

Article

Wormhole Solutions and Modified Tolman–Oppenheimer–Volkoff Equation

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Abstract

This article investigates wormhole geometry in modified $f(R, \phi, X)$ gravity, where R , ϕ , and X denote the Ricci scalar, scalar potential, and kinetic term, respectively. We also introduce charge to examine its influence on the wormhole geometry. In the present scenario, the Karmarkar condition is employed to determine the metric potentials. A key feature of this work is the derivation of the modified Tolman–Oppenheimer–Volkoff equation, which is used to analyze the stability of the wormhole geometry. We have analyzed the behavior of the matter-plus-charge energy conditions and found that the null energy condition is satisfied for the considered model, while the strong energy condition is violated. In contrast to general relativity, where the matter NEC must be violated to support a traversable wormhole, the present modified-gravity model fulfils the flare-out requirement through the effective energy–momentum tensor generated by the curvature–scalar–kinetic sector. Hence the required exoticity is shifted from ordinary matter to the effective modified-gravity sector. We also discuss the behavior of the equation-of-state parameters and anisotropy. Moreover, we perform an additional stability analysis based on the adiabatic index. Our analysis reveals that the wormhole solutions are stable for the considered model in modified $f(R, \phi, X)$ gravity.

Keywords: modified gravity; modified tolman–oppenheimer–volkoff equation; karmarkar condition



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1. Introduction

The idea of wormholes as bridges was first introduced by Einstein and Rosen in 1935 [1]. The “bridge” essentially connects different paths, thereby reducing travel time and distance. Einstein’s field equations provide solutions for wormholes, and the notion that gravity behaves like “tunnels” is even more thought-provoking. A wormhole is formed when large amounts of matter are placed in two parallel domains; the presence of such matter generates a wormhole. The first mathematical solution for wormholes in the context of black holes was discovered. These detailed solutions lead to two geometric interpretations of black hole geometry. Brane tension creates a force that offsets gravitational attraction in dense objects with negative pressure. Sufficiently strong interactions between branes can

lead to deformation and the formation of a wormhole [2]. Modified gravity theories offer alternative explanations for cosmological phenomena such as dark energy, dark matter, and cosmic expansion. Several theoretical models aim to uncover the secrets underlying the cosmos's accelerating expansion. According to Einstein's theories of relativity, the laws of physics are uniform and unchanging for all observers in inertial motion [3]. The terms "drain hole" (Ellis) and "geon" (Wheeler) describe theoretical constructs related to wormholes and spacetime topology [4,5]. Morris and Thorne were the first to suggest that humans might travel through wormholes, establishing the concept of traversable wormholes and Kar found that non-static Lorentzian wormholes can exist for finite or half-infinite time intervals [6–8]. Researchers have investigated the impact of exotic matter on wormhole formation and evolution. Research by Visser et al. [9] has established that through appropriate wormhole geometry configurations, violations of the null energy condition (NEC) can be spatially localized to arbitrarily small regions. Within general relativity (GR), the existence of traversable wormholes fundamentally requires exotic matter that violates the NEC through its stress–energy properties. In modified gravity, however, this statement must be interpreted for the total effective source rather than only the material source: the matter-plus-charge sector may satisfy the NEC, while the additional curvature, scalar-field, and kinetic contributions in the modified field equations provide the effective support needed by the flare-out condition.

The investigation of wormhole solutions within modified gravity theories has consistently attracted significant research attention. A broader investigation of wormhole solutions within modified gravity theories has revealed that higher-order curvature terms and matter–geometry couplings can significantly relax or even eliminate the need for exotic matter. Morris and Thorne's original traversability conditions were re-examined in $f(R)$ gravity by Lobo and Oliveira [10] who showed that wormhole throats can be supported by ordinary matter if the effective gravitational coupling becomes negative in certain curvature regimes. Moraes and Sahoo [11] constructed simple phantom-free wormhole solutions in $f(R, T)$ gravity and noted that the coupling parameter can mimic the effect of exotic matter. Shamir and Fayyaz [12] derived the field equations for spherically symmetric wormholes in $f(R)$ gravity and developed the shape functions that satisfy the energy conditions at the throat without violating NEC. Sharif and Rani [13] investigated non-commutative wormhole geometry in $f(T)$ theory of gravity. In the context of Gauss-Bonnet gravity, Kanti et al. [14] discussed the properties of Lorentzian wormholes in dilatonic Einstein–Gauss–Bonnet theory in four spacetime dimensions. Shweta et al. [15] discussed the wormhole solutions in $f(R, T)$ modified theory of gravity. Jianbo et al. [16] investigated the behavior of wormholes without exotic matter violation in $f(R, T)$ gravity. Shamir and Zia [17] studied the wormhole solutions in $f(R, G)$ theory of gravity. Sharif and Ikram [18] analyzed wormhole solutions in $f(G, T)$ theory of gravity by using shape functions. Malik and collaborators investigated traversable wormhole solutions in the context of $f(R, \phi)$ modified gravity. Their study demonstrated that the proposed shape function constructs a valid wormhole geometry connecting two asymptotically flat regions of spacetime while satisfying all necessary physical conditions for traversability [19–21]. Malik and collaborators studied stellar structures using the embedding approach within the $f(R, \phi, X)$ modified gravity framework, analyzing spherically symmetric spacetimes with anisotropic fluid matter distributions [22]. Within $f(R)$ modified gravity, Golchin and Mehdizadeh [23] verified energy conditions for three classes of wormhole solutions (spherical, flat, hyperbolic). Harko et al. [24] discussed that energy conditions can be satisfied through ordinary matter. Mustafa et al. [25] investigated the wormhole solutions in modified symmetric teleparallel gravity. Lobo and his collaborators [26] discussed the traversable wormhole solutions through equations of state and specific shape function functions.

