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The Geometric Proca–Weyl Field as a Candidate for Dark Matter

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Abstract: We consider the Weyl invariant theory of gravity as an alternative approach to the problem of the origin of dark matter. According to this theory, the geometric Weyl 1-form effectively behaves as a Proca field. In this work, our starting point is to consider the existence of a gas of Weyl–Proca particles in a Bose–Einstein condensate and investigate its behavior in a cosmological context. The results obtained show that, for appropriate values of the free parameter of the model, the Weyl field behaves approximately as a dust fluid in the matter-dominated era as expected for a dark matter candidate.

Keywords: Weyl unified theory; dark matter; Proca field

1. Introduction

Proca's theory has been known since 1936, when it was proposed as a model to describe the weak interaction by assuming the existence of spin-1 mesons [1,2]. The model, however, was abandoned soon and gave way to other proposals, being subsequently almost entirely forgotten. Nevertheless, one could say that the quanta of the Proca field reappeared today as the massive gauge bosons Z, W^+ and W^- of the Standard Model of particle physics [3]. In recent years, motivation from other areas of physics has led to a new interest in Proca's theory. This mainly comes from the current research in astrophysics and cosmology, and, in particular, from the dark matter problem, one of the most important and dramatic open questions in modern cosmology [4,5]. As we know, according to the Lambda-CDM model, this hypothetical kind of matter constitutes 26.4% of the total energy density of the universe. It practically does not interact with electromagnetic radiation. We can infer its existence only from the gravitational effects it causes, which cannot be explained by general relativity when Einstein's equations are sourced by ordinary baryonic matter. Because of this, some astrophysicists have argued in favor of modifications of the standard Einstein's theory [6]. Many attempts to explain dark matter have appeared in the last two decades; among them, we can quote WIMPS, primordial black holes, and axions. Proca fluids have also been considered a viable model for explaining dark matter [7–13]. In this direction, it has been conjectured that the massive vector field considered earlier by nuclear physicists could play a role in modeling this kind of matter. Indeed, some extensions of the Standard Model in the gauge sector suggest the existence of an additional massive spin-1 boson [14–16]. This particle could couple with fermionic matter through a strongly attenuated gauge charge. In this case, the interaction with matter will be so weak that the new particle can be appropriately called a dark photon. There are some studies about the role of dark photons in cosmological evolution. In particular, it has been examined the conditions that this particle should satisfy in order to behave as dark matter [17,18]. Another motivation comes from a recent proposed modified gravity theory inspired by the original Weyl's unified field theory, in which a massive vector field appears from considerations of an entirely

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geometrical nature [19]. It follows from the theory that this vector field does not couple directly with the ordinary matter but interacts only through the gravitational field. This is an interesting characteristic for a candidate for dark matter. Here, we investigate the contribution of this massive vector field for the cosmological evolution. The theory contains two free parameters. One parameter plays the role of a cosmological constant. The other one is related to the effective mass of the geometrical Weyl field. As we shall see, for certain values of the mass, the Weyl field can mimic the cosmological behavior of dark matter in the matter-dominated era.

It should also be mentioned that massive vector fields in general relativity theory have been considered by some authors [20,21]. For instance, it has been shown that cosmic inflation could, in principle, be driven by a vector field [22]. More recently, primordial gravitational waves in the Weyl invariant theory of gravity were examined, pointing to the result that, in addition to metric waves, it predicts the propagation of Proca–Weyl waves [23].

The present article is organized as follows. In Section 2, we briefly review some basic features of the original Weyl unified field theory. In Section 3, we consider another theory, a modified gravity theory, namely, the Weyl invariant gravity, whose field equations are written in a form in which the identification of the Weyl vector field with the Proca field becomes clear [23]. We then proceed to Section 4, where we consider the Proca field as a possible candidate for explaining the existence of dark matter. In Section 5, we consider the existence of a gas of Proca–Weyl particles in a Bose–Einstein condensate. In Section 6, we investigate its equation of state and its consistency with the standard cosmological model. We conclude with some remarks in Section 7.

