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Spherically Symmetric C^3 Matching in General Relativity

Hernando Quevedo

Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, AP 70543, Ciudad de México 04510, Mexico; quevedo@nucleares.unam.mx

Abstract: We study the problem of matching interior and exterior solutions to Einstein's equations along a particular hypersurface. We present the main aspects of the C^3 matching approach that involve third-order derivatives of the corresponding metric tensors in contrast to the standard C^2 matching procedures known in general relativity, which impose conditions on the second-order derivatives only. The C^3 alternative approach does not depend on coordinates and allows us to determine the matching surface by using the invariant properties of the eigenvalues of the Riemann curvature tensor. As a particular example, we apply the C^3 procedure to match the exterior Schwarzschild metric with a general spherically symmetric interior spacetime with a perfect fluid source and obtain that on the matching hypersurface, the density and pressure should vanish, which is in accordance with the intuitive physical expectation.

Keywords: exact solutions; matching conditions; curvature eigenvalues

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1. Introduction

One important problem in astrophysics consists of describing the gravitational field of compact objects. Consider the gravitational field of a compact object, whose surface is denoted as Σ. Let U^+ and U^- represent the Newtonian gravitational potential outside and inside the object, respectively. This means that the potentials should be solutions to the Poisson equation (in Cartesian coordinates)

$$\Delta U^- = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) U^- = 4\pi G\rho, \tag{1}$$

and the Laplace equation

$$\Delta U^+ = 0, \tag{2}$$

respectively. Here, ρ is the density of the matter distribution that generates the gravitational field. The exterior (interior) potential describes the field outside (inside) the mass distribution. In general, the problem of finding solutions to the Laplace and Poisson equations is considered in the framework of potential theory. Usually, it is assumed that the mass distribution fulfills certain symmetry conditions that allow us to simplify the complexity of the corresponding differential equations. Consider, for instance, the case of a spherically symmetric mass distribution with radius R. Then, the Laplace equation reduces to

$$\Delta U^+ = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U^+}{\partial r}\right) = 0\,, \tag{3}$$

where r is the radial coordinate, and the solution for the exterior potential can be expressed as

$$U^+ = \frac{M}{r}\,, \tag{4}$$

where M is a constant of integration. The corresponding Poisson equation for an arbitrary function $\rho(r)$ can be solved by using the Green function

$$U^{-} = -G \int \frac{\rho(r')}{|r - r'|} dr' \,. \tag{5}$$

The matching consists in demanding that on the surface Σ, which in this case corresponds to $r = R$, the potentials U^{-} and U^{+} coincide. This is easily done, and we obtain as a result that M can be written in terms of $\rho(r)$ and corresponds to the total mass of the body.

Another practical example is that of an axially symmetric mass distribution. In this case, the Laplace equation becomes

$$\Delta U^{-} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U^{-}}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial U^{-}}{\partial \theta}\right) = 0 \,, \tag{6}$$

where θ is the azimuthal angle. This is a linear differential equation whose general solution can be represented as

$$U^{+} = \sum_{n=0}^{\infty} \frac{a_n}{r^{\frac{n+1}{2}}} P_n(\cos\theta) \,, \tag{7}$$

where $P_n(\cos\theta)$ are the Legendre polynomials and a_n are constants. As for the internal potential U^{-}, using the Green function formalism, the solution can be expressed as an infinite series, each term of which represents a particular multipole moment. The matching consists in demanding that the interior and exterior potentials coincide on Σ. This can be reached by calculating the explicit value of the series of the interior potential, which is given in terms of the density of the mass distribution, and demanding that it coincides term by term with the exterior potential (7) on Σ. As a result, the exterior multipoles become fixed by the values of the interior multipoles on the matching surface. This means that in Newtonian gravity, the matching problem can be solved uniquely by using multipole moments.

Consider now the matching problem in Einstein's theory of gravity, where the gravitational field of a mass distribution must be described by a metric $g_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$, satisfying Einstein's equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \,, \tag{8}$$

in the interior part of the mass ($T_{\mu\nu} \neq 0$) as well as outside in empty space ($T_{\mu\nu} = 0$). For concreteness, let us consider a mass distribution whose internal structure is described by a perfect fluid

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \,, \tag{9}$$

where $\rho(x^\mu)$ is the density, $p(x^\mu)$ is the pressure, and u^μ is the 4-velocity of a particle inside the fluid. As in Newtonian gravity, the complexity of the corresponding partial differential equations can be reduced by imposing symmetry conditions on the gravitational source. For instance, if we limit ourselves to spherically symmetric gravitational fields, the field equations reduce to a set of ordinary differential equations that can be solved analytically. Furthermore, in the case of vacuum gravitational fields, the differential equations can be solved in general and, by virtue of Birkhoff's theorem [1], the solution turns out to be unique and is known as the Schwarzschild spacetime [2], which describes the gravitational field of a static, spherically symmetric mass distribution. In the case of the interior gravitational field of compact objects, the situation is much more complicated. In the literature, there exists a reasonable number of interior spherically symmetric solutions [3], which are candidates to be matched with the exterior Schwarzschild metric. In this work, we will obtain the conditions under which the Schwarzschild spacetime can be matched with an interior spherically symmetric perfect-fluid solution. To this end, we will apply the C^3 approach, which is based upon the use of the eigenvalues of the curvature tensor.

