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Gravitational Wormholes

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Abstract: Spacetime wormholes are evidently an essential component of the construction of a time machine. Within the context of general relativity, such objects require, for their formation, exotic matter—matter that violates at least one of the standard energy conditions. Here, we explore the possibility that higher-curvature gravity theories might permit the construction of a wormhole without any matter at all. In particular, we consider the simplest form of a generalized quasi topological theory in four spacetime dimensions, known as Einsteinian Cubic Gravity. This theory has a number of promising features that make it an interesting phenomenological competitor to general relativity, including having non-hairy generalizations of the Schwarzschild black hole and linearized equations of second order around maximally symmetric backgrounds. By matching series solutions near the horizon and at large distances, we find evidence that strong asymptotically AdS wormhole solutions can be constructed, with strong curvature effects ensuring that the wormhole throat can exist.

Keywords: wormhole; quasi-topological gravity; Einsteinian Cubic Gravity

1. Introduction

It has long been known that if spacetime is to have closed timelike curves in some local regions [1], then wormholes are an essential part of this construction [2,3]. However, a key characteristic of such objects is that they require exotic matter that does not respect the energy conditions. Despite the challenges presented in constructing wormholes [4], the search nevertheless continues in the hopes of evading the constraints imposed by quantum physics in Einsteinian geometries [5].

Much effort has gone into exploring modified gravity to this end [6–14]. The higher-curvature Lovelock theories [15,16] have been of particular interest [17–24] since the field equations are of second order in the metric components. However, non-trivial solutions exist only in spacetime dimensions $D \geq 5$. Curiously, quasi-topological gravities [25–30] could also be considered for wormhole solutions, but none have been obtained to date. However, such objects would also exist only in $D \geq 5$ dimensions.

Both Lovelock and quasi-topological theories have been shown to be particular cases of a more general class called Generalized Quasi-Topological gravity (GQTG) [31–35]. These theories are characterized by second-order linearized equations around maximally symmetric backgrounds and admit single-function ($g_{tt}g_{rr} = -1$) non-hairy generalizations of the Schwarzschild black hole. These theories are ghost-free on constant-curvature backgrounds, but, on a generic background, will have ghosts. However, such ghosts cannot escape to infinity in spacetimes that are asymptotically of constant curvature. The effects of the additional degrees of freedom in GQTGs have not been fully explored, but it is known that they significantly modify the thermodynamics of black holes for small masses [34] and, in the cubic and quartic cases, exhibit an number of interesting features [34,36–41]. A comprehensive list of their properties has been given [35,42]. A key advantage of GQTGs is that they have non-trivial field equations (and solutions) in $D = 4$ dimensions.

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In this paper, we carry out the first investigations of wormhole solutions in $D = 4$ Generalized Quasi-Topological gravity. For specificity, we shall consider the simplest GQTG, a theory known as Einsteinian Cubic Gravity (ECG) [31,43], whose action is of the form

$$\mathcal{I} = \frac{1}{16\pi} \int d^4x \sqrt{-g}(-2\Lambda_0 + R + \alpha\mathcal{P} + \beta\mathcal{C} + \gamma\mathcal{C}'), \tag{1}$$

with α, β and γ being coupling constants, and three densities that are cubic in the Riemann curvature, given by

$$\mathcal{P} = 12R_{a\ b}^{\ c\ d}R_{c\ d}^{\ e\ f}R_{e\ f}^{\ a\ b} + R_{ab}^{\ \ cd}R_{cd}^{\ \ ef}R_{ef}^{\ \ ab} - 12R_{abcd}R^{ac}R^{bd} + 8R^b{}_aR^c{}_bR^a{}_c, \tag{2}$$

$$\mathcal{C} = \frac{1}{2}RR_a{}^bR_b{}^a - 2R^{ac}R^{bd}R_{abcd} - \frac{1}{4}RR_{abcd}R^{abcd} + R^{de}R_{abcd}R^{abc}{}_e, \tag{3}$$

$$\mathcal{C}' = R_a{}^bR_b{}^cR_c{}^a - \frac{3}{4}RR_a{}^bR_b{}^a + \frac{1}{8}R^3. \tag{4}$$

For a general static spherically symmetric ansatz (GSSS)

$$ds^2 = -f(r)dt^2 + \frac{1}{N(r)f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5}$$

the densities $\mathcal{C}$ and $\mathcal{C}'$ do not contribute in a linearly independent way to the field equations. Both of these terms become trivial when the metric function $N(r)$ is a constant. This situation is the one generally considered in ECG and clearly does not admit a vacuum wormhole. Note that, regardless of the values that the higher-curvature couplings take, the Einstein–AdS limit of the theory at large distances is preserved (albeit with a modified cosmological constant), since the contributions from the cubic terms fall off much more rapidly than the contributions from the Einstein–Hilbert part of the action if AdS asymptotic behaviour is imposed as a requirement.

We therefore consider the GSSS ansatz with $N' \neq 0$. We find that this situation is possible if the spacetime has a spherical deficit/surfeit angle in the asymptotic large-r region. We shall specifically consider solutions whose metric functions have the asymptotic form

$$g_{tt} \sim -\left(\frac{r^2}{l^2} + 1 + \delta\right) + \mathcal{O}(r^{-1}), \quad g^{rr} \sim \frac{r^2}{l^2} + 1 + \delta + \mathcal{O}(r^{-1}) \tag{6}$$

where l is the AdS length scale and $\delta \neq 0$ parametrizes the deficit/surfeit angle.

We find that Einsteinian Cubic Gravity—and, by implication, higher-order GQTGs—admits wormhole solutions that are purely gravitational without any exotic matter. The solutions that we obtain are asymptotically anti-de Sitter, with a spherical deficit angle resembling that of a global monopole. Unlike other solutions with radial symmetry, these solutions have non-zero values for the coupling parameter β. We find that including β provides a sufficiently large number of parameters to match the series solutions for the two metric functions over a broad range of radii at which the matching takes place.

2. The Non-Linear ODE System

For the ansatz (5), we find that the two independent field equations are

$$
\begin{aligned}
&-\frac{1}{8r^5}\Bigg\{8r\Big(r^2(-1+r^2\Lambda_0)+6\alpha N^2f'^2(-1+r^2f'N')+rNf'(r^2+12\alpha f'N')\Big)\\
&+2f\Big[4r^2N'(r^2+15\alpha f'N')+rN\Big(3r^2(74\alpha+5\beta)f'^2N'^2+4r(r+48\alpha N'f'')\\
&-24f'\Big((8\alpha-\beta)N'-8r\alpha N''\Big)\Big)+24r\alpha N^3\Big(4f'^2+r^2f''^2+rf'(-4f''+rf''')\Big)\\
&+12N^2\Big(f'\Big(8\alpha+r^3(22\alpha+\beta)N'f''\Big)+8r^2\alpha f'^2\Big(-3N'+rN''\Big)+4r\alpha\Big(-2f''+rf'''\Big)\Big)\Big]\\
&+6f^3\Big[r^2N'^2\Big(-4\alpha N'+r(8\alpha+\beta)N''\Big)+4N^2\Big(2\beta N'-r\Big((-4\alpha+\beta)N''+4r\alpha N'''\Big)\Big)\\
&+rN\Big((8\alpha-7\beta)N'^2+2r^2(8\alpha+\beta)N''^2+2rN'\Big(-20\alpha N''+r(8\alpha+\beta)N'''\Big)\Big)\Big]\\
&+3f^2\Big[rN'\Big(6\beta N'+r^2(28\alpha+3\beta)f'N'^2+16r\alpha N''\Big)-32\alpha N^3\Big(2f'+r(-2f''+rf''')\Big)\\
&+2N\Big(2r^2N'^2\Big(-52\alpha f'+r(26\alpha+3\beta)f''\Big)+N'\Big(-8\beta+r^3(108\alpha+11\beta)f'N''\Big)\\
&+4r\Big((-4\alpha+\beta)N''+4r\alpha N'''\Big)\Big)+4rN^2\Big[r\Big(r(12\alpha+\beta)f''N''+N'\Big(-48\alpha f''+r(8\alpha+\beta)f'''\Big)\Big)\\
&+f'\Big(6(8\alpha-\beta)N'+4r\alpha(-12N''+rN''')\Big)\Big]\Big]\Bigg\}=0
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
&\frac{1}{8r^5}\Bigg\{-8r\Big(r^2(-1+r^2\Lambda_0)+6\alpha N^2f'^2(-1+r^2f'N')+rNf'(r^2+12\alpha f'N')\Big)\\
&+6\beta f^3N\Big(rN'^2+N(-8N'+4rN'')\Big)-2f\Bigg[-12r^2\alpha f'N'^2+rN\Bigg(3r^2(26\alpha+3\beta)f'^2N'^2\\
&+4r\Big(r+12\alpha N'f''\Big)+24f'\Big((-2\alpha+\beta)N'+2r\alpha N''\Big)\Bigg)+24r\alpha N^3\Big(4f'^2+r^2f''^2\\
&+rf'(-4f''+rf''')\Big)+12N^2\Bigg(f'\Big(8\alpha+r^3(12\alpha+\beta)N'f''\Big)+2r^2\alpha f'^2(-8N'+rN'')\\
&+4r\alpha(-2f''+rf''')\Bigg)\Bigg]+3f^2\Bigg[rN'^2\Big(2\beta+r^2(8\alpha+\beta)f'N'\Big)-2N\Bigg(2r^2N'^2\Big(-4\alpha f'\\
&+r(8\alpha+\beta)f''\Big)+4r\beta N''+N'\Big(-8\beta+3r^3(8\alpha+\beta)f'N''\Big)\Bigg)+32\alpha N^3\Bigg(2f'+r\\
&\Big(-2f''+rf'''\Big)\Bigg)-4rN^2\Bigg(2f'\Big((8\alpha-3\beta)N'-8r\alpha N''\Big)+r\Big(r(8\alpha+\beta)f''N''+N'\\
&\Big(-16\alpha f''+r(8\alpha+\beta)f'''\Big)\Big)\Bigg)\Bigg]\Bigg\}=0
\end{aligned}
\tag{8}
$$

where, without loss of generality, we can set $\gamma = 0$ in (1), since its inclusion simply reproduces the preceding equations but with $\beta \to \beta + \gamma/2$.