2. The Weyl Unified Theory

Let us start by recalling that, in 1918, attempting to unify gravitation with electromagnetism, H. Weyl had to develop a new geometry, a generalization of Riemannian geometry [24]. In this geometric setting, in addition to the metric tensor, the spacetime manifold is endowed with a 1-form field σ, which, in a certain way, regulates the process of the parallel transport of vector fields in the manifold. In this new approach, a new notion of curvature emerges, the so-called length curvature (*Streckenkrummung,* in German), which is given by the 2-form $F = d\sigma$, which has great similarity to the Faraday tensor, in electromagnetism. This analogy led Weyl to interpret σ as the electromagnetic 4-potential, and this opened the way to the first attempt to a geometrization of the electromagnetic field. Weyl also considered a torsionless affine connection ∇ and, at the same time, generalized the Levi–Civita compatibility condition between this connection and the metric field g by choosing the new condition $\nabla g = \sigma \otimes g$, and found out that this condition is invariant under the conformal transformation $\overline{g} = e^f g$, provided that σ transforms as $\overline{\sigma} = \sigma + df$, where f is an arbitrary differentiable function. These transformations also leave invariant the 2-form F. This new symmetry was called by Weyl "gauge symmetry", and its discovery is now considered by historians as the birth of modern gauge theories [25,26].

It turned out that the Weyl unified theory, according the opinion of many physicists, including Einstein, was not complete and not suitable as a physical theory [27]. It leads to the so-called *second clock effect*, which, as yet, has not been detected [28]. Moreover, it was left incomplete by Weyl insofar as it does not describe how gravity and electromagnetism couple with matter. Recently, the theory was reformulated and gave rise to a modified theory of gravity, where the guide to obtain the coupling of matter was by realizing that the Weyl conformal structure naturally contains in itself a gauge-invariant metric tensor. Once this was found, it was possible to use the general relativistic prescription of minimum coupling to construct a gauge-invariant energy–momentum tensor [19]. It is worth mentioning that new approaches of the Weyl theory have been considered in various recent studies [29–32].

3. The Weyl Invariant Gravity Theory

Although inspired by the original Weyl unified field theory, the theory we are concerned with in this article is not to be regarded as a unified theory. It has been significantly reframed into a modified theory of gravity. In this new version, the theory leads an unexpected appearance of a vector field in the gravitation sector of the action [19]. This vector field, for some choices of the values of the cosmological constant Λ and ω (a free parameter of the theory), may be formally interpreted as a massive vector field satisfying an equation that is identical to the Proca equation.

3.1. The Field Equations

We start by recalling that the field equations of the Weyl invariant theory are given by:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \frac{\Lambda}{4}g_{\alpha\beta} = \frac{\omega}{\Lambda}T^{(P)}_{\alpha\beta} - \kappa T^{(m)}_{\alpha\beta} \tag{1}$$

$$\frac{1}{\sqrt{-g}}\partial_\beta\left(\sqrt{-g}F^{\alpha\beta}\right) = -m^2\sigma^\alpha, \tag{2}$$

where, here, $R_{\mu\nu}$ and R denote, respectively, the Ricci tensor and the scalar curvature defined with respect to the Riemannian connection, σ is a 1-form field, $T^{(P)}_{\alpha\beta} = F_{\alpha\mu}F^{\mu}_{\beta} + \frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} - \frac{3\Lambda}{2\omega}\left(\sigma_\alpha\sigma_\beta - \frac{1}{2}g_{\alpha\beta}\sigma^\mu\sigma_\mu\right)$, $T^{(m)}_{\mu\nu}$ represents the energy–momentum tensor of matter, $\kappa = 8\pi G/c^4$ (where G is the gravitational constant and c is the speed of light), and we are defining $m = \sqrt{-\frac{3\Lambda}{2\omega}}$ [19]. The Weyl field σ has a dimension of (length)$^{-1}$, and ω is dimensionless. Thus, the parameter m (throughout this paper, we are assuming that ω is negative in order to interpret $m = \sqrt{-\frac{3\Lambda}{2\omega}}$ as the mass of the Proca field), which acts as an effective mass of the Weyl field according to the field Equation (2), has also dimension of (length)$^{-1}$.

