Sometimes we omit discrete indices characterizing the type of particle in these operators. Then, the operators $a_{in}(\mathbf{k}), a_{in}^+(\mathbf{k}), a_{out}(\mathbf{k}), a_{out}^+(\mathbf{k}))$ should be regarded as m-dimensional vectors and the values of corresponding correlation functions as elements of tensor product of m-dimensional spaces.

If we assume that the theory has particle interpretation, the Møller matrices establish unitary equivalence of $a_{in}(\mathbf{k},j), a_{in}^+(\mathbf{k},j), a_{out}(\mathbf{k},j), a_{out}^+(\mathbf{k},j)$ with $b(\mathbf{k},j), b^+(\mathbf{k},j)$, where $b(\mathbf{k},j), b^+(\mathbf{k},j)$ are operator generalized functions in $\mathcal{H}_{as}$ corresponding to $b(f\rho), b^+(f\rho)$.

The matrix elements of scattering matrix can be expressed in terms of *in*- and *out*-operators:

$$\begin{gathered}\langle S(b^+(g_1)\cdots b^+(g_n)|0\rangle),(b^+(h_1)\cdots b^+(h_m)|0\rangle)\rangle = \\ \langle S_-(b^+(g_1)\cdots b^+(g_n)|0\rangle), S_+(b^+(h_1)\cdots b^+(h_m)|0\rangle)\rangle = \\ \langle a_{in}^+(g_1)\cdots a_{in}^+(g_n)\Phi, a_{out}^+(h_1)\cdots a_{out}^+(h_m)\Phi\rangle = \\ \langle \lim_{\tau\to-\infty} B(g_1\phi^{-1},\tau)\cdots B(g_n\phi^{-1},\tau)\Phi, \lim_{\tau\to+\infty} B(h_1\phi^{-1},\tau)\cdots B(h_m\phi^{-1},\tau)\Phi\rangle\end{gathered}$$

hence

$$\begin{gathered}\langle S(b^+(g_1)\cdots b^+(g_n)|0\rangle),(b^+(h_1)\cdots b^+(h_m)|0\rangle)\rangle = \\ \lim_{\tau\to-\infty,\tau'\to+\infty}\omega(B^*(h_m\phi^{-1},\tau')\cdots B^*(h_1\phi^{-1},\tau')B(g_1\phi^{-1},\tau)\cdots B(g_n\phi^{-1},\tau))\end{gathered} \tag{7}$$

and

$$\begin{gathered}\langle a_{in}^+(\mathbf{k}_1,i_1)\ldots a_{in}^+(\mathbf{k}_n,i_n)\Phi, a_{out}^+(\mathbf{k}_1',j_1)\ldots a_{out}^+(\mathbf{k}_m',j_m)\Phi\rangle = \\ \lim_{\tau\to-\infty,\tau'\to+\infty}\omega(B'(\mathbf{k}_m',j_m,\tau')\ldots B'(\mathbf{k}_1',j_1,\tau')B(\mathbf{k}_1,i_1,\tau)\ldots B(\mathbf{k}_n,i_n,\tau))\end{gathered} \tag{8}$$

where

$$\int d\mathbf{k}\sum_i f^i(\mathbf{k})B(\mathbf{k},i,\tau) = B(f\rho\phi^{-1},\tau),$$

$$\int d\mathbf{k}\sum_i f^i(\mathbf{k})B'(\mathbf{k},i,\tau') = B^*(f\rho\phi^{-1},\tau').$$

Omitting discrete indices and using (2), we can write

$$\begin{gathered}\langle a_{in}^+(\mathbf{k}_1)\ldots a_{in}^+(\mathbf{k}_n)\Phi, a_{out}^+(\mathbf{k}_1')\ldots a_{out}^+(\mathbf{k}_m')\Phi\rangle = \\ \lim_{\tau\to-\infty,\tau'\to+\infty}\omega(D(\mathbf{k}_m')B'(\mathbf{k}_m',\tau')\ldots D(\mathbf{k}_1)B'(\mathbf{k}_1',\tau')D(\mathbf{k}_1)B(\mathbf{k}_1,\tau)..D(\mathbf{k}_n)B(\mathbf{k}_n,\tau)) = \\ \lim_{\tau\to-\infty,\tau'\to+\infty}\omega(D(\mathbf{k}_m')e^{i\tau'E(\mathbf{k}_m')}\hat{B}^*(\tau',\mathbf{k}_m')\ldots D(\mathbf{k}_1')e^{i\tau'E(\mathbf{k}_1')}\hat{B}^*(\tau',\mathbf{k}_1') \\ D(\mathbf{k}_1)e^{i\tau E(\mathbf{k}_1)}\hat{B}(\tau,\mathbf{k}_1)..D(\mathbf{k}_n)e^{i\tau E(\mathbf{k}_n)}\hat{B}(\tau,\mathbf{k}_n))\end{gathered} \tag{9}$$

where D stands for the matrix $\rho\phi^{-1}$.

