

Article

Study of He–Mckellar–Wilkins Effect in Noncommutative Space

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Abstract: The He–McKellar–Wilkins (HMW) effect in noncommutative space has been explored through two distinct methodologies. One approach treats the neutral particle, which harbors a permanent electric dipole moment, as an unstructured entity, while the other approach considers the neutral particle as a composite system consisting of a pair of oppositely charged particles. To preserve gauge symmetry, we apply the Seiberg–Witten map. Surprisingly, both of these approaches converge on the same result. They independently confirm that, up to the first order of the noncommutative parameter (NCP), no corrections are observed in the phase of the HMW effect. Remarkably, these two approaches, although founded on fundamentally different mechanisms, yield identical conclusions.

Keywords: He–McKellar–Wilkins effect; noncommutative space; electric dipole moment

1. Introduction

The significance of topological phases in quantum theories has been evident ever since the groundbreaking discovery of the Aharonov–Bohm (AB) effect. In their seminal work [1], Aharonov and Bohm made the remarkable prediction that the interference pattern of electrons in a double-slit experiment would undergo a transformation when a slender elongated solenoid was positioned between the two slits. This prediction entailed that the electron’s wave function would acquire a distinct topological phase as it circuted around the solenoid. Experimental confirmation of this effect was subsequently obtained, as documented in [2,3].

Following the discovery of the AB effect, further developments in the realm of topological phases led to the proposal of the Aharonov–Casher (AC) effect, associated with neutral particles possessing a magnetic dipole moment. This phenomenon was introduced in [4–6] and named after its creators. Additionally, another topological phase linked to neutral particles with an electric dipole moment (such as molecules with a permanent dipole moment) emerged as the He–McKellar–Wilkins (HMW) effect [7,8]. It was noted that the HMW effect exhibited intrinsic connections with $U(1)$ gauge theory, as discussed in [9,10].

Theoretical investigations of the HMW effect adopted two distinct approaches. The first approach treated the neutral particle as a structureless entity, while the second approach regarded it as a composite system consisting of a pair of oppositely charged particles, as discussed in [11]. In the latter approach, the topological phase associated with the neutral particle was characterized as the sum of the phases accrued by these two oppositely charged particles, as elaborated upon in [12,13].

In recent years, substantial research has been dedicated to the exploration of quantum fields and quantum mechanics within the framework of noncommutative space. This intriguing concept of noncommutativity was first introduced in the mid-1940s, as documented in [14,15]. Over time, noncommutative space has gained increasing



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prominence, particularly in the context of superstring theories, as exemplified by [16–19], along with a substantial body of work focused on noncommutativity [20–24]. Remarkably, spatial noncommutativity has been observed to naturally arise in certain elementary quantum mechanical systems, as discussed by authors such as Jackiw and Susskind [25,26]. To date, extensive investigations have been carried out to probe the ramifications of spatial noncommutativity, encompassing corrections at both the quantum field and quantum mechanics levels, as comprehensively addressed in [27–39].

The study of topological phases within the realm of noncommutative space has emerged as a focal point for researchers, drawing considerable attention. Particularly, the investigation into the topological phase of the AB effect has undergone extensive exploration through the application of the Bopp shift method [40–44]. This method serves to directly map noncommutative variables onto commutative ones, and has found applications in the comprehensive analysis of AC and HMW effects as well [45–48]. One noteworthy characteristic of this approach is its tendency to introduce additional terms that are proportional to the noncommutative parameter (NCP) into the topological phases. However, a limitation of this method is its failure to preserve gauge symmetry. Moreover, investigations into the AB effect in noncommutative space have leveraged formulations involving path integrals and Wilson lines. This avenue of study offers promising potential for further exploration within this captivating field [49–51]. These formulations provide an additional lens through which to delve deeper into understanding and probing the intriguing aspects of noncommutative space and its implications on topological phases.