Let us make a short comment on the role of σ. Let us recall that in Weyl's original approach, σ led to a different notion of curvature, a sort of "length curvature" represented by the 2-form $F = d\sigma$ (*Streckenkrummung*), whereas the affine connection alone gives the well-known notion of "direction curvature" (*Richtungkrummung*), the latter encoded by the Riemann tensor [24]. To his surprise, Weyl realized that the length curvature $F = d\sigma$ presents a striking similarity to the electromagnetic tensor, and it was this discovery that showed him the way to geometrize the electromagnetic field. It is worth mentioning here the discovery of this new symmetry.

It should be mentioned that the above equations may be obtained by varying the action

$$S = \int d^4x\sqrt{-g}[R + \frac{\omega}{2\Lambda}F_{\alpha\beta}F^{\alpha\beta} + \frac{3}{2}\sigma_\alpha\sigma^\alpha - \frac{\Lambda}{2} + \kappa L_m], \tag{3}$$

which is identical to the action of Proca's neutral spin-1 field in curved spacetime with the cosmological constant coupled to gravity. Here, L_m denotes the Lagrangian density of matter.

3.2. The Original Proca's Theory

In this theory, Proca considered the Lagrangian

$$L = -\frac{1}{16\pi c}F_{\alpha\beta}F^{\alpha\beta} + \frac{m^2}{8\pi c}A^\alpha A_\alpha, \tag{4}$$

where the 4-potential A^{μ} describes the field of a massive vector boson. The field equations derived from (4) are given by [1,2]

$$\partial_{\mu}F^{\nu\mu} + m^2 A^{\nu} = 0. \tag{5}$$

From the above equation, we obtain the condition

$$\partial_{\nu}A^{\nu} = 0, \tag{6}$$

which is known as the Lorenz gauge, which then leads to

$$\Box A^{\nu} + m^2 A^{\nu} = 0, \tag{7}$$

the solution of which may be given in the form

$$A^{\nu}(x) = \frac{1}{(2\pi)^4}\int \widetilde{A}^{\nu}(k)\, e^{ik_{\mu}x^{\mu}} d^4k \tag{8}$$

4. The Geometric Proca–Weyl Field as a Candidate for Dark Matter

As we have already mentioned, there have appeared in the literature some recent models that consider the Proca field as a possible candidate for explaining the existence of dark matter [7–13]. Proceeding along this direction, we shall apply the Weyl invariant theory of gravity as an alternative approach to the problem of the origin of dark matter. Our starting point is to consider the existence of a gas of Weyl–Proca particles. As the Weyl field is not coupled directly to the matter, the gas is not in thermal equilibrium with the baryonic matter. Thus, we can assume that the gas is constituted by free particles which are found in the ground state, i.e., with no kinetic energy. First, we investigate if this scenario is consistent with the standard cosmological model.

Let us consider, with this purpose, the Friedmann–Robertson–Walker (FRW) metric in terms of the conformal time η, for a universe with flat spatial sections $k = 0$:

$$ds^2 = a^2\left(d\eta^2 - d\Sigma^2\right). \tag{9}$$

Then, the dynamics of the Weyl field σ will be given by the equations

$$\partial_{\mu}\left(\sqrt{-g}F^{\nu\mu}\right) = -m^2\sqrt{-g}\sigma^{v}, \tag{10}$$

where for (9), we can write

$$F^{\mu\nu} = \frac{1}{a^4}\eta^{\mu\alpha}\eta^{\nu\beta}\left(\partial_{\beta}\sigma_{\alpha} - \partial_{\alpha}\sigma_{\beta}\right), \tag{11}$$

where $\eta^{\mu\alpha}$ represents the Minkowki metric. Since $\sqrt{-g} = a^4$, the field Equation (10) can be rewritten as

$$\partial_{\nu}\left(\eta^{\mu\alpha}\partial_{\mu}\sigma_{\alpha}\right) - \Box\sigma_{\nu} = m^2 a^2 \sigma_{\nu} \tag{12}$$

Multiplying the above equation by $\eta^{\nu\beta}\partial_{\beta}$, we obtain

$$\eta^{\nu\beta}\partial_{\beta}\sigma_{\nu} = -2\sigma_0 a'/a. \tag{13}$$

By using this result in the previous equation, we have

$$2\partial_{\nu}(\sigma_0 a'/a) + \Box\sigma_{\nu} = -m^2 a^2 \sigma_{\nu}. \tag{14}$$