The matching problem in general relativity has been the subject of intensive research since Darmois published in 1927 his matching method [4], stating that the first and second fundamental forms should be continuous across the matching surface and implying conditions on the second derivatives of the metric. For this reason, this method is usually called C^2 matching. However, as pointed out by Israel in [5], in practice, the C^2 matching is of limited utility because it requires the use of particular sets of admissible coordinates. Israel also proposed a generalization of Darmois conditions to include the more realistic case in which surface discontinuities are present. This generalization yields the thin-shell approach, which is widely used in the literature.

The C^3 alternative approach is different. We demand that the eigenvalues of the curvature tensor be continuous across the matching surface and use their derivatives to determine the location where the matching can be performed. Since the curvature eigenvalues are scalars, the C^3 matching conditions are invariant. In addition, it is also possible to generalize the C^3 method to include the case of surface discontinuities by using Israel's thin-shell proposal.

This work is organized as follows. In Section 2, we present a method to compute the eigenvalues of the Riemann curvature tensor, which is based upon the use of a local orthonormal basis and the formalism of differential forms with Cartan's equations as the underlying structure. Then, in Section 3, we describe the C^3 method and present the corresponding matching conditions. In Section 4, we apply the C^3 matching procedure in the case of spherically symmetric spacetimes. Finally, in Section 5, we discuss our results and comment on further applications of the C^3 matching.

2. Eigenvalues of the Riemann Curvature Tensor

The eigenvalues of the curvature tensor can be computed in different ways [6]. Here, we use the formalism of differential forms with a set of local orthonormal tetrads. From a physical point of view, an observer would choose a local orthonormal tetrad as the simplest and most natural frame of reference. Indeed, according to the equivalence principle, local measurements of space and time can be performed in a gravity-free environment so it is natural to use locally the flat Minkowski metric of special relativity. On the other hand, the use of local tetrads allows us to perform measurements that are invariant with respect to coordinate transformations. The only freedom remaining in the choice of this local frame is a Lorentz transformation. So, let us choose the orthonormal tetrad as

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{ab} \vartheta^a \otimes \vartheta^b \,, \tag{10}$$

with $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$, and $\vartheta^a = e^a{}_\mu dx^\mu$. Then, the tetrad components ϑ^a can be interpreted as differential one-forms. Furthermore, Cartan's first structure equation

$$d\vartheta^a = -\omega^a{}_b \wedge \vartheta^b \tag{11}$$

can be used to determine explicitly the components of the connection one-form $\omega^a{}_b$, which, in turn, are used to define the curvature two-form $\Omega^a{}_b$ by means of Cartan's second structure equation

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{bcd} \vartheta^c \wedge \vartheta^d \,, \tag{12}$$

where $R^a{}_{bcd}$ are the components of the Riemann curvature tensor in the local orthonormal frame ϑ^a.

The curvature tensor can be represented as a (6×6)-matrix by introducing the bivector indices A, ..., $A = 1, \ldots 6$, which encode the information of two different tetrad indices, i.e., $ab \to A$. A particular choice of this correspondence is [1]

$$01 \to 1 \,, \quad 02 \to 2 \,, \quad 03 \to 3 \,, \quad 23 \to 4 \,, \quad 31 \to 5 \,, \quad 12 \to 6 \,. \tag{13}$$

Then, the Riemann tensor can be represented by the symmetric matrix $\mathbf{R}_{AB}$ with

$$\mathbf{R}_{AB} = \begin{pmatrix} R_{0101} & R_{0102} & R_{0103} & R_{0123} & R_{0131} & R_{0112} \\ R_{0102} & R_{0202} & R_{0203} & R_{0223} & R_{0231} & R_{0212} \\ R_{0103} & R_{0203} & R_{0303} & R_{0323} & R_{0331} & R_{0312} \\ R_{0123} & R_{0223} & R_{0323} & R_{2323} & R_{2331} & R_{1223} \\ R_{0131} & R_{0231} & R_{0331} & R_{2331} & R_{3131} & R_{1231} \\ R_{0112} & R_{0212} & R_{0312} & R_{1223} & R_{1231} & R_{1212} \end{pmatrix}, \tag{14}$$

which possesses 21 independent components. However, the first Bianchi identity

$$R_{a[bcd]} = 0 \Leftrightarrow R_{0123} + R_{0312} + R_{0231} = 0\,, \tag{15}$$

which in bivector representation reads

$$\mathbf{R}_{14} + \mathbf{R}_{25} + \mathbf{R}_{36} = 0\,, \tag{16}$$

imposes an additional relationship between the components of the curvature matrix and, consequently, reduces the number of independent components to 20, as it should be in the case of a 4-dimensional Riemannian manifold.

