universe MDPI

Article

Lévy α-Stable Model for the Non-Exponential Low-$|t|$ Proton–Proton Differential Cross-Section

Tamás Csörgő [1,2], Sándor Hegyi [2] and István Szanyi [1,2,3,*]

1 MATE Institute of Technology, Károly Róbert Campus, Mátrai út 36, H-3200 Gyöngyös, Hungary; tcsorgo@cern.ch
2 Wigner Research Center for Physics, P.O. Box 49, H-1525 Budapest, Hungary; hegyi.physics@gmail.com
3 Department of Atomic Physics, Eötvös University, Pázmány P. s. 1/A, H-1117 Budapest, Hungary
* Correspondence: iszanyi@cern.ch

Abstract: It is known that the Real Extended Bialas–Bzdak (ReBB) model describes the proton–proton (pp) and proton–antiproton ($p\bar{p}$) differential cross-section data in a statistically non-excludible way, i.e., with a confidence level greater than or equal to 0.1% in the center of mass energy range 546 GeV $\leq \sqrt{s} \leq$ 8 TeV and in the squared four-momentum transfer range 0.37 GeV$^2 \leq -t \leq$ 1.2 GeV^2. Considering, instead of Gaussian, a more general Lévy α-stable shape for the parton distributions of the constituent quark and diquark inside the proton and for the relative separation between them, a generalized description of data is obtained, where the ReBB model corresponds to the $\alpha = 2$ special case. Extending the model to $\alpha < 2$, we conjecture that the validity of the model can be extended to a wider kinematic range, in particular, to lower values of the four-momentum transfer $-t$. We present the formal Lévy α-stable generalization of the Bialas–Bzdak model and show that a simplified version of this model can be successfully fitted, with $\alpha < 2$, to the non-exponential, low $-t$ differential cross-section data of elastic proton–proton scattering at $\sqrt{s} = 8$ TeV.

Keywords: elastic scattering; proton-proton; scattering amplitude

Citation: Csörgő, T.; Hegyi, S; Szanyi, I. Lévy α-Stable Model for the Non-Exponential Low-$|t|$ Proton–Proton Differential Cross-Section. *Universe* **2023**, *9*, 361. https://doi.org/10.3390/universe9080361

Academic Editor: Jun Xu

Received: 31 May 2023
Revised: 13 July 2023
Accepted: 25 July 2023
Published: 3 August 2023

1. Introduction

The Bialas–Bzdak (BB) model considers the proton as a bound state of a quark and a diquark, $p = (q, d)$ for short [1]. The diquark in the proton may also be considered to be a weakly bound state of two constituent quarks, leading to the $p = (q,(q,q))$ variant of the BB model; however, in Ref. [2], it was shown that the $p = (q,(q,q))$ variant of the BB model gives two many diffractive minima, whereas, experimentally, only a single minimum is observed in the differential cross-section of proton–proton (pp) collisions. Thus, in recent studies, in Refs. [3,4], the $p = (q, d)$ version of the model was utilized.

Originally, the BB model considers Gaussian shapes for the parton distributions of constituent quarks and diquarks inside the proton and for the relative separation between them. By these considerations based on R. J. Glauber's multiple scattering theory [5,6], the inelastic scattering cross-section of protons at a fixed $\sqrt{s}$ energy and a fixed b impact parameter value is constructed and denoted as $\tilde{\sigma}_{in}(s,\vec{b})$.

The elastic scattering amplitude in the impact parameter representation is written in terms of $\tilde{\sigma}_{in}(s,\vec{b})$ as a solution of the unitarity equation. The imaginary part of the elastic scattering amplitude is the dominant part, whereas the real part can be considered as a smaller correction. Bialas and Bzdak in Ref. [1] neglected the real part of the amplitude and used a fully imaginary amplitude,

$$\tilde{t}_{el}(s,\vec{b}) = i\left(1 - \sqrt{1 - \tilde{\sigma}_{in}(s,\vec{b})}\right), \tag{1}$$

for the calculations of the scattering cross sections. However, in a model where the amplitude does not have a real part, the characteristic minimum–maximum region of the pp

differential cross-section can not be described properly. In Ref. [7], the elastic scattering amplitude was extended with a real part in a way that the unitarity constraint is fulfilled. This amplitude reads as:

$$t_{el}(s,\vec{b}) = i\left(1 - e^{i\,\alpha\,\tilde{\sigma}_{in}(s,\vec{b})}\sqrt{1-\tilde{\sigma}_{in}(s,\vec{b})}\right), \tag{2}$$

where α is a free parameter to be fitted to the data. In the case of $\alpha = 0$, Equation (2) reduces to Equation (1), i.e., to a scattering amplitude that has a vanishing real part.

The model for the elastic proton–proton scattering amplitude, as defined by Equation (2), with $\tilde{\sigma}_{in}(s,\vec{b})$, as defined in Ref. [1], is called the Real Extended Bialas–Bzdak (ReBB) model. In recent studies [3,4], it was shown that the ReBB model describes pp and proton–antiproton ($p\bar{p}$) differential cross-section data in the center of mass energy range of 0.546 TeV $\leq \sqrt{s} \leq$ 8 TeV and in the squared four-momentum transfer range of 0.37 GeV$^2 \leq -t \leq$ 1.2 GeV^2 in a statistically non-excludible manner, i.e., with a confidence level greater than or equal to 0.1%.

The free parameters of the ReBB model are the Gaussian radii of the quark, the diquark, and the separation between them (correspondingly, R_q, R_d, and R_{qd}) and also, the α parameter regulating the real part of the scattering amplitude. Two additional fit parameters could be present: λ, the ratio of the quark and diquark masses, and A_{qq}, the normalization parameter appearing in the inelastic quark–quark cross-section. However, it was shown in Ref. [2] and later confirmed in Ref. [3] that A_{qq} can be fixed at a value of 1.0, whereas λ can be fixed at a value of $1/2$.

The energy dependence of the ReBB model parameters for pp and $p\bar{p}$ scattering were determined in Ref. [3]. It was found that the energy dependencies of the radius parameters are the same for pp and $p\bar{p}$ scattering, whereas the energy dependencies of the α parameter for pp and $p\bar{p}$ scattering are different, i.e., there are different α^{pp} and $\alpha^{p\bar{p}}$ parameters. The energy dependencies of all the five parameters in the energy range of $0.546 \leq \sqrt{s} \leq 8$ TeV are determined by linear logarithmic functions [3,4].