5. A Particular Solution: The Bose–Einstein Condensate

Inspired by the homogeneity and isotropy of the universe at large scales, let us assume a cosmological solution in which the Weyl field depends only on the time coordinate, i.e., $\sigma_\alpha(t)$. Under this condition, it follows from the field Equation (12), more specifically from the component ($\nu = 0$), that $\sigma_0 = 0$ since the field has a non-null mass. Additionally, from the field equations written in the form (14), we obtain

$$\partial_0^2 \sigma_i = -m^2 a^2 \sigma_i, \tag{15}$$

for the spatial components of the Weyl field.

By considering the Fourier modes of these time-dependent functions σ_i, we can interpret this particular solution as representing a collection of particles with zero momentum in the comoving cosmological frame. Of course, this is an accessible state because this field is massive and therefore the associated particles can be found at rest in a certain reference frame. In this state, these particles form the condensate and we want to investigate its cosmological behavior.

To find the solution of the last Equation (15), let us write the field as $\sigma_j = b_j(\eta)e^{i\theta(\eta)}$, following a similar prescription given by Ref. [5]. We have to remember that the Weyl field is a real quantity so that the true solution is the real part of this complex function. In terms of the amplitude and the phase (let us omit the index j), the field Equation (15) assumes the form

$$(b'' - b\theta'^2) + i\big(\theta'' b + 2b'\theta'\big) = -m^2 a^2 b. \tag{16}$$

Considering that the amplitude b and the phase θ are real functions, the above equation can be split into two independent equations:

$$b'' - b\theta'^2 = -m^2 a^2 b, \tag{17}$$

$$\theta'' b + 2b'\theta' = 0. \tag{18}$$

Assuming that the amplitude varies more slowly than the phase oscillation (WKB-condition), or more specifically,

$$\left| b''/b \right| \ll \theta'^2 \tag{19}$$

we obtain, from the first equation,

$$\theta'^2 = m^2 a^2, \tag{20}$$

that is,

$$\theta(\eta) = m \int a d\eta = mt, \tag{21}$$

where t is the cosmological time ($dt = ad\eta$). Using this result in the second equation, we find

$$b(\eta) = \frac{B}{a^{1/2}}, \tag{22}$$

where B is a constant. Therefore, the solution according to the WKB-approximation method is given by

$$\sigma_j^{WKB} = \frac{mC_j}{a^{1/2}} \exp\left(im \int a d\eta \right), \tag{23}$$

where C_j is an arbitrary dimensionless complex constant.

Let us now check the WKB-approximation condition (19). In terms of the cosmological time t, the conditions can be written as

$$\frac{1}{4}\left| H^2 - 2\frac{\ddot{a}}{a} \right| \ll m^2, \tag{24}$$

where $H = \dot{a}/a$ is the Hubble parameter with respect to the cosmological time (t). The restriction for the geometrical mass depends on the initial epoch that we demand for the WKB-solution to be valid. Here, let us examine the matter-dominated epoch. In this era, which starts at a redshift $z_m \sim 3400$ approximately, $\ddot{a}/a = -H^2/2$ and H is a decreasing parameter. Thus, for the WKB-solution to be valid since the beginning of the matter-dominated era until our days, the mass should satisfy $m \gg H(z_m)$. To make an estimate of this lower bound, let us consider that during this era, the content of universe, at the cosmological scale, is that of a dust fluid. Under this condition, $H(z) \simeq H_0(1+z)^{3/2}$. The current value of the Hubble parameter is $H_0 \sim 10^{-18}$ s^{-1}. Thus, the constraint for m is not too stringent. In fact, in energy units, we should have $m \gg 10^{-27}$eV/c^2 .

As we have seen previously, according to the solution we found above, the phase θ oscillates with a frequency proportional to m. This means that the WKB condition requires this oscillation frequency to be at least 10^6 times greater than the present expansion rate of the universe.

