

Article

Second-Post-Newtonian Energy and Angular Momentum Fluxes for Eccentric Inspirals in Effective-One-Body Formalism via Coordinate Transformation

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Abstract

The effective-one-body (EOB) formalism accurately describes the conservative dynamics of general binary orbits, but current implementations of radiation reaction remain largely limited to quasi-circular inspirals. Extending EOB to eccentric orbits currently requires the corresponding post-Newtonian (PN) energy fluxes in EOB coordinates, which are only known to 1PN order. In this paper, we compute the instantaneous gravitational-wave energy flux in EOB coordinates to 2PN accuracy using a systematic coordinate transformation between the Arnowitt–Deser–Misner (ADM) and EOB phase-space variables. We derive the 2PN-accurate transformation laws for the relative coordinates and velocities between the two coordinate systems and re-express the 2PN instantaneous energy flux entirely in EOB variables. Working within the EOB test-particle framework (with finite mass ratio ν) for an equatorial elliptic orbit, we adopt a Keplerian reparameterization in terms of the semilatus rectum p , eccentricity e , and two phase variables (ξ, ϕ) associated with the radial and azimuthal motion, which makes the calculations more transparent and facilitates the subsequent computation of gravitational waveforms. Using the conservative orbital angular frequency, we compute the orbit-averaged energy flux. In addition to the energy flux, we also compute the corresponding 2PN angular momentum flux in EOB coordinates using the same transformation method. Our results, expressed in gauge-invariant variables $x = (M\omega)^{2/3}$ and eccentricity e_t , agree with known PN results and show improved accuracy compared to the 1PN EOB fluxes of Hinderer and Babak.



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1. Introduction

The observation of gravitational waves (GWs) from merging compact binaries by the LIGO–Virgo–KAGRA collaboration has inaugurated the era of gravitational-wave astronomy [1–6]. With the sensitivity of ground-based detectors steadily improving and space-based missions such as LISA, Taiji, and TianQin nearing realization [7–9], the demand for highly accurate waveform models has become increasingly pressing. These future

instruments will probe a broader parameter space, including lower frequencies, higher mass ratios, and, importantly, orbital eccentricities—a feature that is often negligible for binaries formed via isolated stellar evolution but can be significant for dynamical formation channels in dense stellar environments [10–13]. Orbital eccentricity leaves a distinct imprint on the GW signal: it introduces additional harmonic content, modifies the phase evolution, and, if not properly modeled, can bias parameter estimation and tests of general relativity [14–18].

The effective-one-body (EOB) formalism was originally developed in [19–22] to describe the evolution of the binary system. Among the various modeling strategies, EOB formalism stands out as one of the most successful frameworks for describing the conservative and dissipative dynamics of coalescing compact binaries, combining analytical post-Newtonian (PN) results with strong-field information from black hole perturbation theory and numerical relativity [23–27]. While the EOB Hamiltonian accurately describes the conservative dynamics of general orbits (quasi-circular, elliptic, or hyperbolic), current state-of-the-art implementations of the radiation-reaction force and the emitted waveform remain largely limited to quasi-circular inspirals [28–31]. Extending the EOB formalism to accurately describe eccentric binaries is therefore an important and necessary endeavor.

The computation of GW energy fluxes for eccentric binaries has a long history. The GW energy flux or luminosity from a system of two point masses in elliptic motion was first computed by Peters and Mathews at Newtonian order [11,12]. PN corrections at 1PN and 1.5PN were provided in [23–26,32] and used to study the associated evolution of orbital elements using the 1PN “quasi-Keplerian” representation of the binary’s orbit [28]. Gopakumar and Iyer [29,33] further extended these results to 2PN order using the generalized quasi-Keplerian representation developed in [34–36]. The results for the energy flux and waveform presented in [33] were in perfect agreement with those obtained by Will and Wiseman using a different formalism [27]. Damour, Gopakumar and Iyer [37] later discussed an analytic method for constructing high-accuracy templates for GW signals from the inspiral phase of compact binaries on quasi-elliptical orbits, using an improved “method of variation of constants” to combine the three time scales involved (orbital period, periastron precession, and radiation reaction) without treating the radiative time scale as adiabatic. A major advance was achieved by Arun, Blanchet, Iyer, and Qusailah [38–40], who computed the complete 3PN energy flux for compact binaries on quasi-elliptical orbits. This work included both instantaneous contributions and hereditary (tail, tail-of-tail, and tail-squared) terms, and provided the orbital average of the flux using the 3PN quasi-Keplerian parametrization of the binary’s orbital motion [41].

A significant step toward an EOB-based eccentric model was taken by Hinderer and Babak [42], who laid the foundations for describing eccentric inspirals within the EOB framework. Using a Keplerian-type parametrization of the orbit in terms of semilatus rectum p and eccentricity e , they derived the EOB equations of motion for the orbital phase variables and computed the instantaneous energy and angular momentum fluxes to 1PN order in EOB coordinates. Their work also addressed the treatment of hereditary contributions and outlined a strategy for attaching merger-ringdown waveforms. The resulting 1PN-accurate fluxes represented a crucial proof of principle but left room for improvement at higher PN orders.

In this paper, we extend the work of Hinderer and Babak by computing the instantaneous gravitational-wave energy flux in EOB coordinates to 2PN accuracy. Our calculation is based on a systematic coordinate transformation between the Arnowitt–Deser–Misner (ADM) and EOB phase-space variables. The ADM coordinates are widely used in PN calculations of gravitational radiation [38–40], while the EOB coordinates are adapted to the resummed, strong-field dynamics. Although the transformation between ADM and EOB coordinates has been known at 2PN order for the conservative sector [25,39], its appli-

cation to the dissipative sector—specifically to the fluxes—has not been fully worked out at 2PN. We fill this gap by deriving the 2PN-accurate transformation laws for the velocity components (radial and transverse) between the two coordinate systems.