The construction of traversable wormhole geometries has traditionally faced a critical obstacle: the unavoidable violation of classical energy conditions due to the presence of exotic matter. In recent years, the Karmarkar embedding class-I condition—originally devised for stellar interiors—has been repurposed as a systematic tool for generating wormhole shape functions, thereby reducing the degree of ad hoc assumptions. Sutar et al. [27] first extended this approach to include a cosmological constant in GR, demonstrating that stable, non-exotic wormhole solutions become possible when the cosmological constant is tuned appropriately. Malik et al. [28] introduced electric charge into the Karmarkar framework and found that the repulsive Coulomb effect can help satisfy the null energy condition near the throat. In modified gravity, Turkoglu et al. [29] derived a generalized shape function in $f(R, T)$ gravity and, for the first time, linked the Karmarkar condition with the Casimir effect, suggesting a possible geometric evolution of the wormhole throat. Mustafa et al. [30] performed a similar analysis in $f(R, G)$ gravity and reported that the Karmarkar-derived shape function significantly reduces the required amount of exotic matter for realistic model parameters. Paramanik et al. [31] went a step further by incorporating dark matter halo profiles (URC, NFW, and SFDM) into $f(T, T)$ gravity under the Karmarkar condition, obtaining wormhole solutions that remain traversable without any exotic matter. Sarkar et al. [32] proposed a new Karmarkar-based shape function in $f(R)$ gravity that not only avoids exotic matter but also allows calculation of the weak gravitational lensing deflection angle. Chanda and Sharma [33] showed that simultaneously imposing the Karmarkar condition and conformal flatness leads to overly restrictive constraints, potentially limiting the method's applicability unless time-dependent factors are considered. Kiroriwal et al. [34] discussed the wormhole solutions under tidal force constraints in $f(Q)$ gravity. Sanjay et al. [35] examined the non-commutative finserian wormhole solution admitting conformal motion. Parsaei and Rastgoo [36] examined exact wormhole solutions within the framework of Rastall theory. Gul et al. [37] explored viable traversable wormhole geometries in non-metric gravity theory. Kiroriwal et al. [38] investigated traversable wormhole configurations sourced by dark matter halos modeled through the Burkert density profile. They also [39] investigated the impact of the tangential velocity and Karmarkar condition in constructing wormhole solutions within Rastall–Rainbow gravity. Ditta et al. [40] discussed the embedded wormhole solutions and thermodynamics in extended teleparallel gravity. Sarkar et al. [32] discussed the traversable wormholes using Karmarkar condition in $f(R)$ modified theory of gravity.

Modified gravity theories offer a natural framework to circumvent the exotic matter problem that plagues traversable wormholes in GR. Among these, $f(R, \phi, X)$ gravity stands out as a particularly powerful and unified approach. This theory generalizes both $f(R)$ gravity and scalar-tensor models by allowing the Lagrangian to depend arbitrarily on the Ricci scalar R , scalar potential ϕ , and its kinetic term $X = -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$. The inclusion of the kinetic term introduces additional degrees of freedom that can mimic the effective stress–energy required to keep the wormhole throat open, potentially reducing or even eliminating the need for exotic matter. Moreover, the non-minimal coupling between curvature and the scalar sector gives rise to a rich phenomenology: the effective gravitational coupling becomes scale-dependent, and the null energy condition can be violated by purely geometric contributions even when ordinary matter satisfies all energy conditions. Previous studies have shown that $f(R)$ and $f(R, T)$ gravity can sustain wormholes with less exotic matter, but the additional flexibility of $f(R, \phi, X)$ gravity allows for more precise control of the energy condition violation and the stability profile. Consequently, $f(R, \phi, X)$ gravity provides a compelling arena to construct traversable wormhole solutions that are both physically plausible and stable, while also connecting to broader topics such as inflation, dark energy, and the early universe dynamics. The present work leverages this flexibility

to derive, for the first time, the complete set of wormhole properties including equation of state parameters, anisotropy, and adiabatic stability analysis within this theory. Recently, Li et al. [41] investigated the charged wormhole solutions in $f(R, \phi, X)$ modified and we have extended the analysis in our current work by taking different model. The paper is structured as: Section 2 gives the basic formalism of $f(R, \phi, X)$ gravity. Karmarkar condition is discussed in Section 3. Section 4 presents the detailed physical analysis of wormhole solutions. Section 5 concludes the final remarks.