We now consider Einstein's equations with cosmological constant in the orthonormal frame ϑ^a,

$$R_{ab} - \frac{1}{2}R\eta_{ab} + \Lambda\eta_{ab} = \kappa T_{ab}\,, \quad R_{ab} = R^c{}_{acb}\,, \tag{17}$$

which represent a relationship between the components of the curvature tensor, the cosmological constant, and the components of the energy-momentum tensor.

By writing the Ricci tensor R_{ab} and the curvature scalar R explicitly in terms of the components of the Riemann tensor in the bivector representation, Einstein's equations reduce to a set of ten algebraic equations that relate the components of the matrix $\mathbf{R}_{AB}$. This means that we can express ten of the components $\mathbf{R}_{AB}$ in terms of the remaining ten components. For concreteness, we choose as independent components the following: $\mathbf{R}_{11}$, $\mathbf{R}_{12}$, $\mathbf{R}_{13}$, $\mathbf{R}_{14}$, $\mathbf{R}_{15}$, $\mathbf{R}_{16}$, $\mathbf{R}_{22}$, $\mathbf{R}_{23}$, $\mathbf{R}_{25}$, and $\mathbf{R}_{26}$. Introducing the resulting equations into the matrix $\mathbf{R}_{AB}$, only ten components remain independent. Then, the curvature matrix can be represented as [7,8]

$$\mathbf{R}_{AB} = \begin{pmatrix} \mathbf{M}_1 & \mathbf{L} \\ \mathbf{L} & \mathbf{M}_2 \end{pmatrix}, \tag{18}$$

where

$$\mathbf{L} = \begin{pmatrix} \mathbf{R}_{14} & \mathbf{R}_{15} & \mathbf{R}_{16} \\ \mathbf{R}_{15} - \kappa T_{03} & \mathbf{R}_{25} & \mathbf{R}_{26} \\ \mathbf{R}_{16} + \kappa T_{02} & \mathbf{R}_{26} - \kappa T_{01} & -\mathbf{R}_{14} - \mathbf{R}_{25} \end{pmatrix},$$

and $\mathbf{M}_1$ and $\mathbf{M}_2$ are 3×3 symmetric matrices

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \mathbf{R}_{13} \\ \mathbf{R}_{12} & \mathbf{R}_{22} & \mathbf{R}_{23} \\ \mathbf{R}_{13} & \mathbf{R}_{23} & -\mathbf{R}_{11} - \mathbf{R}_{22} - \Lambda + \kappa\left(\frac{T}{2} + T_{00}\right) \end{pmatrix},$$

$$\mathbf{M}_2 = \begin{pmatrix} -\mathbf{R}_{11} + \kappa\left(\frac{T}{2} + T_{00} - T_{11}\right) & -\mathbf{R}_{12} - \kappa T_{12} & -\mathbf{R}_{13} - \kappa T_{13} \\ -\mathbf{R}_{12} - \kappa T_{12} & -\mathbf{R}_{22} + \kappa\left(\frac{T}{2} + T_{00} - T_{22}\right) & -\mathbf{R}_{23} - \kappa T_{23} \\ -\mathbf{R}_{13} - \kappa T_{13} & -\mathbf{R}_{23} - \kappa T_{23} & \mathbf{R}_{11} + \mathbf{R}_{22} + \Lambda - \kappa T_{33} \end{pmatrix},$$

where T is the trace of the energy-momentum tensor, $T = \eta^{ab}T_{ab}$. Accordingly, this is the most general form of a curvature tensor that satisfies Einstein's equations with cosmological constant and arbitrary energy-momentum tensor.

We note that the traces of the above matrices turn out to be of particular importance. Indeed,

$$\mathrm{Tr}(\mathbf{L}) = 0\,, \tag{19}$$

$$\mathrm{Tr}(\mathbf{M_1}) = -\Lambda + \kappa\left(\frac{T}{2} + T_{00}\right), \quad \mathrm{Tr}(\mathbf{M_2}) = +\Lambda + \kappa T_{00}\,. \tag{20}$$

As shown above, the first equation follows from the Bianchi identities. The second and third equations can be proved by direct computation. Consequently, the trace of the curvature matrix can be expressed as

$$\mathrm{Tr}(\mathbf{R_{AB}}) = \kappa\left(\frac{T}{2} + 2T_{00}\right). \tag{21}$$

Thus, we see that all the relevant traces depend on the components of the energy-momentum tensor only.