It is easy to describe the joint spectrum of momentum and energy operators in $\mathcal{H}_{as}$ (of infinitesimal generators of spatial and time translations). It consists of points $(\epsilon_{j_1}(\mathbf{k}_1)+\ldots+\epsilon_{j_r}(\mathbf{k}_r), \mathbf{k}_1+\ldots+\mathbf{k}_r)$. For $r=0$, we obtain the point $(0,0)$ corresponding to the vacuum vector. The points with $r=1$ constitute a one-particle spectrum, the points with

$r > 1$ belong to a multi-particle spectrum. If the theory has particle interpretation, the same formulas describe the joint spectrum of momentum and energy operators in $\bar{\mathcal{H}}$.

The decomposition of the spectrum in a one-particle spectrum and a multi-particle spectrum induces the decomposition of $\bar{\mathcal{H}}$ into a direct sum of one-dimensional space $\mathcal{H}_0$ containing Φ, one-particle space $\mathcal{H}_1$ (the closure of the image of the map $\mathfrak{h} \to \mathcal{H}$), and a multi-particle space $\mathcal{M}$.

3. LSZ Formula

The scattering matrix can be expressed in terms of Green functions. These functions can be defined by the formula

$$G_r(\tau_1, \mathbf{x}_1, i_1, \dots, \tau_r, \mathbf{x}_r, i_r) = \omega(T(B_{i_1}(\tau_1, \mathbf{x}_1) \dots B_{i_r}(\tau_r, \mathbf{x}_r))$$

where T stand for the chronological product. We defined Green functions in $(\tau, \mathbf{x})$-representation. Taking the Fourier transform with respect to $\mathbf{x}$, we obtain Green functions in $(\tau, \mathbf{k})$- representation; taking an inverse Fourier transform with respect to τ, we obtain Green functions in $(\varepsilon, \mathbf{k})$-representation. For simplicity, we are assuming that that B_i are self-adjoint (otherwise we should consider not only B_i, but also B_i^* under the sign of T-product).

We have defined Green functions using the operators B_i (good operators). However, one can modify this definition replacing B_i by other operators $A_i \in \mathcal{A}$.

It is easy to calculate the two-point Green function G_2. We start with a two-point correlation function

$$w_2(\tau_1, \mathbf{x}_1, i, \tau_2, \mathbf{x}_2, j) = \omega(B_i(\tau_1, \mathbf{x}_1) B_j(\tau_2, \mathbf{x}_2)) =$$

$$\langle B_j(\tau_2, \mathbf{x}_2)\Phi, B_i(\tau_1, \mathbf{x}_1)\Phi\rangle = \int d\mathbf{k} e^{i\mathbf{k}(\mathbf{x}_2 - \mathbf{x}_1)} \langle e^{-i\tau_2 E(\mathbf{k})} \phi_j(\mathbf{k}), e^{-i\tau_1 E(\mathbf{k})} \phi_i(\mathbf{k})\rangle$$

Expressing G_2 in terms of w_2, we obtain that in $(\varepsilon, \mathbf{k})$-representation

$$G_2(\varepsilon_1, \mathbf{k}_1, i, \varepsilon_2, \mathbf{k}_2, j) = G^{i,j}(\varepsilon_1, \mathbf{k}_1)\delta(\varepsilon_1 + \varepsilon_2)\delta(\mathbf{k}_1 + \mathbf{k}_2)$$

where for fixed $\mathbf{k}$ the function $G^{i,j}(\varepsilon, \mathbf{k})$ has poles with respect to ε at the points $\pm\epsilon_s(\mathbf{k})$ (here $\epsilon_s(\mathbf{k})$ are eigenvalues of the matrix $E(\mathbf{k})$)). Namely,

$$G^{i,j}(\varepsilon, \mathbf{k}) = A^{i,j}(\varepsilon, \mathbf{k}) + A^{j,i}(-\varepsilon, -\mathbf{k})$$

where

$$A^{i,j}(\varepsilon, \mathbf{k}) = \sum_s \frac{i}{\varepsilon + \epsilon_s(\mathbf{k}) - i0} \langle a_s(\mathbf{k})\phi_j(\mathbf{k}), \phi_i(\mathbf{k})\rangle \tag{10}$$

We have used the fact that the matrix $e^{iE(\mathbf{k})\tau}$ can be expressed as a linear combination of exponents $e^{i\epsilon_j(\mathbf{k})\tau}$ with matrix coefficients depending on $\mathbf{k}$:

$$e^{iE(\mathbf{k})\tau} = \sum_s a_s(\mathbf{k}) e^{i\epsilon_s(\mathbf{k})\tau}.$$

Using the same fact and (7) or (9), it is easy to check that the scattering matrix can be expressed in terms of asymptotic behavior of Green functions in $(\tau, \mathbf{k})$ representation. (One should divide the arguments of the Green function in two groups; in one group, we should take the times tending to $-\infty$, in the second group to $+\infty$. The ordering of times in every group is irrelevant due to asymptotic commutativity of factors in (7)).