Considering, instead of Gaussian, a more general Lévy α-stable shape for the parton distributions of the constituent quark and diquark inside the proton and for the relative separation between them, an improved description to the data in a wider kinematic range ($\sqrt{s} < 0.546$ TeV, $\sqrt{s} > 8$ TeV, $-t < 0.37$ GeV^2, $-t > 1.2$ GeV^2) is anticipated.

The 0.37 GeV$^2 \leq -t \leq$ 1.2 GeV^2 interval at LHC energies includes the region of the characteristic minimum–maximum structure of the pp elastic differential cross-section. In the 0.01 GeV$^2 \lesssim -t \lesssim$ 0.15 GeV^2 interval, another characteristic structure, a non-exponential behavior is observed. A significant non-exponential behavior was measured by TOTEM at CERN LHC at 8 and 13 TeV center of mass energies [8,9]. Similar behavior was observed also at the CERN ISR accelerator in the 1970s [10], where measurements were made in the 20 GeV $\lesssim \sqrt{s} \lesssim$ 60 GeV energy region.

In Ref. [11], the model-independent Lévy imaging method is successfully employed to describe the pp and $p\bar{p}$ differential cross-section data both at the low and the high $-t$ region simultaneously. In Ref. [12], the model-independent Lévy imaging method was employed to reconstruct the proton inelasticity profile function. This method established a statistically significant proton hollowness effect [13–17], well beyond the 5σ discovery limit at $\sqrt{s} = 13$ TeV. These results suggest that Lévy α-stable models are efficient tools in describing pp and $p\bar{p}$ differential cross-section data, and the ReBB model needs to be Lévy α-stable generalized to have a stronger non-exponential feature at low $-t$ and to accommodate the new features of the differential cross-section data such the hollowness effect at $\sqrt{s} = 13$ TeV or larger energies. In the present work, we complete the formal Lévy α-stable generalization of the Bialas–Bzdak model.

This paper is organized as follows. In Section 2, we deduce the formal Lévy α-stable generalization of the Bialas–Bzdak model and discuss the technical difficulties preventing us to perform an efficient fitting procedure of the model parameters to the experimental data with the full Lévy α-stable generalized Bialas-Bzdak model. In Section 3, we show successful fits to the low $-t$ differential cross-section data at LHC energies with a simple Lévy α-stable model deduced by approximations from the Lévy α-stable generalized Bialas–Bzdak (LBB) model. In Section 4, the parameters of the LBB model is related to the $t = 0$ measurable quantities and to the parameters of the simple Lévy α-stable model. Finally, we summarize and conclude in Section 5.

2. From Gaussian to Lévy α-Stable $p = (q, d)$ BB Model

First, we recapitulate the BB model using normalized Gaussian distributions and introduce some reinterpretations of some of its parts. Then, we change the normalized Gaussian distributions to normalized Lévy α-stable distributions, resulting in the Lévy α-stable generalized BB model.

The inelastic scattering cross-section at a fixed $\vec{b}$ impact parameter value is given as [1]:

$$\tilde{\sigma}_{in}(\vec{b}) = \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} d^2\vec{s}_q d^2\vec{s}_q^{\,\prime} d^2\vec{s}_d d^2\vec{s}_d^{\,\prime} D(\vec{s}_q, \vec{s}_d) D(\vec{s}_q^{\,\prime}, \vec{s}_d^{\,\prime}) \sigma(\vec{s}_q, \vec{s}_d; \vec{s}_q^{\,\prime}, \vec{s}_d^{\,\prime}; \vec{b}), \tag{3}$$

where $D(\vec{s}_q^{\,\prime}, \vec{s}_d^{\,\prime})$ is the quark–diquark distribution inside one of the colliding protons, $\sigma(\vec{s}_q, \vec{s}_d; \vec{s}_q^{\,\prime}, \vec{s}_d^{\,\prime}; \vec{b})$ is the probability of inelastic collision, and the variables we integrate over are the transverse positions of the quarks and diquarks inside the two colliding protons. Note that the energy dependence of $\tilde{\sigma}_{in}(\vec{b})$ is not written out here for clarity reasons; however, through the $\sqrt{s}$ dependence of the model parameters, $R_q(s)$, $R_d(s)$, and $R_{qd}(s)$, $\tilde{\sigma}_{in}(\vec{b})$ has an $\sqrt{s}$ dependence too.

The quark–diquark distribution is considered to be Gaussian:

$$D(\vec{s}_q, \vec{s}_d) = \frac{1 + \lambda^2}{R_{qd}^2 \pi} e^{-(s_q^2 + s_d^2)/R_{qd}^2} \delta^2(\vec{s}_d + \lambda \vec{s}_q), \tag{4}$$

where $\lambda = m_q / m_d$, the ratio of the quark and diquark masses, and R_{qd} are free parameters of the model. The two-dimensional Dirac δ function fixes the center-of-mass of the proton and reduces the dimension of the integral in Equation (3) from 8 to 4. Accordingly, the diquark positions can be expressed by that of the quarks:

$$\vec{s}_d = -\lambda\, \vec{s}_q, \;\; \vec{s}_d^{\,\prime} = -\lambda\, \vec{s}_q^{\,\prime}. \tag{5}$$

After integration over $\vec{s}_d$, $D(\vec{s}_q, \vec{s}_d)$ becomes a Gaussian in $\vec{s}_q$; then, after the integration, also over $\vec{s}_q$, we obtain unity:

$$\int d^2\vec{s}_d D(\vec{s}_q, \vec{s}_d) = G\left(\vec{s}_q | R_{qd} / \sqrt{2(1 + \lambda^2)}\right), \tag{6}$$

$$\int d^2\vec{s}_q d^2\vec{s}_d D(\vec{s}_q, \vec{s}_d) = 1, \tag{7}$$

where:

$$G(\vec{x}|R_G) = \frac{1}{(2\pi)^2} \int d^2q e^{i\vec{q}^T\vec{x}} e^{-\frac{1}{2}q^2 R_G^2} = \frac{1}{2\pi R_G^2} e^{-\frac{x^2}{2R_G^2}} \tag{8}$$

is the normalized bivariate Gaussian distribution.