To maintain gauge symmetry, in [52,53] the Seiberg–Witten (S-W) map was employed as a versatile tool to investigate the AB effect in the context of noncommutative space. The S-W map is a fundamental method for examining gauge theories in such noncommutative settings. In light of the inherent connections between the AC and HMW effects and the $U(1)$ gauge theory, the authors of [54] extended their application of the S-W map to explore the AC and HMW effects within the realm of noncommutative space. This comprehensive approach allowed for a deeper understanding of these intriguing phenomena.

While extensive research has delved into the realm of topological phases within noncommutative space, the findings have at times yielded conflicting results. The majority of studies, as documented in a number of references [40–51,53,54], have indicated that spatial noncommutativity introduces corrections up to the first order of the NCP. These corrections have been observed to affect various aspects of the phenomena under investigation.

Intriguingly, a distinct perspective emerged in [52]. This investigation, conducted within a non-relativistic model, revealed that there were no corrections to the phase of the AB effect, up to the first order of the NCP. A similar conclusion was reached in [55], where the authors analyzed a relativistic spin- $\frac{1}{2}$ model, aligning with the findings in [52]. These divergent results underscore the complexity and nuanced nature of the interplay between topological phases and noncommutative space.

Our analysis of the HMW effect in noncommutative space encompasses both approaches, i.e., taking the neutral particle as a structureless particle and a compound particle with opposite charges. ¹ To uphold gauge symmetry, we employ the SW map. Intriguingly, both approaches yield consistent outcomes, indicating that, up to the first order of the NCP, no correction is introduced to the phase of the HMW effect within the noncommutative space.

The rest of our paper is structured as follows. In the subsequent section, we analyze the dynamics of a relativistic neutral particle with spin $1/2$ and an anomalous electric dipole moment. After introducing the effective gauge potential, we find that this particle engages with an effective electromagnetic field as delineated within the context of $U(1)$ gauge theory. We then extend this model to noncommutative space by replacing the conventional product with the Moyal product. Our analysis reveals that the phase of the HMW effect remains unaltered up to the first order of the NCP. In Section 3, we delve into the model by considering the electric dipole moment as a pair of oppositely charged particles. We establish that, in the HMW effect configuration, spatial noncommutativity

exerts no influence on the phase of the HMW effect. Despite the consistent conclusions reached through both approaches, it is noteworthy that the underlying mechanisms differ significantly. Finally, we conclude our work with additional discussion and remarks in the final section.

2. Approach I: Identifying the Model as the $U(1)$ Gauge Theory

In this section, we study the HMW effect by taking the neutral particle as a structureless particle, then identifying the model with $U(1)$ gauge theory. We begin our investigation with the commutative case and then generalize it to noncommutative space.

The model is a spin- $\frac{1}{2}$ particle possessing an anomalous electric dipole moment that interacts with an electromagnetic field in $2 + 1$ -dimensional spacetime. It is described by the following Lagrangian (the summation convention is applied, and we set $\hbar = c = 1$):

$$\mathcal{L} = i\bar{\psi}g_{\mu\nu}\gamma^\mu\partial^\nu\psi - m\bar{\psi}\psi + \frac{1}{2}\mu_e\bar{\psi}\sigma^{\mu\nu}\tilde{F}_{\mu\nu}\psi, \quad (1)$$

where $g^{\mu\nu} = g_{\mu\nu} = \text{diag}(+, -, -)$ is the metric, $\gamma^\mu = (\gamma^0, \vec{\gamma}) = (\gamma^0, \gamma^1, \gamma^2)$ are Dirac gamma matrices, μ_e is the magnitude of the electric dipole moment, and $\tilde{F}_{\mu\nu}$ is the dual of the electromagnetic tensor. The explicit expression is

$$\tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & B^1 & B^2 \\ -B^1 & 0 & E^3 \\ -B^2 & -E^3 & 0 \end{pmatrix} \quad (2)$$

with B^1, B^2 being the magnetic field along the x and y directions on the $x - y$ plane and E^3 being the electric field normal to the $x - y$ plane.