The eigenvalues of the curvature tensor correspond to the eigenvalues of the matrix $\mathbf{R}_{AB}$. In general, they are functions λ_i, with $i = 1, 2, ..., 6$,which depend on the parameters and coordinates entering the tetrads ϑ^a.

As a particular example of the bivector representation of the curvature, consider now the case of a perfect fluid energy-momentum tensor with density ρ and pressure p, i.e.,

$$T_{ab} = (\rho + p)u_a u_b + p\eta_{ab}\,, \tag{22}$$

where $u^a = (-1, 0, 0, 0)$ is the comoving 4-velocity of the fluid. Then,

$$T_{ab} = \mathrm{diag}(\rho, p, p, p) \tag{23}$$

and the curvature matrix reduces to

$$\mathbf{R}_{AB} = \begin{pmatrix} \mathbf{M}_1 & \mathbf{L} \\ \mathbf{L} & \mathbf{M}_2 \end{pmatrix}, \tag{24}$$

with

$$\mathbf{L} = \begin{pmatrix} \mathbf{R}_{14} & \mathbf{R}_{15} & \mathbf{R}_{16} \\ \mathbf{R}_{15} & \mathbf{R}_{25} & \mathbf{R}_{26} \\ \mathbf{R}_{16} & \mathbf{R}_{26} & -\mathbf{R}_{14} - \mathbf{R}_{25} \end{pmatrix},$$

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \mathbf{R}_{13} \\ \mathbf{R}_{12} & \mathbf{R}_{22} & \mathbf{R}_{23} \\ \mathbf{R}_{13} & \mathbf{R}_{23} & -\mathbf{R}_{11} - \mathbf{R}_{22} - \Lambda + \frac{\kappa}{2}(3p + \rho) \end{pmatrix},$$

$$\mathbf{M}_2 = \begin{pmatrix} -\mathbf{R}_{11} + \frac{\kappa}{2}(\rho + p) & -\mathbf{R}_{12} & -\mathbf{R}_{13} \\ -\mathbf{R}_{12} & -\mathbf{R}_{22} + \frac{\kappa}{2}(\rho + p) & -\mathbf{R}_{23} \\ -\mathbf{R}_{13} & -\mathbf{R}_{23} & \mathbf{R}_{11} + \mathbf{R}_{22} + \Lambda - \kappa p \end{pmatrix}.$$

Thus, in the case of a perfect fluid solution, the curvature eigenvalues are related by

$$\sum_{i=1}^{6} \lambda_i = \frac{3\kappa}{2}(\rho + p)\,. \tag{25}$$

Finally, in the particular case of vacuum fields, $R_{ab} = 0$, with vanishing cosmological constant, $\Lambda = 0$, the curvature matrix reduces to

$$\mathbf{R}_{AB} = \begin{pmatrix} \mathbf{M} & \mathbf{L} \\ \mathbf{L} & -\mathbf{M} \end{pmatrix}, \tag{26}$$

where

$$\mathbf{L} = \begin{pmatrix} \mathbf{R}_{14} & \mathbf{R}_{15} & \mathbf{R}_{16} \\ \mathbf{R}_{15} & \mathbf{R}_{25} & \mathbf{R}_{26} \\ \mathbf{R}_{16} & \mathbf{R}_{26} & -\mathbf{R}_{14} - \mathbf{R}_{25} \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \mathbf{R}_{13} \\ \mathbf{R}_{12} & \mathbf{R}_{22} & \mathbf{R}_{23} \\ \mathbf{R}_{13} & \mathbf{R}_{23} & -\mathbf{R}_{11} - \mathbf{R}_{22} \end{pmatrix}, \quad (27)$$

so that the (3×3) matrices $\mathbf{L}$ and $\mathbf{M}$ are symmetric and trace free,

$$\mathrm{Tr}(\mathbf{L}) = 0\,, \qquad \mathrm{Tr}(\mathbf{M}) = 0\,, \quad \text{i.e.,} \quad \mathrm{Tr}(\mathbf{R}_{AB}) = 0\,. \quad (28)$$

Furthermore, the eigenvalues must satisfy the condition

$$\sum_{i=1}^{6} \lambda_i = 0 \quad (29)$$

as a consequence of the curvature matrix being traceless.