2. Basic Formalism of $f(R, \phi, X)$ Modified Gravity

The gravitational action for $f(R, \phi, X)$ gravity is defined as

$$S = \int \sqrt{-g} \left(\frac{1}{16\pi G} f(R, \phi, X) + L_m + L_e \right) d^4x, \quad (1)$$

where L_m denotes the matter Lagrangian, g represents the determinant of the metric tensor and L_e is the electromagnetic Lagrangian. The field equations are given as follows:

$$f_R G_{\mu\nu} - \frac{1}{2} (f - R f_R) g_{\mu\nu} - \nabla_\mu \nabla_\nu f_R + g_{\mu\nu} \nabla_\alpha \nabla^\alpha f_R - \frac{\epsilon}{2} f_X (\nabla_\mu \phi) (\nabla_\nu \phi) = k^2 T_{\mu\nu}, \quad (2)$$

where $f_R = \frac{\partial f}{\partial R}$, $f_X = \frac{\partial f}{\partial X}$, and ∇_μ denotes the covariant derivative. Moreover, the energy–momentum tensor $T_{\mu\nu}$ is given as

$$T_{\mu\nu} = T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(n)}. \quad (3)$$

The stress–energy tensors for ordinary matter $T_{\mu\nu}^{(n)}$ and the electromagnetic field $T_{\mu\nu}^{(m)}$ combine to form the total energy–momentum tensor. Now, $T_{\mu\nu}^{(n)}$ is written as

$$T_{\mu\nu}^{(n)} = (\rho + p_t) u_\mu u_\nu - (p_r - p_t) v_\mu v_\nu, \quad (4)$$

and $T_{\mu\nu}^{(m)}$ is written as

$$T_{\mu\nu}^{(m)} = \frac{-1}{4\pi} \left(F^{\mu\lambda} F_{\nu\lambda} - \frac{1}{4} \delta_\nu^\mu F^{\gamma\chi} F_{\gamma\chi} \right). \quad (5)$$

The electromagnetic field tensor $F_{\mu\nu}$ can be expressed in terms of the four-potential φ_μ as

$$F_{\mu\nu} = \nabla_\mu \varphi_\nu - \nabla_\nu \varphi_\mu, \quad (6)$$

where ∇_μ is the covariant derivative operator. When expressed in a local coordinate basis, this relation simplifies to

$$F_{\mu\nu} = \frac{\partial \varphi_\nu}{\partial x^\mu} - \frac{\partial \varphi_\mu}{\partial x^\nu}. \quad (7)$$

The static, spherically symmetric electromagnetic field is completely characterized by the four-potential

$$\varphi_\mu = (\phi(r), 0, 0, 0), \quad (8)$$

where $\phi(r)$ denotes the electric potential dependent solely on the radial coordinate r . In this configuration, the electromagnetic field tensor $F_{\mu\nu}$ possesses exactly two independent non-vanishing components,

$$F_{01} = -F_{10} = -\phi'(r), \quad (9)$$

with $\phi'(r) \equiv d\phi/dr$. The antisymmetric property of the field tensor ($F_{\mu\nu} = -F_{\nu\mu}$) is manifest in these non-zero components, while the conditions of time-independence and rotational invariance enforce the vanishing of all remaining components. This configu-

ration describes a purely radial electric field, the field tensor components being entirely determined by the radial derivative of the electric potential. Now the well-known Maxwell equations are

$$\frac{\partial}{\partial x^\nu}(\sqrt{-g}F^{\chi\lambda}) = -4\pi\sqrt{-g}j^\chi, \quad F_{\chi\gamma,\omega} + F_{\gamma\omega,\chi} + F_{\omega\chi,\gamma} = 0. \quad (10)$$

The four-current density j^χ is given by

$$j^\chi = \varrho u^\chi, \quad (11)$$

where ϱ denotes the proper charge density. The electric field intensity E is given as

$$E^2 = -F^{01}F_{10} = \frac{q^2}{r^4}. \quad (12)$$

In the present analysis, we choose $q(r)$ as [42]

$$q(r) = vr^3, \quad (13)$$

where v is a constant. Using the field equations, we obtain

$$k^2\rho - \frac{E^2}{8\pi} = -e^{-b}f_R'' - \frac{1}{2}f - 2e^{-b}\left(\frac{b'}{4} + \frac{1}{r}\right)f_R + \frac{e^{-b}}{4}\left(a'' + a'^2 - a'b' + \frac{4a'}{r}\right)f_R, \quad (14)$$