We may reinterpret $D(\vec{s}_q, \vec{s}_d)$ as the distribution of the relative separation between the quark and the diquark in a single proton, namely:

$$D(\vec{s}_q, \vec{s}_d) = (1 + \lambda)^2 G\left(\vec{s}_q - \vec{s}_d | R_{qd} / \sqrt{2}\right) \delta^2(\vec{s}_d + \lambda \vec{s}_q) \tag{9}$$

which is correctly normalized as follows:

$$\int d^2\vec{s}_d D(\vec{s}_q, \vec{s}_d) = G\Big(\vec{s}_q | R_{qd*}/\sqrt{2}\Big), \tag{10}$$

$$\int d^2\vec{s}_q d^2\vec{s}_d D(\vec{s}_q, \vec{s}_d) = 1, \tag{11}$$

where:

$$R_{qd*} = \frac{R_{qd}}{1+\lambda}. \tag{12}$$

Here, we have rescaled the parameter R_{qd} of the original Bialas–Bzdak model to the parameter that characterizes the uncertainty of the location of a dressed quark inside the proton. The advantage of this interpretation is that we prepare the ground for the generalization to the case of Levy α-stable distributions and instead of taking the product of two Gaussians, as in Equation (4), we had an equivalent rewrite where the relative coordinate distribution of a quark and a diquark is Gaussian, with rescaled parameters. This rewrite is very advantageous, as the product of two Levy distributions is not a Levy distribution, with the exception of the $\alpha_L = 2$ Gaussian case. As such, to have only one Gaussian in the relative coordinate avoids the problem of having products of Levy α-stable distributions in the formulas.

The term $\sigma(\vec{s}_q, \vec{s}_d; \vec{s}_q', \vec{s}_d'; \vec{b})$ is the probability of inelastic interactions at a fixed impact parameter and transverse positions of all constituents and given by a Glauber expansion as follows:

$$\sigma(\vec{s}_q, \vec{s}_d; \vec{s}_q', \vec{s}_d'; \vec{b}) = 1 - \Big[1 - \sigma_{qq}(\vec{s}_q, \vec{s}_q'; \vec{b})\Big]\Big[1 - \sigma_{qd}(\vec{s}_q, \vec{s}_d'; \vec{b})\Big] \times \\ \times \Big[1 - \sigma_{dq}(\vec{s}_q', \vec{s}_d; \vec{b})\Big]\Big[1 - \sigma_{dd}(\vec{s}_d, \vec{s}_d'; \vec{b})\Big], \tag{13}$$

where:

$$\sigma_{qq}(\vec{s}_q, \vec{s}_q'; \vec{b}) \equiv \sigma_{qq}(\vec{b} + \vec{s}_q{}' - \vec{s}_q),$$

$$\sigma_{qd}(\vec{s}_q, \vec{s}_d'; \vec{b}) \equiv \sigma_{qd}(\vec{b} + \vec{s}_d{}' - \vec{s}_q),$$

$$\sigma_{dq}(\vec{s}_d, \vec{s}_q'; \vec{b}) \equiv \sigma_{dq}(\vec{b} + \vec{s}_q{}' - \vec{s}_d),$$

and:

$$\sigma_{dd}(\vec{s}_d, \vec{s}_d'; \vec{b}) \equiv \sigma_{dd}(\vec{b} + \vec{s}_d{}' - \vec{s}_d)$$

are the inelastic differential cross-sections of the binary collisions of the constituents. They have Gaussian shapes:

$$\sigma_{ab}(\vec{x}) = A_{ab} e^{-\vec{s}^2/S_{ab}^2} \tag{14}$$

with $S_{ab}^2 = R_a^2 + R_b^2$ and $a, b \in \{q, d\}$. Equation (14) can be rewritten in terms of normalized bivariate Gaussian distribution:

$$\sigma_{ab}(\vec{s}) = A_{ab}\pi S_{ab}^2 G\Big(\vec{s}|S_{ab}/\sqrt{2}\Big). \tag{15}$$

We can reinterpret the inelastic constituent–constituent collisions by assuming that the constituent quark and the constituent diquark have Gaussian parton distributions, characterized by $G(\vec{s}_q|R_q/\sqrt{2})$ and $G(\vec{s}_d|R_d/\sqrt{2})$. Then, the probability of inelastic collisions at a given impact parameter b is proportional to their convolution:

$$\sigma_{ab}(\vec{s}) = A_{ab}\pi S_{ab}^2 \int d^2 s_a G(\vec{s}_a|R_a/\sqrt{2}) G(\vec{s} - \vec{s}_a|R_b/\sqrt{2}) \\ \equiv A_{ab}\pi S_{ab}^2 G\Big(\vec{s}|S_{ab}/\sqrt{2}\Big). \tag{16}$$

The inelastic quark–quark, quark–diquark, and diquark–diquark cross-sections are obtained by integration:

$$\sigma_{ab,\text{inel}} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \sigma_{ab}(\vec{s})\,\mathrm{d}^2 s = A_{ab}\pi S_{ab}^2\,. \tag{17}$$

The number of the free parameters of the model can be reduced by demanding that the ratios of the cross-sections are:

$$\sigma_{qq,\text{inel}} : \sigma_{qd,\text{inel}} : \sigma_{dd,\text{inel}} = 1:2:4\,, \tag{18}$$

expressing the idea that the constituent diquark contains twice as many partons than the constituent quark and also that the colliding constituents do not "shadow" each other.

Then, the probabilities of inelastic constituent–constituent collisions can be written in the following form:

$$\sigma_{qq}(\vec{s}_q,\vec{s}_q^{\,\prime};\vec{b}) = 2\pi A_{qq}R_q^2 G(\vec{b}+\vec{s}_q^{\,\prime}-\vec{s}_q|R_q), \tag{19}$$

$$\sigma_{qd}(\vec{s}_q,\vec{s}_d^{\,\prime};\vec{b}) = 4\pi A_{qq}R_q^2 G\left(\vec{b}+\vec{s}_d^{\,\prime}-\vec{s}_q\middle|\sqrt{\frac{R_q^2+R_d^2}{2}}\right), \tag{20}$$

$$\sigma_{dq}(\vec{s}_q^{\,\prime},\vec{s}_d;\vec{b}) = 4\pi A_{qq}R_q^2 G\left(\vec{b}+\vec{s}_q^{\,\prime}-\vec{s}_d\middle|\sqrt{\frac{R_q^2+R_d^2}{2}}\right), \tag{21}$$

$$\sigma_{dd}(\vec{s}_d,\vec{s}_d^{\,\prime};\vec{b}) = 8\pi A_{qq}R_q^2 G(\vec{b}+\vec{s}_d^{\,\prime}-\vec{s}_d|R_d). \tag{22}$$