If $N(r)$ is constant, then these equations become linearly dependent, and a wormhole solution is not possible since there will be a single metric function whose largest root corresponds to the horizon of a black hole.

We can see this by considering the series expansions

$$f_\infty = 1 - \frac{\Lambda}{3}r^2 + \sum_{i=1} \frac{\tilde{f}_i}{r^i}, \quad N_\infty = 1 + \sum_{i=1} \frac{\tilde{N}_i}{r^i} \tag{9}$$

in the asymptotically distant region at large r, where asymptotically flat solutions have $\Lambda = 0$. Inserting these into the field Equations (7) and (8) yields

$$h(\Lambda) = 0 \quad \tilde{N}_i = 0 \tag{10}$$

$$f_\infty = 1 - \frac{\Lambda}{3}r^2 - \frac{r_h}{r} + \frac{(54 - 28\Lambda r^2)\alpha r_h^2}{h'(\Lambda) r^6} - \frac{138\alpha r_h^3 + 20192\Lambda^2\alpha^2 r_h^3}{3h'^2 r^7} + \mathcal{O}(r^{-9}) \tag{11}$$

in the limit $r \to \infty$, where

$$h(x) \equiv \frac{8\alpha}{9}x^3 + x - \Lambda_0 \tag{12}$$

and

$$\Lambda = -\frac{3}{l^2} \equiv -3L, \quad h'(\Lambda) = \frac{dh}{dx}\bigg|_\Lambda = \frac{8\alpha}{3}\Lambda^2 + 1 = 24\alpha L^2 + 1 \tag{13}$$

and all β-dependent terms vanish.

From these formulae, we observe several aspects. First, a power-series solution in $1/r$ implies only a single independent function $f(r)$, which is the hallmark of GQTGs. Second, for asymptotically flat solutions $\Lambda = 0$, which in turn implies $\Lambda_0 = 0$, we also have $h' = 1$, and so the asymptotically flat solution can be immediately obtained from (11) by setting $\Lambda = 0$. However, note that the converse is not true: even if $\Lambda_0 = 0$, it is possible to have asymptotically de Sitter solutions with $\Lambda = 3/\sqrt{8|\alpha|}$ provided that $\alpha < 0$.

We seek solutions that have the asymptotic form (6), where $N(r)$ is not constant so as to obtain wormhole solutions. The presence of the wormhole needs to be manifest at large-r in a way that differs from that of a spherically symmetric star or black hole. To this end, we consider the ansatz

$$f_\infty = K + Lr^2 + \sum_{i=1} \frac{a_i}{r^i}, \quad N_\infty = 1 + \sum_{i=1} \frac{b_i}{r^i} \tag{14}$$

where the quantity $K = 1 + \delta$ parameterizes a spherical deficit/surfeit angle produced by the wormhole. The effect is analogous to that produced by a global monopole [44,45]. Far from the wormhole, all light rays are deflected by the same angle regardless of their impact parameter.

Inserting the ansatz (14) into the field equations yields

$$b_1 L(24L^2\alpha + 1) = 0 \tag{15}$$

from both equations to leading order. This equation is satisfied by choosing either $b_1 = 0$ or $(24L^2\alpha + 1) = 0$. However, it is straightforward to show that the next order forces $b_1 = 0$ regardless of the value of $(24L^2\alpha + 1)$. However, if this latter quantity is non-zero, then it is straightforward to show that there is no deficit angle, and all b_i coefficients must vanish as per the discussion above.

Setting $(24L^2\alpha + 1) = 0$ then yields the non-trivial solution

$$\begin{aligned} N_\infty &= 1 - \frac{5a_1}{2Lr^3(9\beta L^2 - 2)} + \frac{135a_1^2\left(\beta L^2\left(99\beta L^2 - 49\right) + 6\right)}{16(K-1)Lr^4(9\beta L^2 - 2)^3} \\ &+ \frac{a_1\left(192(K-1)^2K\left(2 - 9\beta L^2\right)^2 - 27a_1^2 L\left(3\beta L^2\left(3972\beta L^2 - 2207\right) + 919\right)\right)}{64(K-1)^2L^2r^5(9\beta L^2 - 2)^3} + \cdots \end{aligned} \tag{16}$$

$$
\begin{aligned}
f_\infty &= K + Lr^2 + \frac{a_1}{r} - \frac{9a_1^2\left(828\beta^2L^4 - 363\beta L^2 + 37\right)}{32(K-1)r^2\left(9\beta L^2 - 2\right)^2} \\
&+ \frac{a_1\left(9a_1^2L\left(3\beta L^2\left(9\beta L^2\left(9932\beta L^2 - 6829\right) + 13340\right) - 2641\right)\right)}{224(K-1)^2Lr^3\left(9\beta L^2 - 2\right)^3} \\
&+ \frac{a_1\left(-32(K-1)^2K\left(2 - 9\beta L^2\right)^2\left(9\beta L^2 - 6\right)\right)}{224(K-1)^2Lr^3\left(9\beta L^2 - 2\right)^3} + \cdots
\end{aligned}
\tag{17}
$$

where $\Lambda_0 = -2L$ from (10) and (12).

We now see that the condition $h'(\Lambda) = 0$ from (13) allows for N to be a non-constant function, opening up the possibility of obtaining a wormhole solution. We pursue this in the next section.

3. Series Solutions

Anticipating the asymptotic behaviour (14), we rewrite the general static spherically symmetric (GSSS) ansatz (5) in the form

$$
ds^2 = -\frac{r_0^2 g(x)}{(1-x)^2}d\tilde{t}^2 + \frac{r_0^2 dx^2}{n(x)g(x)(1-x)^2} + \frac{r_0^2}{(1-x)^2}\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{18}
$$

using the coordinate transformation

$$
x = 1 - r_0/r \qquad \tilde{t} = t/r_0
$$

where the metric functions $n(x)$ and $g(x)$, defined on $x \in [0,1]$, are

$$
n = N \quad g = \frac{fr_0^2}{r^2} \tag{19}
$$

with r_0 a positive constant.

For wormhole solutions [1], these continuous functions must be everywhere positive in the interior of the domain, with n vanishing and g having a finite positive value at $x = 0$, which locates the position of the wormhole throat. Under this map, $r \to \infty$ is compactified to $x = 1$. With this new ansatz, the boundary condition (6) is equivalent to

$$
g \sim \frac{r_0^2}{l^2} + \frac{(1+\delta)r_0^2}{r^2} + \mathcal{O}(r^{-3}), \quad n \sim 1 + \mathcal{O}(r^{-3}) \tag{20}
$$

as x approaches 1. The effect of δ is analogous to that produced by a global monopole [44,45], which deflects light rays by the same angle regardless of their impact parameter.

The advantage of the ansatz (18) is clear—it compactifies the domain so that numerical and semi-analytic solutions become more easily attainable. We now employ this ansatz to obtain series solutions for the functions n and g. The field equations for $g(x)$ and $n(x)$ are given in Appendix A.

3.1. Large-r Solution

To obtain solutions asymptotic to (20), we substitute the formal series

$$
g = a_0 + \sum_{n=2}^{\infty} a_n(1-x)^n, \quad n = 1 + \sum_{n=3}^{\infty} b_n(1-x)^n, \quad a_0 = r_0^2/l^2 \equiv Lr_0^2, \quad a_2 = 1 + \delta \tag{21}
$$

into the equations and solve them order by order in $(1-x)$. Note that we have set $b_1 = b_2 = 0$ due to the discussion following condition (15). The lowest two orders yield two constraints $h(a_0) = 0$ and $(24\alpha L^2 + 1)(a_2 - 1) = 0$. The first of these simply defines Λ_0 in terms of the other parameters. Solutions with a vanishing deficit $a_2 = 1$ satisfy the second constraint but force $b_{n\geq 3}=0$ or, in other words, $n=1$. The only alternative non-trivial solution occurs when $a_2\neq 1$, yielding