6. The Energy–Momentum Tensor for the Particular Solution

The energy distribution of a fluid, represented by the tensor $T_{\alpha\beta}$, measured by a comoving observer with proper velocity V^μ, is characterized by the following components:

1. Energy density of the fluid: $\rho = T_{\alpha\beta}V^\alpha V^\beta$;
2. Isotropic pressure of the fluid: $p = -\frac{1}{3}T_{\alpha\beta}h^{\alpha\beta}$,where $h^{\alpha\beta} = (g^{\alpha\beta} - V^\alpha V^\beta)$;
3. Energy-flux vector: $q_\mu = -h^\alpha_\mu T_{\alpha\beta}V^\beta$;
4. Anisotropic pressure and stress tensor: $\pi_{\mu\nu} = T_{\alpha\beta}h^\alpha_\mu h^\beta_\nu + ph_{\mu\nu}$.

The energy–momentum tensor associated to the Weyl field is similar to that of the Proca field and can be written as the sum of two tensors:

$$T^{(P)}_{\alpha\beta} = T^{EM}_{\alpha\beta} + m^2\theta_{\alpha\beta}, \tag{25}$$

where $T^{EM}_{\mu\nu}$ is formally identical to the symmetric energy–momentum tensor of the electromagnetic field, and $\theta_{\mu\nu}$ is the additional term proportional to the mass of the field.

Let us calculate each tensor separately. For the electromagnetic tensor, we have in the conformal-time coordinate

$$T^{EM}_{\alpha\beta} = \frac{1}{a^2}\left(\eta^{\lambda\mu}F_{\alpha\lambda}F_{\mu\beta} + \frac{1}{4}\eta_{\alpha\beta}\eta^{\mu\lambda}\eta^{\nu\kappa}F_{\mu\nu}F_{\lambda\kappa}\right). \tag{26}$$

The additional tensor that is associated with the mass is

$$\theta_{\alpha\beta} = \left[\sigma_\alpha\sigma_\beta - \frac{1}{2}\eta_{\alpha\beta}(\eta^{\mu\nu}\sigma_\mu\sigma_\nu)\right] \tag{27}$$

With the WKB-solution, we can now calculate the electromagnetic tensor $F_{\alpha\beta}$. Due to the condition that the field depends only on the time coordinate, i.e., $\sigma_\mu = \sigma_\mu(t)$, the only non-null components of $F_{\mu\nu}$ are the "electric" field components $E_i = F_{0i}$.

Using the real part of the solution (23), i.e., $\sigma_j = mC_j\cos\theta(\eta)/a^{1/2}$, we find

$$E_j = -\frac{mC_j}{a^{1/2}}\left[\frac{1}{2}\frac{a'}{a}\cos\theta(\eta) + ma\sin\theta(\eta)\right], \tag{28}$$

where $\theta(\eta)$ now includes a constant phase that comes from the original complex constant C_j, which, in turn, is now just a real number.

The energy–momentum tensor depends on σ_j^2 and E_j^2. Since these quantities oscillate with frequency m, which is much higher than the expanding rate H of the universe during the matter-dominated era, the influence of these fields on the cosmic evolution depends basically on their mean values. Taking the field average in an interval corresponding to several complete oscillations of the phase θ, we obtain

$$\left\langle \sigma_j^2 \right\rangle = \frac{C_j^2 m^2}{2a},$$
$$\left\langle E_j^2 \right\rangle = \frac{C_j^2 a\, m^4}{2}\left[1 + \frac{1}{4}\frac{H^2}{m^2}\right], \tag{29}$$

since the scale factor a and its derivatives practically do not change in this time interval. Based on these results, we can now analyze the energy, the pressure, and stress distribution of the Weyl field in the condensate state.

6.1. The Energy Density

Now let us evaluate the energy density of this fluid as measured by a comoving observer, whose 4-velocity is $V^\mu = \frac{1}{a}\delta_0^\mu$ in this coordinate system. First let us notice that the tensor $T_{\alpha\beta}^{(P)}$ has dimension of (length)$^{-4}$; hence, the associated energy density, the pressure and the stress will have the same dimension. Only in the next section are we going to consider the appropriate dimensions of these quantities in order to compare them with the available data about dark matter density.