$$k^2p_r + \frac{E^2}{8\pi} = \frac{1}{2e^b}\left(a' + 2b' + \frac{4a'}{r}\right)f_R' - \frac{1}{2e^b}\left(a'' + \frac{a'}{2} - \frac{a'b'}{2} - \frac{2b'}{2}\right)f_R - \frac{1}{2}f - e^{-b}\frac{\epsilon}{2}f_X(\phi')^2, \quad (15)$$

$$k^2p_t - \frac{E^2}{8\pi} = e^{-b}f_R'' + \frac{1}{2e^b}\left(a' + b' + \frac{2}{r}\right)f_R' - \frac{1}{2re^b}\left(a' - b' - \frac{2e^b}{r} + \frac{2}{r}\right)f_R + \frac{1}{2}f. \quad (16)$$

Here, prime denotes differentiation with respect to the radial coordinate r . For computational convenience, we adopt the units $k^2 = 8\pi G = 1$. This choice allows us to simplify the field equations when analyzing wormhole solutions in extended gravity theories. Notably, by fixing the redshift function, we can significantly reduce the complexity of the system. It should be emphasized that the complete field Equations (14)–(16) remain considerably more complex, making analytical solutions particularly challenging to obtain. In what follows, we explore general classes of modified gravity theories to derive exact wormhole solutions. For our current analysis, we consider $f(R, \phi, X)$ gravity as

$$f(R, \phi, X) = R + \alpha R^2 + \phi(r) + h(X), \quad (17)$$

where

$$h(X) = k_1 + k_2X,$$

and the kinetic term is defined as

$$X = \frac{\eta}{2}e^{-b(r)}(\phi'(r))^2.$$

After substituting the above functional forms into Equations (14)–(16), we obtain the following expressions for the energy density and pressure components:

$$\rho - \frac{E^2}{8\pi} = \frac{1}{4}e^{-b}\left(\frac{4a'}{r} + a'^2 - a'b' + 2a''\right) + \frac{1}{2}\left(-k_2 + \frac{2}{r^2} - \frac{2e^{-b}}{r^2} - \phi(r) - \frac{2e^{-b}a'}{r} - \frac{1}{2}e^{-b}a'^2 + \frac{2e^{-b}b'}{r} + \frac{1}{2}e^{-b}a'b' + \frac{1}{2}e^{-b}k_1\eta(\phi')^2 - e^{-b}a''\right), \quad (18)$$

$$p_r + \frac{E^2}{8\pi} = -\frac{1}{2}e^{-b}\left(\frac{1}{2}a'^2 - b' - \frac{1}{2}a'b' + a''\right) + \frac{1}{2}\left(k_2 + \frac{2}{r^2} - \frac{2e^{-b}}{r^2} - \phi(r) - \frac{2e^{-b}a'}{r} - \frac{1}{2}e^{-b}a'^2 + \frac{2e^{-b}b'}{r} + \frac{1}{2}e^{-b}a'b' + \frac{1}{2}e^{-b}k_1\eta(\phi')^2 - e^{-b}a''\right). \quad (19)$$

$$p_t + \frac{E^2}{8\pi} = -\frac{1}{2}e^{-b}r\left(\frac{2}{r} - \frac{2e^{-b}}{r} + a' - b'\right) + \frac{1}{2}\left(k_2 - \frac{2}{r^2} + \frac{2e^{-b}}{r^2} + \phi(r) + \frac{2e^{-b}a'}{r} + \frac{1}{2}e^{-b}a'^2 - \frac{2e^{-b}b'}{r} - \frac{1}{2}e^{-b}a'b' - \frac{1}{2}e^{-b}k_1\eta(\phi')^2 + e^{-b}a''\right). \quad (20)$$

These field Equations (18)–(20) are essential for discussing the behavior of the wormhole geometry. For clarity, the quantities ρ , p_r , and p_t , together with the charge contribution, are used in the energy-condition plots. The wormhole throat, however, is governed by the effective source appearing in the modified field equations. Therefore, satisfaction of the NEC for the matter-plus-charge sector does not contradict the flare-out condition. In the present solution, the effective geometric contribution generated by the αR^2 , $\phi(r)$, and $h(X)$ terms provides the required support at the throat.

