

## Article

# The Geometric Proca Field in Weyl Gauge-Invariant Theory

M. Duarte , T. A. T. Sanomiya , F. Dahia  and C. Romero 

Departamento de Física, Universidade Federal da Paraíba, Caixa Postal 5008, João Pessoa 58059-970, PB, Brazil; mauro.parnaiba@estudantes.ufpb.br (M.D.); sanomiya@fisica.ufpb.br (T.A.T.S.); fdahia@fisica.ufpb.br (F.D.)

\* Correspondence: cromero@fisica.ufpb.br

## Abstract

We present a detailed study on the geometrization of the Proca field in the so-called Weyl Gauge-Invariant Theory, shedding new light on the physical interpretation of the Weyl field. We first describe the field equations of the theory. We then obtain a solution for the weak field using a spherically symmetric and static approximate metric. Our analysis revealed that the Weyl field, in the weak field approximation, exhibits a behavior identical to the Yukawa potential, similar to the Proca field. Furthermore, the obtained metric solution is equivalent to the Einstein–Proca case, demonstrating that the description of the Weyl field in the Weyl Gauge-Invariant Theory is consistent with Proca theory in the context of General Relativity. Finally, we conclude that the Weyl field can be formally interpreted as a Proca field of geometrical nature.

**Keywords:** Gauge-Invariant Weyl Theory; proca field; metric solution

## 1. Introduction

In this paper, we revisit the problem of geometrizing the electromagnetic field in Weyl’s unified theory, shedding new light on a possible new interpretation of the Weyl field. It has long been argued since the theory appeared that the electrodynamics originally proposed by Weyl does not coincide with Maxwell’s electrodynamics. By re-examining Weyl’s theory, with a particular focus on some problems that have rendered it incomplete as a physical theory, we were led to construct a (gauge) invariant version of Weyl’s theory, which we have called *Weyl Gauge-Invariant Theory* [1,2]. In this new version, with the aim of making it more complete, we formulate a prescription of how to implement an invariant way of carrying out the coupling of geometry with matter in the theory, using a new concept of invariant metric. This step is essential in order, for instance, to study cosmology within the framework of the theory.

As highlighted in [1,2], we observe that the action that leads to the field equations in Gauge-Invariant Weyl Theory bears remarkable similarities to the action that describes the interaction of the geometry with the Proca field. As we know, the dynamics of the original Proca field are described by the so-called Proca equations (named after the Romanian physicist Alexandru Proca), first developed to describe the dynamics of massive spin-1 bosons [3–5]. In the context of this new version, we are led to the view that the Weyl field might be identified, not as representing the electromagnetic field, but rather the Proca field, which is, in a natural way, incorporated into the gravitational scenario. Therefore, the now called the Gauge-Invariant Weyl Theory can be seen as a modified approach to gravitation, in which a vector field, formally similar to the Proca field, is considered to have a purely geometric nature.



Academic Editor: Lorenzo Iorio

Received: 22 December 2025

Revised: 8 February 2026

Accepted: 14 February 2026

Published: 18 February 2026

**Copyright:** © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

Finally, to have a deeper understanding of the theory, we investigate the field equations in the weak field approximation. We then show that the results obtained are consistent with Proca theory in the context of general relativity. In the next section, we give a brief summary of Weyl geometry.

## 2. A Very Brief Summary of Weyl Geometry

Weyl geometry is considered to be one of the simplest generalizations of Riemannian geometry. The only difference is that in the former, the covariant derivative of the metric tensor  $g$  does not vanish, but instead is given by the condition of non-metricity:

$$\nabla_\alpha g_{\beta\lambda} = \sigma_\alpha g_{\beta\lambda}, \quad (1)$$

where  $\sigma_\alpha$  indicates the components of a one-form field  $\sigma$  in local coordinate. The new compatibility condition is equivalent to demanding that the length of a vector field may change if parallel-transported along a curve. We refer to the triple  $(M, g, \sigma)$  consisting of a differentiable manifold  $M$  endowed with both a metric  $g$  and a 1-form field  $\sigma$  as a *Weyl gauge*. An important discovery made by Weyl was the following: let us assume that we perform the conformal transformation,

$$\bar{g} = e^f g, \quad (2)$$

where  $f$  is an arbitrary scalar function defined on  $M$ . It then follows that the Weyl compatibility condition (1) still holds as long as we let the Weyl field  $\sigma$  transform as

$$\bar{\sigma} = \sigma + df. \quad (3)$$

In other words, the Weyl compatibility condition is invariant when we go from one gauge  $(M, g, \sigma)$  to another gauge  $(M, \bar{g}, \bar{\sigma})$  by the simultaneous transformations in  $g$  and  $\sigma$ . The collection of all triples  $(M, g, \sigma)$  is referred to as a *Weyl conformal structure*  $\mathcal{W}$ .

Let us now consider on the manifold an arbitrary closed curve  $\alpha : [a, b] \in \mathbb{R} \rightarrow M$ , i.e., with  $\alpha(a) = \alpha(b)$ . Then, it can easily be shown that if we transport a vector  $V$  along  $\alpha$ , then it follows that its length will change according to  $L = L_0 e^{\frac{1}{2} \oint \sigma_\alpha dx^\alpha}$ , where  $L_0$  and  $L$  denotes the values of the length at  $a$  and  $b$ , respectively. Then, from Stokes's theorem, we can write  $L = L_0 e^{\frac{1}{4} \int \int F_{\mu\nu} dx^\mu \wedge dx^\nu}$ ,<sup>1</sup> where  $F_{\mu\nu} = \partial_\mu \sigma_\nu - \partial_\nu \sigma_\mu$ . Therefore, according to the axioms of Weyl geometry, the necessary and sufficient condition for a vector to have its original length preserved after being parallel transported along any closed trajectory is that the 2-form  $F = d\sigma = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$  vanishes.

In this way, Weyl realized that in his new geometry, there appear two kinds of curvature, a *direction curvature* (*Richtungskrummung*) and a *length curvature* (*Streckenkrummung*). The first determines a change in the direction of parallel-transported vectors, and this change is determined locally by the curvature tensor  $R^\alpha_{\beta\mu\nu}$ , while the other is responsible for the change in their length, and is given by  $F_{\mu\nu}$ . Weyl's other important discovery was that the 2-form  $F$  is invariant under the gauge transformation (3). The analogy with the electromagnetic field is clear and becomes even more so when we take into account that  $F$  satisfies the Bianchi identity for the electromagnetic field strength, that is,  $dF = 0$ .