Equivalently, one can work in $(\varepsilon, \mathbf{k})$- representation taking an inverse Fourier transform with respect to τ in $(\tau, \mathbf{k})$-representation. Then, the scattering matrix can be expressed in terms of poles of Green functions with respect to ε and residues in these poles. This is the famous LSZ formula. One can derive it from the following statements:

Let E denote a Hermitian matrix with eigenvalues ϵ_j. Then, the matrix $e^{i\tau E}$ can be written in the form $\sum_j a_j e^{i\tau\epsilon_j}$ where a_j are constant matrices.

Let us assume there exist limits $A_{\pm} = \lim_{\tau\to\pm\infty} e^{i\tau E}\rho(\tau)$ where ρ is a column vector. Then, $\rho(\tau)$ has asymptotic behavior

$$\rho(\tau) \sim e^{-i\tau E} A_{\pm} = \sum_k e^{-i\epsilon_k \tau} a_k A_{\pm}$$

as $\tau \to \pm\infty$.

This implies that the (inverse) Fourier transform $\rho(\varepsilon)$ of $\rho(\tau)$ has poles at the points $\pm(\epsilon_k + i0)$ with residues $\mp 2\pi i a_k A_{\pm}$.

We can say that

The asymptotic behavior of $\rho(\tau)$ is determined by the polar part of $\rho(\varepsilon)$.

In what follows, we use these statements in a little bit different form. We represent $\rho(\tau)$ as $e^{-i\tau E}A_-\Phi(-\tau) + e^{-i\tau E}A_+\Phi(\tau) + \sigma(\tau)$ where $\sigma(\tau) \to 0$ as $\tau \to \infty$. We assume that $\sigma(\tau)$ is a summable function. Then,

$$\rho(\varepsilon) = (E - \varepsilon + i0)^{-1}C_- + (E + \varepsilon + i0)^{-1}C_+ + \sigma(\varepsilon)$$

where $\sigma(\varepsilon)$ is continuous, and C_- and C_+ do not depend on ε. We say that the first two summands constitute the polar part of $\rho(\varepsilon)$. We need the following statement

If $f(\varepsilon)$ is a smooth function, then

$$f(\varepsilon)\rho(\varepsilon) = (E - \varepsilon - i0)^{-1}C'_- + (E + \varepsilon - i0)^{-1}C'_+\sigma'(\varepsilon) \tag{11}$$

where $C'_- = f(E)C_-, C'_+ = f(-E)C_+$ do not depend on ε, and σ' is a continuous function.

Notice that in the LSZ formula, the operators B_i transforming the vector Φ into an element of $\Phi(\mathfrak{h})$ (of one-particle space) can be replaced by asymptotically commuting smooth operators $A_i \in \mathcal{A}$ obeying a weaker condition. Namely, one should require that the projections of vectors $A_i\Phi$ on the one-particle space are linearly independent, and the projection of $A_i\Phi$ on Φ vanishes. This can be proved if the theory has particle interpretation. (See Section 4.6 of [2] for the proof of this fact in a less general case; this proof can be generalized to our setting). Instead of this condition, we can require the existence of smooth fast decreasing functions $\alpha_i^j(\tau, \mathbf{x})$ such that the operators $B_i = \int d\tau d\mathbf{x} \alpha_i^j(\tau, \mathbf{x}) A_j(\tau, \mathbf{x})$ are good operators. (This condition is always satisfied if the joint spectrum of $H, \mathbf{P}$ in $\mathcal{H}_0 \oplus \mathcal{H}_1$.

Using the formula

$$B_i(\tau, \mathbf{x}) = \int d\tau' d\mathbf{x}' \alpha_i^j(\tau' - \tau, \mathbf{x}' - \mathbf{x}) A_j(\tau', \mathbf{x}')$$

we can express the correlation functions for operators B_i in terms of correlation functions for operators A_i. The expression looks very simple in $(\varepsilon, \mathbf{k})$-representation. For example, if $\alpha_i^j = \alpha_i \delta_i^j$, one should multiply the correlation functions of operators A_i by the product of Fourier transforms of functions α_i with respect to $\tau, \mathbf{x}$. The corresponding expression for Green functions is more complicated due to factors $\theta(\tau_i - \tau_j)$ entering the definition of the chronological product. However, in scattering theory, we are interested in asymptotic behavior of Green functions in $(\tau, \mathbf{k})$ representation or in the behavior of polar parts of Green functions in $(\varepsilon, \mathbf{k})$-representation. For $\mathbf{k}$ in a dense open set, the behavior of the polar parts of Green functions for operators A_i in $(\varepsilon, \mathbf{k})$ representation can be described in the same way as for correlation functions. (To prove this statement we use asymptotic commutativity of operators A_i and the assumption before Theorem 1. In the calculation of scattering matrix, we decompose the arguments of Green functions in two groups; we use the fact that due to asymptotic commutativity, the time ordering inside every group is irrelevant).