$$
\begin{aligned}
g = \; & Lr_0^2 + a_2(1-x)^2 - \frac{2}{5}b_3 L r_0^2 (1-x)^3\big(9\beta L^2 - 2\big) \\
& - \frac{9b_3^2 L^2 r_0^4 (1-x)^4 \big(828\beta^2 L^4 - 363\beta L^2 + 37\big)}{200(a_2-1)} \\
& + \frac{3b_3(1-x)^5}{3500(a_2-1)^2}\Bigg\{\big(200a_2^3\big(3\beta L^2-2\big) - 400a_2^2\big(3\beta L^2-2\big) + 200a_2\big(3\beta L^2-2\big)\big) \\
& \quad + \big(3b_3^2 L^3 r_0^6\big(-268164\beta^3 L^6 + 184383\beta^2 L^4 - 40020\beta L^2 + 2641\big)\big)\Bigg\} \\
& - \frac{b_3^2 L r_0^2 (1-x)^6}{4480000(a_2-1)^3}\Bigg\{\big(-43200(a_2-1)^2(1803a_2+85)\beta^2 L^4 \\
& + 13200(a_2-1)^2(3063a_2+545)\beta L^2 - 400(a_2-1)^2(13371a_2+3535) \\
& + 72626980752\beta^4 b_3^2 L^{11} r_0^6 - 68916687624\beta^3 b_3^2 L^9 r_0^6 + 23598134433\beta^2 b_3^2 L^7 r_0^6 \\
& - 3385644966\beta b_3^2 L^5 r_0^6 + 164339937 b_3^2 L^3 r_0^6\big)\Bigg\} \\
& + \frac{b_3(1-x)^7}{392000000(a_2-1)^4 L r_0^2}\Bigg\{36000a_2 b_3^2 L^3 r_0^6\big(5699484\beta^3 L^6 - 3438357\beta^2 L^4 \\
& + 659159\beta L^2 - 39732\big) + 800a_2^3\big(295678404\beta^3 b_3^2 L^9 r_0^6 - 211323051\beta^2 b_3^2 L^7 r_0^6 \\
& + 49375923\beta b_3^2 L^5 r_0^6 - 3749468 b_3^2 L^3 r_0^6 + 336000\beta L^2 - 448000\big) \\
& - 800a_2^2\big(571755996\beta^3 b_3^2 L^9 r_0^6 - 394347609\beta^2 b_3^2 L^7 r_0^6 + 88894962\beta b_3^2 L^5 r_0^6 \\
& - 6518172 b_3^2 L^3 r_0^6 + 84000\beta L^2 - 112000\big) - 22400000a_2^6\big(3\beta L^2 - 4\big) \\
& + 89600000a_2^5\big(3\beta L^2 - 4\big) - 134400000a_2^4\big(3\beta L^2 - 4\big) \\
& + b_3^2 L^3 r_0^6\big(-170858662247952\beta^5 b_3^2 L^{13} r_0^6 + 207845496209160\beta^4 b_3^2 L^{11} r_0^6 \\
& - 98339490752265\beta^3 b_3^2 L^9 r_0^6 + 22351263412410\beta^2 b_3^2 L^7 r_0^6 - 2386812017025\beta b_3^2 L^5 r_0^6 \\
& + 91235810916 b_3^2 L^3 r_0^6 + 15680649600\beta^3 L^6 - 22638794400\beta^2 L^4 + 7885507200\beta L^2 \\
& - 784611200\big)\Bigg\} + \cdots
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
n = \quad & 1 + b_3(1-x)^3 + \frac{27 b_3^2 L r_0^2 (1-x)^4 \left(11\beta L^2 - 3\right)}{20(a_2-1)} \\
& + \frac{3 b_3 (1-x)^5 \left(-400 a_2^3 + 800 a_2^2 - 400 a_2 + 9 b_3^2 L^3 r_0^6 \left(11916\beta^2 L^4 - 6621\beta L^2 + 919\right)\right)}{1000(a_2-1)^2 L r_0^2} \\
& + \frac{b_3^2 (1-x)^6}{3500(a_2-1)^3} \Big\{ \big(-750(a_2-1)^2(105a_2+16)\beta L^2 + 125(a_2-1)^2(175a_2+22) \\
& + 28954908\beta^3 b_3^2 L^9 r_0^6 - 24304212\beta^2 b_3^2 L^7 r_0^6 + 6795981\beta b_3^2 L^5 r_0^6 - 633051 b_3^2 L^3 r_0^6\big) \Big\} \\
& + \frac{9 b_3 (1-x)^7}{6272000(a_2-1)^4 L^2 r_0^4} \Big\{ -160 a_2^3 \big(2795796\beta^2 b_3^2 L^7 r_0^6 - 1577271\beta b_3^2 L^5 r_0^6 \\
& + 222099 b_3^2 L^3 r_0^6 + 22400\big) + 160 a_2^2 \big(5138172\beta^2 b_3^2 L^7 r_0^6 - 2934417\beta b_3^2 L^5 r_0^6 \\
& + 417978 b_3^2 L^3 r_0^6 + 5600\big) - 480 a_2 b_3^2 L^3 r_0^6 \big(629652\beta^2 L^4 - 379007\beta L^2 + 56553\big) \\
& + 896000 a_2^6 - 3584000 a_2^5 + 5376000 a_2^4 + 3 b_3^2 L^3 r_0^6 \big(54758424816\beta^4 b_3^2 L^{11} r_0^6 \\
& - 61487988408\beta^3 b_3^2 L^9 r_0^6 + 25878434823\beta^2 b_3^2 L^7 r_0^6 - 4838200650\beta b_3^2 L^5 r_0^6 \\
& + 339035463 b_3^2 L^3 r_0^6 - 24182400\beta^2 L^4 + 11740000\beta L^2 - 1398400\big) \Big\} + \cdots
\end{aligned}
\tag{23}
$$

where

$$24\alpha L^2 + 1 = 0 \tag{24}$$

and Λ_0 and α are replaced by expressions in terms of L via the two constraints $h(a_0) = 0$ and (24).

The parameters $a_0 = L r_0^2$, a_2 and b_3 are the only free variables in this solution; we also have $n(1) = n_0 = 1$. Furthermore β is an independent coupling parameter. It can happen in a non-linear system that fewer constants of integration appear in a solution than the differential order of the equations. Due to the non-linearity, a spontaneous singularity could appear, which means that the radii of convergence for g and n depend on the initial conditions, namely the values of these parameters. We do not expect vanishing radii of convergence for all values of the parameters, since the series with $b_3 = 0$ converges to the AdS solution with a deficit.

3.2. Near-Throat Solution

There is likewise a near-throat solution for $(24L^2\alpha + 1) = 0$. Local solutions near $x = 0$ compatible with (22) and (23) are necessarily Taylor series. In this case, the desired boundary conditions at the throat require the ansatz to be

$$n = \sum_{n=1}^{\infty} A_n x^n \qquad g = B_0 + \sum_{n=1}^{\infty} B_n x^n, \quad B_0 \neq 0, \tag{25}$$

which yields two series whose coefficients are fully determined by A_1, B_0 and $a_0 = L r_0^2$. We obtain

$$
\begin{aligned}
n_{th} &= A_1 x + \frac{1}{8A_1B_0^2(A_1B_0(3\beta L^2-1)-1)^2} \\
&\Big\{ 2A_1^2B_0^2(1-3\beta L^2)\left(A_1^2B_0^2(5-12\beta L^2)+9A_1B_0+4\right) \\
&-8L^2{r_0}^4\left(2A_1^2B_0^2(3\beta L^2-1)+A_1B_0(7-24\beta L^2)+12\right) \\
&+48L^3{r_0}^6\left(A_1B_0(8\beta L^2-3)-4\right)\Big\} x^2 + \cdots
\end{aligned} \tag{26}
$$

$$
g_{th} = B_0 - \frac{2L^2\left(3A_1^2B_0^2(A_1B_0+1)\left(\beta-\frac{1}{3L^2}\right)+8L{r_0}^6+4{r_0}^4\right)}{A_1^2B_0(A_1B_0(3\beta L^2-1)-1)} x + \cdots \tag{27}
$$

Higher-order terms are very lengthy and cumbersome to write; we present some of them in Appendix B.

Note that in both series solutions (near $x=1$ and near $x=0$), the independent parameters r_0 and β always appear as Lr_0^2 and $L^2\beta$. Consequently, we can set $L=1$ and regard r_0 and β as independent parameters without loss of generality.

4. Matching the Solutions

As a consequence of the uniqueness of Taylor series, we expect that the Taylor expansions of (22) and (23) at $x=0$ can be matched with (26) and (27) as long as a wormhole solution (analytic on $[0,1]$) exists for some values of $(\beta, r_0, a_2, b_3, A_1, B_0)$. We achieve this matching by minimizing the quantity

$$
\Delta(x_0) \equiv \left(\frac{\Delta g}{0!}\right)^2 + \left(\frac{\Delta g'}{1!}\right)^2 + \left(\frac{\Delta g''}{2!}\right)^2 + \left(\frac{\Delta n}{0!}\right)^2 + \left(\frac{\Delta n'}{1!}\right)^2 + \left(\frac{\Delta n''}{2!}\right)^2 \tag{28}
$$

where x_0 is the matching point, as a function of the parameters $(\beta, r_0, a_2, b_3, A_1, B_0)$, where $\Delta F \equiv F_\infty - F_{th}$. By matching the second derivatives, we ensure that there are no discontinuities in the Riemann curvature.

The presence of the coupling parameter β, irrelevant for asymptotically AdS solutions (with $K=1$), has a profound effect insofar as it yields a sufficient amount of freedom in the parameter space to minimize Δ to high precision. The precision of our matching is accurate to one part in 10^{15} at worst. Note from (24) that each solution appears for a specific choice of α. We have found a broad range of wormhole solutions using this method. These are illustrated in Figures 1–3 and, respectively, correspond to matching for small x_0, mid-range x_0, and large x_0.

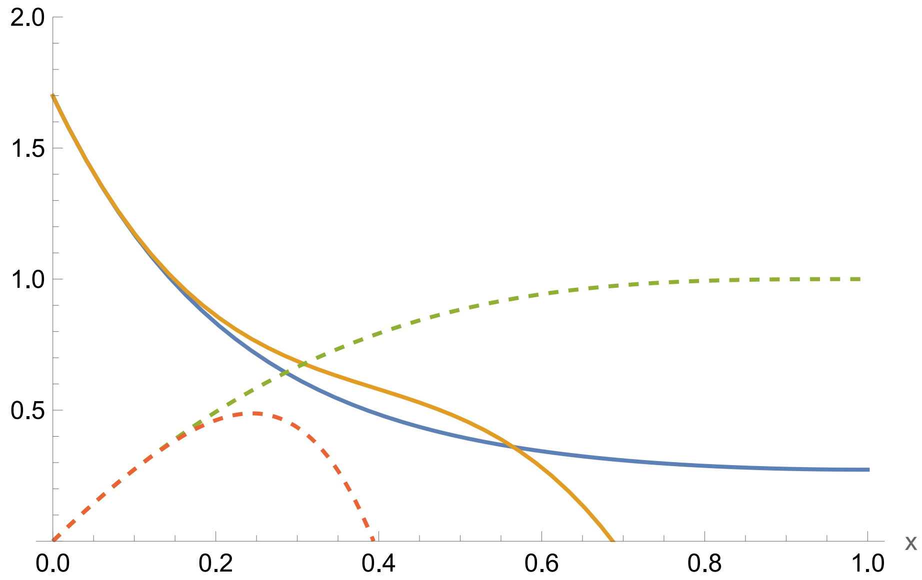

Figure 1. Plots of the series solutions $[g(x), n(x)]$ for both large-r [(solid blue, dashed green)] and near-throat [(solid orange, dashed red)]. The solutions smoothly match with $\Delta = 1.03924 \times 10^{-19}$ at $x = x_0 = 0.00505304$ for the parameters $\beta = 0.505553, a_2 = 0.318022, b_3 = -0.680002$, $r_0 = 0.522375, A_1 = 3.03285, B_0 = 1.69888$.