Now, calculating $\rho = T_{\alpha\beta}^{(P)} V^\alpha V^\beta$, we can express the density in terms of squared fields:

$$\rho_W = \frac{1}{2a^4}E^2 + \frac{m^2}{2a^2}\sigma^2, \tag{30}$$

where $E^2 = \sum E_j^2$ and $\sigma^2 = \sum \sigma_j^2$.

Taking the time average and utilizing the solution (29), we obtain

$$\langle \rho_W \rangle = \frac{m^4 C^2}{2a^3}\left[1 + \frac{1}{8}H^2/m^2\right], \tag{31}$$

where $C^2 = C_1^2 + C_2^2 + C_3^2$. In the matter-dominated epoch in which the condition $m \gg H$ is satisfied, then $\langle \rho_W \rangle \sim \frac{m^4}{2a^3}C^2 \propto a^{-3}$, which corresponds to the characteristic energy density of non-relativistic matter in an expanding universe. Therefore, the Weyl field in this condensate state has the expected behavior of dark matter in this era, at least in the order below H^2/m^2.

6.2. The Isotropic Pressure

The isotropic pressure is obtained from the energy–momentum tensor (25). For this particular solution, in which the Weyl field depends only on time t, the pressure of the fluid can be written in term of the electric and Weyl field in the following way:

$$p_W = \frac{1}{3}\frac{1}{2a^4}E^2 - \frac{1}{3}\frac{m^2}{2a^2}\sigma^2.$$

Calculating the time average of this quantity, we find

$$\langle p_W \rangle = \frac{1}{24}\left(\frac{C^2 m^4}{2a^3}\right)\left[H^2/m^2\right].$$

Therefore, in comparison with the energy density (31), the pressure of the condensate is given by

$$\frac{\langle p_W \rangle}{\langle \rho_W \rangle} = \frac{1}{24}\left[H^2/m^2\right].$$

Considering the condition $m \gg H$, we can verify that, in this condensate state, the Weyl field does not generate significant pressure. In other words, the Weyl field satisfies a state equation which is characteristic of a dust fluid as expected from a candidate for a dark matter.

6.3. The Energy Flux, the Shear Stress and the Anisotropic Pressure

In this section, we initiate studying non-diagonal components of $T^{(P)}_{\mu\nu}$. Clearly, $T^{(P)}_{i0} = 0$, which means that, in the condensate state of the Weyl field, there is no energy flux $(q_\mu = 0)$ with respect to the comoving observers.

Now, let us consider the tensor $\pi_{\mu\nu}$, which contains information about anisotropic pressure and the shear stress of the fluid. For the FRW metric in the standard coordinates, we have $h^i_j = \delta^i_j$ for the spatial–spatial components, while all mixed components $h_{\mu 0}$ are null. Thus, we have

$$\pi_{\mu 0} = \pi_{0\mu} = 0, \tag{32}$$

$$\pi_{ij} = T^{(P)}_{ij} - p\left(a^2\delta_{ij}\right). \tag{33}$$

With respect to the spatial components with $i \neq j$, we can write

$$\pi_{ij} = T^{(P)}_{ij} = -\frac{1}{a^2}E_iE_j + m^2\sigma_i\sigma_j. \tag{34}$$

Let us consider the time average of each term separately. The product of the components of the Weyl field in different directions has the following time average:

$$\langle\sigma_i\sigma_j\rangle = \frac{m^2C_iC_j}{a}\langle\cos\theta_i\cos\theta_j\rangle, \tag{35}$$

where $\theta_i(t)$ and $\theta_j(t)$ oscillate with the same frequency m but may have different phases ϕ_i and ϕ_j. The mean value of the product of cosine functions in time depends on the difference of the phases. Defining $\delta\phi_{ij} = \phi_i - \phi_j$, we can write

$$\langle\sigma_i\sigma_j\rangle = \frac{m^2C_iC_j}{2a}\cos\left(\delta\phi_{ij}\right), \tag{36}$$

In its turn, the product of distinct electric field components gives us

$$\langle E_iE_j\rangle = \frac{m^2C_iC_j}{2a}\left[\left(\frac{1}{4}\left(\frac{a'}{a}\right)^2 + m^2a^2\right)\cos\left(\delta\phi_{ij}\right)\right]. \tag{37}$$