Using these transformations, we re-express the 2PN instantaneous energy flux—originally known in ADM coordinates from the work [38]—entirely in terms of EOB variables. We then specialize to the case of a test particle moving on an equatorial elliptic orbit in a spherically symmetric spacetime. For such orbits, we compute the orbit-averaged energy flux by integrating over the radial period, making use of the orbital angular frequency derived from the conservative EOB dynamics. We verify that our resulting orbit-averaged flux, when expressed in terms of gauge-invariant quantities such as the orbital frequency parameter $x = (M\omega)^{2/3}$ and the eccentricity e_t , agrees with known PN results, thereby providing a stringent consistency check. Finally, we compare our 2PN EOB fluxes with the 1PN results of Hinderer and Babak [42] and demonstrate the quantitative improvement in accuracy.

The remainder of this paper is organized as follows. In Section 2, we derive the 2PN coordinate transformations between ADM and EOB phase-space velocity components. In Section 3, we present the 2PN instantaneous energy flux in EOB coordinates, compute the orbit-averaged flux for equatorial elliptic orbits, validate our results against known PN expressions, and compare with the 1PN EOB fluxes of [42] to highlight the improved accuracy. Finally, Section 4 contains our conclusions and an outlook on remaining tasks for future work.

Throughout this paper we will use geometric units $G = c = 1$, and the units of time and length is the mass of system M , and the unit of linear and angular momentum are μ and μM , respectively, where μ is the reduced mass of the effective body.

2. Coordinate Transformations in Phase Space

In post-Newtonian theory, three coordinate systems are commonly employed: harmonic (h), ADM (a), and EOB (e). Each serves a distinct purpose, and we discuss their transformation laws at 2PN order. In the standard center-of-mass description, the motion of two bodies with masses m_1 and m_2 on general orbits is expressed in terms of their relative separation $\mathbf{x}$ and relative velocity $\mathbf{v}$. By contrast, the effective-one-body (EOB) approach maps this two-body problem into the motion of a single effective particle of mass $\mu = \nu M$ (where $M = m_1 + m_2$ and $\nu = m_1 m_2 / M^2$) moving in an external metric. To work with these formalisms, we introduce scaled position variables $\mathbf{q}_h = \mathbf{x}_h / M$, $\mathbf{q}_a = \mathbf{x}_a / M$, $\mathbf{q}_e = \mathbf{x}_e / M$, and similarly for velocities or momenta per unit reduced mass, which we denote simply by $\mathbf{p}_h, \mathbf{p}_a, \mathbf{p}_e$ (omitting tildes). For harmonic coordinates, the phase-space description is given solely by $(\mathbf{q}_h, \mathbf{v}_h)$ —no ordinary Hamiltonian exists in this gauge. For the ADM and EOB formulations, however, one can work with either coordinate-velocity or coordinate-momentum representations, i.e., $(\mathbf{q}_a, \mathbf{v}_a)$, or $(\mathbf{q}_a, \mathbf{p}_a)$ for ADM, and analogously $(\mathbf{q}_e, \mathbf{v}_e)$, or $(\mathbf{q}_e, \mathbf{p}_e)$ for EOB. Each choice of phase-space variables naturally defines a set of fundamental scalars, for example

$$(\mathbf{q}_h, \mathbf{v}_h) \rightarrow X_1^h = v_h^2, X_2^h = (\mathbf{n}_h \cdot \mathbf{v}_h)^2 = \dot{r}_h^2, X_3^h = \frac{1}{q_h},$$

where $\mathbf{n}_h = \mathbf{q}_h / q_h$, etc. The main transformation laws among phase-space vectors have been derived by Bini and Damour [43]. In the following, we list the specific relations needed to perform the transformation of variables from ADM to EOB coordinates:

$$(1) \quad \mathbf{q}_a \rightarrow \mathbf{q}_e$$

$$\begin{aligned} \mathbf{q}_a &= \mathbf{q}_e + \epsilon^2 \left[\left(\frac{\nu}{2} p_e^2 - \frac{1}{q_e} \left(1 + \frac{\nu}{2} \right) \right) \mathbf{q}_e + \nu (\mathbf{q}_e \cdot \mathbf{p}_e) \mathbf{p}_e \right] \\ &+ \epsilon^4 \left\{ \left[-\frac{\nu}{8} (1 + \nu) p_e^4 + \frac{3}{4} \nu \left(\frac{\nu}{2} - 1 \right) \frac{p_e^2}{q_e} - \nu \left(2 + \frac{5}{8} \nu \right) \frac{(\mathbf{q}_e \cdot \mathbf{p}_e)^2}{q_e^3} + \frac{-\nu^2 + 7\nu - 1}{4q_e^2} \right] \mathbf{q}_e \right. \\ &\left. + (\mathbf{q}_e \cdot \mathbf{p}_e) \left[\frac{\nu(\nu - 1)}{2} p_e^2 + \frac{\nu}{2} \left(-5 + \frac{\nu}{2} \right) \frac{1}{q_e} \right] \mathbf{p}_e \right\}, \end{aligned} \quad (1)$$

$$(2) \quad \mathbf{p}_e \rightarrow \mathbf{v}_e$$

$$\begin{aligned} \mathbf{p}_e &= \mathbf{v}_e + \epsilon^2 \left[2\mathbf{n}_e \frac{(\mathbf{n}_e \cdot \mathbf{v}_e)}{q_e} + \frac{\mathbf{v}_e}{2} \left(v_e^2 (1 + \nu) - 2(\nu - 1) \frac{1}{q_e} \right) \right] \\ &+ \epsilon^4 \left[\mathbf{n}_e \frac{(\mathbf{n}_e \cdot \mathbf{v}_e)}{q_e} \left(-2 \frac{(4\nu - 3)}{q_e} + (1 + \nu) v_e^2 \right) + \frac{\mathbf{v}_e}{8} \left(\frac{-4\nu^2 - 12\nu + 12}{q_e^2} \right. \right. \\ &\left. \left. + \frac{-4\nu^2 - 4\nu + 12}{q_e} v_e^2 + \frac{8(1 + \nu)(\mathbf{n}_e \cdot \mathbf{v}_e)^2}{q_e} + 3v_e^4 (\nu^2 + 3\nu + 1) \right) \right], \end{aligned} \quad (2)$$