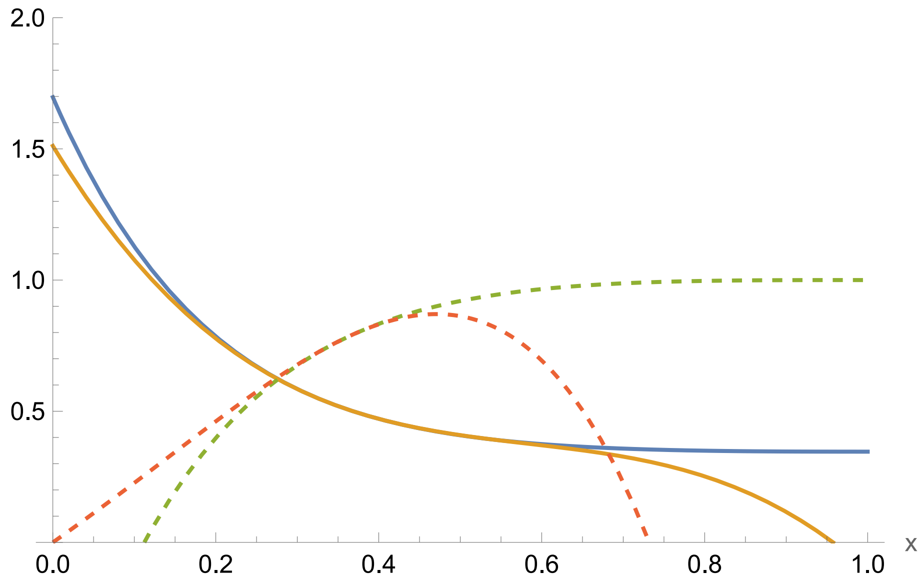

Figure 2. Plots of the series solutions $[g(x), n(x)]$ for both large-r [(solid blue, dashed green)] and near-throat [(solid orange, dashed red)]. The solutions smoothly match with $\Delta = 3.22544 \times 10^{-15}$ at $x = x_0 = 0.35156$ for the parameters $\beta = 0.792546, a_2 = 0.0368723, b_3 = -0.327935$, $r_0 = 0.588074, A_1 = 2.1819, B_0 = 1.51153$.

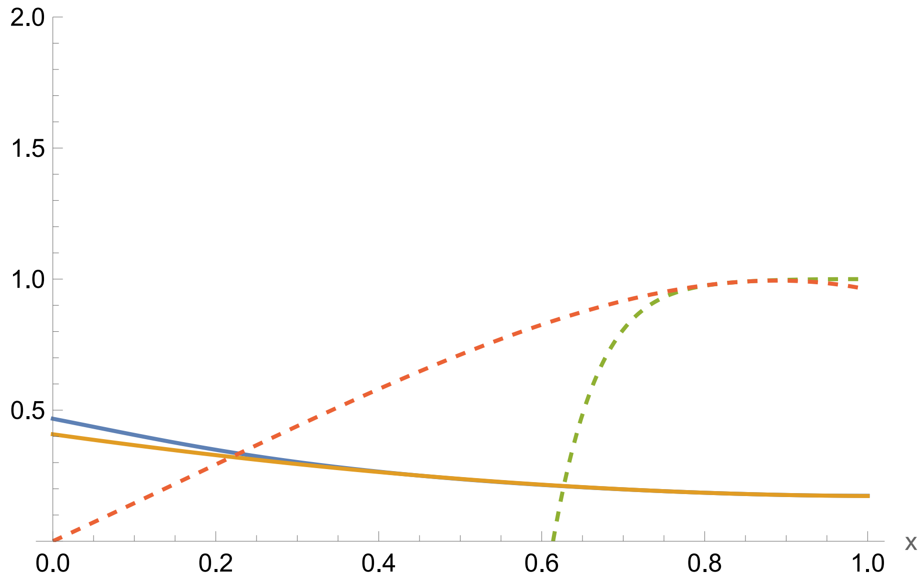

Figure 3. Plots of the series solutions $[g(x), n(x)]$ for both large-r [(solid blue, dashed green] and near-throat [(solid orange, dashed red]. The solutions smoothly match with $\Delta = 3.2155 \times 10^{-16}$ at $x = x_0 = 0.83693$ for the parameters $\beta = 0.0706116, a_2 = 0.34649, b_3 = -3.53033, r_0 = 0.41619, A_1 = 1.43351, B_0 = 0.408057$.

5. Conclusions

We have shown that Einsteinian Cubic Gravity contains wormhole solutions that are purely gravitational. Unlike wormholes obtained in generic higher-curvature gravity theories, our solutions are in $(3+1)$-dimensions and require no exotic matter in their construction.

In contrast to previous solutions obtained in the theory, our wormhole solutions require three special characteristics. One is that their asymptotic behaviour is that of AdS spacetime with a global monopole deficit. The second is that the the coupling parameter α is related to the effective cosmological constant via (24). The third is that the coupling parameter $\beta \neq 0$. Our wormhole solutions have no horizons or singularities and so are traversable in principle. Imposing more stringent traversability requirements (such as requiring that the gravity at the throat not exceed Earth's gravity) will reduce the range of allowed solutions; we have not imposed this constraint on the solutions that we have obtained.

Although our series-matching approach has yielded candidate gravitational wormholes, a full wormhole solution to the field Equations (A1) and (A2) remains to be obtained. This can be done numerically, but it presents a computational challenge. A solution in terms of Tchebyshev polyonomials requires many coefficients to obtain high accuracy. We were not able to achieve this using the computational resources available. While it is straightforward to apply the shooting method to an ODE system, here, we have a double shooting problem, which is considerably more challenging. The solution will necessarily depend on tuning the constants of integration appearing in the local series expansions of the metric functions near $x = 1$ such that the integrated solution from $x = 1$ satisfies the boundary condition at the other end (or vice versa, beginning at $x = 0$). We did attempt such solutions but found that we typically encountered at least one spontaneous singularity for g between $x = 0$ and $x = 1$. Some of these cases might be indicative of a new class of black holes, which merit further investigation.

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Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

dS	de Sitter
AdS	Anti-de Sitter
ECG	Einsteinian Cubic Gravity
GR	General Relativity
GQTG	Generalized Quasitopological Gravity
GSSS	General Static Spherically Symmetric
ODE	Ordinary Differential Equation

Appendix A. On-Shell Field Equations

Using the ansatz (18), field Equations (7) and (8) become, for $\gamma \neq 0$,

$$\begin{aligned}
0 = \quad & -6\big(-\big((16\alpha+2\beta+\gamma)n'^2(n'-(x-1)n'')(x-1)^3\big) \\
& +(16\alpha+2\beta+\gamma)n\Big(n'^2-2(x-1)\Big(n''-(x-1)n^{(3)}\Big)n' \\
& +2(x-1)^2n''^2\big)(x-1)^2-64\alpha n^2n'(x-1)+64\alpha n^3\big)g^3 \\
& +3(x-1)\big(-n'\big((56\alpha+6\beta+3\gamma)g'n'^2+6(16\alpha+2\beta+\gamma)n'-32(x-1)\alpha n''\big)(x-1)^3 \\
& +2n\big(2(5(16\alpha+2\beta+\gamma)g'-(x-1)(52\alpha+6\beta+3\gamma)g'')n'^2 \\
& +(-128\alpha-(x-1)(216\alpha+11(2\beta+\gamma))g'n'')n' \\
& -4(x-1)\Big((8\alpha+2\beta+\gamma)n''-8(x-1)\alpha n^{(3)}\Big)\Big)(x-1)^2 \\
& -4n^2\big(24\alpha g''n''x^2+2\beta g''n''x^2+\gamma g''n''x^2-48\alpha g''n''x-4\beta g''n''x-2\gamma g''n''x \\
& -32\alpha+24\alpha g''n''+2\beta g''n''+\gamma g''n''+(x-1)(16\alpha+2\beta+\gamma)n'\Big((x-1)g^{(3)}-2g''\Big) \\
& +2g'\Big((16\alpha+2\beta+\gamma)n'-(x-1)\Big((16\alpha+2\beta+\gamma)n''-4(x-1)\alpha n^{(3)}\Big)\Big)(x-1) \\
& +128\alpha n^3g'\big)g^2 \\
& -2\Big(48\alpha n^3\Big(g''^2+g'g^{(3)}\Big)(x-1)^4-12n^2\big(2((16\alpha+2\beta+\gamma)n'-8(x-1)\alpha n'')g'^2 \\
& +(-32\alpha-(x-1)(44\alpha+2\beta+\gamma)n'g'')g'+8(x-1)\alpha\Big(2g''+(x-1)g^{(3)}\Big)\Big)(x-1)^3 \\
& -8n'\big(r_0^4+15(x-1)^4\alpha g'n'\big)(x-1)+3n\big((148\alpha+5(2\beta+\gamma))g'^2n'^2(x-1)^4 \\
& +8g'((2\beta+\gamma)n'-16(x-1)\alpha n'')(x-1)^4+8\big(r_0^4-16(x-1)^5\alpha n'g''\big)\big)\big)g \\
& -16\big(\big(\Lambda_0r_0^2-x^2+2x-1\big)r_0^4+6(x-1)^4\alpha n^2g'^2(g'n'-1) \\
& -(x-1)ng'\big(r_0^4+12(x-1)^4\alpha g'n'\big)\big)
\end{aligned} \tag{A1}$$