Substituting these expression in Equation (34), and writing $a' = \dot{a}a$, we obtain:

$$\pi_{ij} = -\frac{m^4C_iC_j}{2a}\left[\frac{1}{4}\left(\frac{H}{m}\right)^2\cos\left(\delta\phi_{ij}\right)\right]. \tag{38}$$

The shear stress measured by the comoving observers can be obtained from tensor π_{ij} by a contraction with the spacelike vectors $\hat{e}_I = e^\mu_I\partial_\mu = \frac{1}{a}\delta^\mu_I\partial_\mu$ of the tetrad associated to the comoving observer. Thus, the non-diagonal elements $(I \neq J)$ of the tensor are given by

$$\pi_{IJ} = \pi_{ij}e^i_Ie^j_J = -\frac{m^4C_IC_J}{2a^3}\left[\frac{1}{4}\left(\frac{H}{m}\right)^2\cos\delta\phi_{IJ}\right]. \tag{39}$$

Let us now compare the shear stress with the energy density of the Weyl field in the condensate state. It follows from (31) and (39) that, in the leading order, we have

$$\frac{|\pi_{IJ}|}{\rho} \simeq \frac{C_I C_J}{C^2}\left[\frac{1}{4}\left(\frac{H}{m}\right)^2 \cos\delta\phi_{IJ}\right]. \tag{40}$$

According to this result, the shear stress is maximal if the oscillations of the Weyl field along orthogonal directions are in phase $(\phi_I = \phi_J)$. A non-null stress could be a potential challenge for the present model since this kind of stress feeds the shear strain of the cosmological fluid, inducing, consequently, deviations from an isotropic evolution. In principle, the condition $m \gg H$ can mitigate the problem in the matter-dominated era. Perhaps, for a sufficiently high mass, the level of deviation could be made compatible with the low degree of the anisotropy in the CMB radiation ($\delta T/T \sim 10^{-5}$), even in the case of a polarized classical field solution. This issue should be analyzed in a future work.

Here, we also would like to highlight two other possibilities. The shear stress will be zero when the Weyl field is in an unpolarized state. It happens that, under certain physics conditions, the condensate can be find in an unpolarized state. In fact, treating σ_μ as a quantum field, we can speculate that it is in a state with a well-defined number of particles with zero momentum, equally distributed in the three polarization modes of this massive field. Thus, if n is the total number of the particles, then, in this solution, one third of them $(n/3)$ are oscillating in each one of the three directions ($I = 1, 2$ and 3). Since each mode has a well-defined number of particles then, as it is known (see [33]), the corresponding state can be represented by a classical phasor with well-defined modulus C_I (proportional to $\sqrt{(n/3) + 1/2}$) but random phase ϕ_I uniformly distributed between 0 and 2π. In other words, the phase is undetermined when the state has a well-defined number of particles. Therefore, the average of the non-diagonal components of the stress tensor is null, which means that in this state, the condensate has no shear stress.

Another possibility to have an unpolarized condensate is to consider that the particles with zero-momentum are in a quasi-classical state produced by random processes in a finite interval of the primordial universe. In this scenario, it is reasonable to assume the random phases hypothesis, i.e., that the phases of the field describing the modes that are produced in different instants have no correlation with each other. In this case, the state of the Weyl field can be described as an incoherent mixture of null-momentum modes. Under this assumption, the average in the corresponding ensemble gives also $\langle \pi_{IJ} \rangle = 0$, for $I \neq J$.

To finish this section, let us analyze the diagonal elements of the tensor π_{ij}, which gives us the anisotropic pressure of the fluid. Following the same reasoning, we find

$$\frac{\left\langle \pi_{(I)(I)} \right\rangle}{\langle \rho_W \rangle} = \left(-\frac{C_I^2}{C^2} + \frac{1}{3}\right)\left[\frac{H^2}{m^2}\right].$$

Again, the anisotropic pressure is attenuated by the factor H^2/m^2 in the matter-dominated era. Moreover, if we have the same number of particles in each mode, i.e., $C_I^2 = C^2/3$, then, the condensate has no anisotropic pressure, and, therefore, it will be exactly compatible with the standard cosmological model in all epochs of evolution.