## 3. The Field Equations

We begin by defining, in this new approach, a *gauge-invariant metric*, a sort of representative metric of the conformal structure  $\mathcal{W}$ , which then allows us to develop an invariant procedure, i.e., gauge-independent, to carry out the coupling between matter and geometry,

a question that was not dealt with in the original Weyl unified theory [6]. With this idea in mind, we then choose, from the conformal structure  $\mathcal{W}$  of Weyl geometry, the invariant metric  $\gamma_{\alpha\beta} = \frac{R}{\Lambda} g_{\alpha\beta}$ , where  $R$  is the scalar curvature constructed with the metric  $g_{\alpha\beta}$ , chosen among any metric field of  $\mathcal{W}$ , and  $\Lambda$  is a non-zero constant (originally associated by Weyl with the cosmological constant, and which defines the so-called *natural gauge*). (It is quite easy to verify that  $\gamma_{\alpha\beta}$  is gauge-invariant).<sup>2</sup>

It should be mentioned that the 1-form field given by  $\xi_\alpha = \sigma_\alpha + (\ln R)_{,\alpha}$ , where  $\sigma_\alpha$  is any field of an arbitrary gauge of  $\mathcal{W}$ , is also gauge-invariant, and may be viewed as a representative 1-form of the conformal structure [1,2].<sup>3</sup>

With these two invariant quantities, we can derive the complete field equations in the Weyl Gauge-Invariant Theory, considering the usual prescription of the interaction action with matter, in other words, by defining an invariant energy-momentum tensor of matter, using the standard procedure (minimal coupling principle) of general relativity. In this way, the matter action will be given by

$$\delta S_{(m)} = \delta \left( \chi \int \sqrt{-\gamma} L_M(\psi, \nabla \psi) d^4x \right) = \chi \int \sqrt{-\gamma} T_{\alpha\beta}^{(m)} \delta \gamma^{\alpha\beta} d^4x, \quad (4)$$

where  $L_M$  denotes the Lagrangian density of matter,  $\psi$  generically represents the matter fields,  $\nabla$  is the covariant derivative operator defined with respect to  $\gamma_{\mu\nu}$ , and  $\chi$  is the coupling constant. It is worth noting that the form taken by  $L_M(\psi, \nabla \psi)$  is built according to the principle of minimal coupling adopted in general relativity. Then, the total action will be given by  $S = S_{(g)} + S_{(m)}$ , where  $S_{(g)} = \int [R^2 + \omega F_{\alpha\beta} F^{\alpha\beta}] \sqrt{-\gamma} d^4x$  corresponds to the field action proposed by Weyl. The field equations are derived from the minimal action principle, that is,

$$\delta S = \delta \left( \int [R^2 + \omega F_{\alpha\beta} F^{\alpha\beta}] \sqrt{-\gamma} d^4x \right) + \delta \left( \chi \int L_M(\psi, \nabla \psi) \sqrt{|g|} d^4x \right) = 0. \quad (5)$$

By carrying out the variation of  $S$  with respect to  $\gamma^{\alpha\beta}$  we have

$$\delta S = \int [2R\delta R \sqrt{-\gamma} + R^2 \delta \sqrt{-\gamma}] d^4x + \int \omega \delta (F_{\alpha\beta} F^{\alpha\beta} \sqrt{-\gamma}) d^4x + \delta \left( \chi \int L_M(\psi, \nabla \psi) \sqrt{|g|} d^4x \right) = 0$$

Now let us choose to work in the *Weyl gauge* (also called the *natural gauge*), by setting  $R = \Lambda$ . Then, the above equation becomes<sup>4</sup>

$$\delta S = 2\Lambda \delta \int \left( R - \frac{\Lambda}{2} + \frac{\omega}{2\Lambda} F_{\alpha\beta} F^{\alpha\beta} + \frac{\chi}{2\Lambda} L_M \right) \sqrt{-g} d^4x = 0.$$

We thus see that in this gauge, proposed by Weyl, the field equations get enormously simplified. By expressing the Weylian scalar curvature  $R$  in terms of the Riemannian scalar curvature  $\tilde{R}$  we obtain [7]

$$\delta \int \left( \tilde{R} + \frac{3}{2} (\sigma_\alpha \sigma^\alpha) - \frac{3}{\sqrt{-g}} (\sqrt{-g} \sigma^\alpha)_{|\alpha} - \frac{\Lambda}{2} + \frac{\omega}{2\Lambda} F_{\alpha\beta} F^{\alpha\beta} + \frac{\chi}{2\Lambda} L_M \right) \sqrt{-g} d^4x = 0.$$

Then, it follows that

$$\delta \int \left( \tilde{R} + \frac{3}{2} (\sigma_\alpha \sigma^\alpha) - \frac{\Lambda}{2} + \frac{\omega}{2\Lambda} F_{\alpha\beta} F^{\alpha\beta} + \frac{\chi}{2\Lambda} L_M \right) \sqrt{-g} - 3 \int \partial_\alpha (\delta (\sqrt{-g} \sigma^\alpha)) d^4x = 0 \quad (6)$$

Clearly, the last term of the above integral is a surface term, which is then set to zero. Therefore, we have

$$\delta S = 2\Lambda \delta \int \left( \tilde{R} + \frac{3}{2} \sigma_\alpha \sigma^\alpha - \frac{\Lambda}{2} + \frac{\omega}{2\Lambda} F_{\alpha\beta} F^{\alpha\beta} + \frac{\chi}{2\Lambda} L_M \right) \sqrt{-g} d^4x.$$

By taking variations with respect to the metric  $g_{\mu\nu}$  and the field  $\sigma_\alpha$ , one is led to the field equations

$$\tilde{R}_{\alpha\beta} - \frac{1}{2} \tilde{R} g_{\alpha\beta} + \frac{\Lambda}{4} g_{\alpha\beta} = \frac{\omega}{\Lambda} T_{\alpha\beta} - \kappa T_{\alpha\beta}^{(m)}, \quad (7)$$

$$\frac{1}{\sqrt{-g}} (\sqrt{-g} F^{\alpha\beta})_{,\beta} = \frac{3\Lambda}{2\omega} \sigma^\alpha, \quad (8)$$

where  $T_{\alpha\beta} = F_\alpha{}^\nu F_{\nu\beta} + \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} - \frac{3\Lambda}{2\omega} (\sigma_\alpha \sigma_\beta - \frac{1}{2} g_{\alpha\beta} \sigma_\mu \sigma^\mu)$  is interpreted as the energy-momentum tensor of the Weyl field, while  $T_{\alpha\beta}^{(m)}$  is the energy-momentum tensor of matter, and  $\kappa = \frac{\chi}{2\Lambda}$ . The Equations (7) and (8) are the field equations which describe the dynamics of both the gravitational field and the Weyl field, as well as their interaction with matter.