Substituting these into Equation (13), then substituting $\sigma(\vec{s}_q,\vec{s}_d;\vec{s}_q^{\,\prime},\vec{s}_d^{\,\prime};\vec{b})$ into Equation (3), we get a sum of eleven integral terms (with proper sign) for $\tilde{\sigma}_{in}(\vec{b})$:

$$\begin{aligned}\tilde{\sigma}_{in}(\vec{b}) = {}& \tilde{\sigma}_{in}^{qq}(\vec{b}) + 2\tilde{\sigma}_{in}^{qd}(\vec{b}) + \tilde{\sigma}_{in}^{dd}(\vec{b}) - [2\tilde{\sigma}_{in}^{qq,qd}(\vec{b}) + \tilde{\sigma}_{in}^{qd,dq}(\vec{b}) + \tilde{\sigma}_{in}^{qq,dd}(\vec{b}) + \\ &+ 2\tilde{\sigma}_{in}^{qd,dd}(\vec{b})] + [\tilde{\sigma}_{in}^{qq,qd,dq}(\vec{b}) + 2\tilde{\sigma}_{in}^{qq,qd,dd}(\vec{b}) + \tilde{\sigma}_{in}^{dd,qd,dq}(\vec{b})] - \tilde{\sigma}_{in}^{qq,qd,dq,dd}(\vec{b}).\end{aligned} \tag{23}$$

Let us have a look for the most general fourth-order term, $\tilde{\sigma}_{in}^{qq,qd,dq,dd}(\vec{b})$. After making use of the presence of the Dirac δ function in Equation (9), we have to calculate a four-dimensional integral of products of normalized bivariate Gaussian distributions:

$$\begin{aligned}\tilde{\sigma}_{in}^{qq,qd,dq,dd}(\vec{b}) = {}& \int d^2 s_q d^2 s_q^{\prime} G(\vec{s}_q|R_{qd*}/\sqrt{2}) G(\vec{s}_q^{\,\prime}|R_{qd*}/\sqrt{2}) \times \\ &\times \sigma_{qq}(\vec{s}_q,\vec{s}_q^{\,\prime};\vec{b})\sigma_{qd}(\vec{s}_q,-\lambda\vec{s}_q^{\,\prime};\vec{b})\sigma_{dq}(\vec{s}_q^{\,\prime},-\lambda\vec{s}_q;\vec{b})\sigma_{dd}(-\lambda\vec{s}_q,-\lambda\vec{s}_q^{\,\prime};\vec{b}).\end{aligned} \tag{24}$$

Such an integral results in an expression having a Gaussian shape. The lower-order terms can be obtained from Equation (24) by excluding the proper σ_{ab} term/terms from the integrand. Thus, after computing the integrals in all order, we get the sum of eleven different Gaussian-shaped terms, i.e., the BB model as introduced in Ref. [1].

Now, we perform the Lévy α-stable generalization of the BB model.

Let us introduce the normalized bivariate symmetric Lévy α-stable distribution,

$$L(\vec{x}|\alpha_L,R_L) = \frac{1}{(2\pi)^2}\int d^2 q e^{i\vec{q}^{\,T}\vec{x}} e^{-|q^2 R_L^2|^{\alpha_L/2}}, \tag{25}$$

which, for $\alpha_L = 2$, gives exactly the bivariate Gaussian distribution:

$$L(\vec{x}|\alpha_L = 2, R_L = R_G/\sqrt{2}) \equiv G(\vec{x}|R_G). \tag{26}$$

Note that the Lévy index of stability α_L, that controls the power-law tails of the inelastic cross-sections, is a different parameter from the α parameter of the ReBB model, that controls the opacity or the real part of the scattering amplitude. Due to historic reasons, both were denoted by α originally, but in this work, we add a subscripted L to distinguish the Lévy parameter α_L from the opacity parameter α.

Since we work with here with symmetric Lévy α-stable distribution, the skewness parameter $\beta_L = 0$ of the Lévy stable source distributions is implicit and are assumed to have zero values. The shift parameter δ_L of the Lévy stable source distribution is explicitely written out when considering the impact parameter picture, while the overall shift of the impact parameter cancels from the final results hence it is assumed to have a vanishing value.

We then consider that the relative separation between the quark and the diquark in a single proton follows Lévy α-stable distribution:

$$D(\vec{s}_q, \vec{s}_d) = (1+\lambda)^2 L\Big(\vec{s}_q - \vec{s}_d|\alpha_L, R_L = R_{qd}/2\Big)\delta^2(\vec{s}_d + \lambda\vec{s}_q) \tag{27}$$

with:

$$\int d^2\vec{s}_d D(\vec{s}_q, \vec{s}_d) = L\Big(\vec{s}_q|\alpha_L, R_{qd*}/2\Big), \tag{28}$$

$$\int d^2\vec{s}_q d^2\vec{s}_d D(\vec{s}_q, \vec{s}_d) = 1, \tag{29}$$

similarly to the original case with Gaussian distributions.

As the next step in the generalization, we consider, instead of Gaussian, Lévy α-stable parton distributions for the constituent quark and the constituent diquark: $L(\vec{s}_q|\alpha_L, R_q/2)$ and $L(\vec{s}_d|\alpha_L, R_d/2)$. Then, as in the Gaussian case above, the probability of inelastic collisions at a given impact parameter b is proportional to their convolution:

$$\begin{aligned}\sigma_{ab}(\vec{s}) &= A_{ab}\pi S_{ab}^2 \int d^2 s_a L(\vec{s}_a|\alpha_L, R_a/2) L(\vec{s} - \vec{s}_a|\alpha_L, R_b/2) \\ &= A_{ab}\pi S_{ab}^2 L(\vec{s}|\alpha_L, S_{ab}/2),\end{aligned} \tag{30}$$

where now:

$$S_{ab} = \left(R_a^{\alpha_L} + R_b^{\alpha_L}\right)^{1/\alpha_L}, \tag{31}$$

i.e., in this case, after making use of the convolution theorem, the radii add up not quadratically, but at the power of α_L.