In $2 + 1$ -dimensional spacetime, we are able to use two-component Dirac spinors; correspondingly, Dirac gamma matrices can be realized by Pauli matrices ($\sigma_x, \sigma_y, \sigma_z$). The explicit expression of the Dirac matrices are

$$\begin{aligned} \gamma^0 = \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ \gamma^2 &= -i\sigma_x = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \end{aligned} \quad (3)$$

and they satisfy the algebra

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\epsilon^{\mu\nu\rho}\gamma_\rho, \quad (4)$$

where $\epsilon^{\mu\nu\rho}$ is the anti-symmetric tensor with $\epsilon^{012} = \epsilon_{012} = 1$. In this representation, the operator $\sigma^{\mu\nu}$ is

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \epsilon^{\mu\nu\rho}\gamma_\rho. \quad (5)$$

Substituting Equation (5) into the Lagrangian Equation (1) and introducing an effective gauge potential

$$T^\mu = \frac{1}{2}\epsilon^{\mu\alpha\beta}\tilde{F}_{\alpha\beta}, \quad (6)$$

we can rewrite the Lagrangian Equation (1) in the following form:

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu D_\mu^{\text{eff}} - m)\psi(x) \quad (7)$$

where

$$D_\mu^{\text{eff}} = \partial_\mu - i\mu_e T_\mu \quad (8)$$

is the covariant derivative with μ_e acting as the coupling constant. Obviously, the Lagrangian Equation (7) is invariant under the gauge transformation

$$\begin{aligned}\psi(x) &\rightarrow \psi(x)e^{i\mu_e\chi(x)}, \\ T_\mu &\rightarrow T_\mu(x) + \partial_\mu\chi(x),\end{aligned}\quad (9)$$

with $\chi(x)$ being an arbitrary differentiable function. This shows that the Lagrangian Equation (1) can be identified as the $U(1)$ gauge theory when the effective gauge potential from Equation (6) is introduced.

The dynamical equation is derived from the Lagrangian Equation (7), and is

$$(i\gamma^\mu D_\mu^{\text{eff}} - m)\psi(x) = 0. \quad (10)$$

The solution to this dynamical equation is

$$\psi(x) = \psi_0(x) \exp\left\{i\mu_e \int_{x_0^\mu}^{x^\mu} T_\mu(z) dz^\mu\right\}, \quad (11)$$

in which $\psi_0(x)$ is the solution to Equation (10) in the absence of the effective gauge potential, while x_0^μ and x^μ are a fixed point and an arbitrary point, respectively, in $2 + 1$ -dimensional spacetime.

In the HMW setting, there is no electric field and the magnetic field is produced by a uniformly distributed magnetic charges on the long filament. The magnetic field produced by the filament is obvious along the radial direction. Using Gauss's theorem, we have

$$\vec{B} = \frac{\mu_0\lambda_m\vec{r}}{2\pi r^2}, \quad (12)$$

where μ_0 and λ_m are the permeability of the vacuum and the line density of the magnetic charges on the filament, respectively, and $r^2 = x^2 + y^2$ is the distance between the particle and the filament. Substituting Equation (12) into Equation (6), we obtain the effective gauge potential T^μ in the HMW setting, which is

$$T^\mu = (T^0, \vec{T}) = (0, -\frac{\mu_0\lambda_my}{2\pi r^2}, \frac{\mu_0\lambda_mx}{2\pi r^2}). \quad (13)$$

The effective field strength is

$$W^{\mu\nu} = \partial^\mu T^\nu - \partial^\nu T^\mu. \quad (14)$$

Because the effective gauge potential is static and because $T^0 = 0$, the corresponding effective field strength only has a magnetic component. Furthermore, a straightforward calculation shows that $B^{\text{eff}} = \mu_0\lambda_m\delta^2(r)$, meaning that $W^{\mu\nu} = 0$ in the area $r \neq 0$.

Thus, if the particle moves around the filament, it develops a topological phase. This topological phase is

$$\phi = \mu_e \oint \vec{T} \cdot d\vec{r} = \mu_0\mu_e\lambda_m. \quad (15)$$

The phase in Equation (15) coincides with the findings of previous studies [9]. This result shows that the phase obtained by identifying the model in Equation (1) with $U(1)$ gauge theory is equivalent to the traditional one.