#### 4. The Geometrization of the Proca Field

In the original Weyl unified theory, both the gravitational and the electromagnetic fields are geometrized. In his proposal,  $\sigma_\mu$  corresponds to a particular kind of non-metricity and may be viewed as the field that regulates parallel transport in the spacetime manifold. As we have already pointed out, the so-called *length curvature* leads naturally to the appearance of a 2-form field, namely  $F = d\sigma$ , which has rather unexpected algebraic and invariant properties analogous to the Faraday tensor of electromagnetic theory. However, it turns out that this analogy is not complete. Indeed, if we look at the Equation (8), we observe that the field  $\sigma_\mu$  interacts with itself, in contrast to the electromagnetic 4-potential, say  $A_\mu$ , which has as its only source the 4-current [8]. Moreover, we have no way to distinguish between the motions of positively and negatively charged particles. The only curves that remain invariant under Weyl transformations are the affine geodesics, which, in turn, provide no information about the motion of particles influenced by both the gravitational field and the electromagnetic field, which would be analogous to the Lorentz force.

On the other hand, as noted in [1,2], when we consider the variation of the action in the Weyl gauge

$$\delta \int \left[ \tilde{R} + \frac{3}{2} (\sigma_\alpha \sigma^\alpha) - \frac{\Lambda}{2} + \frac{\omega}{2\Lambda} F_{\alpha\beta} F^{\alpha\beta} \right] \sqrt{-g} d^4x = 0 \quad (9)$$

We can formally identify the above equation as describing the dynamics of a massive Proca field in a curved spacetime in the presence of the cosmological constant [9].

As we know, the Proca equation describes the dynamics of a spin-1 massive boson [3–5]. Here we see that the equation of the Weyl field corresponds to the Proca equation in curved spacetime. On the other hand, the energy-momentum tensor of the Weyl field looks the same as the energy-momentum tensor of the Proca field. We therefore may interpret the Weyl field not as a geometric electromagnetic field as originally proposed by Weyl, but rather as a geometric Proca field. We thus can regard this approach as a modified gravity theory which introduces a massive vector field of geometric nature. If we restrict ourselves to negative values of the coupling constant  $\omega$ , then we may consider in Equation (8),  $m = \sqrt{\frac{-3\Lambda}{2\omega}}$  as its mass. Let us recall that we are considering  $\Lambda$  to be interpreted as the cosmological constant according to Weyl's original ideas; hence, we take  $\Lambda > 0$  and  $\omega < 0$ .

It should be mentioned that recently the Proca field has been well investigated in the framework of general relativity. The case of a spacetime generated by a source consisting of a single pointlike Proca charge has been studied with interest, leading to solutions obtained

in the weak field regime (see, for instance, [9–21]). In the present section, we will obtain a simple solution to the field equations in the context of the Gauge-Invariant Weyl Theory.

## 5. Solving the Field Equations in the Weak Field Approximation

Due to the presence of mass, the energy-momentum tensor of the Weyl–Proca has a non-null trace, and thus the field equations of the Weyl Gauge-Invariant Theory in the absence of matter, i.e., with  $T_{\alpha\beta}^{(m)} = 0$ , may be written in the following form:

$$\tilde{R}_{\alpha\beta} = \frac{\Lambda}{4} g_{\alpha\beta} + \frac{\omega}{\Lambda} \left( T_{\alpha\beta}^{(P)} - \frac{1}{2} g_{\alpha\beta} T^{(P)} \right) \quad (10)$$

$$\frac{1}{\sqrt{-g}} \partial_\beta (\sqrt{-g} F^{\alpha\beta}) = -m^2 \sigma^\alpha, \quad (11)$$

where  $\tilde{R}_{\alpha\beta}$  denotes the (Riemannian) Ricci tensor,  $T_{\alpha\beta}^{(P)} = F_\alpha{}^\nu F_{\nu\beta} + \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} + m^2 \left( \sigma_\alpha \sigma_\beta - \frac{1}{2} g_{\alpha\beta} \sigma_\mu \sigma^\mu \right)$  corresponds to the energy-momentum tensor of the Weyl–Proca field, while  $m^2 = \left( \frac{3\Lambda}{2\omega} \right)$  is its mass and  $T^{(P)} = -m^2 \sigma_\mu \sigma^\mu$  gives the trace of  $T_{\alpha\beta}^{(P)}$ .

We shall now consider the case of a static and spherically symmetric spacetime, whose metric in the weak-field regime takes the usual form

$$ds^2 = (1 + \varepsilon\nu) dt^2 - (1 + \varepsilon\lambda) dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad (12)$$

where  $\nu = \nu(r)$  and  $\lambda = \lambda(r)$ , with  $\varepsilon$  being a first-order parameter in the linear approximation. The components of the Ricci tensor in the above approximation are then given by

$$\begin{aligned} \tilde{R}_{00} &= -\frac{\nu''\varepsilon}{2} - \frac{\nu'\varepsilon}{r}, \quad \tilde{R}_{11} = \frac{\nu''\varepsilon}{2} - \frac{\lambda'\varepsilon}{r}, \\ \tilde{R}_{22} &= \frac{1}{2} r \nu' \varepsilon - \frac{1}{2} r \lambda' \varepsilon - \lambda \varepsilon, \quad \tilde{R}_{33} = \sin^2 \theta R_{22}. \end{aligned}$$

Clearly, for consistency with the weak-field approximation, both the terms corresponding to the cosmological constant and the Weyl–Proca energy-momentum which appear in Equation (10) must also be of first-order in  $\varepsilon$ . This means that due to the assumed symmetries, the Weyl–Proca field may be written as  $\sigma_\mu = (\sqrt{\varepsilon}\varphi(r), 0, 0, 0)$ , which then guarantees that its energy-momentum tensor has all its components of order  $\varepsilon$ .