Then:

$$\sigma_{qq}(\vec{s}_q, \vec{s}_q'; \vec{b}) = \pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L} L\Big(\vec{b} + \vec{s}_q' - \vec{s}_q|\alpha_L, \left(2R_q^{\alpha_L}\right)^{1/\alpha_L}/2\Big), \tag{32}$$

$$\sigma_{qd}(\vec{s}_q, \vec{s}_d'; \vec{b}) = 2\pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L} L\left(\vec{b} + \vec{s}_d' - \vec{s}_q\middle|\alpha_L, \left(R_q^{\alpha_L} + R_d^{\alpha_L}\right)^{1/\alpha_L}/2\right), \tag{33}$$

$$\sigma_{dq}(\vec{s}_q', \vec{s}_d; \vec{b}) = 2\pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L} L\left(\vec{b} + \vec{s}_q' - \vec{s}_d\middle|\alpha_L, \left(R_q^{\alpha_L} + R_d^{\alpha_L}\right)^{1/\alpha_L}/2\right), \tag{34}$$

and

$$\sigma_{dd}(\vec{s}_d, \vec{s}_d'; \vec{b}) = 4\pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L} L\Big(\vec{b} + \vec{s}_d' - \vec{s}_d)|\alpha_L, \left(2R_d^{\alpha_L}\right)^{1/\alpha_L}/2\Big). \tag{35}$$

Equation (3) with Equation (13), Equation (27), and Equations (32)–(35) define the Lévy α_L-stable generalized Bialas–Bzdak (LBB) model for $\tilde{\sigma}_{in}(b)$. Now, in Equation (23), instead of a sum of integrals of products of normalized Gaussian distributions, there are a sum of integrals of products of normalized Lévy α_L-stable distributions. Though integrals of products of Gaussian distributions can be calculated, the calculation of integrals of products of Lévy α_L-stable distributions is an issue. Integrals of products of Lévy α_L-stable distributions can be easily calculated if the integral can be written in a convolution form. This is the case for the first three terms in Equation (23). The results can be written in terms of Lévy α_L-stable distributions:

$$\begin{aligned}\tilde{\sigma}^{qq}_{in}(\vec{b}) &= \pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L}\times \\ &\times\int d^2s_q d^2s_q' L(\vec{s}_q|\alpha_L, R_{qd*}/2)L(\vec{s}_q\,'|R_{qd*}/2)L\left(\vec{b}+\vec{s}_q\,'-\vec{s}_q|\left(2R_q^{\alpha_L}\right)^{1/\alpha_L}/2\right) \\ &= \pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L}L\left(\vec{b}\middle|\alpha_L, \left(2R_{qd*}^{\alpha_L}+2R_q^{\alpha_L}\right)^{1/\alpha_L}/2\right),\end{aligned} \tag{36}$$

$$\begin{aligned}\tilde{\sigma}^{qd}_{in}(\vec{b}) &= 2\pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L}\times \\ &\times\int d^2s_q d^2s_q' L(\vec{s}_q|R_{qd*}/2)L(\vec{s}_q\,'|R_{qd*}/2)L\left(\vec{b}-\lambda\vec{s}_q\,'-\vec{s}_q\middle|\alpha_L, \left(R_q^{\alpha_L}+R_d^{\alpha_L}\right)^{1/\alpha_L}/2\right) \\ &= 2\pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L}L\left(\vec{b}\middle|\alpha_L, \left((1+\lambda^{\alpha_L})R_{qd*}^{\alpha_L}+R_q^{\alpha_L}+R_d^{\alpha_L}\right)^{1/\alpha_L}/2\right),\end{aligned} \tag{37}$$

$$\begin{aligned}\tilde{\sigma}^{dd}_{in}(\vec{b}) &= 4\pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L}\times \\ &\times\int d^2s_q d^2s_q' L(\vec{s}_q|R_{qd*}/2)L(\vec{s}_q\,'|R_{qd*}/2)L\left(\vec{b}+\lambda(\vec{s}_q-\vec{s}_q\,')|\alpha_L, (2R_d^{\alpha_L})^{1/\alpha_L}/2\right) \\ &= 4\pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L}L\left(\vec{b}\middle|\alpha_L, \left(2\lambda^{\alpha_L}R_{qd*}^{\alpha_L}+2R_d^{\alpha_L}\right)^{1/\alpha_L}/2\right).\end{aligned} \tag{38}$$

The results of the remaining eight integrals, corresponding to higher-order terms in the BB model, are yet to be determined in terms of analytic formulas.

Whereas univariate and multivariate Gaussian distributions have closed forms in terms of elementary functions, univariate and multivariate Lévy α_L-stable distributions have forms in terms of special functions. This makes it hard to perform a numerical fitting procedure of the model parameters to the experimental data. To complete this work in the future, a relatively high computing capacity or improved analytic insight will be needed. In this work, we have chosen another approach, limiting the domain of the applicability of the calculations in the squared four-momentum transfer $-t$. This allows for certain simplifications and results in an increased analytic insight to certain properties of the LBB model.

A possible alternative to the Lévy α-stable generalization of the BB model could be its Tsallis or q-exponential generalization, since data from high-energy collisions have shown such distribution. The presence of the Tsallis distribution was explained in Ref. [18] using the fractal approach to the non-perturbative QCD, and also, the q index was expressed in terms of the number of colors and the number of flavors. The validity of the derived relation was reinforced later in Ref. [19]. These results suggest that the investigation of the Tsallis generalization of the BB model is worthwhile. This will be done in a future study. In this manuscript, we investigate the Lévy α-stable generalization of the BB model.

3. A Simple Lévy α-Stable Model

Now, we check if the Lévy α-stable generalization of the BB model has an enhanced potential, as compared to the ReBB model, or not. The mathematical and computing difficulties discussed in the previous section can be bypassed by introducing new approximations that are valid at low $-t$, in the domain where the original ReBB model had difficulties to describe the strongly non-exponential features of the experimental data of elastic proton-proton scattering at the TeV energy scale. Our aim is, thus, to deduce a model for the differential cross-section which is valid at the low-$|t|$ region.

Since the elastic scattering amplitude is predominantly imaginary in this kinematic region, we approximate it by an imaginary part, as given by Equation (1). Low-$|t|$ scattering corresponds to high-b scattering and, at high b values $\tilde{\sigma}_{in}(s,b)$, is small. Thus, the leading order term in the Taylor expansion of Equation (1), i.e.,

$$\tilde{t}_{el}(s,\vec{b}) = \frac{i}{2}\tilde{\sigma}_{in}(s,\vec{b}), \tag{39}$$

should be a reasonable approximation at low $-t$ values in the $\alpha = 0$ (vanishing real part) case.

As discussed in Section 3, low-$|t|$ scattering corresponds to high-b scattering and, at high b values $\tilde{\sigma}_{in}(s,b)$, is small. Thus, the leading order term in the Taylor expansion of Equation (2), i.e.,

$$\tilde{t}_{el}(s,\vec{b}) = \left(\alpha + \frac{i}{2}\right)\tilde{\sigma}_{in}(s,\vec{b}), \tag{40}$$

should be a reasonable approximation at low $-t$ values if the opacity parameter α is small.