We now generalize previous studies to noncommutative space. The noncommutative spacetime we are considering here is characterized by the algebra

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad (16)$$

where $\theta^{\mu\nu}$ is an anti-symmetric constant tensor. In order to avoid the unitarity problem, we only concentrate on the case in which the noncommutativity only happens among spatial coordinates, i.e., $\theta^{0\mu} = 0$; thus, the above noncommutative relation reduces to the form

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}. \quad (17)$$

In two-dimensional space, $\theta^{ij} = \theta\epsilon^{ij}$, with ϵ^{ij} being the Levi-Civita symbol in two dimensional space.

Quantum theories in the noncommutative spacetime Equation (16) can be constructed from the commutative ones. To do this, it is necessary to replace the ordinary product in commutative spacetime with the Moyal or ‘*’ product:

$$f(x) * g(x) = \exp\left\{\frac{i}{2}\theta^{\mu\nu}\partial_{y^\mu}\partial_{z^\nu}\right\}f(y)g(z)|_{y=z=x} \quad (18)$$

where $f(x)$ and $g(x)$ are arbitrary infinitely differentiable and rapidly decreasing functions in commutative 2 + 1-dimensional spacetime.

After introducing the effective gauge potential Equation (6), the Lagrangian of a dipole moment interacting with an electromagnetic field in the noncommutative spacetime Equation (16) is

$$\mathcal{L} = \bar{\psi}(x)i\gamma^\mu D_\mu^{\text{eff}} * \psi(x) - m\bar{\psi}(x) * \psi(x). \quad (19)$$

Following [54], we can apply the S-W map to map the model Equation (19) from the noncommutative space Equation (16) to the commutative one. Up to the first order of the NCP, the S-W map is [17,56]

$$\begin{aligned} \psi(x) &\rightarrow \psi(x) + \frac{\mu_e}{2}\theta^{\alpha\beta}T_\alpha\partial_\beta\psi, \\ T_\mu(x) &\rightarrow T_\mu(x) + \frac{\mu_e}{2}\theta^{\alpha\beta}A_\alpha(\partial_\beta T_\mu + W_{\beta\mu}). \end{aligned} \quad (20)$$

The Lagrangian Equation (19) becomes

$$\begin{aligned} \mathcal{L} &= (1 + \frac{\mu_e}{4}\theta^{\alpha\beta}W_{\alpha\beta})\bar{\psi}(x)(i\gamma^\mu D_\mu^{\text{eff}} - m)\psi(x) \\ &\quad - \frac{1}{2}i\mu_e\theta^{\alpha\beta}\bar{\psi}(x)\gamma^\mu W_{\mu\alpha}D_\beta^{\text{eff}}\psi(x). \end{aligned} \quad (21)$$

The effective field strength $W_{\mu\nu}$ and effective covariant derivative D_μ^{eff} are invariant and covariant under the gauge transformation Equation (9), i.e.,

$$\begin{aligned} W_{\mu\nu} &\rightarrow W'_{\mu\nu} = W_{\mu\nu}, \\ D_\mu^{\text{eff}}\psi(x) &\rightarrow D_\mu^{\text{eff}'}\psi'(x) = e^{i\chi(x)}D_\mu^{\text{eff}}\psi(x). \end{aligned} \quad (22)$$

Therefore, the invariance of the Lagrangian Equation (19) under the gauge transformation Equation (9) is obvious.

Neglecting the higher orders of the NCP, we next obtain the dynamical equation from the Lagrangian Equation (21):

$$(i\gamma^\mu D_\mu^{\text{eff}} - m)\psi(x) - \frac{1}{2}i\mu_e\theta^{\alpha\beta}\gamma^\mu W_{\mu\alpha}D_\beta^{\text{eff}}\psi(x) = 0. \quad (23)$$

It is important to remember that the region that the particle traverses has no effective field strength, i.e., $W^{\mu\nu} = 0$ for $r \neq 0$, which naturally leads to the conclusion that the dynamical equation takes the same form as the commutative space Equation (10). As a result, the particle receives the same phase as in the commutative space Equation (15) when it moves around the filament. Therefore, the spatial noncommutativity does not influence the phase of the HMW effect.