and

$$
\begin{aligned}
0 = \;& -16\big(-(x-1)ng'\big(12\alpha(x-1)^4g'n' + r_0^4\big) \\
&+6\alpha(x-1)^4n^2g'^2(g'n'-1) + r_0^4\big(\Lambda_0 r_0^2 - x^2 + 2x - 1\big)\big) \\
&-6g\big(8\alpha(x-1)^5g'n'^2 + n\big(8\big(r_0^4 - 4\alpha(x-1)^5g''n'\big) + (x-1)^4(52\alpha+6\beta+3\gamma)g'^2n'^2 \\
&+8(x-1)^4g'((12\alpha+2\beta+\gamma)n' - 4\alpha(x-1)n'')) \\
&-4(x-1)^3n^2\Big(2g'^2((16\alpha+2\beta+\gamma)n' - 2\alpha(x-1)n'') + 8\alpha(x-1)\Big((x-1)g^{(3)} + 2g''\Big) \\
&+g'(-32\alpha - (x-1)(24\alpha+2\beta+\gamma)g''n')) + 16\alpha(x-1)^4n^3\Big(g''^2 + g^{(3)}g'\Big)\Big) \\
&+3(x-1)g^2\big((x-1)^3(16\alpha+2\beta+\gamma)n'^2(g'n'+2) + 128\alpha n^3g' \\
&+2(x-1)^2n(16\alpha+2\beta+\gamma)\big(g'n'(10n' - 3(x-1)n'') - 2(x-1)\big(g''n'^2 + 2n''\big)\big) \\
&-4(x-1)n^2\big(-32\alpha + 16\alpha x^2g''n'' + 2\beta x^2g''n'' + \gamma x^2g''n'' - 32\alpha xg''n'' \\
&+16\alpha g''n'' - 4\beta xg''n'' + 2\beta g''n'' - 2\gamma xg''n'' + \gamma g''n'' \\
&+2(16\alpha+2\beta+\gamma)g'(n' - (x-1)n'') + (x-1)(16\alpha+2\beta+\gamma)\Big((x-1)g^{(3)} - 2g''\Big)n'\Big)\Big) \\
&-6g^3\big((x-1)^3(16\alpha+2\beta+\gamma)n'^3 + 3(x-1)^2n(16\alpha+2\beta+\gamma)n'(n' - 2(x-1)n'') + 64\alpha n^3\big)
\end{aligned}
\tag{A2}
$$

Appendix B. Higher-Order Terms of Near-Throat Solutions

Here, we present the first few coefficients of the near-throat solutions (26) and (27) beyond linear order; note that the present A_2 and B_1 appear in the main text.

$$
\begin{aligned}
A_2 = \;& \frac{1}{8A_1B_0^2\left(A_1B_0(3\beta L^2-1)-1\right)^2}\Big\{2A_1^2B_0^2\big(1-3\beta L^2\big)\big(A_1^2B_0^2\big(5-12\beta L^2\big) + 9A_1B_0 + 4\big) \\
& -8L^2{r_0}^4\big(2A_1^2B_0^2\big(3\beta L^2-1\big) + A_1B_0\big(7-24\beta L^2\big) + 12\big)\Big\}
\end{aligned}
\tag{A3}
$$

$$
B_1 = -\frac{2L^2\left(3A_1^2B_0^2(A_1B_0+1)\left(\beta - \dfrac{1}{3L^2}\right) + 8L{r_0}^6 + 4{r_0}^4\right)}{A_1^2B_0(A_1B_0(3\beta L^2-1)-1)}
\tag{A4}
$$

$$
\begin{aligned}
A_3 = \quad & \frac{1}{72A_1^3B_0^4((3L^2\beta-1)A_1B_0-1)^5}\Big\{221184L^{12}\beta^3A_1^3B_0^3{r_0}^{12} \\
& +8L^3A_1^2B_0^2\big(315A_1^4B_0^4+1421A_1^3B_0^3+2374A_1^2B_0^2+1724A_1B_0+456\big){r_0}^6 \\
& +6912L^{11}\beta^3A_1^3B_0^3\big(32{r_0}^4+3\beta A_1^2B_0^2(8A_1B_0+9)\big){r_0}^6 \\
& +576L^9\beta^2A_1^2B_0^2\big({r_0}^4\big(5A_1^2B_0^2-392A_1B_0-448\big) \\
& -3\beta A_1^2B_0^2\big(133A_1^2B_0^2+257A_1B_0+120\big)\big){r_0}^6 \\
& -24L^7\beta A_1B_0\big(8{r_0}^4\big(10A_1^3B_0^3-391A_1^2B_0^2-936A_1B_0-592\big) \\
& -3\beta A_1^2B_0^2\big(1675A_1^3B_0^3+4642A_1^2B_0^2+4056A_1B_0+1024\big)\big){r_0}^6 \\
& +8L^5\big(8{r_0}^4\big(5A_1^4B_0^4-127A_1^3B_0^3-488A_1^2B_0^2-656A_1B_0-336\big) \\
& -3\beta A_1^2B_0^2\big(1182A_1^4B_0^4+4295A_1^3B_0^3+5470A_1^2B_0^2+2744A_1B_0+480\big)\big){r_0}^6 \\
& -3A_1^4B_0^4(A_1B_0+1)^2\big(37A_1^3B_0^3+98A_1^2B_0^2+84A_1B_0+24\big) \\
& -144L^{10}\beta^2A_1^2B_0^2\big(16(101A_1B_0+112){r_0}^{12}+12\beta A_1B_0\big(3A_1^2B_0^2-32\big){r_0}^8 \\
& +9\beta^2A_1^3B_0^3\big(15A_1^2B_0^2-44A_1B_0-72\big){r_0}^4-27\beta^3A_1^5B_0^5\big(5A_1^2B_0^2-4A_1B_0-6\big)\big) \\
& +12L^8\beta A_1B_0\big(64\big(108A_1^2B_0^2+247A_1B_0+148\big) \\
& {r_0}^{12}+24\beta A_1B_0\big(18A_1^3B_0^3+17A_1^2B_0^2-190A_1B_0-224\big){r_0}^8 \\
& +18\beta^2A_1^3B_0^3\big(125A_1^3B_0^3-301A_1^2B_0^2-960A_1B_0-480\big){r_0}^4 \\
& +27\beta^3A_1^5B_0^5\big(-105A_1^3B_0^3-30A_1^2B_0^2+123A_1B_0+32\big)\big) \\
& -3L^6\big(256\big(13A_1^3B_0^3+46A_1^2B_0^2+58A_1B_0+28\big){r_0}^{12} \\
& +32\beta A_1B_0\big(18A_1^4B_0^4+34A_1^3B_0^3-169A_1^2B_0^2-442A_1B_0-296\big){r_0}^8 \\
& +12\beta^2A_1^3B_0^3\big(390A_1^4B_0^4-741A_1^3B_0^3-4112A_1^2B_0^2-4016A_1B_0-1024\big){r_0}^4 \\
& +9\beta^3A_1^5B_0^5\big(-891A_1^4B_0^4-1245A_1^3B_0^3+110A_1^2B_0^2+404A_1B_0+80\big)\big) \\
& +L^2A_1^2B_0^2(A_1B_0+1)\big(\big(-280A_1^4B_0^4+540A_1^3B_0^3+4064A_1^2B_0^2+5344A_1B_0+2016\big){r_0}^4 \\
& +3\beta A_1^2B_0^2\big(513A_1^4B_0^4+1332A_1^3B_0^3+1156A_1^2B_0^2+420A_1B_0+56\big)\big) \\
& +L^4\big(32\big(6A_1^5B_0^5+17A_1^4B_0^4-43A_1^3B_0^3-212A_1^2B_0^2-308A_1B_0-168\big){r_0}^8 \\
& +12\beta A_1^2B_0^2\big(270A_1^5B_0^5-380A_1^4B_0^4-3631A_1^3B_0^3-5402A_1^2B_0^2-2808A_1B_0-480\big){r_0}^4 \\
& -9\beta^2A_1^4B_0^4\big(953A_1^5B_0^5+2390A_1^4B_0^4+1823A_1^3B_0^3+430A_1^2B_0^2+16\big)\big)\Big\}
\end{aligned} \tag{A5}
$$

$$\begin{aligned}
B_2 \quad &= \frac{1}{6A_1^4B_0^3((3L^2\beta-1)A_1B_0-1)^4}\Big\{4608L^{10}\beta^2A_1^2B_0^2{r_0}^{12} \\
&-8L^3A_1^2B_0^2\big(15A_1^3B_0^3+39A_1^2B_0^2+34A_1B_0+10\big){r_0}^6 \\
&-576L^9\beta^2A_1^2B_0^2\big({r_0}^4(A_1B_0-8)-6\beta A_1^3B_0^3\big){r_0}^6 \\
&+24L^7\beta A_1B_0\big(16{r_0}^4\big(A_1^2B_0^2-8A_1B_0-12\big)-3\beta A_1^2B_0^2\big(47A_1^2B_0^2+42A_1B_0+16\big)\big){r_0}^6 \\
&-8L^5\big(8{r_0}^4\big(A_1^3B_0^3-8A_1^2B_0^2-24A_1B_0-12\big) \\
&-3\beta A_1^2B_0^2\big(46A_1^3B_0^3+81A_1^2B_0^2+50A_1B_0+8\big)\big){r_0}^6 \\
&+6A_1^4B_0^4(A_1B_0+1)^4-24L^8\beta A_1B_0\big(8(17A_1B_0+24){r_0}^{12}+12\beta A_1B_0(A_1B_0-4){r_0}^8 \\
&+9\beta^2A_1^4B_0^4(A_1B_0-7){r_0}^4-27\beta^3A_1^5B_0^5\big(A_1^2B_0^2+3A_1B_0+1\big)\big) \\
&+3L^6\big(64\big(3A_1^2B_0^2+8A_1B_0+4\big){r_0}^{12}+16\beta A_1B_0\big(4A_1^2B_0^2-15A_1B_0-24\big){r_0}^8 \\
&+12\beta^2A_1^3B_0^3\big(6A_1^3B_0^3-33A_1^2B_0^2-36A_1B_0-16\big){r_0}^4 \\
&-27\beta^3A_1^5B_0^5\big(10A_1^3B_0^3+33A_1^2B_0^2+27A_1B_0+8\big)\big) \\
&-L^2A_1^2B_0^2(A_1B_0+1)\big(\big(-8A_1^3B_0^3+28A_1^2B_0^2+56A_1B_0+24\big){r_0}^4 \\
&+3\beta A_1^2B_0^2\big(26A_1^3B_0^3+73A_1^2B_0^2+64A_1B_0+18\big)\big) \\
&+2L^4\big(8\big(-2A_1^3B_0^3+7A_1^2B_0^2+24A_1B_0+12\big){r_0}^8 \\
&-6\beta A_1^2B_0^2\big(6A_1^4B_0^4-24A_1^3B_0^3-57A_1^2B_0^2-42A_1B_0-8\big){r_0}^4 \\
&+9\beta^2A_1^4B_0^4\big(21A_1^4B_0^4+75A_1^3B_0^3+87A_1^2B_0^2+41A_1B_0+6\big)\big)\Big\}
\end{aligned} \tag{A6}$$