In Section 2, we discussed that in the Lévy α-stable generalized case of the BB model, the leading order terms in $\tilde{\sigma}_{in}(s,b)$ are Lévy-α-stable-shaped terms. Motivated by this fact in our simplified model, we approximate $\tilde{\sigma}_{in}(s,b)$ with a single Lévy-αstable-shaped term, i.e.,

$$\tilde{\sigma}_{in}(s,\vec{b}) = \tilde{c}(s)L(\vec{b}|\alpha_L(s),r(s)) \tag{41}$$

where $\tilde{c}(s)$ is in general a complex-valued and s dependent function, while $\alpha_L(s)$, and $r(s)$ are adjustable parameters determined at a given $\sqrt{s}$ energy,

Then, by Equation (39), we have:

$$\tilde{t}_{el}(s,\vec{b}) = ic(s)L(\vec{b}|\alpha_L(s),r(s)), \tag{42}$$

where $c(s) = \tilde{c}(s)/2$ is a rescaled and complex valued parameter. Now, we transform the impact parameter amplitude into momentum space:

$$t(s,t) = \int d^2b e^{i\vec{\Delta}^T\vec{b}}\tilde{t}(s,\vec{b}) = ic(s)e^{-|tr^2(s)|^{\alpha_L(s)/2}}, \tag{43}$$

where $|\vec{\Delta}| \simeq \sqrt{-t}$. The resulting differential cross-section is:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}(s,t) = \frac{1}{4\pi}|t(s,t)|^2 = a(s)e^{-|tb(s)|^{\alpha_L(s)/2}}, \tag{44}$$

where $a(s) = \frac{|c(s)|^2}{4\pi}$ and $b(s) = 2^{2/\alpha_L(s)}r^2(s)$. Thus, finally, this simple model for the differential cross-sections has three adjustable parameters, α_L, a, and b, to be determined at a given energy.

The result of a fit to the TOTEM pp elastic differential cross-section data at $\sqrt{s} = 8$ TeV by the model defined by Equation (44) is shown in Figure 1. One can see that the non-exponential model with $\alpha_L = 1.953 \pm 0.004$ represents the low-$|t|$ differential cross-section data with a confidence level of 55%.

Figure 2 shows the ratio, $(d\sigma/dt - ref)/ref$, with $ref = Ae^{-Bt}$, used by the TOTEM collaboration [8] to make the relatively small, but significant, low-$|t|$ non-exponential behavior visible. One can clearly see that our model successfully describes the low-$|t|$ data.

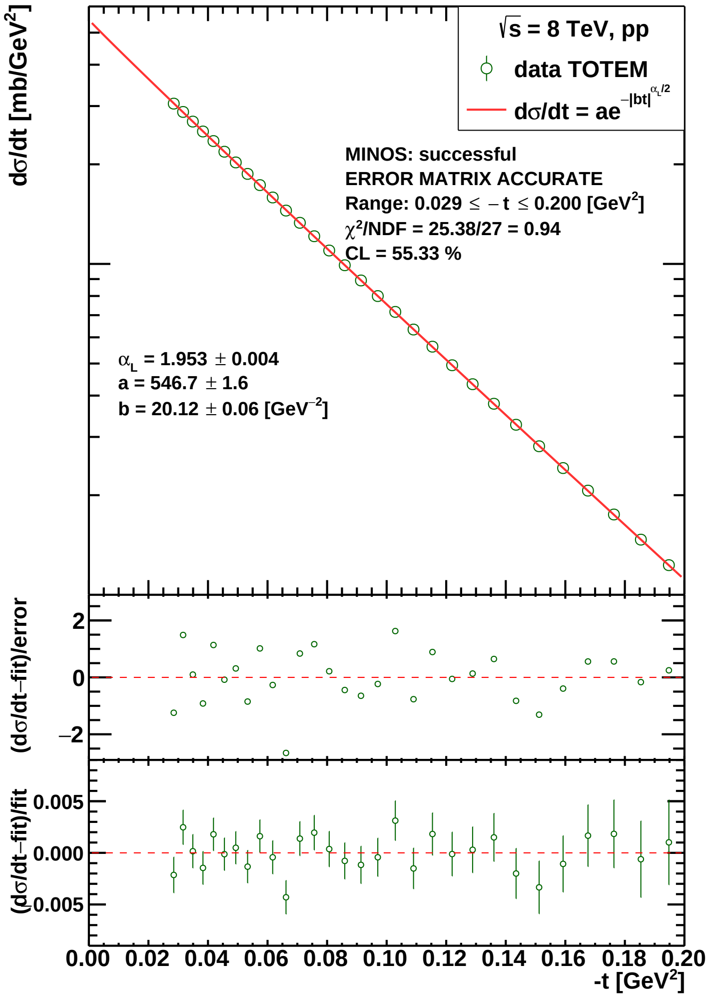

Figure 1. Fit-to-the-TOTEM pp elastic differential cross-section data at $\sqrt{s} = 8$ TeV [8] by the model defined by Equation (44).

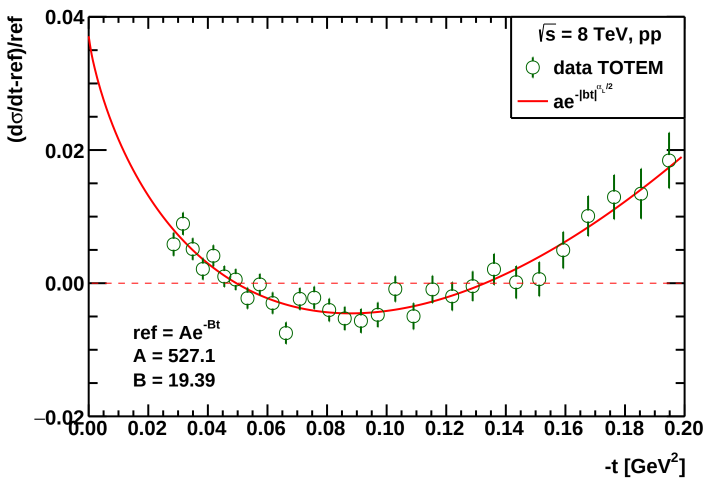

Figure 2. The ratio, $(d\sigma/dt - ref)/ref$, evaluated from the TOTEM pp elastic differential cross-section data at $\sqrt{s} = 8$ TeV [8]. The curve corresponds to the fitted model defined by Equation (44).