3. Approach II: Taking the Dipole Moment as a Pair of Oppositely Charged Particles

In this section, we study the noncommutative HMW effect from the other approach, i.e., taking the electric dipole as a pair of oppositely charged particles [11]. Thus, unlike the previous section, in which the particle was taken to be structureless and we worked in $2 + 1$ -dimensional spacetime, in the present approach we need to work in $3 + 1$ -dimensional spacetime.

We start from the model of a charged spin- $\frac{1}{2}$ particle interacting with an electromagnetic field in $3 + 1$ -dimensional commutative spacetime. The Lagrangian is

$$\mathcal{L} = \bar{\psi}(ig_{\mu\nu}\gamma^\mu D^\nu - m)\psi, \quad (24)$$

where $g^{\mu\nu} = g_{\mu\nu} = \text{diag}(-, +, +, +)$ is the metric of the $3 + 1$ -dimensional spacetime, $\gamma^\mu = (\gamma^0, \gamma^i)$, $i = 1, 2, 3$ are 4×4 dimensional matrices for which the explicit expressions are $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$, and $D^\mu = \partial^\mu + iqA^\mu$ is the covariant derivative, with q being the charges carried by the particle and A^μ the electromagnetic potential, and ψ the four-component Dirac spinor. The dynamical equation is obtained from the Lagrangian Equation (24):

$$(ig_{\mu\nu}\gamma^\mu D^\nu - m)\psi = 0. \quad (25)$$

The solution to this equation of motion is

$$\psi(x) = \psi_0(x) \exp\left\{-iq \int_{x_0}^{x^\mu} A_\mu(z) dz^\mu\right\}, \quad (26)$$

where $\psi_0(x)$ is the solution to Equation (25) in the absence of the electromagnetic potential.

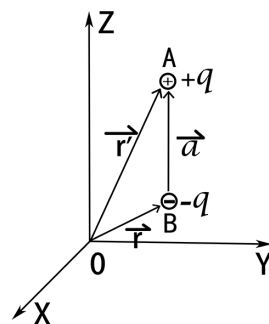


Figure 1. A schematic drawing of the HMW configuration in which the electric dipole is taken as a pair of oppositely charged particles. This electric dipole moves around the Z-axis, keeping the electric dipole moment parallel to the Z-axis, where the long filament carried by the magnetic charge is localized. The motions of particles A and B are parallel to the XOY plane, while the radial vectors of particles A and B are $\vec{r}'$ and $\vec{r}$, respectively.

In the setting of the HMW effect, there is no electric field and the magnetic field is produced by the uniformly distributed magnetic charges in the filament Equation (12). The electromagnetic potential takes the following form:

$$A^\mu = (0, \vec{A}), \quad (27)$$

where $\vec{A}$ is the magnetic potential. In the symmetric gauge, this is [11]

$$\vec{A} = \left(\frac{\mu_0 \lambda_m y z}{2\pi \rho^2}, -\frac{\mu_0 \lambda_m x z}{2\pi \rho^2}, 0\right), \quad (28)$$

where $\rho = \sqrt{x^2 + y^2}$ is the distance between the electric dipole and the filament.

For definiteness, we denote these two particles as A and B, which carry electric charges $+q$ and $-q$, respectively. Thus, when these two particles circle around the filament, they acquire the phases

$$\phi_A = -q \oint \vec{A}(\vec{r}') \cdot d\vec{r}' \quad (29)$$

and

$$\phi_B = q \oint \vec{A}(\vec{r}) \cdot d\vec{r}, \quad (30)$$

where $\vec{r}'$, $\vec{r}$ are the respective radius vectors of particles A and B; here $\vec{r}' = \vec{r} + \vec{a}$, with $\vec{a}$ being the vector between particles B and A. Therefore, the phase of the electric dipole is $\phi = \phi_A + \phi_B$. Noting that the electric dipole is parallel to the filament and that $a = |\vec{a}|$ is small, we find that this phase becomes

$$\phi = \phi_A + \phi_B = q \oint \frac{a\mu_0\lambda_m}{2\pi r^2} (ydx - xdy) = aq\mu_0\lambda_m. \quad (31)$$

Because $\mu_e = qa$, we find that this phase is equivalent to the previous one (15) obtained by identifying the model Equation (1) with $U(1)$ gauge theory.