3. Karmarkar Condition

The static, spherically symmetric spacetime is defined as

$$ds^2 = -e^{a(r)}dt^2 + e^{b(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (21)$$

requires specific conditions on its metric functions for physical viability. The redshift function $a(r)$ must remain finite and differentiable everywhere to prevent horizon formation and ensure traversability. The metric function $b(r)$ must satisfy $e^{-b(r_0)} = 0$ at the throat radius r_0 . Non-zero Riemann curvature can be written as

$$\begin{aligned} R_{1313} &= R_{3131}, & R_{1212} &= R_{2121}, & R_{1221} &= R_{2112}, \\ R_{1331} &= R_{3113}, & R_{2332} &= R_{3223}, & R_{2323} &= R_{3232}, \\ R_{1414} &= R_{4141}, & R_{4242} &= R_{2424}, & R_{4114} &= R_{1441}, \\ R_{4224} &= R_{2442}, & R_{4334} &= R_{3443}, & R_{4343} &= R_{3434}. \end{aligned} \quad (22)$$

The well-established Karmarkar relation is defined as

$$R_{1414} = \frac{R_{1212}R_{3434} + R_{1221}R_{4334}}{R_{2323}}. \quad (23)$$

The Karmarkar condition establishes a relation as

$$\frac{1}{2}a''e^a + \frac{1}{4}a'^2e^a - \frac{1}{4}a'b'e^b = \frac{\frac{1}{2}(rb')\left(\frac{1}{2}ra'e^{a-b}\sin^2\theta\right) + \frac{1}{2}(rb')\left(\frac{1}{2}a'e^{a-b}\sin^2\theta\right)}{-e^br^2\sin^2\theta + r^2\sin^2\theta}. \quad (24)$$

After some simplification, we obtain

$$b'a' - (a'')^2 - a'^2 = \frac{2b'a'}{1 - e^b}, \quad (25)$$

under the condition $e^{b(r)} \neq 1$. Consequently, we obtain

$$e^b = Be^a(a')^2 + 1, \quad (26)$$

where B is a constant of integration. The shape function of the wormhole was derived using the Morris–Thorne metric

$$ds^2 = -e^{a(r)} dt^2 + \frac{1}{1 - \frac{\varepsilon(r)}{r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (27)$$

Comparing Equations (21) and (27), we obtain

$$e^b = \frac{1}{1 - \frac{\varepsilon(r)}{r}}. \quad (28)$$

Furthermore, we choose $a(r)$ as

$$a(r) = -\frac{2\eta}{r}, \quad (29)$$

where η acts as a constant parameter. By simplifying Equations (26), (28) and (29), we obtain the wormhole shape function as

$$\varepsilon(r) = r - \frac{r^5}{r^4 + 4B\eta^2 e^{-\frac{2\eta}{r}}}. \quad (30)$$

After applying the throat condition to Equation (30), we obtain a trivial solution. To avoid this issue, we introduce a constant C in (30) as

$$\varepsilon(r) = r - \frac{r^5}{r^4 + 4B\eta^2 e^{-\frac{2\eta}{r}}} + C, \quad (31)$$

Applying the throat condition to Equation (31), we obtain

$$B = \frac{r_0^4(r_0 - C)}{4C\eta^2 e^{-\frac{2\eta}{r_0}}}. \quad (32)$$

Finally, Equation (31) becomes

$$\varepsilon(r) = r - \frac{Cr^5}{Cr^4 + r_0^4(r_0 - C)} + C. \quad (33)$$

This $\varepsilon(r)$ must satisfy the following conditions:

- **Throat condition:** $\varepsilon(r_0) = r_0$ at r_0 .
- **Flaring-out condition:** $[\varepsilon(r) - r\varepsilon'(r)]/\varepsilon^2(r) > 0$ at $r = r_0$.
- **Derivative constraint:** $\varepsilon'(r_0) < 1$ at the throat and $\varepsilon'(r) < 1$ for all $r \geq r_0$.
- **Asymptotic flatness:** $\lim_{r \rightarrow \infty} \varepsilon(r)/r = 0$.

As shown in Figures 1 and 2, $\varepsilon(r)$ satisfies all the necessary conditions.

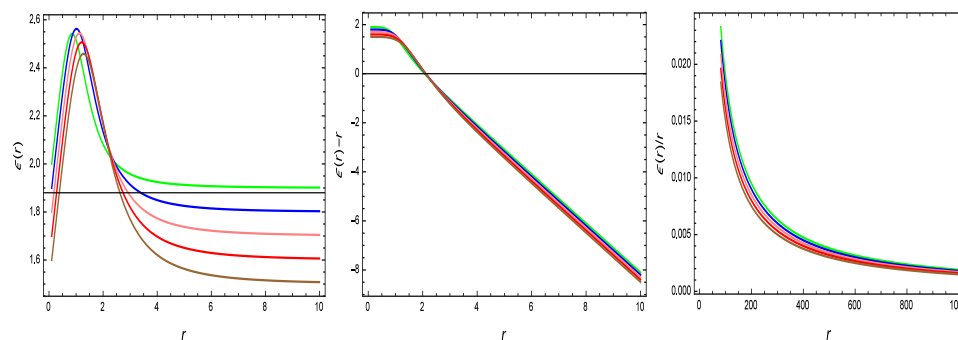


Figure 1. Graphical representation of $\varepsilon(r)$, $\varepsilon(r) - r$ and $\varepsilon(r)/r$.