The explicit form of curvature eigenvalues λ_i depends on the components of the Riemann curvature tensor and behaves as scalars under coordinate transformations. They can, therefore, be used to formulate invariant statements in general relativity. In particular, the properties of λ_i are used to formulate the Petrov classification of gravitational fields [6]. Additionally, the eigenvalue properties have been used to propose an invariant definition of repulsive gravity [9] and alternative cosmological models [10]. Here, we use this idea to propose an invariant formulation of the matching problem in which only curvature eigenvalues are involved.

3. C^3 Matching

The matching between two different spacetimes along a surface Σ is usually performed by using the Darmois and Lichnerowicz conditions (C^2 conditions), which have been shown to be equivalent in a particular coordinate system [4,11–14]. The C^2 conditions state that in certain coordinates, the first fundamental form, i.e., the metrics induced on the matching surface, and the second fundamental form, i.e., the corresponding extrinsic curvatures, must be continuous across Σ. Darmois conditions are represented in a covariant way, which implies that no preference should be given to any particular coordinate system. However, these conditions turn out to be very restrictive in concrete examples, in particular, because the choice of coordinates is a very important step in the sense that the so-called admissible coordinates have to be found in order to apply the matching procedure (for more details see [14]). For instance, in the case of spherically symmetric spacetimes, several options are possible and, therefore, a detailed analysis of each coordinate system should be performed before proceeding with the matching itself [15–18]. One of the advantages of using the C^3 matching procedure is that the results do not depend on the choice of coordinates because we will use only quantities that behave as scalars under a coordinate transformation [9].

Furthermore, an alternative approach was proposed by Israel in [5], which is applied when the extrinsic curvature is not continuous. In fact, in this case, Σ is replaced by a thin shell with an effective energy-momentum tensor, which is defined in terms of the difference of the extrinsic curvature evaluated inside and outside the hypersurface Σ. Since the above matching approaches involve second-order derivatives of the metric, they are known, in general, as C^2 matching.

The C^2 matching is usually difficult to implement because it requires knowing a priori the location of Σ in a particular coordinate system. In the case of compact objects, Σ is identified with the surface of the source of gravity. In general, however, it is quite complicated to find the equation that determines the matching surface, except in cases with a high number of symmetries, such as spherical symmetry, in which the surface is simply a sphere of constant radius.

The main objective of the C^3 procedure is to provide matching conditions that do not depend on the choice of a particular coordinate system and allow us to obtain information about the matching surface Σ. To this end, the C^3 matching uses as a starting point the eigenvalues of the Riemann curvature tensor, which are independent of the choice of coordinate system [6]. In fact, we will also consider the derivatives of the eigenvalues, which involve third-order derivatives of the metric, in order to obtain information about the location of the matching surface. For this reason, we denote our method as C^3 matching.

One of the first applications of the formalism presented above was to formulate an invariant definition of repulsive gravity [9]. The idea of this definition is as follows. In the case of an isolated mass distribution, the corresponding spacetime should be asymptotically flat, and, consequently, all the eigenvalues should vanish at infinity, i.e.,

$$\lim_{r\to\infty} \lambda_i = 0 \; \forall \; i \; , \tag{30}$$

where r is a spatial coordinate that measures the distance to the source of gravity. Then, as the mass distribution is approached, the intensity of the gravitational field should increase, and, correspondingly, the eigenvalues are expected to increase. If an eigenvalue happens to change its sign as the source is approached, we interpret this behavior as an indication of the presence of repulsive gravity. Furthermore, since the eigenvalue vanishes at infinity and increases its value as the object is approached, it should pass through an extremum before changing its sign. To realize this intuitive idea in concrete examples, we proceed as follows. Let the set

$$\{r_l\}, \; l = 1, 2, ... \quad \text{with} \quad 0 < r_l < \infty \tag{31}$$

represents the set of solutions to the equation

$$\left.\frac{\partial \lambda_i}{\partial r}\right|_{r=r_l} = 0 \; , \quad \text{with} \quad r_{rep} = \max\{r_l\} \; , \tag{32}$$

i.e., r_{rep} is the location of the first extremum that is found when approaching the source from infinity. We call r_{rep} repulsion radius because at $r = r_{rep}$, the maximum value of attractive gravity is reached, and repulsive gravity starts to play an important role.

The main point now is to use this definition of repulsive gravity in the context of realistic compact objects. In fact, since in the case of compact mass distributions, no repulsive gravity has been detected so far, the idea of the C^3 approach is to replace the region of repulsion $(r < r_{rep})$ with an interior solution of Einstein equations as follows. Indeed, regions of repulsive gravity have been shown to exist in Reissner–Nordström, Kerr, and Kerr–Newman black holes [9] as well as in gravitational fields generated by a mass distribution with quadrupole moment [19]. In such cases, the matching with an interior solution should be performed in such a way that the matching surface is located outside the region of repulsive gravity.