Let us give more precise formulations of the above statements.

We are starting with asymptotically commuting operators $A_1, \ldots, A_m$ obeying $\langle A_i\Phi, \Phi\rangle = 0$. We introduce the notation $T_i^A(\mathbf{k})$ for projections of vectors $A_i\Phi \in \mathcal{H}$ on $\Phi(\mathfrak{h})$ (on one-particle subspace of $\mathcal{H}$) considered as elements of $\mathfrak{h}$ in momentum representation. We assume that these projections are linearly independent (the matrix $T^A(\mathbf{k}) = (T^A(\mathbf{k}))_i^\alpha$ is non-degenerate).

We consider the Green function

$$G_r(\tau_1, \mathbf{x}_1, i_1, \ldots, \tau_r, \mathbf{x}_r, i_r) = \omega(T(A_{i_1}(\tau_1, \mathbf{x}_1) \ldots A_{i_r}(\tau_r, \mathbf{x}_r))$$

and their Fourier transforms (Green functions in $(\tau, \mathbf{k})$- and $(\varepsilon, \mathbf{k})$- representations. Notice that due to translation invariance, the Green function in $(\varepsilon, \mathbf{k})$- representation contains a delta-function $\delta(\sum \varepsilon_i)\delta(\sum \mathbf{k}_i)$; talking about two-point function (r = 2), we always omit this delta-function. (Hence, the two-point Green function is a matrix-valued function of $(\varepsilon, \mathbf{k})$). We can write the two-point Green function in $(\varepsilon, \mathbf{k})$-representation as a sum of the polar part (having first order poles with respect to variables ε) and a regular part. The polar part

$$(E(\mathbf{k}) - \varepsilon + i0)^{-1}C_-(\mathbf{k}) + (E(\mathbf{k}) + \varepsilon + i0)^{-1}C_+(\mathbf{k})$$

governs the behavior of the Green function in $(\tau, \mathbf{k})$-representation as $\tau \to \infty$; it is a sum of two summands; one of them (*in*-summand) is responsible for the limit $\tau \to -\infty$, another (*out*-summand) is responsible for the limit $\tau \to \infty$.

Let us consider the Green function G_r in $(\tau, \mathbf{k})$-representation. We assume that the arguments of this function are divided in two groups (with indices i in the interval $1 \le i \le a$ and with indices in the interval $a < i \le r$). We assume that the times τ_i with the indices from the first group tend to $-\infty$, and the remaining times tend to $+\infty$.

In $(\varepsilon, \mathbf{k})$- representation, the Green function G_r can be represented as a product of the amputated Green function and r the two-point Green functions labeled by index i. We change the signs of the variables $\varepsilon_i, \mathbf{k}_i$ where i is the index from the second group to interpret these variables as energies and momenta of outgoing particles. We define the polar part

$$P_r(\varepsilon_1, \mathbf{k}_1, j_1, \ldots, \varepsilon_r, \mathbf{k}_r, j_r)$$

of the Green function replacing every two-point Green function in this representation by its *in*-summand of its polar part for indices $i \le a$ and by *out*-summand of the polar part for $i > a$.

Let us define operators A_i' by the formula $A_i' = \int d\tau d\mathbf{x} \alpha_i^j(\tau, \mathbf{x}) A_j(\tau, \mathbf{x})$, where $\alpha_i^j(\tau, \mathbf{x})$ are smooth fast decreasing functions. Polar parts of the corresponding Green functions are denoted by P_r'.

It is easy to express the projections $T_i^{A'}(\mathbf{k})$ of $A_i'\Phi$ on one-particle space in terms of projections $T_j^A(\mathbf{k})$. We obtain

$$T_i^{A'}(\mathbf{k}) = \alpha_i^j(E(\mathbf{k}), \mathbf{k}) T_j^A(\mathbf{k}) \tag{12}$$

where $\alpha_i^j(E(\mathbf{k}), \mathbf{k}) = \int d\tau d\mathbf{x} e^{i\mathbf{k}\mathbf{x} - iE(\mathbf{k})\tau} \alpha_i^j(\tau, \mathbf{x})$

Equivalently

$$T^{A'}(\mathbf{k}) = \alpha(E(\mathbf{k}), \mathbf{k}) T^A(\mathbf{k}). \tag{13}$$

To prove (12), we represent $A_j(\tau, \mathbf{x})\Phi$ as $\Phi(T_j^{A(\tau, \mathbf{x})}) + \rho(\tau, \mathbf{x})$ where ρ belongs to the multiparticle space. We obtain

$$A_i'\Phi = \int d\tau d\mathbf{x} \alpha_i^j(\tau, \mathbf{x}) \Phi(T_j^{A(\tau, \mathbf{x})}) + \int d\tau d\mathbf{x} \alpha_i^j(\tau, \mathbf{x}) \rho(\tau, \mathbf{x}).$$

The second summand lies in the multiparticle space and that for $\phi \in \mathfrak{h}$ we have $\Phi(e^{i\mathbf{P}\mathbf{x} - H\tau}\phi) = \Phi(e^{i\mathbf{k}\mathbf{x} - iE(\mathbf{k})\tau}\phi)$. This implies (12).