From the above considerations, it follows that the only non-trivial equation of (11) is

$$\partial_1 (\sqrt{-g} F^{01}) = -m^2 \sigma^0 \sqrt{-g}. \quad (13)$$

In this approximation, we have  $\sqrt{-g} = r^2 \sin \theta \left[ 1 + \frac{\varepsilon}{2} (\nu + \lambda) \right]$ , and the Equation (13) becomes:<sup>5</sup>

$$-\frac{d}{dr} \left( r^2 \sin \theta \left[ 1 + \frac{\varepsilon}{2} (\nu + \lambda) \right] \frac{\sqrt{\varepsilon} \varphi'(r)}{\left[ 1 + \varepsilon(\lambda + \nu) \right]} \right) = -m^2 r^2 \sin \theta \left[ 1 + \frac{\varepsilon}{2} (\nu + \lambda) \right] (1 - \varepsilon\nu) \sqrt{\varepsilon} \varphi(r),$$

which, as  $\varepsilon \ll 1$ , may be written as

$$-\frac{d}{dr} \left( r^2 \left\{ 1 + \frac{\varepsilon}{2} (\nu + \lambda) \right\} [1 - \varepsilon(\lambda + \nu)] \sqrt{\varepsilon} \varphi'(r) \right) = -m^2 r^2 \left[ 1 + \frac{\varepsilon}{2} (\nu + \lambda) \right] (1 - \varepsilon\nu) \sqrt{\varepsilon} \varphi(r).$$

Now, after neglecting the terms of order  $\varepsilon^{\frac{3}{2}}$  the above equation becomes

$$-\frac{d}{dr} \left( r^2 \sqrt{\varepsilon} \varphi'(r) \right) = -m^2 r^2 \sqrt{\varepsilon} \varphi(r).$$

Therefore, the equation of the Weyl–Proca field takes the simpler form

$$\varphi''(r) + \frac{2}{r}\varphi'(r) - m^2\varphi(r) = 0, \quad (14)$$

whose general solution is

$$\varphi(r) = \frac{C_1}{r}e^{-mr} + \frac{C_2}{r}e^{mr},$$

where  $C_1$  and  $C_2$  are arbitrary. By choosing these constants to be  $C_2 = 0$  and  $C_1 = q$ ,  $q$  being interpreted as a kind of *geometric charge*, we finally obtain the final form of the Weyl field:

$$\varphi(r) = \frac{q}{r}e^{-mr} \quad (15)$$

At this point, it is worth noting that the only non-trivial component of the Weyl–Proca field presents an identical behavior with the Yukawa potential. For the other components we have  $\sigma_\mu = \sqrt{\varepsilon} \frac{q}{r} e^{-mr} \delta_{0\mu}$ .

Let us now turn our attention to the metric field by considering Equation (10). The non-null components of the energy-momentum tensor are diagonal and are given by

$$\begin{aligned} T_{00} &= \frac{1}{2} \frac{\varepsilon \varphi'(r)^2}{(1 + \varepsilon\lambda)} + \frac{m^2}{2} \varepsilon \varphi(r)^2, \\ T_{11} &= -\frac{1}{2} \frac{\varepsilon \varphi'(r)^2}{(1 + \varepsilon\nu)} + \frac{m^2}{2} \frac{\varepsilon \varphi(r)^2 (1 + \varepsilon\lambda)}{(1 + \varepsilon\nu)}, \\ T_{22} &= \frac{1}{2} r^2 \frac{\varepsilon \varphi'(r)^2}{(1 + \varepsilon\lambda)(1 + \varepsilon\nu)} + \frac{m^2}{2} \frac{\varepsilon r^2 \varphi(r)^2}{(1 + \varepsilon\nu)}, \\ T_{33} &= \sin^2 \theta T_{22}. \end{aligned}$$

Then the field equations for (10)  $\mu = \nu = 0, 1$  lead to

$$\begin{aligned} -\frac{\nu''\varepsilon}{2} - \frac{\nu'\varepsilon}{r} &= \frac{\Lambda}{4}(1 + \varepsilon\nu) + \frac{\omega}{\Lambda} \left( \frac{1}{2} \frac{\varepsilon \varphi'(r)^2}{(1 + \varepsilon\lambda)} + \frac{m^2}{2} \varepsilon \varphi(r)^2 + (1 + \varepsilon\nu) \frac{m^2}{2} (1 - \varepsilon\nu) \varepsilon \varphi(r)^2 \right) \\ \frac{\nu''\varepsilon}{2} - \frac{\lambda'\varepsilon}{r} &= -\frac{\Lambda}{4}(1 + \varepsilon\lambda) + \frac{\omega}{\Lambda} \left( -\frac{1}{2} \frac{\varepsilon \varphi'(r)^2}{(1 + \varepsilon\nu)} + \frac{m^2}{2} \frac{\varepsilon \varphi(r)^2 (1 + \varepsilon\lambda)}{(1 + \varepsilon\nu)} - (1 + \varepsilon\lambda) \frac{m^2}{2} (1 - \varepsilon\nu) \varepsilon \varphi(r)^2 \right), \end{aligned}$$

Keeping only terms of first order in  $\varepsilon$  and recalling that we have assumed  $\Lambda$  to be of order of  $\varepsilon$  the equations become

$$-\frac{\nu''\varepsilon}{2} - \frac{\nu'\varepsilon}{r} = \frac{\Lambda}{4} + \frac{\omega}{\Lambda} \left( \frac{1}{2} \varepsilon \varphi'(r)^2 + \frac{m^2}{2} \varepsilon \varphi(r)^2 + \frac{m^2}{2} \varepsilon \varphi(r)^2 \right) \quad (16)$$

$$\frac{\nu''\varepsilon}{2} - \frac{\lambda'\varepsilon}{r} = -\frac{\Lambda}{4} + \frac{\omega}{\Lambda} \left( -\frac{1}{2} \varepsilon \varphi'(r)^2 + \frac{m^2}{2} \varepsilon \varphi(r)^2 - \frac{m^2}{2} \varepsilon \varphi(r)^2 \right), \quad (17)$$