4. The $t = 0$ Measurable Quantities and the BB Model Parameters

In this section, we relate the $t = 0$ measurable quantities and the LBB parameters. First, we work with the original BB model with Gaussian distributions and then derive the formulas for the Levy α-stable generalized case. We note again that to avoid confusion with the α parameter of the ReBB model regulating the real part of the amplitude and that of the Lévy α-stable distribution, the latter we have denoted in this manuscript as α_L. For the $\alpha_L = 2$ limiting case, the relations obtained in the original BB model are recovered.

Now, we consider only the leading order terms in $\tilde{\sigma}_{in}(s, b)$, i.e., $\tilde{\sigma}_{in}^{qq}(\vec{b})$, $\tilde{\sigma}_{in}^{qd}(\vec{b})$, and $\tilde{\sigma}_{in}^{dd}(\vec{b})$, which give the dominant contribution at $t = 0$. We get the amplitude in momentum space by Fourier transformation as in Equation (43). As discussed in the Introduction, the parameter A_{qq} can be fixed at a value of 1.0, whereas λ can be fixed at a value of $1/2$. We use these specific values below.

With Gaussian distributions in the BB model, in the low-$|t|$ approximation, σ_{tot} is related to the square of the quark radius R_q,

$$\sigma_{tot} = 2Imt(s, t = 0) = 18\pi R_q^2, \tag{45}$$

whereas the ratio of the real to the imaginary part of the forward scattering amplitude is related to the α parameter of the ReBB model,

$$\rho_0 = \frac{Ret(s, t = 0)}{Imt(s, t = 0)} = 2\alpha. \tag{46}$$

Note that this result for ρ_0 holds also in the Levy α-stable generalized case.

The low-$|t|$ pp differential cross-section is in the form [8]:

$$\frac{d\sigma}{dt} = \frac{1}{4\pi}|t(s, t)|^2 = ae^{-b_1 t + b_2 t^2} \tag{47}$$

where:

$$a = \left.\frac{d\sigma}{dt}\right|_{t=0} \tag{48}$$

is the optical point,

$$b_1 = \left.\left(\frac{d}{dt}\ln\frac{d\sigma}{dt}\right)\right|_{t=0} \tag{49}$$

is the slope parameter, and

$$b_2 = \frac{1}{2}\left.\left(\frac{d}{dt^2}\ln\frac{d\sigma}{dt}\right)\right|_{t=0} \tag{50}$$

is the curvature parameter. These measurable quantities can be expressed in terms of the ReBB model parameters:

$$a = \frac{81}{4}\pi Rq^4\left(1 + 4\alpha^2\right), \tag{51}$$

$$b_1 = \frac{2}{9}R_{qd}^2 + \frac{2}{3}R_d^2 + \frac{1}{3}R_q^2, \tag{52}$$

and

$$b_2 = \frac{1}{324}\left(R_{qd}^2 - 3R_d^2 + 3R_q^2\right)^2. \tag{53}$$

Now, we turn to the LBB model. Using the Levy α-stable generalized forms of the leading order terms in $\tilde{\sigma}_{in}(s, b)$, i.e., Equations (36)–(38), the total cross-section is:

$$\sigma_{tot} = 9\pi\left(2R_q^{\alpha_L}\right)^{2/\alpha_L}. \tag{54}$$

Furthermore, we consider now that the differential cross-section has the form as written in Equation (44). Now, the optical point is:

$$a = \frac{81}{16}\pi(2Rq^{\alpha_L})^{4/\alpha_L}\left(1+4\alpha^2\right), \tag{55}$$

whereas the slope parameter is:

$$b = \frac{1}{36}\left(\frac{4}{3}\right)^{2/\alpha_L}\left((2+2^{\alpha_L})R_{qd}^{\alpha_L} + 3^{\alpha_L}\left(2R_d^{\alpha_L}+R_q^{\alpha_L}\right)\right)^{2/\alpha_L}. \tag{56}$$

One can easily check that for $\alpha_L = 2$, Equation (54) reduces to Equation (45), Equation (55) to Equation (51), and Equation (56) to Equation (52). Since the function in Equation (44) is not an analytic function of t at $t = 0$, Equation (56) was obtained by a Taylor expansion in $t^{\alpha_L/2}$ around zero and by keeping only the leading order term.

As discussed in Section 3, the Levy scale parameter r in our simple Lévy α-stable model is related to the slope parameter. The relation can be rewritten as $r = \sqrt{b}/2^{1/\alpha_L}$. Then, this r parameter can be expressed in terms of the LBB model parameters:

$$r = \frac{1}{6}\left(\frac{2}{3}\right)^{1/\alpha_L}\left((2+2^{\alpha_L})R_{qd}^{\alpha_L} + 3^{\alpha_L}\left(2R_d^{\alpha_L}+R_q^{\alpha_L}\right)\right)^{1/\alpha_L}. \tag{57}$$

Thus, we have shown that the parameters of our simple Lévy α-stable model, namely, a and b (or equivalently, r), can be approximately expressed in terms of those of the LBB model.

In Ref. [20], the three-dimensional radius of the proton is defined and its relation to the slope parameter is derived. In our work, we related the Levy scale parameter r in our simple Lévy α-stable model to the elastic slope parameter and expressed it in terms of the radii of the constituents of the proton (R_q and R_d) and their typical separation (R_{qd}).

Finally, we note that there are five measurable parameters at the forward region: the total cross-section, the ratio of the real to the imaginary part of the forward scattering amplitude, the optical point, the slope parameter, and the curvature parameter. The ReBB model has four free parameters, whereas the LBB model has five. This naturally suggests that the LBB can give a better description to the data than the ReBB model.

5. Summary

The ReBB model turned out to be an efficient tool in describing pp and $p\bar{p}$ differential cross-section data, but in a limited $\sqrt{s}$ and $-t$ range. The validity range of the ReBB model in $\sqrt{s}$ does not include 13 TeV, possibly due to the significant hollowness effect observed at that energy. The validity range of the ReBB model in $-t$ includes the minimum–maximum structure of the differential cross-section, but does not include the significant non-exponential behavior at low $-t$ values. To overcome these shortcomings of the ReBB model, in this paper, we introduce the Lévy α-stable generalized Real Extended Bialas–Bzdak (LBB) model. The fitting of the parameters of the LBB model to the experimental data, however, requires the solution of difficult and complex technical (mathematical and computational) problems. However, in the low four-momentum transfer region, based on our novel approximations and the idea of the Levy-α-stable-shaped inelastic scattering probability suggested by the LBB model, we deduced and fitted a highly simplified Levy α-stable model of the pp differential cross section to the measured data at $\sqrt{s} = 8$ TeV. The results show that our simple model represents the low-$|t|$ experimental data in a statistically acceptable manner. This is a promising prospect for the future utility of the Lévy α-stable generalized Real Extended Bialas–Bzdak (LBB) model.