As we noticed, the polar part governs the asymptotic behavior in $(\tau, \mathbf{k})$- representation. Using this fact, one can prove that $P'_r(\varepsilon_1, \mathbf{k}_1, i_1, \dots, \varepsilon, \mathbf{k}_r, i_r)$ is equal to the polar part of

$$\alpha^{j_1}_{i_1}(\varepsilon_1, \mathbf{k}_1) \dots \alpha^{j_r}_{i_r}(\varepsilon_r, \mathbf{k}_r) P_r(\varepsilon_1, \mathbf{k}_1, j_1, \dots, \varepsilon_r, \mathbf{k}_r, j_r). \tag{14}$$

Using (11), we obtain that

$$P'_r(\varepsilon_1, \mathbf{k}_1, i_1 \dots, \varepsilon, \mathbf{k}_r, i_r) = \alpha^{j_1}_{i_1}(E(\mathbf{k}_1), \mathbf{k}_1) \dots \alpha^{j_r}_{i_r}(E(\mathbf{k}_r), \mathbf{k}_r) P_r(\varepsilon_1, \mathbf{k}_1, j_1, \dots, \varepsilon_r, \mathbf{k}_r, j_r)$$

or equivalently

$$P'_r(\varepsilon_1, \mathbf{k}_1, \dots, \varepsilon, \mathbf{k}_r) = \alpha(E(\mathbf{k}_1), \mathbf{k}_1) \otimes \dots \otimes \alpha(E(\mathbf{k}_r), \mathbf{k}_r) P_r(\varepsilon_1, \mathbf{k}_1, \dots, \varepsilon_r, \mathbf{k}_r) \tag{15}$$

In this formula and in what follows, we consider P'_r and P_r as functions taking values in r-th tensor power of m-dimensional space (we consider discrete variables in P_r and in P'_r as tensor indices).

Let us now define the normalized polar part of the Green function (closely related to the Green function on shell) by the formula

$$\tilde{P}_r(\varepsilon_1, \mathbf{k}_1, \dots, \varepsilon_r, \mathbf{k}_r) = (T^A(\mathbf{k}_1))^{-1} \otimes \dots \otimes (T^A(\mathbf{k}_r))^{-1} P_r(\varepsilon_1, \mathbf{k}_1, \dots, \varepsilon_r, \mathbf{k}_r).$$

This function takes values in r-th tensor power of $\mathfrak{h}$.

It follows immediately from (13) and (15) that

$$\tilde{P}'_r(\varepsilon_1, \mathbf{k}_1 \dots, \varepsilon_r, \mathbf{k}_r) = \tilde{P}_r(\varepsilon_1, \mathbf{k}_1 \dots, \varepsilon_r, \mathbf{k}_r). \tag{16}$$

This means that normalized polar parts of the Green functions for $A_1, \dots, A_m$ and $A'_1, \dots, A'_m$ coincide.

Let us define the Green functions on shell taking the residues of normalized polar parts of the Green functions.

Now, we can formulate the LSZ formula in the following way:

Matrix elements of the scattering matrix coincide with the Green functions on shell (up to sign change in outgoing momenta).

To prove this fact, it is sufficient to verify it for good operators $B_1, \dots, B_m$ using (7) and (9).

Notice that the matrices T^A entering the definition of normalized polar part are closely related to the polar part of the two-point Green function.

4. Inclusive Scattering Matrix

An element B of the algebra $\mathcal{A}$ specifies two operators on linear functionals on $\mathcal{A}$. The first operator is denoted by the same symbol B; it transforms the functional σ into the functional $(B\sigma)(A) = \sigma(AB)$. The second one (denoted $\tilde{B}$) transforms σ into the functional $(\tilde{B}\sigma)(A) = \sigma(B^*A)$. The vectors in the space $\mathcal{H}$ correspond to excitations of the state ω. The vector $B\Phi$ corresponds to the state $\tilde{B}B\omega$. Let us introduce notations

$$B(f) = B(f, 0), L(g) = \tilde{B}(g\phi^{-1})B(g\phi^{-1}), L(g, \tau) = T_\tau L(T_{-\tau} g) = \tilde{B}(f, \tau)B(f, \tau)$$

where $g = f\phi \in \mathfrak{h}$. Instead of working with vectors

$$\Psi(f_1, \tau_1, \dots, f_n, \tau_n) = B(f_1, \tau_1) \dots B(f_n, \tau_n)\Phi$$

we can work with corresponding states

$$\Lambda(g_1, \tau_1, \dots, g_n, \tau_n) = L(g_1, \tau_1), \dots L(g_n, \tau_n)\omega.$$

It follows from Lemma 2 that these states have a limit as $\tau_j \to -\infty$ if $f_j = g_j\phi^{-1}$ do not overlap. These states will be necessary in a geometric approach (see [13]); however, they are also useful in an algebraic approach. Namely, we use these states to construct the inclusive scattering matrix.