Adding the above equations leads to

$$\nu' = -\lambda' - \frac{\omega}{\Lambda} m^2 r \varphi(r)^2. \quad (18)$$

For  $\mu = \nu = 2$  (10) reads

$$\frac{1}{2} r \nu' \varepsilon - \frac{1}{2} r \lambda' \varepsilon - \lambda \varepsilon = -\frac{\Lambda}{4} r^2 + \frac{\omega}{\Lambda} \left( \frac{1}{2} r^2 \frac{\varepsilon \varphi'(r)^2}{(1 + \varepsilon\lambda)(1 + \varepsilon\nu)} + \frac{m^2}{2} \frac{\varepsilon r^2 \varphi(r)^2}{(1 + \varepsilon\nu)} - \frac{m^2}{2} r^2 (1 - \varepsilon\nu) \varepsilon \varphi(r)^2 \right),$$

which, after keeping only first-order terms in  $\varepsilon$ , becomes

$$\frac{1}{2}rv'\varepsilon - \frac{1}{2}r\lambda'\varepsilon - \lambda\varepsilon = -\frac{\Lambda}{4}r^2 + \frac{\omega}{\Lambda}\frac{1}{2}r^2\varepsilon\varphi'(r)^2.$$

Now by using (18) we readily obtain

$$-r\lambda'\varepsilon - \lambda\varepsilon - \frac{\omega}{2\Lambda}m^2r^2\varepsilon\varphi(r)^2 = -\frac{\Lambda}{4}r^2 + \frac{\omega}{2\Lambda}r^2\varepsilon\varphi'(r)^2.$$

On the other hand, the equation for the metric function  $\lambda(r)$  will be given by

$$(r\lambda)' = \frac{\Lambda}{4\varepsilon}r^2 - \frac{\omega}{\Lambda}r^2\varphi'(r)^2 + \frac{\omega}{2\Lambda}m^2r^2\varphi(r)^2. \quad (19)$$

In this way, by substituting  $\varphi(r)$  in the above Equation (19) we will obtain:<sup>6</sup>

$$(r\lambda)' = \frac{\Lambda}{4\varepsilon}r^2 - \frac{\omega}{2\Lambda}\frac{q_g^2}{r^2}e^{-2mr} - \frac{\omega}{\Lambda}m\frac{q_g^2}{r}e^{-2mr} - \frac{\omega}{\Lambda}m^2q_g^2e^{-2mr}.$$

Integrating this equation in the variable  $r$  gives us

$$\lambda(r) = \frac{C_3}{r} + \frac{\Lambda}{12\varepsilon}r^2 + \frac{\omega}{2\Lambda}\frac{q_g^2}{r^2}e^{-2mr} + \frac{\omega}{2\Lambda}m\frac{q_g^2}{r}e^{-2mr}, \quad (20)$$

where  $C_3$  is an integration constant. From (18), we finally obtain

$$\nu(r) = -\frac{C_3}{r} - \frac{\Lambda}{12\varepsilon}r^2 - \frac{\omega}{2\Lambda}\frac{q_g^2}{r^2}e^{-2mr} - \frac{\omega}{2\Lambda}m\frac{q_g^2}{r}e^{-2mr} + \frac{\omega}{\Lambda}m^2q_g^2\int\frac{e^{-2mr}}{r}dr$$

Thus, in the weak field regime, the spacetime in the Gauge-Invariant Weyl Theory is described by the following metric:

$$\begin{aligned} ds^2 = & \left(1 - \frac{2m_g}{r} - \frac{\Lambda}{12}r^2 - \frac{\varepsilon\omega}{2\Lambda}\frac{q_g^2}{r^2}e^{-2mr} + \frac{\varepsilon\omega}{2\Lambda}m\frac{q_g^2}{r}e^{-2mr} + \frac{\varepsilon\omega}{\Lambda}m^2q_g^2\int_r^\infty\frac{e^{-2mr}}{r}dr\right)dt^2 \\ & - \left(1 + \frac{2m_g}{r} + \frac{\Lambda}{12}r^2 + \frac{\varepsilon\omega}{2\Lambda}\frac{q_g^2}{r^2}e^{-2mr} + \frac{\varepsilon\omega}{2\Lambda}m\frac{q_g^2}{r}e^{-2mr}\right)dr^2 \\ & - r^2(d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \quad (21)$$

where the constant  $C_3$  is determined by taking  $C_3 = \frac{2m_g}{\varepsilon}$ ,  $m_g$  denoting the *geometric mass* and  $m$  corresponding to the mass of the Weyl–Proca field.

If we set  $\Lambda = 0$ , then this result reproduces the solution obtained in the context of general relativity, which investigates the dynamics of a spacetime generated by a Proca field sourced by a charged point particle [17]. Moreover, the solution found above has great similarities with the result obtained in [19], which examined the effect of the Proca field in Reissner–Nordstrom–de Sitter spacetime.

However, in the present case, the massive vector field has a geometrical nature. In other words, it is part of the spacetime geometry, and should be more appropriately called a *geometrical Proca field*. We can thus conclude that the Weyl field in the Gauge-Invariant Weyl Theory presents an identical behavior as the Proca field in general relativity.

Let us now examine the special case when the mass  $m$  of the Weyl–Proca field is very small.<sup>7</sup> By carrying out a Taylor expansion of  $e^{-mr}$ , we may consider the approximation

$$e^{-mr} \approx 1 - mr + \frac{1}{2}m^2r^2.$$

It is easy to verify that the Weyl–Proca vector takes the form of a Coulomb potential when the order of the field is  $\sqrt{\varepsilon}$ , that is,  $\sigma_\mu = \sqrt{\varepsilon} \frac{q_g}{r} \delta_{0\mu}$ , while the metric, to first order in  $\varepsilon$  will be given by

$$ds^2 = \left(1 - \frac{2m_g}{r} - \frac{\Lambda}{12}r^2 - \frac{\varepsilon\omega}{2\Lambda} \frac{q^2}{r^2}\right) dt^2 - \left(1 + \frac{2m_g}{r} + \frac{\Lambda}{12}r^2 + \frac{\varepsilon\omega}{2\Lambda} \frac{q^2}{r^2}\right) dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 e^{-mr}. \quad (22)$$

As is well-known, the above solution corresponds to the Reissner–Nordström–de Sitter spacetime. Let us also mention that, to recover the Schwarzschild–de Sitter solution, we could admit that the charge  $q_g$  has a very small value.