6.4. Mass and Abundance of the Proca–Weyl Particles

For the Weyl field to be able to explain the gravitational effect attributed to the dark matter component of the universe in the cosmological scale, its energy density should be equal to the measured dark matter density. To compare the theoretical value with the available data about dark matter, we need first to convert the tensor $T^{(P)}_{\alpha\beta}$ to the appropriate dimensions by the following transformation:

$$\tilde{T}^{(P)}_{\alpha\beta} = -\frac{1}{\kappa}\frac{\omega}{\Lambda}T^{(P)}_{\alpha\beta} = \frac{c^4}{8\pi G}\frac{3}{2m^2}T^{(P)}_{\alpha\beta},$$

thus, according to (31), in the leading approximation order, the energy density of the Weyl in the present epoch is given by

$$\langle\tilde{\rho}_W\rangle = \frac{3}{2}\frac{c^4}{8\pi G}\frac{m^2C^2}{2a_0^3}. \tag{41}$$

Dividing by the critical density in the present epoch ($\rho_{c,0} = (3H_0^2c^2)/(8\pi G)$), we find the parameter of density of the Weyl field in the condensate state:

$$\Omega_{W,0} = \frac{m^2c^2C^2}{4a_0^3H_0^2}, \tag{42}$$

where, here, H_0 is in units of s^{-1}. If the Weyl condensate corresponds to all the dark matter content of the universe, then $\Omega_{W,0}$ should be equal to the parameter density of dark matter, whose present value is $\Omega_{DM,0} = 0.264$. Of course, the assumption that $\Omega_{W,0} = \Omega_{DM,0}$ imposes conditions for the mass and the constant C^2.

As we have mentioned previously, the amplitude C^2 is related to the abundance of the Weyl particles produced in the primordial universe. Of course, the abundance depends on the mechanism which generates the Weyl–Proca particles. Making an estimate for the abundance is an important challenge for the model. This issue will probably require that we consider new kinds of coupling between the Weyl field and the spacetime curvature.

7. Final Remarks

The origin of the so-called dark matter has been the main subject of a lot of recent research in theoretical cosmology and astrophysics. There are many proposals to explain its nature ranging from WIMPS, primordial black holes, axions, dark photons, and modified gravity. A modification of the original Weyl unified theory allows us to approach the dark matter problem in a purely geometrical way. In order to carry out this program, we assume the existence of a gas of Weyl–Proca particles in a state of the Bose–Einstein condensate and investigate its behavior at a large scale. The results obtained show that the present model with the presence of the Weyl field is compatible with the standard cosmological model up to the order of H^2/m^2 in the matter-dominated era. In fact, we verify that, for a mass that is sufficiently high, the shear stress and the anisotropic pressure are negligible, even when the field are in a classical state with a well-defined polarization. Moreover, we find that, under the condition $m \gg H$, the Weyl condensate behaves like a pressureless fluid as expected from a candidate for dark matter. In this scenario, the Weyl field has a pure geometrical nature and, as it does not interact directly with the ordinary matter, except through the gravitational field, the laboratory bounds for dark matter candidates can be evaded.

The main restriction for the model comes from the assumption that the Weyl condensate corresponds to the total dark matter content of the Universe. In this case, a constraint must be satisfied by the mass and the present amplitude of the field. This amplitude is related to the abundance of the Weyl–Proca particles produced in the primordial universe which, in its turn, is strongly connected with the generating process of the Weyl particles in the very early phase of the Universe. This is an aspect that warrants further investigation, and that will probably demand the adoption of new coupling between the Weyl field and the curvature. This question is one we intend to analyze in a future work.

Many physicists believe that dark matter must be constituted by some unknown kind of particle, and in this respect, there are many candidates to be considered. On the other hand, another school of thought prefers to believe that perhaps the effects attributed to dark matter may be explained by new theories of gravity. Who is right? This is a question to be decided only by the conjunction of precise observational techniques and new theoretical developments.

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Abbreviations

The following abbreviations are used in this manuscript:

CDM	Cold Dark Matter
WIMPS	Weakly Interacting Massive Particles
FRW	Friedmann–Lemaître–Robertson–Walker (cosmological model)
WKB	Wentzel–Kramers–Brillouin (approximation)
CMB	Cosmic Microwave Background

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