Let us consider the state

$$L(g_1',\tau_1')\dots L(g_{n'}',\tau_{n'}')L(g_1,\tau_1)\dots L(g_n,\tau_n)\omega$$

considered as a linear functional on $\mathcal{A}$ (as an element of the cone $\mathcal{C}$). We assume that g_i' as well as g_j are not overlapping. Then, this state has a limit as $\tau_i'\to+\infty,\tau_j\to-\infty$; we denote this limit by Q. Notice that Q does not change if we permute $g_1,\dots,g_n$ (in the limit $\tau_j\to-\infty$ the operators $L(g_j,\tau_j)$ commute). Similarly, Q does not change if we permute $g_1',\dots,g_{n'}'$.

More generally, we can consider a linear functional $\tilde{Q}$ on $\mathcal{A}$ defined as a limit of

$$L(\tilde{g}_1',g_1',\tau_1')\dots L(\tilde{g}_{n'}',g_{n'}',\tau_{n'}')L(\tilde{g}_1,g_1,\tau_1)\dots L(\tilde{g}_n,g_n,\tau_n)\omega$$

as $\tau_i'\to+\infty,\tau_j\to-\infty$. (We introduced the notation $L(\tilde{g},g,\tau)=\tilde{B}(\tilde{f},\tau)B(f,\tau)$).

Then we define

$$\sigma(\tilde{g}_1',g_1',\dots,\tilde{g}_{n'}',g_{n'}',\tilde{g}_1,g_1,\dots,\tilde{g}_n,g_n).$$

as $\tilde{Q}(1)$. This functional is linear with respect to its arguments $\tilde{g}_i',g_i,\tilde{g}_j,g_j$. It is well-defined if each of four families $\tilde{g}_i',g_i,\tilde{g}_j,g_j$ is non-overlapping. In bra-ket notations

$$\begin{gathered}\sigma(\tilde{g}_1',g_1',\dots,\tilde{g}_{n'}',g_{n'}',\tilde{g}_1,g_1,\dots,\tilde{g}_n,g_n)=\\ \langle 1|\lim_{\tau_i'\to+\infty,\tau_j\to-\infty}L(\tilde{g}_1',g_1',\tau_1')\dots L(\tilde{g}_{n'}',g_{n'}',\tau_{n'}')L(\tilde{g}_1,g_1,\tau_1)\dots L(\tilde{g}_n,g_n,\tau_n)|\omega\rangle\end{gathered}\tag{17}$$

By definition, the functional σ is *an inclusive scattering matrix*.

To justify this definition, we notice that

$$\begin{gathered}\sigma(\tilde{g}_1',g_1',\dots,\tilde{g}_{n'}',g_{n'}',\tilde{g}_1,g_1,\dots,\tilde{g}_n,g_n)=\\ \lim_{\tau_j\to-\infty',\tau_i'\to+\infty}(L(\tilde{g}_1,g_1,\tau_1)\dots L(\tilde{g}_n,g_n,\tau_n)\omega)(B(f_1',\tau_1')\dots B(f_{n'}',\tau_{n'}')B(f_{n'}',\tau_{n'}')^*\dots B(f_1',\tau_1')^*)=\\ (\lim_{\tau_j\to-\infty}(L(\tilde{g}_1,g_1,\tau_1)\dots L(\tilde{g}_n,g_n,\tau_n)\omega))(a_{out}^+(g_1')\dots a_{out}^+(g_{n'}')a_{out}(\tilde{g}_{n'}')\dots(a_{out}(\tilde{g}_1')))\end{gathered}$$

We have used the relation $(\tilde{M}N\rho)(X)=\rho(M^*XN)$ in this derivation.