$$\begin{aligned}
A_4 =&\frac{1}{2880A_1^5B_0^6((3L^2\beta-1)A_1B_0-1)^8}\Big\{891813888r_0^{18}\beta^5A_1^5B_0^5L^{19} \\
&-5971968r_0^{12}\beta^5A_1^5B_0^5\Big(r_0^4(26A_1B_0-224)-3\beta A_1^2B_0^2(33A_1B_0+40)\Big)L^{18} \\
&-41472r_0^6\beta^4A_1^4B_0^4\Big(32(1109A_1B_0+1108)r_0^{12}-24\beta A_1B_0\Big(35A_1^2B_0^2-156A_1B_0+672\Big)r_0^8 \\
&-36\beta^2A_1^3B_0^3\Big(15A_1^2B_0^2+466A_1B_0+480\Big)r_0^4 \\
&-27\beta^3A_1^5B_0^5\Big(201A_1^2B_0^2+358A_1B_0+288\Big)\Big)L^{17} \\
&+5184\beta^4A_1^4B_0^4\Big(64\Big(749A_1^2B_0^2-5978A_1B_0-6648\Big)r_0^{16} \\
&-24\beta A_1B_0\Big(9767A_1^3B_0^3+20248A_1^2B_0^2+11272A_1B_0-896\Big)r_0^{12} \\
&-18\beta^2A_1^3B_0^3\Big(181A_1^3B_0^3+232A_1^2B_0^2-2144A_1B_0-1920\Big)r_0^8 \\
&-54\beta^3A_1^5B_0^5\Big(75A_1^3B_0^3-164A_1^2B_0^2-462A_1B_0-576\Big)r_0^4 \\
&+81\beta^4A_1^7B_0^7\Big(61A_1^3B_0^3-84A_1^2B_0^2-260A_1B_0-168\Big)\Big)L^{16} \\
&+1728r_0^6\beta^3A_1^3B_0^3\Big(32\Big(17575A_1^2B_0^2+34964A_1B_0+16864\Big)r_0^{12} \\
&-24\beta A_1B_0\Big(1309A_1^3B_0^3-5084A_1^2B_0^2+21208A_1B_0+26592\Big)r_0^8 \\
&-36\beta^2A_1^3B_0^3\Big(611A_1^3B_0^3+22896A_1^2B_0^2+42456A_1B_0+21920\Big)r_0^4
\end{aligned}$$

$$-27\beta^3 A_1^5 B_0^5 \Big(11647 A_1^3 B_0^3 + 29660 A_1^2 B_0^2 + 30012 A_1 B_0 + 10224\Big)\Big) L^{15}$$
$$- 216\beta^3 A_1^3 B_0^3 \Big(128\Big(5740 A_1^3 B_0^3 - 42811 A_1^2 B_0^2 - 101468 A_1 B_0 - 50592\Big) r_0^{16}$$
$$-192\beta A_1 B_0 \Big(25148 A_1^4 B_0^4 + 74151 A_1^3 B_0^3 + 76846 A_1^2 B_0^2 + 23300 A_1 B_0 - 4432\Big) r_0^{12}$$
$$-72\beta^2 A_1^3 B_0^3 \Big(2263 A_1^4 B_0^4 + 4289 A_1^3 B_0^3 - 24448 A_1^2 B_0^2 - 44136 A_1 B_0 - 21920\Big) r_0^8$$
$$-324\beta^3 A_1^5 B_0^5 \Big(757 A_1^4 B_0^4 - 1268 A_1^3 B_0^3 - 6766 A_1^2 B_0^2 - 9244 A_1 B_0 - 3408\Big) r_0^4$$
$$+81\beta^4 A_1^7 B_0^7 \Big(4009 A_1^4 B_0^4 - 1277 A_1^3 B_0^3 - 18230 A_1^2 B_0^2 - 18420 A_1 B_0 - 4240\Big)\Big) L^{14}$$
$$- 144 r_0^6 \beta^2 A_1^2 B_0^2 \Big(128\Big(17445 A_1^3 B_0^3 + 51924 A_1^2 B_0^2 + 49948 A_1 B_0 + 15472\Big) r_0^{12}$$
$$-96\beta A_1 B_0 \Big(2436 A_1^4 B_0^4 - 8220 A_1^3 B_0^3 + 33313 A_1^2 B_0^2 + 98044 A_1 B_0 + 50592\Big) r_0^8$$
$$-36\beta^2 A_1^3 B_0^3 \Big(4955 A_1^4 B_0^4 + 234088 A_1^3 B_0^3 + 632164 A_1^2 B_0^2 + 610608 A_1 B_0 + 211072\Big) r_0^4$$
$$-27\beta^3 A_1^5 B_0^5 \Big(144887 A_1^4 B_0^4 + 483300 A_1^3 B_0^3 + 634826 A_1^2 B_0^2 + 367064 A_1 B_0 + 72000\Big)\Big) L^{13}$$
$$+ 18\beta^2 A_1^2 B_0^2 \Big(512\Big(5484 A_1^4 B_0^4 - 38677 A_1^3 B_0^3 - 146716 A_1^2 B_0^2 - 149004 A_1 B_0 - 46416\Big) r_0^{16}$$
$$-192\beta A_1 B_0 \Big(138699 A_1^5 B_0^5 + 532871 A_1^4 B_0^4$$
$$+795648 A_1^3 B_0^3 + 505582 A_1^2 B_0^2 + 66424 A_1 B_0 - 33728\Big) r_0^{12}$$
$$-144\beta^2 A_1^3 B_0^3 \Big(11770 A_1^5 B_0^5 + 29857 A_1^4 B_0^4$$
$$-110436 A_1^3 B_0^3 - 329450 A_1^2 B_0^2 - 307400 A_1 B_0 - 105536\Big) r_0^8$$
$$-108\beta^3 A_1^5 B_0^5 \Big(29377 A_1^5 B_0^5 - 33804 A_1^4 B_0^4 - 337692 A_1^3 B_0^3 - 577166 A_1^2 B_0^2 - 362040 A_1 B_0$$
$$-72000) r_0^4 + 81\beta^4 A_1^7 B_0^7 \Big(57749 A_1^5 B_0^5 + 43531 A_1^4 B_0^4 - 217888 A_1^3 B_0^3$$
$$-360546 A_1^2 B_0^2 - 172264 A_1 B_0 - 27456\Big)\Big) L^{12}$$
$$+ 24 r_0^6 \beta A_1 B_0 \Big(128\Big(17386 A_1^4 B_0^4 + 69063 A_1^3 B_0^3 + 100482 A_1^2 B_0^2 + 64648 A_1 B_0 + 16320\Big) r_0^{12}$$
$$-384\beta A_1 B_0 \Big(1127 A_1^5 B_0^5 - 3318 A_1^4 B_0^4 + 12928 A_1^3 B_0^3 + 68448 A_1^2 B_0^2 + 74082 A_1 B_0 + 23208\Big) r_0^8$$
$$-144\beta^2 A_1^3 B_0^3 \Big(2490 A_1^5 B_0^5 + 159306 A_1^4 B_0^4$$
$$+570715 A_1^3 B_0^3 + 803940 A_1^2 B_0^2 + 524024 A_1 B_0 + 129504\Big) r_0^4$$
$$-27\beta^3 A_1^5 B_0^5 \Big(502259 A_1^5 B_0^5 + 2084564 A_1^4 B_0^4 + 3461556 A_1^3 B_0^3$$
$$+2775856 A_1^2 B_0^2 + 1022544 A_1 B_0 + 148544\Big)\Big) L^{11}$$
$$- 3\beta A_1 B_0 \Big(512\Big(5224 A_1^5 B_0^5 - 35378 A_1^4 B_0^4 - 190741 A_1^3 B_0^3$$
$$-297058 A_1^2 B_0^2 - 192792 A_1 B_0 - 48960\Big) r_0^{16}$$
$$-384\beta A_1 B_0 \Big(108185 A_1^6 B_0^6 + 514663 A_1^5 B_0^5 + 1001953 A_1^4 B_0^4$$
$$+950766 A_1^3 B_0^3 + 368616 A_1^2 B_0^2 - 21096 A_1 B_0 - 30944\Big) r_0^{12}$$
$$-288\beta^2 A_1^3 B_0^3 \Big(16300 A_1^6 B_0^6 + 52208 A_1^5 B_0^5 - 122595 A_1^4 B_0^4$$