Finally, for reasons of consistency, we would like to show explicitly that our solution satisfies the Weyl gauge condition, a requirement which has played an essential role in the new version of Weyl’s theory and has been extensively used throughout the development of the present formalism.<sup>8</sup>

As we have mentioned previously,  $\tilde{R}_{\alpha\beta}$  refers to the Riemannian Ricci tensor, and so it does not correspond to Weylian Ricci  $R_{\alpha\beta}$ . On the other hand, the Riemannian and Weylian Ricci scalars are related by the equation

$$R = \tilde{R} + \frac{3}{2}\sigma_\mu\sigma^\mu. \quad (23)$$

As we have already shown, in the weak field approximation, in which the Weyl field and the line element are given, respectively, by  $\sigma_\mu = (\sqrt{\varepsilon}\varphi(r), 0, 0, 0)$  and  $ds^2 = (1 + \varepsilon\nu)dt^2 - (1 + \varepsilon\lambda)dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2$ , the components of  $\tilde{R}_{\alpha\beta}$  are, to first order in  $\varepsilon$ , given by  $\tilde{R}_{00} = -\frac{\nu''\varepsilon}{2} - \frac{\nu'\varepsilon}{r}$ ,  $\tilde{R}_{11} = \frac{\nu''\varepsilon}{2} - \frac{\lambda'\varepsilon}{r}$ ,  $\tilde{R}_{22} = \frac{1}{2}r\nu'\varepsilon - \frac{1}{2}r\lambda'\varepsilon - \lambda\varepsilon$ ,  $\tilde{R}_{33} = \sin^2 \theta \tilde{R}_{22}$ . It then follows that

$$\tilde{R} = -\varepsilon\nu'' - \frac{2\varepsilon}{r}\nu' + \frac{2\varepsilon}{r}\lambda' + \frac{2\varepsilon}{r^2}\lambda.$$

Now, substituting  $\lambda(r)$  and  $\nu(r)$  by their expressions

$$\begin{aligned} \lambda(r) &= \frac{C_3}{r} + \frac{\Lambda}{12\varepsilon}r^2 + \frac{\omega}{2\Lambda} \frac{q^2}{r^2} e^{-2mr} + \frac{\omega}{2\Lambda} m \frac{q^2}{r} e^{-2mr}, \\ \nu(r) &= -\frac{C_3}{r} - \frac{\Lambda}{12\varepsilon}r^2 - \frac{\omega}{2\Lambda} \frac{q_g^2}{r^2} e^{-2mr} - \frac{\omega}{2\Lambda} m \frac{q_g^2}{r} e^{-2mr} + \frac{\omega}{\Lambda} m^2 q_g^2 \int \frac{e^{-2mr}}{r} dr, \end{aligned}$$

it is not difficult to verify that

$$\tilde{R} = \Lambda + \frac{\omega}{\Lambda} \frac{\varepsilon m^2 q_g^2}{r^2} e^{-2mr}.$$

Moreover, since  $\sigma_\mu\sigma^\mu = g^{00}(\sigma_0)^2 = (1 - \varepsilon\nu)\varepsilon\varphi(r)^2 = \varepsilon\varphi(r)^2$  and  $m^2 = -\frac{3\Lambda}{2\omega}$  we finally obtain

$$R = \Lambda + \frac{\omega}{\Lambda} \left(-\frac{3\Lambda}{2\omega}\right) \frac{\varepsilon q_g^2}{r^2} e^{-2mr} + \frac{3}{2} \frac{\varepsilon q_g^2}{r^2} e^{-2mr} = \Lambda,$$

which then shows explicitly that the solution obtained is consistent with the Weyl gauge condition, i.e.,  $R = \Lambda$ .

## 6. Conclusions

In this work, we analyze the problem of the geometrization of the electromagnetic field in Weyl’s theory, offering a new perspective based on his program. To this end, we argue that the electrodynamics originally obtained by Weyl does not coincide with Maxwell’s electrodynamics, but rather with a Proca electrodynamics, in which the massive vector field is understood as possessing a purely geometric nature.

We also present, as an application of the field equations of Weyl Gauge-Invariant Theory, a solution for the weak field in empty space. Comparing the solution obtained with the result already known in the literature, we note a great similarity between them when we take  $\Lambda = 0$  [17]. Furthermore, we also observe a remarkable similarity between the solution found and that obtained in [19] Shi in the study of the gravitational interaction of the Proca field in Reissner–Nordstrom–de Sitter spacetime in the context of general relativity. Therefore, the description of the geometric Proca field in Weyl’s invariant theory proves to be, in a certain way, consistent with the Proca theory in the context of general relativity.

## 7. Final Remark

We would like to add, as a kind of special addendum to the present article, an epistemological discussion motivated by the methodology inherent to Weyl’s approach to a problem traditionally examined in the context of another theory, namely, general relativity. The text which follows, although lying somewhat outside of the main development of the paper, touches on an interesting point raised by one of the referees, who kindly gave us authorization for including it as part of the article: *The geometrization of the Proca field within the Weyl Invariant Theory, as revealed by the weak field analysis yielding a Yukawa type potential for the Weyl field and a static, spherically symmetric metric equivalent to the Einstein–Proca solution opens a conceptual layer that goes far beyond a technical matching of equations of motion. What is at stake is not merely the fact that two formalisms reproduce the same classical dynamics in an appropriate limit, but the manner in which physical meaning is encoded, generated, and rendered expressible within a given theoretical language. In ordinary General Relativity, the Proca field represents a massive vector degree of freedom whose defining feature, namely the mass term, must be introduced by hand at the level of the action. The mass parameter is external to the geometric structure of spacetime. It is not dictated by curvature, connection, or metric compatibility, but rather appended as an additional physical input. Consequently, the interpretation of mass and finite interaction range in Einstein Proca theory remains conceptually extrinsic to the purely Riemannian framework.*