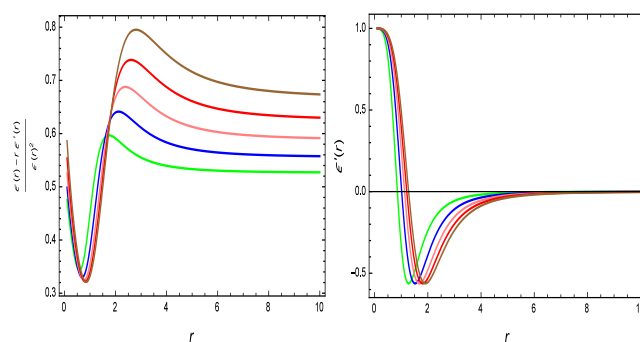


Figure 2. Graphical representation of $\frac{\varepsilon(r) - r\varepsilon'(r)}{\varepsilon^2(r)} > 0$ and $\varepsilon'(r)$ showing $\varepsilon'(r_0) < 1$.

4. Physical Analysis of Wormhole Solutions

In this section, we examine the physical analysis of wormhole solutions in $f(R, \phi, X)$ theory of gravity. We also investigate the equation of state (EoS) parameters and anisotropy. Moreover, we examine the behavior of the energy conditions and identify the relevant violations, which are also a critical part of these solutions. We discuss the stability of wormhole geometry through the modified Tolman–Oppenheimer–Volkoff (TOV) equation and adiabatic index. Detailed discussion of all these aspects can be seen in upcoming subsections. For all numerical analyses and graphical representations throughout this section, we adopt the following parameter values: $\beta = 0.08$, $\alpha = 0.000009$, $\gamma = 0.5$, $k_1 = 1.25$, $\eta = 1$, $k_2 = -0.8$, and $C = 1.5$.

4.1. Energy Density and Pressure Components

After substituting the metric potentials into the field Equations (18)–(20), we obtain the expressions for the energy density and pressure components (see Appendix A (A1)–(A3)), which help us to analyze the behavior of wormhole geometry. The structural properties of the wormhole are determined by the material composition, characterized by the energy density $\rho + \frac{E^2}{8\pi}$, radial pressure $p_r + \frac{E^2}{8\pi}$, and tangential pressure $p_t + \frac{E^2}{8\pi}$, whose radial profiles are depicted in Figure 3. The energy density remains positive throughout the spacetime and decreases gradually as the radial coordinate r increases. The graphical analysis of radial pressure $p_r + \frac{E^2}{8\pi}$ is positive, but the graphical representation of tangential pressure $p_t + \frac{E^2}{8\pi}$ is negative. This persistent negativity is a common feature in wormhole physics, as it is directly linked with exotic matter.

4.2. Energy Conditions

In this subsection, we discuss the energy conditions, which play a fundamental role in determining the physical viability of wormhole geometries. The inequalities displayed below refer to the matter sector together with the charge/electromagnetic contribution. This distinction is important because the Morris–Thorne flare-out condition is controlled

by the effective source in the modified field equations. Consequently, satisfaction of the NEC for the matter-plus-charge sector does not contradict the existence of the wormhole; it indicates that the required NEC violation is produced by the effective geometric sector rather than by ordinary matter. These conditions are defined as follows:

$$\text{Null Energy Condition (NEC)} : \rho + p_r + \frac{E^2}{8\pi} \geq 0, \quad \rho + p_t + \frac{E^2}{8\pi} \geq 0.$$

$$\text{Weak Energy Condition (WEC)} : \rho + \frac{E^2}{8\pi} \geq 0, \quad \rho + p_r + \frac{E^2}{8\pi} \geq 0, \quad \rho + p_t + \frac{E^2}{8\pi} \geq 0.$$

$$\text{Strong Energy Condition (SEC)} : \rho + p_r + \frac{E^2}{8\pi} \geq 0, \quad \rho + p_t + \frac{E^2}{8\pi} \geq 0, \quad \rho + p_r + 2p_t + \frac{E^2}{8\pi} \geq 0.$$

$$\text{Dominant Energy Condition (DEC)} : \rho + \frac{E^2}{8\pi} \geq 0, \quad \rho - |p_r| + \frac{E^2}{8\pi} \geq 0, \quad \rho - |p_t| + \frac{E^2}{8\pi} \geq 0.$$

The quantities $\rho + p_r + \frac{E^2}{8\pi}$ and $\rho + p_t + \frac{E^2}{8\pi}$ are positive, as shown in Figure 4, indicating that the NEC is satisfied. The quantity $\rho + p_r + 2p_t + \frac{E^2}{8\pi}$ is violated, whereas the other two SEC components are satisfied; hence, the SEC is violated. Similarly, $\rho - p_r + \frac{E^2}{8\pi}$ (Figure 4) and $\rho - p_t + \frac{E^2}{8\pi}$ (Figure 5) remain positive, indicating that the DEC is satisfied. The violation of these energy conditions, especially SEC, indicates the presence of effective exotic behavior and justifies the existence of wormhole geometry. Based on the above energy conditions, we analyze how the energy components—especially the null and weak energy conditions—contribute to the understanding of wormhole formation. In the present modified-gravity model, the matter-plus-charge NEC remains satisfied; therefore, the flare-out requirement is supplied by the effective geometric sector rather than by direct NEC violation in ordinary matter. The validity and violation of energy conditions in the context of wormhole solutions have been extensively examined in the literature using various frameworks of modified gravity [43–45]. Such effective violations imply the presence of exotic behavior, which is essential for maintaining wormhole geometries.