$$-583985A_1^3B_0^3 - 815324A_1^2B_0^2 - 526312A_1B_0 - 129504\Big)r_0^8$$
$$-108\beta^3A_1^5B_0^5\Big(105250A_1^6B_0^6 - 65163A_1^5B_0^5 - 1456552A_1^4B_0^4$$
$$-3164156A_1^3B_0^3 - 2774160A_1^2B_0^2 - 1040688A_1B_0 - 148544\Big)r_0^4$$
$$+81\beta^4A_1^7B_0^7\Big(238435A_1^6B_0^6 + 440055A_1^5B_0^5 - 453246A_1^4B_0^4$$
$$-1597446A_1^3B_0^3 - 1279464A_1^2B_0^2 - 411216A_1B_0 - 43456\Big)\Big)L^{10}$$
$$-\ 48r_0^6\Big(64\Big(1162A_1^5B_0^5 + 5805A_1^4B_0^4 + 11508A_1^3B_0^3 + 11860A_1^2B_0^2 + 7080A_1B_0 + 2400\Big)r_0^{12}$$
$$-16\beta A_1B_0\Big(2072A_1^6B_0^6 - 5416A_1^5B_0^5 + 19450A_1^4B_0^4 + 171097A_1^3B_0^3$$
$$+291582A_1^2B_0^2 + 191640A_1B_0 + 48960\Big)r_0^8$$
$$-24\beta^2A_1^3B_0^3\Big(1225A_1^6B_0^6 + 121734A_1^5B_0^5 + 549834A_1^4B_0^4$$
$$+1024407A_1^3B_0^3 + 973526A_1^2B_0^2 + 457784A_1B_0 + 77120\Big)r_0^4$$
$$-9\beta^3A_1^5B_0^5\Big(262288A_1^6B_0^6 + 1309168A_1^5B_0^5 + 2665786A_1^4B_0^4$$
$$+2758629A_1^3B_0^3 + 1478542A_1^2B_0^2 + 389976A_1B_0 + 37696\Big)\Big)L^9$$
$$+\ \Big(512\Big(992A_1^6B_0^6 - 6574A_1^5B_0^5 - 47061A_1^4B_0^4 - 100384A_1^3B_0^3$$
$$-103684A_1^2B_0^2 - 61560A_1B_0 - 21600\Big)r_0^{16}$$
$$-384\beta A_1B_0\Big(45317A_1^7B_0^7 + 258249A_1^6B_0^6 + 621826A_1^5B_0^5$$
$$+782965A_1^4B_0^4 + 491757A_1^3B_0^3 + 88426A_1^2B_0^2 - 39784A_1B_0 - 16320\Big)r_0^{12}$$
$$-288\beta^2A_1^3B_0^3\Big(12680A_1^7B_0^7 + 49342A_1^6B_0^6 - 65489A_1^5B_0^5$$
$$-541303A_1^4B_0^4 - 1039649A_1^3B_0^3 - 986034A_1^2B_0^2 - 462744A_1B_0 - 77120\Big)r_0^8$$
$$-216\beta^3A_1^5B_0^5\Big(56410A_1^7B_0^7 - 4766A_1^6B_0^6 - 901913A_1^5B_0^5$$
$$-2459528A_1^4B_0^4 - 2817077A_1^3B_0^3 - 1556554A_1^2B_0^2 - 406680A_1B_0 - 37696\Big)r_0^4$$
$$+81\beta^4A_1^7B_0^7\Big(308945A_1^7B_0^7 + 914075A_1^6B_0^6 + 342886A_1^5B_0^5$$
$$-1512538A_1^4B_0^4 - 2124802A_1^3B_0^3 - 1129076A_1^2B_0^2 - 255472A_1B_0 - 19072\Big)\Big)L^8$$
$$-\ 24r_0^6\Big(32\Big(126A_1^7B_0^7 - 304A_1^6B_0^6 + 900A_1^5B_0^5 + 13403A_1^4B_0^4$$
$$+32076A_1^3B_0^3 + 33548A_1^2B_0^2 + 19800A_1B_0 + 7200\Big)r_0^8$$
$$+16\beta A_1^2B_0^2\Big(220A_1^7B_0^7 + 49502A_1^6B_0^6 + 273371A_1^5B_0^5$$
$$+639529A_1^4B_0^4 + 803617A_1^3B_0^3 + 555864A_1^2B_0^2 + 187736A_1B_0 + 23712\Big)r_0^4$$
$$+3\beta^2A_1^4B_0^4\Big(330418A_1^7B_0^7 + 1935576A_1^6B_0^6 + 4709353A_1^5B_0^5$$
$$+6032726A_1^4B_0^4 + 4276472A_1^3B_0^3 + 1641704A_1^2B_0^2 + 305232A_1B_0 + 20928\Big)\Big)L^7$$
$$+\ \Big(128\Big(7971A_1^8B_0^8 + 53165A_1^7B_0^7 + 153225A_1^6B_0^6 + 240997A_1^5B_0^5$$
$$+207437A_1^4B_0^4 + 76308A_1^3B_0^3 - 10988A_1^2B_0^2 - 19080A_1B_0 - 7200\Big)r_0^{12}$$

$$+96\beta A_1^2B_0^2\Big(5254A_1^8B_0^8+24166A_1^7B_0^7-11913A_1^6B_0^6$$
$$-253171A_1^5B_0^5-643372A_1^4B_0^4-818867A_1^3B_0^3-569832A_1^2B_0^2-192040A_1B_0-23712\Big)r_0^8$$
$$+36\beta^2A_1^4B_0^4\Big(72380A_1^8B_0^8+32574A_1^7B_0^7-1300710A_1^6B_0^6$$
$$-4376623A_1^5B_0^5-6291458A_1^4B_0^4-4661792A_1^3B_0^3-1800328A_1^2B_0^2-325744A_1B_0-20928\Big)r_0^4$$
$$-27\beta^3A_1^6B_0^6\Big(257494A_1^8B_0^8+1053494A_1^7B_0^7+1361749A_1^6B_0^6$$
$$+68595A_1^5B_0^5-1349132A_1^4B_0^4-1271758A_1^3B_0^3-490352A_1^2B_0^2-77520A_1B_0-2880\Big)\Big)L^6$$
$$+16r_0^6A_1^2B_0^2(A_1B_0+1)\Big(4\Big(-21A_1^6B_0^6+16753A_1^5B_0^5+94051A_1^4B_0^4$$
$$+221145A_1^3B_0^3+276244A_1^2B_0^2+180688A_1B_0+46656\Big)r_0^4$$
$$+3\beta A_1^2B_0^2\Big(58141A_1^6B_0^6+334207A_1^5B_0^5+782546A_1^4B_0^4$$
$$+936061A_1^3B_0^3+590704A_1^2B_0^2+186008A_1B_0+22752\Big)\Big)L^5$$
$$+A_1^2B_0^2(A_1B_0+1)\Big(-32\Big(906A_1^7B_0^7+3919A_1^6B_0^6-2822A_1^5B_0^5$$
$$-44173A_1^4B_0^4-111314A_1^3B_0^3-142304A_1^2B_0^2-95260A_1B_0-25104\Big)r_0^8$$
$$-24\beta A_1^2B_0^2\Big(12869A_1^7B_0^7-231A_1^6B_0^6-254792A_1^5B_0^5-786785A_1^4B_0^4$$
$$-1037178A_1^3B_0^3-678848A_1^2B_0^2-214240A_1B_0-25344\Big)r_0^4$$
$$+9\beta^2A_1^4B_0^4\Big(134892A_1^7B_0^7+571752A_1^6B_0^6+863639A_1^5B_0^5$$
$$+465790A_1^4B_0^4-102004A_1^3B_0^3-213928A_1^2B_0^2-78008A_1B_0-7840\Big)\Big)L^4$$
$$-8r_0^6A_1^4B_0^4(A_1B_0+1)^2\Big(17643A_1^5B_0^5+99838A_1^4B_0^4+224838A_1^3B_0^3$$
$$+248712A_1^2B_0^2+133336A_1B_0+27744\Big)L^3$$
$$+15A_1^6B_0^6(A_1B_0+1)^3\Big(359A_1^5B_0^5+1632A_1^4B_0^4+2908A_1^3B_0^3+2544A_1^2B_0^2+1104A_1B_0+192\Big)$$
$$-A_1^4B_0^4(A_1B_0L+L)^2\Big(4\Big(-3914A_1^6B_0^6+1919A_1^5B_0^5+84470A_1^4B_0^4$$
$$+241480A_1^3B_0^3+291320A_1^2B_0^2+162560A_1B_0+34560\Big)r_0^4$$
$$+3\beta A_1^2B_0^2\Big(40623A_1^6B_0^6+178461A_1^5B_0^5+298868A_1^4B_0^4$$
$$+232906A_1^3B_0^3+78672A_1^2B_0^2+3464A_1B_0-2720\Big)\Big)\Bigg\} \tag{A7}$$

$$B_3=-\frac{L^2}{180A_1^6B_0^5((3L^2\beta-1)A_1B_0-1)^7}\Bigg\{9289728L^{15}\beta^4A_1^4B_0^4r_0^{18}$$
$$-497664L^{14}\beta^4A_1^4B_0^4\Big(4(2A_1B_0-7)r_0^4+\beta A_1^2B_0^2(19-6A_1B_0)\Big)r_0^{12}$$
$$-8LA_1^4B_0^4(A_1B_0+1)^2\Big(63A_1^4B_0^4+378A_1^3B_0^3+602A_1^2B_0^2+384A_1B_0+88\Big)r_0^6$$
$$-3456L^{13}\beta^3A_1^3B_0^3\Big(16(237A_1B_0+296)r_0^{12}-36\beta A_1B_0\Big(3A_1^2B_0^2-32A_1B_0+56\Big)r_0^8$$