Let us consider an exterior spacetime $(M^+, g^+_{\mu\nu})$ and an interior spacetime $(M^-, g^-_{\mu\nu})$ with curvature eigenvalues $\{\lambda^+_i\}$ and $\{\lambda^-_i\}$, respectively. Then, the C^3 matching approach consists of two steps:

(i) Define the matching surface Σ by means of the matching radius r_{match}, defined as

$$r_{match} \in [r_{rep}, \infty) \; , \quad \text{with} \quad r_{rep} = \max\{r_l\} \; , \quad \left.\frac{\partial \lambda^+_i}{\partial r}\right|_{r=r_l} = 0 \; . \tag{33}$$

This means that the repulsion radius is determined by the location of the first extremum that is found when approaching the source of gravity from infinity.

(ii) Perform the matching of the spacetimes $(M^+, g^+_{\mu\nu})$ and $(M^-, g^-_{\mu\nu})$ at Σ by imposing the conditions

$$\left.\lambda^+_i\right|_\Sigma = \left.\lambda^-_i\right|_\Sigma \quad \forall i \; . \tag{34}$$

In other words, the C^3 matching consists in demanding that the curvature eigenvalues be continuous across the matching surface Σ, which should be located anywhere between the repulsion radius and infinity. Thus, the idea of the C^3 matching is to avoid the presence of repulsive gravity in the case of gravitational compact objects. Conditions (33) and (34) turn out to be very restrictive in the sense that they do not allow the case of discontinuities across the matching surface. We will see in Section 4 that the C^3 matching procedure can be generalized to include this case too.

4. The Spherically Symmetric Matching

In the case of spherically symmetric gravitational fields, the exterior spacetime is unique and is described by the Schwarzschild line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{35}$$

where M represents the mass of the gravitational source. The orthonormal tetrad can be chosen in the canonical form

$$\vartheta^0 = \left(1 - \frac{2M}{r}\right)^{1/2} dt\,, \quad \vartheta^1 = \left(1 - \frac{2M}{r}\right)^{-1/2} dr\,, \quad \vartheta^2 = rd\theta\,, \quad \vartheta^3 = r\sin\theta d\varphi. \tag{36}$$

A straightforward computation shows that, in this case, the curvature matrix has the form

$$\mathbf{R}_{AB} = \begin{pmatrix} -\frac{2M}{r^3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{M}{r^3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{M}{r^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2M}{r^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{M}{r^3} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{M}{r^3} \end{pmatrix}, \tag{37}$$

Then, the eigenvalues are determined by the diagonal elements of the matrix $\mathbf{R}_{AB}$ and we obtain

$$\lambda_1^+ = -\lambda_4^+ = -\frac{2M}{r^3},\ \lambda_2^+ = \lambda_3^+ = -\lambda_5^+ = -\lambda_6^+ = \frac{M}{r^3}. \tag{38}$$

For the investigation of the interior spacetime M^-, we consider the general spherically symmetric line element

$$ds^2 = -e^\nu dt^2 + e^\phi dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{39}$$

where ν and ϕ are functions that depend on r only. It then follows that the orthonormal tetrad can be chosen as

$$\vartheta^0 = e^{\nu/2}dt\,, \quad \vartheta^1 = e^{\phi/2}dr\,, \quad \vartheta^2 = rd\theta\,, \quad \vartheta^3 = r\sin\theta d\varphi\,. \tag{40}$$

Using Cartan's structure equations, we obtain the following non-vanishing components of the curvature matrix:

$$\mathbf{R}_{11} = -\frac{1}{4}(\phi_{,r}\nu_{,r} - \nu_{,r}^2 - 2\nu_{,rr})e^{-\phi},\ \mathbf{R}_{22} = \frac{1}{2r}\nu_{,r}e^{-\phi},\ \mathbf{R}_{33} = \frac{1}{2r}\nu_{,r}e^{-\phi}, \tag{41}$$

$$\mathbf{R}_{44} = \frac{1}{r^2}(1 - e^{-\phi}),\ \mathbf{R}_{55} = \frac{1}{2r}\phi_{,r}e^{-\phi},\ \mathbf{R}_{66} = \frac{1}{2r}\phi_{,r}e^{-\phi}, \tag{42}$$

where we have used Einstein's equations in the form

$$\nu_{,rr} + \frac{1}{2}\nu_{,r}^2 - \frac{\nu_{,r}}{2r}(2 + r\phi_{,r}) - \frac{\phi_{,r}}{r} - \frac{2}{r^2}(1 - e^\phi) = 0\,, \tag{43}$$

$$\kappa\rho = \frac{1}{r^2}[1 + e^{-\phi}(r\phi_{,r} - 1)],\ \ \kappa p = -\frac{1}{r^2}[1 - e^{-\phi}(1 + r\nu_{,r})]. \tag{44}$$