The inclusive cross section can be expressed in terms of the inclusive S-matrix defined above. To verify this statement, we consider the expectation value

$$\nu(a_{out,k_1}^+(\mathbf{p}_1)a_{out,k_1}(\mathbf{p}_1)\dots a_{out,k_m}^+(\mathbf{p}_m)a_{out,k_m}(\mathbf{p}_m))\tag{18}$$

where ν is an arbitrary state. This quantity is the probability density in momentum space for finding m outgoing particles of the types $k_1,\dots,k_n$ with momenta $\mathbf{p}_1,\dots,\mathbf{p}_m$ plus other unspecified outgoing particles. It gives an inclusive cross section if

$$\nu=\lim_{\tau_i\to-\infty}L(g_1,\tau_1)\dots L(g_n,\tau_n)\omega$$

The inclusive scattering matrix can be expressed in terms of the generalized Green functions (GGreen functions). These functions appear naturally in the formalism of L-functionals [1,11,12]; their relation to inclusive cross sections is analyzed in [1,2,10]. They also appear in Keldysh formalism and in thermo-field dynamics [7–9]. GGreen functions can be defined by the formula

$$\begin{gathered}G(\tau_1,\mathbf{x}_1,i_1,\dots,\tau_r,\mathbf{x}_r,i_r,\tau_1',\mathbf{x}_1',i_1',\dots,\tau_{r'}',\mathbf{x}_{r'}',i_{r'}')=\\ (T(B_{i_1}(\tau_1,\mathbf{x}_1)\dots B_{i_r}(\tau_r,\mathbf{x}_r,)\tilde{B}_{i_1'}(\tau_1',\mathbf{x}_1')\dots\tilde{B}_{i_{r'}'}(\tau_r',\mathbf{x}_r'))\omega)(1)\end{gathered}$$

where T stands for the chronological product. More precisely, we defined the GGreen functions in $(\tau, \mathbf{x})$-representation, taking the Fourier transform with respect to $\mathbf{x}$, we obtain the GGreen functions in $(\tau, \mathbf{k})$- representation. Using the fact that the matrix $e^{iE(\mathbf{k})\tau}$ can be expressed as a linear combination of exponents $e^{i\epsilon_j(\mathbf{k})\tau}$ with matrix coefficients depending on $\mathbf{k}$, it is easy to check that the inclusive scattering matrix can be expressed in terms of asymptotic behavior of the GGreen functions in $(\tau, \mathbf{k})$ representation. (One should take $r = r'$ and assume that $\tau_i \to +\infty, \tau'_i \to +\infty$ for $i \leq m$ and $\tau_j \to -\infty, \tau'_j \to -\infty$ for $j > m$. The ordering of times in every group is irrelevant due to asymptotic commutativity of factors).

Equivalently, one can work in $(\varepsilon, \mathbf{k})$- representation taking the inverse Fourier transform with respect to τ in $(\tau, \mathbf{k})$-representation. Then, the inclusive scattering matrix can be expressed in terms of poles of GGreen functions with respect to ε and residues in these poles.

As in the LSZ formula for a scattering matrix, we can work with operators $A_1, \ldots, A_m$ requiring the existence of fast decreasing functions α_i^j such that the operators $B_i = \int d\tau d\mathbf{x}$ $\alpha_i^j(\tau, \mathbf{x}) A_j(\tau, \mathbf{x})$ are good operators. Using Kållén–Lehmann representation of the two-point GGreen function, we define polar part and normalized polar part of the GGreen function. (We represent GGreen functions in terms of amputated GGreen functions and use the Kållén-Lehmann representation of the two-point GGreen function in the proof).

5. Fermions

We assumed that operators B_i asymptotically commute (4). One can replace this assumption with the assumption of asymptotic anticommutativity (we replace the commutator in (4) by anticommutator). Then, we should also modify the definition of truncated correlation functions including some signs.

One can repeat all considerations of the present paper in this situation. Slight modifications are necessary. In particular, instead of bosonic Fock space one should consider fermionic Fock space (Fock representation of canonical anticommutation relations). The particles obey Fermi statistics. To define an inclusive scattering matrix, we again consider states

$$\Lambda(g_1, \tau_1, \ldots, g_n, \tau_n) = L(g_1, \tau_1), \ldots L(g_n, \tau_n)\omega$$

and prove that these states have a limit as $\tau_i \to -\infty$ under the same conditions on $g_1, \ldots, g_n$. It is important to notice that under these conditions it follows from the asymptotic *anticommutativity* of operators B_i that the operators $L(g_i, \tau_i)$ *commute* in the limit $\tau_i \to -\infty$.

6. BRST Formalism

Methods of homological algebra (=BRST formalism) can be applied in scattering theory. Recall that in homological algebra together with modules (algebras, etc.) one considers differential graded modules (algebras, ...). It is sufficient to have $\mathbb{Z}_2$-grading. A module is $\mathbb{Z}_2$-graded if it is represented as a direct sum of even and odd parts. A differential can be defined as parity reversing homomorphism Q obeying $Q^2 = 0$. Homology is defined as $KerQ/ImQ$ (as a quotient of the submodule consisting of Q-closed elements with respect to the submodule consisting of Q-exact elements).

The main idea is to replace a module by a simpler (for example, free) differential graded module. (The new module should be quasi-isomorphic to the original module, considered as a differential module with trivial grading and trivial differential. Quasi-isomorphism is defined as a homomorphism commuting with the differential and inducing an isomorphism on homology).