$$+9\beta^2A_1^3B_0^3\Big(45A_1^2B_0^2-136A_1B_0+304\Big)r_0^4-27\beta^3A_1^5B_0^5\Big(3A_1^2B_0^2+16A_1B_0+30\Big)\Big)r_0^6$$
$$+144L^{11}\beta^2A_1^2B_0^2\Big(512\Big(93A_1^2B_0^2+224A_1B_0+129\Big)r_0^{12}$$
$$-288\beta A_1B_0\Big(12A_1^3B_0^3-121A_1^2B_0^2+117A_1B_0+296\Big)r_0^8$$
$$+72\beta^2A_1^3B_0^3\Big(225A_1^3B_0^3-405A_1^2B_0^2+758A_1B_0+1072\Big)r_0^4$$
$$-27\beta^3A_1^5B_0^5\Big(69A_1^3B_0^3+572A_1^2B_0^2+1998A_1B_0+624\Big)\Big)r_0^6$$
$$-24L^9\beta A_1B_0\Big(128\Big(514A_1^3B_0^3+1797A_1^2B_0^2+1986A_1B_0+592\Big)r_0^{12}$$
$$-192\beta A_1B_0\Big(54A_1^4B_0^4-513A_1^3B_0^3+70A_1^2B_0^2+2324A_1B_0+1548\Big)r_0^8$$
$$+144\beta^2A_1^3B_0^3\Big(450A_1^4B_0^4-255A_1^3B_0^3+727A_1^2B_0^2+2868A_1B_0+1032\Big)r_0^4$$
$$-27\beta^3A_1^5B_0^5\Big(9A_1^4B_0^4+1362A_1^3B_0^3+9700A_1^2B_0^2+7744A_1B_0+2096\Big)\Big)r_0^6$$
$$+48L^3A_1^2B_0^2(A_1B_0+1)\Big(4\Big(30A_1^5B_0^5+66A_1^4B_0^4+118A_1^3B_0^3+297A_1^2B_0^2+276A_1B_0$$
$$+80)r_0^4+\beta A_1^2B_0^2\Big(153A_1^5B_0^5+870A_1^4B_0^4+1036A_1^3B_0^3+309A_1^2B_0^2-132A_1B_0-56\Big)\Big)r_0^6$$
$$+48L^7\Big(64\Big(44A_1^4B_0^4+199A_1^3B_0^3+318A_1^2B_0^2+188A_1B_0+40\Big)r_0^{12}$$
$$-16\beta A_1B_0\Big(72A_1^5B_0^5-642A_1^4B_0^4-482A_1^3B_0^3+3979A_1^2B_0^2+5742A_1B_0+1776\Big)r_0^8$$
$$+24\beta^2A_1^3B_0^3\Big(450A_1^5B_0^5+305A_1^4B_0^4+672A_1^3B_0^3+3174A_1^2B_0^2+2478A_1B_0+624\Big)r_0^4$$
$$+9\beta^3A_1^5B_0^5\Big(192A_1^5B_0^5+361A_1^4B_0^4-4181A_1^3B_0^3-6493A_1^2B_0^2-3478A_1B_0-496\Big)\Big)r_0^6$$
$$+8L^5\Big(32\Big(18A_1^6B_0^6-150A_1^5B_0^5-256A_1^4B_0^4$$
$$+1107A_1^3B_0^3+2646A_1^2B_0^2+1676A_1B_0+360\Big)r_0^8$$
$$-24\beta A_1^2B_0^2\Big(450A_1^6B_0^6+870A_1^5B_0^5+1350A_1^4B_0^4$$
$$+4001A_1^3B_0^3+4632A_1^2B_0^2+2128A_1B_0+240\Big)r_0^4$$
$$-9\beta^2A_1^4B_0^4\Big(546A_1^6B_0^6+2738A_1^5B_0^5+213A_1^4B_0^4$$
$$-5766A_1^3B_0^3-5964A_1^2B_0^2-1928A_1B_0-144\Big)\Big)r_0^6$$
$$+1296L^{12}\beta^3A_1^3B_0^3\Big(32\Big(127A_1^2B_0^2-354A_1B_0-592\Big)r_0^{16}$$
$$+16\beta A_1B_0\Big(-243A_1^3B_0^3+386A_1^2B_0^2+488A_1B_0+56\Big)r_0^{12}$$
$$+12\beta^2A_1^3B_0^3\Big(9A_1^3B_0^3-40A_1^2B_0^2+88A_1B_0-152\Big)r_0^8$$
$$+9\beta^3A_1^5B_0^5\Big(3A_1^3B_0^3-A_1^2B_0^2+48A_1B_0+120\Big)r_0^4$$
$$+27\beta^4A_1^7B_0^7\Big(5A_1^3B_0^3+19A_1^2B_0^2-4A_1B_0-14\Big)\Big)$$
$$-18L^{10}\beta^2A_1^2B_0^2\Big(256\Big(567A_1^3B_0^3-1177A_1^2B_0^2-5012A_1B_0-3096\Big)r_0^{16}$$
$$-96\beta A_1B_0\Big(1965A_1^4B_0^4-328A_1^3B_0^3-4686A_1^2B_0^2-2292A_1B_0-1184\Big)r_0^{12}$$
$$+288\beta^2A_1^3B_0^3\Big(45A_1^4B_0^4-146A_1^3B_0^3+190A_1^2B_0^2-381A_1B_0-536\Big)r_0^8$$
$$+108\beta^3A_1^5B_0^5\Big(33A_1^4B_0^4+11A_1^3B_0^3+455A_1^2B_0^2+1886A_1B_0+624\Big)r_0^4$$

$$
\begin{aligned}
&+81\beta^4A_1^7B_0^7\Big(225A_1^4B_0^4+969A_1^3B_0^3+446A_1^2B_0^2-230A_1B_0-272\Big)\Big)\\
&+A_1^5B_0^5(A_1B_0+1)^2\Big(4\Big(6A_1^4B_0^4+23A_1^3B_0^3-56A_1^2B_0^2-104A_1B_0-40\Big)r_0^4\\
&+3\beta A_1B_0\Big(50A_1^5B_0^5+223A_1^4B_0^4+206A_1^3B_0^3-94A_1^2B_0^2-208A_1B_0-72\Big)\Big)\\
&+3L^8\beta A_1B_0\Big(512\Big(375A_1^4B_0^4-512A_1^3B_0^3-4685A_1^2B_0^2-5850A_1B_0-1776\Big)r_0^{16}\\
&-192\beta A_1B_0\Big(1983A_1^5B_0^5+2233A_1^4B_0^4-2814A_1^3B_0^3-3722A_1^2B_0^2-4104A_1B_0-2064\Big)r_0^{12}\\
&+144\beta^2A_1^3B_0^3\Big(360A_1^5B_0^5-736A_1^4B_0^4+7A_1^3B_0^3-2440A_1^2B_0^2-6096A_1B_0-2064\Big)r_0^8\\
&+108\beta^3A_1^5B_0^5\Big(150A_1^5B_0^5+163A_1^4B_0^4+1618A_1^3B_0^3+9462A_1^2B_0^2+7656A_1B_0+2096\Big)r_0^4\\
&+81\beta^4A_1^7B_0^7\Big(1050A_1^5B_0^5+5035A_1^4B_0^4+4985A_1^3B_0^3+1740A_1^2B_0^2-416A_1B_0+80\Big)\Big)\\
&-L^2A_1^2B_0^2(A_1B_0+1)\Big(16\Big(36A_1^6B_0^6+20A_1^5B_0^5-181A_1^4B_0^4\\
&-824A_1^3B_0^3-1724A_1^2B_0^2-1368A_1B_0-360\Big)r_0^8\\
&+24\beta A_1^2B_0^2\Big(21A_1^6B_0^6+68A_1^5B_0^5-54A_1^4B_0^4+183A_1^3B_0^3+454A_1^2B_0^2+268A_1B_0+32\Big)r_0^4\\
&+9\beta^2A_1^4B_0^4\Big(330A_1^6B_0^6+1684A_1^5B_0^5+2347A_1^4B_0^4+1052A_1^3B_0^3-176A_1^2B_0^2-152A_1B_0+8\Big)\Big)\\
&-L^6\Big(512\Big(93A_1^5B_0^5-61A_1^4B_0^4-1449A_1^3B_0^3-2754A_1^2B_0^2-1684A_1B_0-360\Big)r_0^{16}\\
&-192\beta A_1B_0\Big(999A_1^6B_0^6+2314A_1^5B_0^5+688A_1^4B_0^4\\
&-1410A_1^3B_0^3-3814A_1^2B_0^2-3996A_1B_0-1184\Big)r_0^{12}\\
&+144\beta^2A_1^3B_0^3\Big(360A_1^6B_0^6-304A_1^5B_0^5-1019A_1^4B_0^4\\
&-3745A_1^3B_0^3-8620A_1^2B_0^2-5604A_1B_0-1248\Big)r_0^8\\
&+216\beta^3A_1^5B_0^5\Big(90A_1^6B_0^6+176A_1^5B_0^5+655A_1^4B_0^4\\
&+5086A_1^3B_0^3+6755A_1^2B_0^2+3498A_1B_0+496\Big)r_0^4\\
&+81\beta^4A_1^7B_0^7\Big(1300A_1^6B_0^6+6840A_1^5B_0^5+9665A_1^4B_0^4\\
&+6281A_1^3B_0^3+2056A_1^2B_0^2+996A_1B_0+352\Big)\Big)\\
&+L^4\Big(-64\Big(201A_1^7B_0^7+689A_1^6B_0^6+768A_1^5B_0^5+244A_1^4B_0^4\\
&-886A_1^3B_0^3-1812A_1^2B_0^2-1112A_1B_0-240\Big)r_0^{12}\\
&+48\beta A_1^2B_0^2\Big(180A_1^7B_0^7+64A_1^6B_0^6-779A_1^5B_0^5-3226A_1^4B_0^4\\
&-7216A_1^3B_0^3-6710A_1^2B_0^2-2624A_1B_0-240\Big)r_0^8\\
&+36\beta^2A_1^4B_0^4\Big(120A_1^7B_0^7+358A_1^6B_0^6+442A_1^5B_0^5+4453A_1^4B_0^4\\
&+8680A_1^3B_0^3+6528A_1^2B_0^2+1848A_1B_0+144\Big)r_0^4\\
&+27\beta^3A_1^6B_0^6\Big(900A_1^7B_0^7+5130A_1^6B_0^6+8965A_1^5B_0^5\\
&+7163A_1^4B_0^4+2994A_1^3B_0^3+1170A_1^2B_0^2+560A_1B_0+112\Big)\Big)\Bigg\}
\end{aligned}
\tag{A8}
$$

The coefficients B_4 and A_5 require about 5 and 10 pages, respectively, to present, so we omit them here.

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