$$(3) \quad \mathbf{v}_h \rightarrow \mathbf{p}_e$$

$$\begin{aligned} \mathbf{v}_h &= \mathbf{p}_e + \epsilon^2 \left[-\frac{3\nu + 2}{2q_e} (\mathbf{n}_e \cdot \mathbf{p}_e) \mathbf{n}_e + \left(-\frac{\nu + 4}{2q_e} + \frac{2\nu - 1}{2} p_e^2 \right) \mathbf{p}_e \right] \\ &+ \epsilon^4 \left[\left(\frac{3\nu(5\nu + 1)}{8q_e} (\mathbf{n}_e \cdot \mathbf{p}_e)^2 - \frac{7\nu^2 + 23\nu - 4}{8q_e} p_e^2 - \frac{3\nu^2 - 9\nu - 4}{4q_e^2} \right) (\mathbf{n}_e \cdot \mathbf{p}_e) \mathbf{n}_e \right. \\ &\left. + \left(-\frac{15\nu^2 - 29\nu - 8}{8q_e} (\mathbf{n}_e \cdot \mathbf{p}_e)^2 + \frac{3 - 8\nu}{8} p_e^4 + \frac{7\nu^2 - 41\nu + 8}{8q_e} p_e^2 - \frac{\nu^2 - 15\nu - 1}{2q_e^2} \right) \mathbf{p}_e \right], \end{aligned} \quad (3)$$

$$(4) \quad \mathbf{v}_h \rightarrow \mathbf{v}_a$$

$$\begin{aligned} \mathbf{v}_a &= \mathbf{v}_h - \epsilon^4 \left\{ \left[\frac{3\nu}{8q_h} (\mathbf{n}_h \cdot \mathbf{v}_h)^2 - \frac{19\nu + 2}{4q_h^2} - \frac{7\nu}{8q_h} v_h^2 \right] (\mathbf{n}_h \cdot \mathbf{v}_h) \mathbf{n}_h \right. \\ &\left. + \left[\frac{17\nu}{8} \frac{(\mathbf{n}_h \cdot \mathbf{v}_h)^2}{q_h} + \frac{1 + 21\nu}{4q_h^2} - \frac{13\nu}{8q_h} v_h^2 \right] \mathbf{v}_h \right\}. \end{aligned} \quad (4)$$

Using the transformations in (1) and (2), we obtain the transformation between the EOB (denoted by subscript e) and ADM (subscript a) coordinates truncated at 2PN order:

$$x_a^i = x_e^i + \delta x_e^i, \quad (5a)$$

$$\begin{aligned} \delta x_e^i &= \left(\frac{\nu v_e^2}{2} - \frac{2 + \nu}{2q} - \frac{1 - 7\nu + \nu^2}{4q^2} + \frac{(16 - 5\nu)r_e \dot{r}_e^2 + (2 - 5\nu + 3(1 + \nu)r_e v_e^2)r_e v_e^2}{8q^2} \right) x_e^i \\ &+ \left(\left(1 + \frac{1 + 3\nu}{2} v_e^2 \right) r_e + \left(\frac{3}{2} - \frac{7\nu}{4} \right) M \right) \nu \dot{r}_e v_e^i, \end{aligned} \quad (5b)$$

where $q = r_e/M$, $\dot{r}_e = \mathbf{n}_e \cdot \mathbf{v}_e$. Next, using the transformations in (2), (3), and (4), we obtain the velocity ($\mathbf{v}_e = \dot{\mathbf{q}}_e$) transformation between the EOB and ADM frameworks

$$v_a^i = v_e^i + \delta v_e^i, \quad (6a)$$

$$\begin{aligned} \delta v_e^i &= \left(\frac{2 - 3\nu}{2q} + \frac{\nu(15\nu \dot{r}_e^2 - (2 + 13\nu)v^2)}{8q} + \frac{2 - 22\nu + 3\nu^2}{4q^2} \right) \dot{r}_e \frac{x^i}{r_e} \\ &+ \left(\frac{3\nu}{2} v_e^2 - \frac{2 + 3\nu}{2q} + \frac{1}{8} (7 + 15\nu) v_e^4 + \frac{(52 - 15\nu)\nu \dot{r}_e^2 - (6\nu + 23\nu^2)v_e^2}{8q} - \frac{1 - 9\nu + 2\nu^2}{4q^2} \right) v_e^i. \end{aligned} \quad (6b)$$

Since the fluxes of energy and angular momentum in the ADM gauge are expressed in terms of the three scalars v_a^2 , $\dot{r}_a^2$, and M/r_a , we can derive their EOB-expressed counterparts via the coordinate transformations derived above.

3. Calculation of the Fluxes

The EOB approach provides a framework to combine post-Newtonian results with strong-field knowledge from the test-particle limit and to further incorporate information from numerical relativity simulations. Since many studies of compact binary coalescences are performed in EOB coordinates, it is desirable for future applications to have explicit expressions for the radiation-reaction fluxes directly in EOB variables. In this section, we take the existing 2PN instantaneous fluxes in ADM coordinates and transform them into EOB variables using the coordinate transformations derived in Section 2. In all expressions, we will keep only terms up to 2PN order without indicating the omission of higher-order terms. We then compute the orbital average of the instantaneous fluxes by employing an explicit solution of the equations of motion. Finally, we validate our results by comparing them with the known 1PN EOB fluxes, demonstrating the improvement achieved at 2PN order.

3.1. Instantaneous Fluxes

We start from the PN results for the instantaneous parts of the fluxes in ADM coordinates given by Equations (5.1) and (6.11) of [38] and Equations (3.4a), (3.4b) and (3.5a) of [40]:

$$\begin{aligned}
 \mathcal{F}_{2\text{PN}} = & \frac{32}{5} \frac{\mu^2 M^2}{r^4} \left[v^2 - \frac{11}{12} \dot{r}^2 + \left(\frac{785}{336} - \frac{71}{28} \nu \right) v^4 + \left(-\frac{1487}{168} + \frac{58}{7} \nu \right) \dot{r}^2 v^2 + \left(-\frac{170}{21} + \frac{10}{21} \nu \right) \frac{M}{r} v^2 \right. \\
 & + \left(\frac{687}{112} - \frac{155}{28} \nu \right) \dot{r}^4 + \left(\frac{367}{42} - \frac{5}{14} \nu \right) \frac{M}{r} \dot{r}^2 + \left(\frac{1}{21} - \frac{4}{21} \nu \right) \frac{M^2}{r^2} \\
 & + v^6 \left(\frac{47}{14} - \frac{5497}{504} \nu + \frac{2215}{252} \nu^2 \right) + \dot{r}^2 v^4 \left(-\frac{573}{56} + \frac{1713}{28} \nu - \frac{1573}{42} \nu^2 \right) \\
 & + \frac{M}{r} v^4 \left(-\frac{247}{14} + \frac{5237}{252} \nu - \frac{199}{36} \nu^2 \right) + \dot{r}^4 v^2 \left(\frac{1009}{84} - \frac{5069}{56} \nu + \frac{631}{14} \nu^2 \right) \\
 & + \frac{M}{r} \dot{r}^2 v^2 \left(\frac{4987}{84} - \frac{8513}{84} \nu + \frac{2165}{84} \nu^2 \right) + \frac{M^2}{r^2} v^2 \left(\frac{281473}{9072} + \frac{2273}{252} \nu + \frac{13}{27} \nu^2 \right) \\
 & + \dot{r}^6 \left(-\frac{2501}{504} + \frac{10117}{252} \nu - \frac{2101}{126} \nu^2 \right) + \frac{M}{r} \dot{r}^4 \left(-\frac{5585}{126} + \frac{60971}{756} \nu - \frac{7145}{378} \nu^2 \right) \\
 & + \frac{M^2}{r^2} \dot{r}^2 \left(-\frac{106319}{3024} - \frac{1633}{504} \nu - \frac{16}{9} \nu^2 \right) + \frac{M^3}{r^3} \left(-\frac{253}{378} + \frac{19}{7} \nu - \frac{4}{27} \nu^2 \right) \Big] \\
 & - \frac{\mu^2 M^3}{r^5} \left[\frac{320}{3} \dot{r}^4 \nu - \frac{736}{5} \dot{r}^2 v^2 \nu + \frac{184}{5} v^4 \nu + \frac{M}{r} \dot{r}^2 \left(-\frac{12}{5} - \frac{56\nu}{15} \right) + \frac{M}{r} v^2 \left(\frac{16}{5} + \frac{48\nu}{5} \right) \right], \quad (7)
 \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{2\text{PN}}^z = & \frac{\mu^2 M \dot{\phi}}{r} \left[\frac{16}{5} v^2 - \frac{24}{5} \dot{r}^2 + \frac{16}{5} \frac{M}{r} + \left(\frac{614}{105} - \frac{1096}{105} \nu \right) v^4 + \left(-\frac{296}{35} + \frac{1108}{35} \nu \right) \dot{r}^2 v^2 + \left(-\frac{464}{105} - \frac{152}{21} \nu \right) \frac{M}{r} v^2 \right. \\ & + \left(\frac{38}{7} - \frac{144}{7} \nu \right) \dot{r}^4 + \left(\frac{496}{35} + \frac{788}{105} \nu \right) \frac{M}{r} \dot{r}^2 + \left(-\frac{596}{21} + \frac{8}{105} \nu \right) \frac{M^2}{r^2} \\ & + v^6 \left[\frac{533}{63} - \frac{353}{9} \nu + \frac{614}{15} v^2 \right] + \dot{r}^2 v^4 \left(-\frac{2246}{105} + \frac{12653}{105} \nu - \frac{15637}{105} v^2 \right) \\ & + \frac{M}{r} v^4 \left(\frac{11}{21} - \frac{1333}{63} \nu + \frac{4022}{315} v^2 \right) + \dot{r}^4 v^2 \left(\frac{715}{21} - \frac{3361}{21} \nu + \frac{448}{3} v^2 \right) \\ & + \frac{M}{r} \dot{r}^2 v^2 \left(\frac{21853}{315} + \frac{2942}{105} \nu + \frac{2551}{315} v^2 \right) + \frac{M^2}{r^2} v^2 \left(-\frac{4210}{63} + \frac{2962}{35} \nu - \frac{6856}{315} v^2 \right) \\ & + \dot{r}^6 \left(-\frac{52}{3} + \frac{652}{9} \nu - \frac{388}{9} v^2 \right) + \frac{M}{r} \dot{r}^4 \left(-\frac{22312}{315} + \frac{1999}{45} \nu - \frac{277}{9} v^2 \right) \\ & \left. + \frac{M^2}{r^2} \dot{r}^2 \left(\frac{5666}{105} - \frac{6938}{45} \nu + \frac{3058}{105} v^2 \right) + \frac{M^3}{r^3} \left(\frac{336188}{2835} + \frac{11878}{315} \nu + \frac{44}{45} v^2 \right) \right], \end{aligned} \quad (8)$$

where $\mathcal{F}$ denotes the energy flux, $\mathcal{G}^z$ denotes the z -component of the angular momentum flux, and $\dot{\phi}$ is linked to the orbital velocity by $v^2 = \dot{r}^2 + r^2 \dot{\phi}^2$.

The flux in EOB coordinates is then obtained from the expression in ADM coordinates (7) by the transformation

$$\mathcal{F}_{\text{EOB}} = \mathcal{F}_{\text{ADM}}(r, \dot{r}, v^2) + \delta \mathcal{F}^{\text{a to e}}, \quad (9a)$$

$$\delta \mathcal{F}^{\text{a to e}} = \delta_{1,r} \frac{\partial \mathcal{F}_{\text{ADM}}}{\partial r} + \delta_{1,\dot{r}} \frac{\partial \mathcal{F}_{\text{ADM}}}{\partial \dot{r}} + \delta_{1,v^2} \frac{\partial \mathcal{F}_{\text{ADM}}}{\partial v^2}. \quad (9b)$$