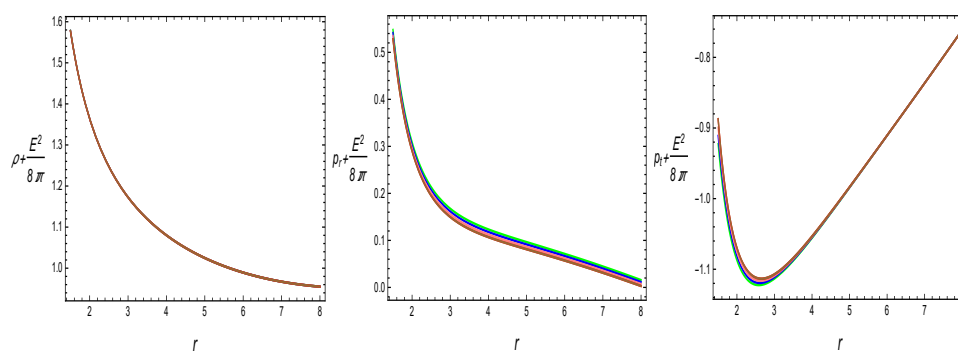


Figure 3. Graphical analysis of energy density, radial pressure, and tangential pressure.

4.3. Equation of State Parameters

The equation of state (EoS) provides a useful description of the matter content supporting the wormhole geometry. In most cases, it is assumed that there is a linear relationship between the radial pressure and the energy density, $p_r = \omega\rho$, where ω is the EoS parameter. For normal matter, ω may be 0 (dust), $1/3$ (radiation), or -1 (vacuum energy). On the other hand, traversable wormholes usually need an effective $\omega < -1$ (or at least negative enough) to break the null energy requirements at the throat. We can tell how “exotic” the fluid is and if the solution is still physically possible by looking at the radial profile $\omega(r)$. In modified gravity theories, the EoS also receives geometric contributions, which may

help satisfy the flare-out requirement without requiring classical exotic matter. Thus, the radial and tangential EoS parameters are defined as

$$w_r = \frac{p_r}{\rho}, \quad w_t = \frac{p_t}{\rho}. \quad (34)$$

After substituting the relevant quantities into Equation (34), we obtain the expressions for w_r and w_t ; see Appendix A (A4) and (A5). Our analysis reveals that the radial EoS parameter is positive throughout the domain and decreases monotonically with increasing radial distance, approaching zero as $r \rightarrow \infty$ (see Figure 6). Since both the energy density ρ and the radial pressure p_r are positive (Figure 3), a positive w_r simply indicates that the fluid behaves in a conventional manner in the radial direction, with pressure and density having the same sign. The gradual decrease in w_r means that the radial pressure becomes less dominant compared to the energy density as we move away from the throat. The asymptotic limit $w_r \rightarrow 0$ corresponds to a pressureless fluid at infinity, which is compatible with asymptotic flatness. This behavior is consistent with the requirement that the exotic matter needed to sustain the throat is predominantly associated with the negative tangential pressure, not with the radial component.

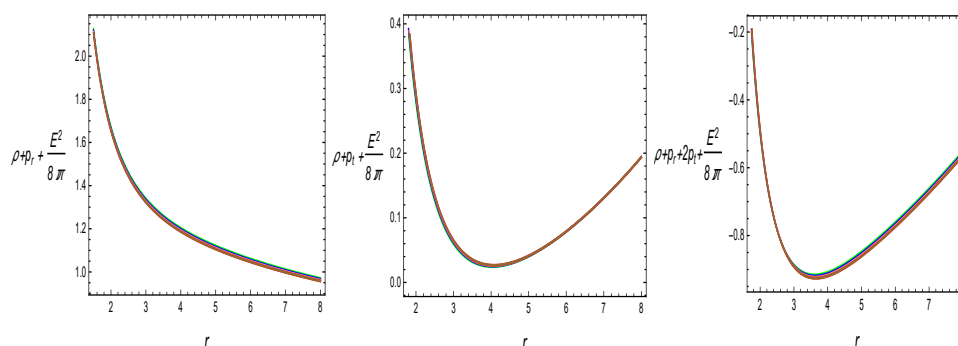


Figure 4. Graphical analysis of NEC and SEC.