*In contrast, in Weyl Invariant Theory, the same physical content emerges intrinsically from geometry itself. The Weyl connection, together with its associated scale structure, naturally gives rise to an effective massive vector degree of freedom whose weak field behavior reproduces a Yukawa potential. Here, the mass scale is no longer an externally imposed constant but a quantity encoded in the geometric data of spacetime through the dynamics of local scale invariance and its fixing or breaking. The passage from Einstein Proca to Weyl invariant geometry thus represents a shift from an external parametrization of physical effects to an internal geometric realization. This shift has a precise conceptual analogue in the foundations of logic and semantics, most notably in the seminal results of Kurt Gödel and Alfred Tarski.*

*Gödel’s 1931 incompleteness theorem established that any sufficiently expressive and consistent axiomatic system inevitably contains true statements that cannot be proven within that system itself. These truths are not false or meaningless. Rather, they transcend the internal deductive resources of the theory. Tarski’s 1933 theorem on the undefinability of truth sharpened this insight by showing that a consistent formal language cannot contain a truth predicate for its own sentences. Any adequate notion of truth must be defined only in a stronger meta-language. Together, these results demonstrate a structural separation between syntax and semantics. Truth is not fully capturable from within the system whose truths are being discussed.*

*In the present physical context, pure Einstein gravity may be viewed as an internally consistent formal system whose language is the geometry of a Riemannian manifold equipped with a Levi Civita connection. Within this language, certain physically meaningful statements, such as the existence of an intrinsic mass scale for a vector degree of freedom with finite interaction range, are*

not naturally expressible as geometric truths. They must be appended as external structures, in close analogy with undecidable but true statements in arithmetic. The Weyl–Proca correspondence then plays the role of a semantic extension analogous to passing from an object language to a meta language. By enlarging the geometric framework to include Weyl invariance and its associated connection, statements that were previously external to the theory become internally meaningful and geometrically encoded.

This analogy can be made precise through the notion of an External Truth Predicate. An External Truth Predicate formalizes the idea that certain propositions acquire a well-defined truth value only relative to a theory that transcends the internal deductive closure of a given framework. In logical terms, it is the semantic device that allows one to speak truthfully about a system from a standpoint that is not confined to that system’s own language. In physical terms, the External Truth Predicate captures how certain physically relevant structures, including mass generation, finite interaction range, and effective Proca dynamics, can be meaningful and true even though they are not internally definable within the original geometric formalism. From this perspective, the statement that a massive vector field with Yukawa suppression is present in the gravitational sector is not internally decidable within pure Riemannian General Relativity. It becomes a well-defined and true statement only when evaluated from the standpoint of the extended Weyl invariant theory.

Within this enlarged framework, the Weyl field itself functions as a geometrically realized external truth predicate for Einstein gravity. It encodes, in geometric language, information about mass, scale, and finite-range interactions that cannot be fully reduced to the internal vocabulary of Riemannian curvature and metric compatibility. Recent developments have shown that such external truth predicates can be introduced consistently, without contradiction, and can provide a rigorous semantic framework for understanding theory extension and interpretation. In particular, the construction and analysis of External Truth Predicates in a physical and logical setting have been developed in recent work, where it is demonstrated that semantic extensions of this kind do not undermine consistency but instead clarify the conditions under which new physical truths become expressible.

Seen from this vantage point, the Weyl–Proca equivalence is not merely a convenient reformulation or a field redefinition. It is an explicit physical realization of Gödel Tarski incompleteness in a geometric setting. The mass of the vector field and its Yukawa-type behavior correspond to truths about the gravitational system that are inaccessible within the internal language of Einstein gravity but become manifest once the theory is semantically completed by a Weyl invariant extension. The equivalence thus illustrates how physical theories, like formal logical systems, may be internally coherent yet semantically incomplete, and how their completion requires an enlargement of the conceptual and mathematical language in which they are formulated.

From this standpoint, incompleteness ceases to be a peculiarity of abstract logic and instead appears as a structural feature of fundamental physical theories. Just as arithmetic requires a meta theory to speak coherently about its own truth, gravitational dynamics formulated purely in Riemannian terms may require geometric extensions to articulate all physically meaningful phenomena. Weyl geometry, in this sense, plays the role of a semantic completion. It does not contradict Einstein gravity, but rather embeds it into a richer framework in which previously external truths, such as massive vector dynamics, are rendered internal, geometric, and intelligible. This perspective suggests a deep unity between logic and physics, in which theory extension, semantic completeness, and geometric enrichment are different manifestations of the same underlying necessity imposed by the limits of internal definability.

**Author Contributions:** Conceptualization, M.D., T.A.T.S. and C.R.; methodology, M.D., T.A.T.S. and C.R.; validation, M.D., F.D. and C.R.; formal analysis, M.D., T.A.T.S., F.D. and C.R.; investigation, M.D. and C.R.; resources, M.D., T.A.T.S., F.D. and C.R.; data curation, F.D.; writing—original draft preparation, M.D. and C.R.; writing—review and editing, M.D. and C.R.; visualization, M.D., F.D.

and C.R.; supervision, C.R.; project administration, C.R.; funding acquisition, M.D. and C.R. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research was funded by National Council for Scientific and Technological Development (grant 310046/2022-0).

**Data Availability Statement:** The original contributions presented in this study are included in the article material. Further inquiries can be directed to the corresponding author.

**Acknowledgments:** This work was partially supported by CNPq and FAPESQ (Brazil). We thank the referees for their relevant and useful comments.