Here, the notation is $r_a = \sqrt{\delta_{ij} x_a^i x_a^j} = r_e + \delta_r$, where δ_r represents the correction at 2PN order, and similarly for $\dot{r}$ and v^2 . The corrections to r , $\dot{r}$, and v^2 are determined from Equation (1) to be

$$\begin{aligned} \delta_r &= \frac{x_e^i}{r_e} \delta x_e^i \\ &= -M + \nu(r\dot{r}^2 + \frac{1}{2}rv^2 - \frac{M}{2}) - \frac{M^2}{4r} + \frac{\nu M}{8} \left(\frac{14-2\nu}{q} + \dot{r}^2(28-19\nu+4(1+3\nu)qv^2) + v^2(2-5\nu+3(1+\nu)qv^2) \right), \end{aligned} \quad (10a)$$

$$\begin{aligned} \delta_{\dot{r}} &= \frac{x_e^i}{r_e} \delta v_e^i + \frac{r_e \dot{x}_e^i - \dot{r} x_e^i}{r_e^2} \delta x_e^i, \\ &= \frac{\dot{r} M^2}{4r^2} + \nu \dot{r} \left[\frac{5}{2} v^2 - \dot{r}^2 - \frac{3M}{r} - \frac{M^2}{8r^2} (26-2\nu-2q\dot{r}^2(20+7\nu-2(1+3\nu)qv^2) - qv^2(4-50\nu+(11+27\nu)qv^2)) \right], \end{aligned} \quad (10b)$$

$$\begin{aligned} \delta_{v^2} &= 2v_e^i (\delta v_e^i), \\ &= \frac{2M}{r} (\dot{r}^2 - v^2) + \nu \left(3v^4 - \frac{3(v^2 + \dot{r}^2)M}{r} \right) + \frac{15\nu^2 \dot{r}^4 M}{4r} \\ &\quad + \frac{M^2}{4r^2} \left[\dot{r}^2 (4-2\nu(22-3\nu-(25-14\nu)qv^2)) - v^2 (2-\nu(18-4\nu-(6+23\nu-(7+15\nu)qv^2)qv^2)) \right], \end{aligned} \quad (10c)$$

where the subscript E has been omitted in the second equalities on the right hand sides. To obtain Equation (10), we have also used the relation

$$r\dot{r} = v_e^i x_e^i. \quad (11)$$

The above equations provide the 2PN generalization of Equation (3.5) of [42]. In what follows, we omit the subscript E on the EOB coordinates for brevity. Unless otherwise indicated, all coordinates and momenta are understood to refer to the EOB gauge.

Using Equation (10) in Equation (9) gives for the additional contribution to the flux in EOB coordinates

$$\begin{aligned} \delta \mathcal{F}^{\text{a to e}} = & \frac{2\nu^2}{105q^7} \left[48(2 - 7\nu - 4\nu^2) + 2q\dot{r}^2(561 - 539\nu + 239\nu^2) \right. \\ & + 4q^2\dot{r}^2(28(1 - \nu) - (128 - 121\nu + 204\nu^2)\dot{r}^2) + 4q^4\dot{r}^2\dot{\phi}^2M^2(84(2 - \nu) - (702 - 7006\nu + 1691\nu^2)) \\ & - q^3(84\nu\dot{r}^4 + 7\nu(7 + 15\nu)\dot{r}^6 + 8(999 - 119\nu - 14\nu^2)\dot{\phi}^2M^2) - 6\nu q^5\dot{r}^2\dot{\phi}^2M^2(266 - (141 - 635\nu)\dot{r}^2) \\ & \left. + 3\nu q^7\dot{\phi}^4M^4(112 - (4123 - 2841\nu)\dot{r}^2) - 4\nu q^6\dot{\phi}^4M^4(1335 - 599\nu) + 52\nu q^9\dot{\phi}^6M^6(62 - 51\nu) \right]. \end{aligned} \quad (12)$$

The angular momentum flux further involves the quantity $r^2\dot{\phi} = \epsilon_{3jk}x^jv^k$. Its transformation is given by

$$\begin{aligned} r_a^2\dot{\phi}_a &= r_e^2\dot{\phi}_e + (v_e^y\delta x_e - v_e^x\delta y_e + x_e\delta v_e^y - y_e\delta v_e^x) \\ &= r^2\dot{\phi} \left[1 + 2\left(\nu v^2 - \frac{(1+\nu)M}{r}\right) - \frac{2 - 16\nu + 3\nu^2 - \nu(5 + 9\nu)q^2v^4 + 2\nu r((5\nu - 17)\dot{r}^2 + (7\nu + 1)v^2)}{4q^2} \right], \end{aligned} \quad (13)$$

then we obtain the transformation connecting two azimuthal angular velocities

$$\begin{aligned} \delta\dot{\phi} &= \dot{\phi}_a - \dot{\phi}_e \\ &= \frac{\nu\dot{\phi}(r^3\dot{\phi}^2 - r\dot{r}^2 - M)}{r} - \frac{4\dot{\phi}}{q^2} \\ &\quad - \frac{\nu\dot{\phi}}{4q^2} \left[22 + 9\nu - 2q\dot{r}^2(21 + 16\nu - (1 + 15\nu)q\dot{r}^2) - q^3\dot{\phi}^2M^2(20 + \nu(7 - 32q\dot{r}^2)) - 2(1 - \nu)q^6\dot{\phi}^4M^4 \right]. \end{aligned} \quad (14)$$

Using these transformations leads to the following contribution to the angular momentum flux

$$\begin{aligned} \delta \mathcal{G}_{\text{a to e}}^z = & \frac{\mu^2\dot{\phi}}{105q^4} \left[-2(5058 + \nu(2245 + 292\nu)) + 2q(336 + (1297 + 6810\nu + 2372\nu^2)\dot{r}^2) \right. \\ & - 2q^2\dot{r}^2(42(2 + 3\nu) - (148 - 4197\nu - 2497\nu^2)\dot{r}^2) - q^4\dot{\phi}^2M^2(168(2 + 7\nu) + (340 - 5468\nu + 9153\nu^2)\dot{r}^2) \\ & - q^3(84\nu\dot{r}^4 - 7\nu(139 + 169\nu)\dot{r}^6 + 2(714 - 1735\nu - 1138\nu^2)\dot{\phi}^2M^2) - 6\nu q^5\dot{r}^2\dot{\phi}^2M^2(238 - (477 - 88\nu)\dot{r}^2) \\ & \left. + 3\nu q^7\dot{\phi}^4M^4(392 + (840 + 2417\nu)\dot{r}^2) - q^6\dot{\phi}^4M^4(1842 + 1087\nu - 3290\nu^2) + \nu q^9\dot{\phi}^6M^6(4621 - 6158\nu) \right]. \end{aligned} \quad (15)$$