In order to study the HMW effect in noncommutative space by taking the electric dipole as a pair of oppositely charged particles, we can calculate the phase of each of these particle in noncommutative space. Because we are working in $3+1$ -dimensional spacetime, it is necessary to relabel the spatial noncommutativity. As before, the algebraic relation of the noncommutative spacetime is characterized by Equation (16) and we only consider the case in which $\theta^{0\mu} = 0$; thus, the algebra which characterizes the noncommutative space becomes $[x^i, x^j] = i\theta^{ij}$. We introduce a vector $\vec{\theta} = (\theta^1, \theta^2, \theta^3)$. The NCP θ^{ij} relates $\vec{\theta}$ by $\theta^{ij} = \epsilon^{ijk}\theta^k$, or equivalently, $\theta^i = \frac{1}{2}\epsilon^{ijk}\theta^{jk}$.

As mentioned above, for the sake of generalizing the above studies to the noncommutative spacetime it is desirable to replace the ordinary product in the Lagrangian Equation (24) with the Moyal one in the noncommutative spacetime Equation (16). Thus, the Lagrangian Equation (24) should be

$$\mathcal{L} = \bar{\psi}(x)ig_{\mu\nu}\gamma^\mu D^\nu * \psi(x) - m\bar{\psi}(x) * \psi(x). \quad (32)$$

We can apply the S-W map to map the Lagrangian (32) to the commutative spacetime. Up to the first order of the NCP, the S-W map is

$$\begin{aligned} \psi(x) &\rightarrow \psi(x) - \frac{q}{2}\theta^{\alpha\beta}A_\alpha\partial_\beta\psi, \\ A_\mu(x) &\rightarrow A_\mu(x) - \frac{q}{2}\theta^{\alpha\beta}A_\alpha(\partial_\beta A_\mu + F_{\beta\mu}) \end{aligned} \quad (33)$$

in which $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the field strength. After the S-W mapping, the Lagrangian Equation (24) becomes

$$\begin{aligned} \mathcal{L} &= (1 - \frac{q}{4}\theta^{\alpha\beta}F_{\alpha\beta})\bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x) \\ &\quad + \frac{1}{2}iq\theta^{\alpha\beta}\bar{\psi}(x)\gamma^\mu F_{\mu\alpha}D_\beta\psi(x). \end{aligned} \quad (34)$$

The dynamical equation can be derived from the above Lagrangian. Up to the first order of the NCP, we obtain

$$(i\gamma^\mu D_\mu - m)\psi(x) + \frac{1}{2}iq\theta^{\alpha\beta}\gamma^\mu F_{\mu\alpha}D_\beta\psi(x) = 0. \quad (35)$$

At first glance, the Lagrangian Equation (34) and dynamical Equation (35) take the same forms as those of Equations (21) and (23). Nevertheless, they are quite different. In Equations (21) and (23), the effective field strength $W_{\mu\nu}$ is vanishing in the region

traversed by the particle. Therefore, we reach the conclusion that there is no correction due to spatial noncommutativity. At this point, the area circled by the compound particle system is not field strength-free; instead, there is a radial magnetic field produced by the magnetic charges on the filament. Thus, it is necessary to examine the noncommutative correction more carefully.

The solution to this dynamical equation is

$$\begin{aligned}\psi(x) = & \psi_0(x) \exp\left\{-i \int_{x_0^\mu}^{x^\mu} (q A_\mu(z) \right. \\ & \left. + i \frac{1}{2} q F_{\mu\alpha}(z) \theta^{\alpha\beta} D_\beta(z)) dz^\mu\right\}.\end{aligned}\quad (36)$$

Therefore, when the charged particle circles around the filament, it receives a phase, which is

$$\begin{aligned}\phi^{NC} &= \phi + \phi^\theta \\ &= -q \oint A_\mu dx^\mu - \frac{1}{2} q \oint F_{\mu\alpha} \theta^{\alpha\beta} p_\beta dx^\mu.\end{aligned}\quad (37)$$

Obviously, the second term is the noncommutative correction.