The above considerations can be applied to differential $\mathbb{Z}_2$- graded algebras (algebras with parity reversing BRST operator Q obeying $Q^2 = 0$). All physical quantities should be BRST-closed (should belong to the kernel of Q); one should neglect the BRST-exact quantities (the elements of the image of Q). The BRST-operator on algebra should satisfy the graded Leibniz rule: $Q(AB) = Q(A)B \pm AQ(B)$ (plus sign if A is even, minus sign if A

is odd. If algebra $\mathcal{A}$ is realized by operators in a differential $\mathcal{A}$-module with differential $\hat{Q}$, then the differential Q on algebra is defined as supercommutator with $\hat{Q}$, i.e., $QA = [\hat{Q}, A]$ if A is even and $QA = [\hat{Q}, A]_+$ if A is odd.

Instead of Hilbert spaces, one can consider differential modules equipped with a structure of pseudo Hilbert space (space with non-degenerate, but indefinite scalar product). However, the indefinite scalar product should descend to a definite scalar product on homology.

These ideas are widely used in gauge theories and in string theory.

The Gelfand–Naimark–Segal (GNS) construction can be generalized to the case when an algebra is not equipped with involution. In this generalization, we start with a unital associative algebra $\mathcal{A}$ and a linear functional ω on $\mathcal{A}$. Then, we can introduce a (not necessarily symmetric) scalar product on $\mathcal{A}$ by the formula $\langle x, y\rangle = \omega(xy)$. We are saying that $a \in \mathcal{A}$ is a right null vector if $\langle x, a\rangle = 0$ for every $x \in \mathcal{A}$. It is easy to check that right null vectors constitute a left ideal in $\mathcal{A}$. Factorizing $\mathcal{A}$ with respect to this ideal, we obtain a right $\mathcal{A}$-module denoted by R. Similarly factorizing with respect to left null vectors, we obtain a left $\mathcal{A}$-module denoted by L. It is easy to define a pairing between L and R; this paring is non-degenerate. If the algebra $\mathcal{A}$ is equipped with involution, we can consider an induced involution on the space of linear functionals; we assume that the functional ω is self-adjoint. Then, R is a complex conjugate to L, and the pairing between L and R can be interpreted as (in general indefinite) a scalar product in L. If ω is a positive functional, we come back to the GNS construction.

Let us now suppose $\mathcal{A}$ is a differential algebra with differential Q. This differential specifies a differential on the space $\mathcal{A}^\vee$ of linear functionals denoted by the same symbol. *We assume that $Q\omega = 0$ (the functional ω is Q-closed).* This assumption implies that the ideals we constructed are Q-invariant; hence, the differential Q descends to the $\mathcal{A}$-modules R and L. The pairing between differential modules R and L respects the differential Q.

We will work with differential algebra $\mathcal{A}$ equipped with involution * that agrees with the differential. We assume that time translations and spatial translations act as automorphisms of $\mathcal{A}$ and commute with the differential Q. We fix a translation-invariant stationary self-adjoint Q-closed linear functional ω that descends to a positive functional on homology of $\mathcal{A}$. Applying the modification of GNS construction to ω, we obtain a differential pseudo pre-Hilbert space $\mathcal{H}$ with the differential (BRST operator) denoted $\hat{Q}$.

We modify the definition of elementary space saying that a differential vector space $\tilde{\mathfrak{h}}$ is an elementary space in the new sense if its homology can be identified with elementary space in the old sense. An elementary excitation of ω is defined as a linear map of $\tilde{\mathfrak{h}}$ in $\mathcal{H}$ commuting with space-time translations and differentials (BRST operators).

We can repeat with minor modifications the construction of Møller matrices and scattering matrices in a new situation. In particular, a scattering matrix $\tilde{S}$ can be defined as an operator in Fock space $\tilde{F}$ corresponding to the space $\tilde{\mathfrak{h}}$. The operator $\tilde{S}$ commutes with the BRST operator; hence, it descends to homology giving the scattering matrix S of physical (quasi)particles. (The operator S acts in the Fock space $\tilde{F}$ corresponding to the space $\mathfrak{h}$).

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Notes

1 The spatial and time translations are naturally defined for a theory in $(d+1)$-dimensional flat space-time. Notice that we do not assume that we consider such a a theory. For example we can consider a theory on $(d+2)$-dimensional anti de Sitter space; the symmetry group of this space has a $(d+1)$-dimensional commutative subgroup that can be interpreted as a group of time and spatial translations. In string field theory we can interpret time and spatial translations as transformations of the target.

2 It is possible that the "elementary space" $\mathfrak{h}$ does not describe all particles existing in the theory (for example, we are missing some composite particles). In this case we have a chance to get a theory with particle interpretation extending the space $\mathfrak{h}$.

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