3.2. Orbit-Averaged Fluxes

For nonspinning binaries confined to the equatorial plane $\theta = \pi/2$, a Keplerian reparameterization of the constants of motion and the equations of motion is already available. This formulation is given in terms of the semilatus rectum p , the eccentricity e , and two phase variables (ξ, ϕ) that encode the radial and azimuthal geometry of the orbit [42]. These variables are defined by expressing the radial motion as

$$r = \frac{pM}{1 + e \cos \xi} \quad (16)$$

so that turning points of the motion correspond to $\xi = (0, \pi) \bmod(2\pi)$. Differentiating Equation (16) leads to the following evolution equation

$$\dot{\xi} = \frac{(1 + e \cos \xi)^2}{epM \sin \xi} \dot{r} + \frac{\cot \xi}{e} \dot{e} - \frac{1 + e \cos \xi}{ep \sin \xi} \dot{p}, \quad (17)$$

where the conservative part

$$\mathcal{P}(e, p, \xi) = \frac{(1 + e \cos \xi)^2}{epM \sin \xi} \dot{r}. \quad (18)$$

Eccentric planar orbits possess two frequencies characterizing the radial librations between the turning points and the azimuthal rotations. The next step is to substitute for $\dot{r}$ and $\dot{\phi}$ from the EOB dynamics. To compare with existing results we perform a 2PN expansion of the conservative EOB dynamics expressed in terms of (e, p, ξ) in Equation (2.15) of [42], which leads to

$$\mathcal{P} = \frac{(1 + e \cos \xi)^2}{Mp^{3/2}} \left[1 - \frac{6(1 + e \cos \xi) - \nu(1 - e^2)}{2p} + \frac{4(3e^2 \cos^2 \xi + 4e^2 - 2e \cos \xi - 9) + \nu(21 + 26e^2 - 3e^4 + 12e^2 \cos(2\xi) + 4(3e^2 + 11)e \cos \xi) + 3(1 - e^2)^2 \nu^2}{8p^2} \right], \quad (19)$$

$$\dot{\phi} = \frac{(1 + e \cos \xi)^2}{Mp^{3/2}} \left[1 - \frac{4e \cos \xi - \nu(1 - e^2)}{2p} + \frac{16e(e - 2 \cos \xi) - (15 - 2e^2 + 3e^4 + 8(1 - e^2)e \cos \xi)\nu + 3(1 - e^2)^2 \nu^2}{8p^2} \right]. \quad (20)$$

Using these expansions in Equations (7) and (12) and truncating at 2PN order yields the instantaneous energy flux in EOB variables. Since the instantaneous fluxes are gauge-dependent, it is more convenient to compare orbit-averaged fluxes across different approaches.

In the Newtonian limit, the radial and azimuthal frequencies coincide, but relativistic effects break this degeneracy. The radial frequency is defined by the time required for ξ to increase from 0 to 2π , i.e., one full radial period between successive periastron passages. The corresponding radial frequency is given by

$$\omega_r = \frac{2\pi}{\int_0^{2\pi} d\xi / \mathcal{P}}. \quad (21)$$

We associate to this frequency an angle variable ψ_r defined by

$$d\psi_r / dt = \omega_r. \quad (22)$$

Any function of r can thus be decomposed in a Fourier series as

$$f(r) = \sum_{k=-\infty}^{\infty} f_k e^{ik\psi_r}, \quad f_k = \frac{1}{2\pi} \int_0^{2\pi} d\psi_r f e^{-ik\psi_r}. \quad (23)$$

The orbit-average $\langle f \rangle$ is the zero-coefficient computed from

$$\langle f \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\psi_r f = f_0. \quad (24)$$

The relation between the two radial phase variables, ψ_r associated with the frequency and ξ with the orbit's spatial geometry is

$$\frac{d\psi_r}{d\xi} = \frac{\omega_r}{\mathcal{P}}. \quad (25)$$

Thus, integrals over ψ_r can also be computed by converting them to integrals over ξ :

$$\int d\psi_r f = \omega_r \int \frac{d\xi}{\mathcal{P}} f. \quad (26)$$

Now, we calculate the fundamental orbital frequencies in terms of the geometric parameters. Firstly, we express the radial frequency ω_r which reflects the period of the

radial motion from the periastron to apastron and back to periastron again, and the orbital period T_r can be calculated by taking ξ from 0 to 2π , then

$$\omega_r = \frac{2\pi}{\int_0^{2\pi} \frac{1}{\mathcal{P}} d\xi}. \quad (27)$$

In Equation (20), we have already written the variation of ϕ with coordinate time t . Rigidly, the radial motion is the real periodic motion. When the particle passes through the periastron twice, $\Delta\phi$ will be larger than 2π because of the periastron procession caused by the relativistic effect. We can then define the frequency of the azimuthal motion by $\Delta\phi/T_r$. So, we compute the azimuthal frequency from the orbit-average of the ϕ motion as

$$\omega_\phi = \langle \dot{\phi} \rangle = \frac{\int_0^{2\pi} \frac{\dot{\phi}}{\mathcal{P}} d\xi}{\int_0^{2\pi} \frac{1}{\mathcal{P}} d\xi}. \quad (28)$$

Just replacing r in these equations with ξ , the above two integrals only contain argument ξ and can be integrated easily. Here, we do not write down the fully expanded expressions of these two integrals, it is direct and trivial.

$$\begin{aligned} M\omega_r &= \frac{(1-e^2)^{3/2}}{p^{3/2}} \left[1 + \frac{(1-e^2)(-6+\nu)}{2p} + \frac{3(1-e^2)(8-20\sqrt{1-e^2}-24e^2 + (8\sqrt{1-e^2}+5e^2-1)\nu + (1-e^2)\nu^2)}{8p^2} \right], \quad (29) \\ M\omega_\phi &= \frac{(1-e^2)^{3/2}}{p^{3/2}} \left[1 + \frac{\nu+e^2(6-\nu)}{2p} + \frac{3}{4p^2}(19-10\nu) \right. \\ &\quad \left. - \frac{3(1-e^2)(18+20\sqrt{1-e^2}+24e^2 - (5e^2+8\sqrt{1-e^2}+7)\nu - (1-e^2)\nu^2)}{8p^2} \right]. \quad (30) \end{aligned}$$