Then, we obtain the following eigenvalues for the curvature tensor of a spherically symmetric interior perfect fluid solution

$$\lambda_1^- = -\frac{1}{4}(\phi_{,r}\nu_{,r} - \nu_{,r}^2 - 2\nu_{,rr})e^{-\phi}, \tag{45}$$

$$\lambda_2^- = \lambda_3^- = \frac{1}{2r}\nu_{,r}e^{-\phi}, \tag{46}$$

$$\lambda_4^- = \frac{1}{4}(\phi_{,r}\nu_{,r} - \nu_{,r}^2 - 2\nu_{,rr})e^{-\phi} + \frac{k(\rho+p)}{2}, \tag{47}$$

$$\lambda_5^- = \lambda_6^- = -\frac{1}{2r}\nu_{,r}e^{-\phi} + \frac{k(\rho+p)}{2}\,. \tag{48}$$

The computation of the C^3 matching condition $d\lambda_i^+/dr = 0$ shows that there is no repulsion radius, implying that the matching can be carried out within the interval $r_{match} \in (0,\infty)$. The second matching condition implies that the exterior (38) and interior eigenvalues (45) coincide on the matching surface. This leads to the following set of independent equations

$$-\frac{1}{4}(\phi_{,r}\nu_{,r} - \nu_{,r}^2 - 2\nu_{,rr})e^{-\phi}, = -\frac{2M}{r^3} \tag{49}$$

$$\frac{1}{2r}\nu_{,r}e^{-\phi} = \frac{M}{r^3}\,, \tag{50}$$

$$\frac{1}{4}(\phi_{,r}\nu_{,r} - \nu_{,r}^2 - 2\nu_{,rr})e^{-\phi} + \frac{k(\rho+p)}{2} = \frac{2M}{r^3}, \tag{51}$$

$$-\frac{1}{2r}\nu_{,r}e^{-\phi} + \frac{k(\rho+p)}{2} = -\frac{M}{r^3}. \tag{52}$$

The above system of algebraic equations has to be satisfied in order for an arbitrary perfect fluid solution to be matched with the exterior Schwarzschild spacetime. It is easy to show that the above set of algebraic conditions allows only one solution, namely,

$$\rho = 0,\; p = 0\,. \tag{53}$$

This result corroborates in an invariant way our physical expectation of vanishing pressure and density on the matching surface. This result contrasts with the one obtained by using the Darmois matching conditions, according to which perfect-fluid interior solutions with non-zero densities and pressures at the matching surface, described by a sphere of constant radius, are configurations that can be matched with the exterior Schwarzschild spacetime [8]. In this sense, the Israel matching conditions offer an additional possibility, according to which the non-zero values of the density and pressure on the matching surface are due to the presence of a thin shell with exactly those values of density and pressure. In the resulting configuration, the matching problem is transferred to the thin shell, which is described by an energy-momentum tensor whose physical meaning has to be established separately [5,8].

In the C^3 approach, it is also possible to generalize the matching conditions to include non-zero values of the energy density and pressure. Indeed, as shown in [8], a discontinuous matching can be performed explicitly if we assume that the Einstein tensor $G_{ij}^{\pm}$ induced on the matching surface Σ satisfies the condition

$$G_{ij}^- - G_{ij}^+ = kS_{ij} \tag{54}$$

where k is a real constant and S_{ij} is a well-defined energy-momentum tensor. This means that Einstein's equations are satisfied across the matching surface. In the case of perfect-fluid solutions, the tensor S_{ij} is physically relevant if it is defined as

$$S_{ij} = T^{-}_{ij} - T^{+}_{ij} = [(\sigma + P)u_i u_j + P\gamma_{ij}] \,, \tag{55}$$

where $\sigma = \rho|_\Sigma$ and $P = p|_\Sigma$ are the non-zero values of the density and pressure at the matching surface, respectively, and γ_{ij} is the spacetime metric induced on Σ. Thus, we see that the non-zero values of the density and pressure at the matching surface can be used to construct an energy-momentum tensor that guarantees the fulfillment of the Einstein equations on the matching conditions so that the matching can be performed explicitly. Several examples of the application of this procedure have been presented in [8]. In particular, the case in which the surface pressure P vanishes can be represented as

$$S_{ij} = 2(\lambda^{-}_{1} - \lambda^{+}_{1})u_i u_j \,, \tag{56}$$

indicating that the surface density σ can be represented invariantly in terms of the eigenvalues. This particular result could be used to apply the matching procedure in the case of strange stars [20].