Bearing in mind that $\theta^{0\mu} = 0$ and that spatial noncommutativity only makes sense on planes perpendicular to z-axis, i.e., $\vec{\theta} = \theta \vec{e}_z$, we find that the noncommutative correction in Equation (37) can be written as

$$\begin{aligned}& -\frac{1}{2} q \oint F_{\mu\alpha} \theta^{\alpha\beta} p_\beta dx^\mu \\ &= -\frac{1}{2} q \oint (\vec{B} \cdot \vec{p})(\vec{\theta} \cdot d\vec{r}) + \frac{1}{2} q \oint (\vec{\theta} \cdot \vec{B})(\vec{p} \cdot d\vec{r}).\end{aligned}\quad (38)$$

Because $\vec{\theta} \perp d\vec{r}$ and $\vec{\theta} \perp \vec{B}$, we find that both of the above terms equal zero.

This result indicates that the spatial noncommutativity does not contribute extra phases to each charged particle. The total phase of the neutral particle in noncommutative space is equal to its commutative counterpart in Equation (31). Therefore, we reach the conclusion that if we take the electric dipole moment as a pair of oppositely charged particles, then there is no correction to the HMW effect up to the first order of the NCP.

4. Conclusions and Remarks

Topological phases in noncommutative space remain controversial topics. Most studies have shown corrections due to spatial noncommutativity. However, other studies have reached different conclusions [52,55,57]. In the present paper, we have investigated the HMW effect in noncommutative space using two different approaches.

Our findings reveal that both approaches converge on a shared conclusion: up to the first order of the noncommutative parameter (NCP), there exists no correction to the HMW effect. In the first approach, the effective field strength within the particle's traversed region diminishes to zero. This outcome implies that neither the dynamical equations nor the wave function of the neutral particle are affected by spatial noncommutativity. Consequently, no corrections to the HMW effect are observed, indicating a lack of influence of spatial noncommutativity on the particle's dynamics and wave function. In the second approach, the noncommutative correction is zero because of the specific configuration, i.e., $\vec{\theta} \perp d\vec{r}$, $\vec{\theta} \perp \vec{B}$; therefore, the conclusion that spatial noncommutativity does not contribute to the phase of the HMW effect up to the first order of the NCP can be obtained naturally. Spatial noncommutativity, if any exists, must be extremely small; thus, any higher-order corrections due to spatial noncommutativity are negligible.

In closing this paper, it should be mentioned that in addition to the standard interpretations of the topological phases, there are several alternative explanations [58–63].

In the standard interpretation, particles are taken as force-free and the topological phases are attributed to quantum non-locality. However, it has been argued that interactions between particles and electric or magnetic fields should change the sources which produce these electric or magnetic fields. When these changes are taken into account, the phases can be explained as the results of local interactions. In [59] it was proved that, in the AB effect, the solenoid induces an electric field which can act locally on the charged particles as they pass by. The induced electric field produces a classical lag, which generates a phase shift identical to that derived by Aharonov and Bohm. In [60], it was argued that when the source of the electromagnetic potential is treated in a quantum mechanical fashion, the AB effect can be explained by local action of the electron field exerted upon the source of the potential. In a recent paper [62], the authors obtained the topological phases of the AB, AC, and Spavieri effects [64] from the actions of local electromagnetic effective forces. Hence, it is of interest to investigate whether it is possible to interpret topological phases in noncommutative space from the point of view of local interactions. This topic may be deserving of further study.

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Conflicts of Interest: The authors declare no conflict of interest.

Note

- ¹ We note that the HMW effect in noncommutative space was investigated in [48]; however, both the methods and conclusions of [48] were different from those of the present study.

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