A check on these results is that the test-particle ($\nu \rightarrow 0$) limit of Equation (29) agrees with Equations (5.1) and (5.2) of [44]. Using Equations (24) and (26), performing the averages of the fluxes, and truncating at 2PN order leads to

$$\begin{aligned} \langle \mathcal{F} \rangle &= \frac{32\mu^2(1-e^2)^{3/2}}{5p^5M^2} \left\{ 1 + \frac{73}{24}e^2 + \frac{37}{96}e^4 + \frac{1}{p} \left[-\frac{1247}{336} - \frac{5\nu}{4} - e^2 \left(\frac{9181}{672} + \frac{325\nu}{24} \right) \right. \right. \\ &\quad \left. \left. + e^4 \left(\frac{809}{128} - \frac{435\nu}{32} \right) + e^6 \left(\frac{8609}{5376} - \frac{185\nu}{192} \right) \right] \right. \\ &\quad \left. + \frac{1}{p^2} \left[\frac{23329}{9072} - \frac{15\sqrt{1-e^2}}{2} + \left(3\sqrt{1-e^2} + \frac{31}{18} \right) \nu + e^8 \left(\frac{108780\nu^2 - 304890\nu + 371627}{64512} \right) \right. \right. \\ &\quad \left. \left. + e^2 \left(-\frac{142187}{2592} - \frac{245\sqrt{1-e^2}}{16} + \left(\frac{49\sqrt{1-e^2}}{8} + \frac{62581}{1008} \right) \nu + \frac{105\nu^2}{8} \right) \right. \right. \\ &\quad \left. \left. + e^4 \left(\frac{5(96390\sqrt{1-e^2} - 690839)}{24192} - \frac{3}{64} (170\sqrt{1-e^2} - 1393) \nu + \frac{1855\nu^2}{32} \right) \right. \right. \\ &\quad \left. \left. + e^6 \left(\frac{7(6660\sqrt{1-e^2} + 14543)}{16128} - 36(518\sqrt{1-e^2} + 17079) \nu + 552720\nu^2 \right) \right] \right\}, \quad (31) \end{aligned}$$

$$\begin{aligned}
\langle \mathcal{G}^z \rangle = & \frac{32(1-e^2)^{3/2}\mu^2}{5Mp^{7/2}} \left\{ 1 + \frac{7}{8}e^2 + \frac{1}{p} \left[-\frac{1247}{336} - \frac{7\nu}{4} - e^2 \left(\frac{425}{336} + \frac{401\nu}{48} \right) + e^4 \left(\frac{10751}{2688} - \frac{205\nu}{96} \right) \right] \right. \\
& + \frac{1}{p^2} \left[\frac{23329}{9072} - \frac{15\sqrt{1-e^2}}{2} + \left(3\sqrt{1-e^2} + \frac{10993}{2016} \right) \nu + \frac{\nu^2}{2} + e^6 \left(\frac{60984\nu^2 - 179115\nu + 209911}{16128} \right) \right. \\
& + e^2 \left(\frac{-662486 + 11340\sqrt{1-e^2} - 9(504\sqrt{1-e^2} - 61277)\nu + 197757\nu^2}{12096} \right) \\
& \left. \left. + e^4 \left(\frac{-1743550 + 317520\sqrt{1-e^2} - 27(4704\sqrt{1-e^2} + 21941)\nu + 1347948\nu^2}{48384} \right) \right] \right\}. \quad (32)
\end{aligned}$$

The quantities (e, p) are gauge dependent as they are defined by the EOB dynamics. We therefore reformulate the expressions in terms of the gauge-invariant variables E , P_ϕ , and ω_ϕ , or equivalently the dimensionless quantities

$$x = (M\omega_\phi)^{2/3}, \quad (33a)$$

$$\epsilon = -\frac{2(E-M)}{\mu}, \quad j = -\frac{2(E-M)\hat{P}_\phi^2}{\mu M^2}. \quad (33b)$$

where E and P_ϕ are expressed in Equation (2.12) of [42]. Existing PN results are usually given in terms of (x, e_t) , where e_t is one of the eccentricities in the QK parameterization. To convert between e_t and the EOB eccentricity e , it is easiest to proceed as follows. First, we compute the quantities from Equation (33) in terms of the EOB parameters. Using Equation (30) in Equation (33a) gives an expression for $x(e, p)$, which can be inverted to obtain

$$x = (1-e^2) \left[\frac{1}{p} + \frac{\nu + e^2(6-\nu)}{3p^2} + \frac{1}{p^3} \left(5 - \frac{3e^2}{2} + 5e^4 - (1-e^2)^{3/2} (5-2\nu) - \frac{39+10e^2+11e^4}{12}\nu + \frac{2(1-e^2)^2}{9}\nu^2 \right) \right]. \quad (34)$$

The PN parameter e_t is given in terms of ϵ and j , e.g., in Equation (7.7e) of [38]. Inserting therein the EOB variables from (2.12a) and (2.12b) of [42] we obtain the following relation between the eccentricity parameters

$$e_t^2 = e^2 \left[1 + \frac{2}{p}(1-e^2)(\nu-3) + \frac{1-e^2}{2p^2} \left(39 - 30\sqrt{1-e^2} - e^2(54 - 26\nu + 6\nu^2) - (29 - 12\sqrt{1-e^2})\nu + 6\nu^2 \right) \right]. \quad (35)$$

Using the relations (34) and (35) to transform the flux results from the PN variables to variables (p, e) , we find agreement: our energy flux matches Equations (8.18)–(8.19c) of [38], and our angular momentum flux matches Equations (4.10)–(4.11c) of [40].

To illustrate the importance of the 2PN corrections, we compare the 1PN and 2PN energy and angular momentum fluxes as functions of the semilatus rectum p for selected values of eccentricity e and mass ratio ν , as shown in Figure 1. The relative differences are defined as

$$\Delta \mathcal{F} = \frac{\langle \mathcal{F}_{1PN} \rangle - \langle \mathcal{F}_{2PN} \rangle}{\langle \mathcal{F}_{2PN} \rangle}, \quad \Delta \mathcal{G}^z = \frac{\langle \mathcal{G}_{1PN}^z \rangle - \langle \mathcal{G}_{2PN}^z \rangle}{\langle \mathcal{G}_{2PN}^z \rangle}. \quad (36)$$

For small p (i.e., orbits closer to the central black hole), the relative differences grow significantly, reaching up to $\sim 50\%$ for high eccentricities near the last stable orbit (LSO). This trend is consistent across different mass ratios, indicating that the 1PN approximation becomes inadequate in the strong-field regime. The results also show that, for a fixed value of p , the deviation increases with eccentricity e , since a more eccentric orbit with the same semilatus rectum lies closer to the LSO, where strong-field effects are more prominent.