**Conflicts of Interest:** The authors declare no conflicts of interest.

## Notes

- <sup>1</sup> Here we are assuming that the region of integration is simply connected.
- <sup>2</sup> In Weyl geometry, the scalar curvature under Weyl transformations, the scalar curvature and the metric transforms as  $\bar{R} = e^{-f(x)}R$ ,  $\bar{g}_{\alpha\beta} = e^{f(x)}g_{\alpha\beta}$ . Then it is clear that  $\bar{\gamma}_{\alpha\beta} = \frac{\bar{R}}{R}g_{\alpha\beta} = \frac{e^{-f}R}{R}e^f g_{\alpha\beta} \Rightarrow \bar{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta}$ .
- <sup>3</sup> Indeed,  $\bar{\xi}_\alpha = \xi_\alpha$  as  $\bar{R} = e^{-f(x)}R$  and  $\bar{\sigma}_\alpha = \sigma_\alpha + f(x)_{,\alpha}$ .
- <sup>4</sup> Note that in the *natural gauge*, the metric  $\gamma_{\alpha\beta}$  and the invariant 1-form  $\xi_\alpha$  become  $g_{\mu\nu}$  and  $\sigma_\alpha$ , respectively.
- <sup>5</sup> Because  $\Lambda$  is of first order in  $\epsilon$ , in order to keep the mass of the Weyl field  $m = \sqrt{\frac{3\Lambda}{2|\omega|}}$  small, the value of  $\omega$  should be chosen great enough for consistency with the weak-field regime.
- <sup>6</sup> The term  $(\varphi'(r))^2$  which appears in the Equation (19), with  $\varphi(r) = \frac{q}{r}e^{-mr}$  is given by  $(\varphi'(r))^2 = \frac{q^2}{r^4}e^{-2mr} + 2m\frac{q^2}{r^3}e^{-2mr} + m^2\frac{q^2}{r^2}e^{-2mr}$ .
- <sup>7</sup> By definition  $m^2 = \frac{3\Lambda}{2|\omega|}$ , and as  $\Lambda$  is of first order in  $\epsilon$ , then we can consider the case in which  $m$  is of order  $\sqrt{\epsilon}$  while  $\omega$  is left arbitrary within a certain range.
- <sup>8</sup> We would like to thank one of the referees for calling our attention to this particular point. His/her pertinent comments have helped us to improve the quality of the paper.

## References

1. Sanomiya, T.A.T.; Lobo, I.P.; Formiga, J.B.; Dahia, F.; Romero, C. Invariant approach to Weyl's unified field theory. *Phys. Rev. D* **2020**, *102*, 124031. <https://doi.org/10.1103/PhysRevD.102.124031>.
2. Romero, C.; Lima, R.G.; Sanomiya, T.A.T. One hundred years of Weyl's (unfinished) unified field theory. *Stud. Hist. Philos. Modern Phys.* **2019**, *66*, 180–185. <https://doi.org/10.1016/j.shpsb.2019.02.005>.
3. Proca, A. Sur la théorie ondulatoire des électrons positifs et négatifs. *J. Phys. Radium* **1936**, *7*, 347–353.
4. Proca, A. Sur la théorie du positon. *C. R. Acad. Sci. Paris* **1936**, *202*, 1366.
5. Poenaru, N.D. Proca equations of a massive vector boson field. *Nucl. Theory* **2006**, *26*, 175–183.
6. Weyl, H. *Gravitation und Elektrizität*; Sitzungsber Deutsch. Akad. Wiss: Berlin, Germany, 1918; p. 465.
7. Adler, R.; Bazin, M.; Schiffer, M.M. *Introduction to General Relativity*; McGraw-Hill: New York, NY, USA, 1965.
8. Landau, L.D.; Lifshitz, E.M. *The Classical Theory of Fields*; Butterworth-Heinemann: Oxford, UK, 1980.
9. Li, G.; Zhang, Y.; Zhang, L.; Feng, Z.; Zu, X. Strong Gravitational Lensing in the Einstein-Proca Theory. *Int. J. Theor. Phys.* **2015**, *54*, 1245–1252. <https://doi.org/10.1007/s10773-014-2321-4>.
10. Romero, C.; Duarte, M.P. The Coming Back of the Proca Field. *Space Time Fundam. Interact.* **2023**, *3–4*, 247–250.
11. Duarte, M.; Dahia, F.; Romero, C. On the Propagation of Gravitational Waves in the Gauge-Invariant Weyl Theory of Gravity. *Universe* **2024**, *10*, 361. <https://doi.org/10.3390/universe10090361>.
12. Duarte, M.; Dahia, F.; Romero, C. The Geometric Proca–Weyl Field as a Candidate for Dark Matter. *Universe* **2025**, *11*, 34. <https://doi.org/10.3390/universe11020034>.
13. Babichev, E.; Charmousis, C.; Hassaine, M. Black holes and solitons in an extended Proca theory. *J. High Energ. Phys.* **2017**, *2017*, 114. [https://doi.org/10.1007/JHEP05\(2017\)114](https://doi.org/10.1007/JHEP05(2017)114).
14. Dereli, T.; Önder, M.; Schray, J.; Tucker, R.W.; Wang, C. Non-Riemannian gravity and the Einstein-Proca system. *Class. Quantum Gravity* **1996**, *13*, L103–L109. <https://doi.org/10.1088/0264-9381/13/8/002>.
15. Tucker, R.; Wang, C. An Einstein-Proca-fluid model for dark matter gravitational interactions. *Nucl. Phys. Proc. Suppl.* **1997**, *57*, 259–262. [https://doi.org/10.1016/S0920-5632\(97\)00399-X](https://doi.org/10.1016/S0920-5632(97)00399-X).
16. Toussaint, M. A Numeric Solution for Einstein's Gravitational Theory with Proca Matter and Metric-Affine Gravity. *Gen. Gravit.* **2000**, *32*, 1689–1709. <https://doi.org/10.1023/a:1001942420350>.

17. Vuille, C.; Ipser, J.; Gallagher, J. Einstein-Proca Model, Micro Black Holes, and Naked Singularities. *Gen. Relativ. Gravit.* **2002**, *34*, 689–696. <https://doi.org/10.1023/A:1015942229041>.
18. Obukhov, Y.N.; Vlachynsky, E.J. Einstein-Proca model: Spherically symmetric solutions. *Ann. Phys.* **1999**, *8*, 497–509. <https://doi.org/10.1002/andp.19995110603>.
19. Shi, C.; Liu, Z. Proca Effect in Reissner–Nordstrom de Sitter Metric. *Int. J. Theor. Phys.* **2005**, *44*, 303–308. <https://doi.org/10.1007/s10773-005-2992-y>.
20. Buchbinder, I.L.; Netto, T.D.; Shapiro, I.L. Massive vector field on curved background: Nonminimal coupling, quantization, and divergences. *Phys. Rev. D* **2017**, *95*, 085009.
21. Gonzales, B.V.; Linares, R.; Maceda, M.; S'anchez-Santos, O. Non-commutative Einstein-Proca Space-time. *arXiv* **2014**, arXiv:1409.3759.

**Disclaimer/Publisher's Note:** The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.