5. Final Remarks and Perspectives

In this work, we presented an invariant formalism to apply matching conditions in general relativity, which is based upon the use of the eigenvalues of the Riemann curvature tensor and its derivatives. In this C^3 approach, we demand that the curvature eigenvalues of the exterior and interior solutions be continuous across the matching surface. In addition, the derivatives of the eigenvalues are used to determine the location of the matching surface. In this work, we limit ourselves to the case of isolated gravitational sources so that the curvature and the eigenvalues vanish at spatial infinity. Then, we look at the behavior of the eigenvalues as the source of gravity is approached from infinity. We argue that if an eigenvalue shows local extrema and changes its sign as the source is approached, this is an effect due to the presence of repulsive gravity. In fact, this behavior has been used to propose an invariant definition of repulsive gravity, which includes the concept of the radius of repulsion as corresponding to the location of the first extremum that appears as the source is approached from spatial infinity. Furthermore, we define the matching radius as the minimum radius where the matching can be performed. In other words, the matching surface can be located anywhere between the location of the repulsion radius and infinity. The goal of fixing a minimum radius for the matching surface is to avoid the presence of repulsive gravity because, so far, it has not been detected in the gravitational field of compact astrophysical objects.

We analyze in detail the case of a spherically symmetric mass distribution, in which the exterior field is described by the Schwarzschild spacetime, and the interior counterpart corresponds to a perfect fluid. It is interesting to note that due to the versatility of the C^3 matching formalism in the sense that the curvature eigenvalues can be calculated in general for any metric without specifying any particular solution, it is not necessary to fix the interior perfect-fluid solution. We use instead the general form of the matrix curvature that satisfies Einstein's equations. First, we notice that the derivatives of the exterior eigenvalues do not have any extrema, a result that we interpret as indicating that there is no repulsion radius and the matching can be performed at any place between the origin of coordinates and spatial infinity. Then, we find the set of algebraic equations that follows from the condition that the interior and exterior eigenvalues coincide at the matching surface. It turns out that this set of equations allows only one solution, namely, that the pressure and density should vanish at the matching surface. We conclude that the C^3 matching procedure in the case of a spherically symmetric gravitational field leads to the results expected from a physical point of view.

Another case of interest is that of stationary axially symmetric fields, which allows the analysis of rotating gravitational fields. In the case of vacuum, the general line element can be written in cylindrical coordinates (t, ρ, z, φ) as [6]

$$ds^2 = e^{2\psi}(dt - \omega d\varphi)^2 - e^{-2\psi}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right], \tag{57}$$

where ψ, ω, and γ are functions of ρ and z, only. The calculation of the corresponding curvature eigenvalues and their derivatives with respect to ρ and z leads to a set of equations that should determine the location of the matching surface. However, it is not easy to interpret the significance of the results, probably because it is necessary to use a different set of coordinates. We expect to investigate this problem in future works. In the particular case of a slowly rotating mass, whose gravitational field can be described by the exterior Lense–Thirring metric and the interior Hartle–Thorne approximate solution [6], the matching conditions lead to a system of equations that must be solved numerically. Work in this direction is in progress.

The particular case of static axially symmetric gravitational $(\omega = 0)$ is interesting because it resembles the case of Newtonian gravity. Indeed, in this case, the field equation that determines the function ψ turns out to be linear, and its general asymptotically flat solution can be written as [21]

$$\psi = \sum_{n=0}^{\infty} \frac{a_n}{(\rho^2 + z^2)^{\frac{n+1}{2}}} P_n(\cos\theta), \qquad \cos\theta = \frac{z}{\sqrt{\rho^2 + z^2}}, \tag{58}$$

where a_n $(n = 0, 1, ...)$ are arbitrary constants, and $P_n(\cos\theta)$ represents the Legendre polynomials of degree n. The solution for the function γ can be obtained from the above expression by quadratures. Interestingly, the solution (58) coincides with the exterior Newtonian potential given in Equation (7). This coincidence could be used to search for an interior line element, in which the function corresponding to ψ could be given as an infinite series in terms of the Green function (5). This has been done in the particular case of a metric with a quadrupole moment in [22,23]. We plan to continue the study of this problem in future works.

Another interesting aspect that has not been explored in the C^3 formalism is the possibility of analyzing the internal structure of compact objects by using thin shells, especially regarding stability properties and phase transition structures [24]. To this end, it will be necessary to investigate the dynamics of thin shells determined by the matching conditions in the presence of discontinuities as given in Equations (54)–(56). This is an interesting open question that deserves further development.

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