We note that the 2PN result is expected to be more accurate than the 1PN one for several reasons. First, the PN expansion for the energy flux of eccentric binaries is known to converge in the weak-field regime, and higher PN orders systematically improve the agreement with exact numerical results in the test-particle limit. Second, the 2PN flux incorporates higher-order nonlinear effects that become increasingly important near the LSO, where the 1PN flux significantly underestimates the energy loss. Finally, the ADM-to-EOB coordinate transformation is derived consistently at 2PN order, ensuring that the improvement is a natural consequence of including the next order in the PN expansion. We therefore consider the 2PN flux to be a more reliable approximation for eccentric EOB inspirals, particularly in the strong-field regime.

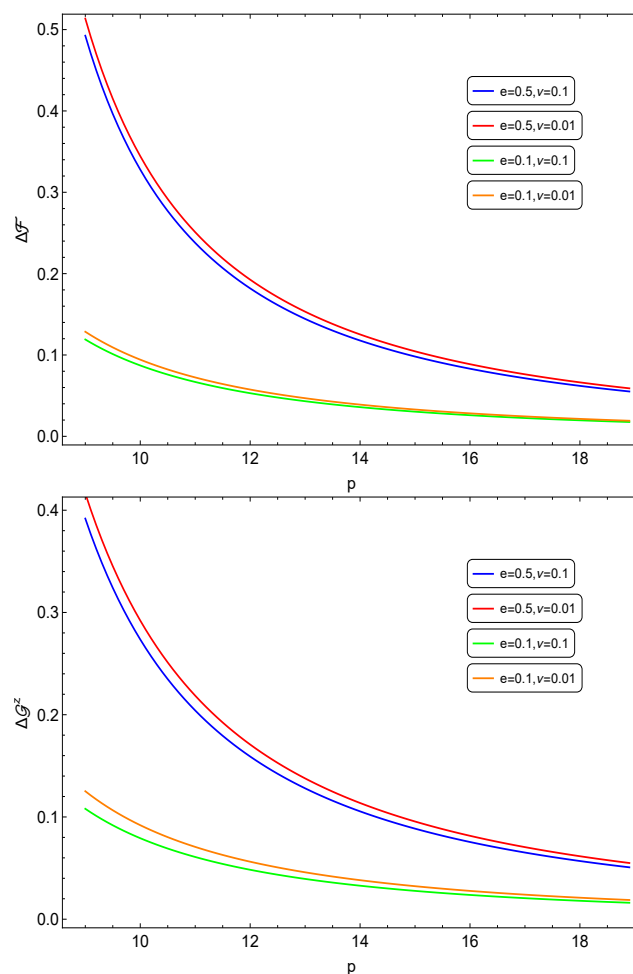


Figure 1. Relative differences between the 1PN and 2PN orbit-averaged energy and angular momentum fluxes, defined as $\Delta\mathcal{F} = (\langle\mathcal{F}_{1\text{PN}}\rangle - \langle\mathcal{F}_{2\text{PN}}\rangle)/\langle\mathcal{F}_{2\text{PN}}\rangle$ and $\Delta\mathcal{G}^z = (\langle\mathcal{G}_{1\text{PN}}^z\rangle - \langle\mathcal{G}_{2\text{PN}}^z\rangle)/\langle\mathcal{G}_{2\text{PN}}^z\rangle$, plotted as functions of the semilatus rectum p for selected values of eccentricity e and mass ratio ν . When p becomes smaller, the deviation could become larger due to the orbit approaching LSO.

4. Conclusions and Outlook

In this paper, we have extended the effective-one-body description of eccentric inspirals by computing the instantaneous gravitational-wave energy flux in EOB coordinates to 2PN accuracy. Our calculation is based on a systematic coordinate transformation between the ADM and EOB phase-space variables. We derived the 2PN-accurate transformation laws for the relative coordinates and velocities between the two coordinate systems, and re-expressed the 2PN instantaneous energy flux entirely in EOB variables. Working within the EOB test-particle framework for an equatorial elliptic orbit, we adopted a Keplerian reparameterization in terms of the semilatus rectum p , eccentricity e , and two phase vari-

ables (ξ, ϕ) associated with the radial and azimuthal motion. Using the conservative orbital angular frequency, we computed the orbit-averaged energy flux. Our result, expressed in gauge-invariant variables $x = (M\omega)^{2/3}$ and eccentricity e_t , agrees with known PN results and shows improved accuracy compared to the 1PN EOB fluxes of Hinderer and Babak. A detailed comparison for four representative orbital configurations, summarized in Figure 1, reveals that the semilatus rectum p is the primary factor affecting the discrepancy between the 1PN and 2PN fluxes. For orbits close to the last stable orbit (LSO), the 1PN flux deviates from the 2PN result by approximately 50%, indicating that the 1PN approximation becomes inadequate in the strong-field regime. In addition to the energy flux, we have also computed the 2PN angular momentum flux in EOB coordinates using the same transformation method, providing a complete set of dissipative fluxes for eccentric inspirals at 2PN order.

The Keplerian reparameterization in terms of semilatus rectum p , eccentricity e , and the phase variables (ξ, ϕ) offers several advantages. It makes the calculations more transparent by cleanly separating the radial and azimuthal degrees of freedom. It also provides a natural framework for orbit averaging and for the Fourier decomposition of the gravitational waveform, as the phase variables evolve linearly with the fundamental frequencies of the motion. This structural clarity considerably facilitates the subsequent computation of gravitational waveforms. The present work thus provides a key ingredient for building high-precision eccentric EOB waveform templates. In future work, we plan to compute more accurate fluxes by incorporating higher-order PN corrections and extending our approach to include the spins of the compact objects and orbital inclination effects, thereby enabling a full three-dimensional description of eccentric EOB inspirals.

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Abbreviations

The following abbreviations are used in this manuscript:

EOB	effective-one-body
PN	post-Newtonian
ADM	Arnowitt–Deser–Misner
GW	gravitational wave
LSO	last stable orbit

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