We have shown also that the parameters of our simple Lévy α-stable model, namely, a and b (or equivalently, r), can be approximately expressed in terms of those of the LBB model, which is based on R. J. Glauber's multiple diffractive scattering theory. We emphasize that there are five measurable parameters at the forward region, whereas the

ReBB model has only four free parameters. Since the LBB model has five free parameters, it is natural to expect that it can give a better description to the data than the ReBB model.

In the next steps of our research, we are planning to extend the fits with our simple model for all the energies where low-$|t|$ experimental data exist, and after solving the technical issues, to fit the full LBB model to all the existing experimental pp and $p\bar{p}$ differential cross-section data.

Author Contributions: Conceptualization, T.C.; methodology, T.C., S.H. and I.S.; investigation, I.S.; writing—original draft preparation, I.S.; writing—review and editing, T.C. and S.H.; supervision, T.C. All authors have read and agreed to the published version of the manuscript.

Funding: NKFIH Grants no. K133046 and 2020-2.2.1-ED-2021-00181; ÚNKP-22-3 New National Excellence Program of the Ministry for Innovation and Technology from the source of the National Research, Development and Innovation Fund.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bialas, A.; Bzdak, A. Constituent quark and diquark properties from small angle proton-proton elastic scattering at high energies. *Acta Phys. Polon.* **2007**, *38* , 159–168.
2. Nemes, F.; Csörgő, T. Detailed Analysis of p^+p Elastic Scattering Data in the Quark-Diquark Model of Bialas and Bzdak from $\sqrt{s} = 23.5$ GeV to 7 TeV. *Int. J. Mod. Phys.* **2012**, *27*, 1250175. [CrossRef]
3. Csörgő, T.; Szanyi, I. Observation of Odderon effects at LHC energies: A real extended Bialas–Bzdak model study. *Eur. Phys. J. C* **2021**, *81*, 611. [CrossRef]
4. Szanyi, I.; Csörgő, T. The ReBB model and its H(x) scaling version at 8 TeV: Odderon exchange is a certainty. *Eur. Phys. J. C* **2022**, *82*, 827. [CrossRef]
5. Glauber, R.J.; Matthiae, G. High-energy scattering of protons by nuclei. *Nucl. Phys. B* **1970**, *21*, 135–157. [CrossRef]
6. Glauber, R.J.; Velasco, J. Multiple Diffraction Theory of $p\bar{p}$ Scattering at 546-GeV. *Phys. Lett. B* **1984**, *147*, 380–384. [CrossRef]
7. Nemes, F.; Csörgő, T.; Csanád, M. Excitation function of elastic pp scattering from a unitarily extended Bialas–Bzdak model. *Int. J. Mod. Phys.* **2015**, *30*, 1550076. [CrossRef]
8. Antchev, G.; Aspell, F.; Atanassov, I.; Avati, V.; Baechler, J.; Baldenegro Barrera, C.; Berardi, V.; Berretti, M.; Bossini, E.; Bottigli, U.; et al. Evidence for non-exponential elastic proton–proton differential cross-section at low $|t|$ and $\sqrt{s}$ = 8 TeV by TOTEM. *Nucl. Phys. B* **2015**, *899*, 527–546. [CrossRef]
9. Antchev, G.; Aspell, F.; Atanassov, I.; Avati, V.; Baechler, J.; Baldenegro Barrera, C.; Berardi, V.; Berretti, M.; Bossini, E.; Bottigli, U.; et al. Elastic differential cross-section measurement at $\sqrt{s} = 13$ TeV by TOTEM. *Eur. Phys. J. C* **2019**, *79*, 861. [CrossRef]
10. Barbiellini, G.; Bozzo, M.; Darriulat, P.; Palazzi, G.; De Zorzi, G.; Fainberg, A.; Ferrero, M.; Holder, M.; McFarland, A.; Maderni, G.; et al. Small-angle proton-proton elastic scattering at very high energies (460 GeV^2 < s < 2900 GeV^2). *Phys. Lett. B* **1972**, *39*, 663–667. [CrossRef]
11. Csörgő, T.; Pasechnik, R.; Ster, A. Odderon and proton substructure from a model-independent Lévy imaging of elastic pp and $p\bar{p}$ collisions. *Eur. Phys. J.* **2019**, *79*, 62. [CrossRef] [PubMed]
12. Csörgő, T.; Pasechnik, R.; Ster, A. Proton structure and hollowness from Lévy imaging of pp elastic scattering. *Eur. Phys. J.* **2020**, *80*, 126. [CrossRef]
13. Dremin, I.M. Critical regime of proton elastic scattering at the LHC. *JETP Lett.* **2014**, *99*, 243–245. [CrossRef]
14. Dremin, I.M. Interaction region of high energy protons. *Phys. Usp.* **2015**, *58*, 61–70. [CrossRef]
15. Albacete, J.L.; Soto-Ontoso, A. Hot spots and the hollowness of proton–proton interactions at high energies. *Phys. Lett.* **2017**, *770*, 149–153. [CrossRef]
16. Troshin, S.M.; Tyurin, N.E. Experimental signatures of hadron asymptotics at the LHC. *Int. J. Mod. Phys.* **2017**, *32*, 1750103. [CrossRef]
17. Broniowski, W.; Jenkovszky, L.; Ruiz Arriola, E.; Szanyi, I. Hollowness in pp and $p\bar{p}$ scattering in a Regge model. *Phys. Rev.* **2018**, *98*, 074012. [CrossRef]
18. Deppman, A.; Megias, E.; Menezes, D.P. Fractals, nonextensive statistics, and QCD. *Phys. Rev. D* **2020**, *101*, 034019. [CrossRef]

19. Megias, E.; Deppman, A.; Pasechnik, R.; Tsallis, C. Comparative study of the heavy-quark dynamics with the Fokker-Planck Equation and the Plastino-Plastino Equation. *arXiv* **2023**, arXiv:2303.03819.
20. Petrov, V.A.; Okorokov, V.A. The size seems to matter or where lies the "asymptopia"? *Int. J. Mod. Phys. A* **2018**, *33*, 1850077. [CrossRef]