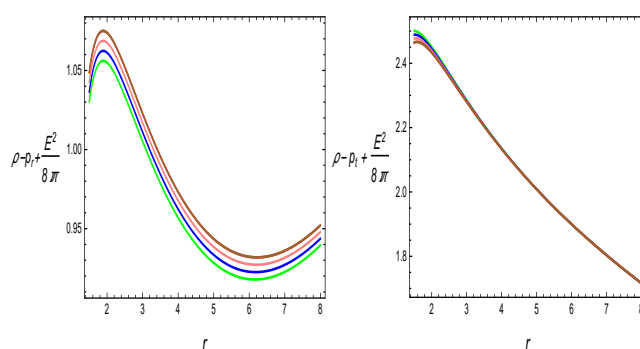


Figure 5. Graphical analysis of DEC.

In contrast, the tangential EoS parameter is negative over the entire range and increases (i.e., becomes less negative) with r , also approaching zero from below (Figure 6). Because $\rho > 0$ and p_t is negative, a negative w_t means that the tangential pressure opposes the energy density. This negative tangential pressure provides the necessary transverse tension to counteract the radial compression, which is precisely the condition required for the flare-out of the wormhole throat. The fact that w_t tends to zero at large radii indicates that the tangential stress vanishes asymptotically, restoring local isotropy.

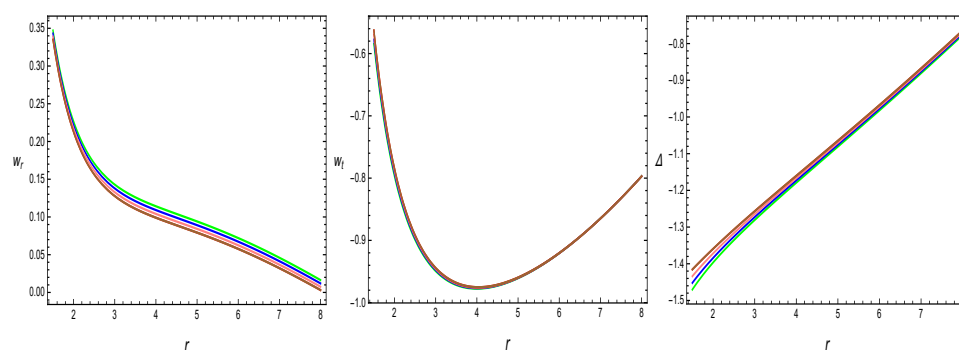


Figure 6. Graphical representation of w_r , w_t and Δ .

4.4. Anisotropy

Anisotropy, defined as the difference between the tangential and radial pressures, plays an important role in wormhole physics. In several traversable wormhole models, the supporting matter is intrinsically anisotropic, exhibiting large differences between p_r and p_t . This anisotropy may facilitate the effective exotic behavior required for the flare-out condition while keeping the energy density positive. The anisotropy factor is a crucial diagnostic; a non-zero Δ often indicates the existence of supplementary forces or geometric factors that stabilize the throat. In modified gravity theories, anisotropy arises inherently from the curvature–matter interactions or the dynamics of the scalar field, enabling more flexibility in the formulation of physically plausible wormhole solutions. Mathematically, it is defined as

$$\Delta = p_t - p_r. \quad (35)$$

The expression for the anisotropy parameter Δ is given in Appendix A (A6), which is found to be negative throughout the domain as shown in Figure 6. This means that the radial pressure dominates over the tangential pressure. In wormhole physics, a negative anisotropy (i.e., $p_t < p_r$) is often associated with the stability of the throat, as the radial tension is balanced by the anisotropy force in the modified TOV equation. The magnitude of Δ decreases with radial distance, indicating that the pressure anisotropy is a near-throat phenomenon and that the fluid becomes nearly isotropic in the asymptotic region.

4.5. Stability Analysis via TOV Equation

This subsection conducts a comprehensive stability analysis through TOV methodology [46–48]. Five main factors work together to create the equilibrium state including gravitational force, hydrostatic force, anisotropic force, electromagnetic force and one extra force due to the modified gravity. The coupling of the $f(R, \phi, X)$ gravity significantly modifies the standard equilibrium conditions, resulting in stability requirements that are fundamentally distinct from those of classical general relativistic wormhole models.

$$f_R G_{\mu\nu} - \frac{1}{2}(f - Rf_R)g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square)f_R - \frac{\epsilon}{2}f_X(\nabla_\mu \phi)(\nabla_\nu \phi) = T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(e)}. \quad (36)$$

Applying the covariant derivative ∇^μ and utilizing the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ yields the generalized conservation equation:

$$\nabla^\mu [T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(e)}] = \nabla^\mu \left[\frac{1}{2}(f - Rf_R)g_{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square)f_R + \frac{\epsilon}{2}f_X(\nabla_\mu \phi)(\nabla_\nu \phi) \right]. \quad (37)$$

After some manipulation, we obtain

$$-\frac{dp_r}{dr} - \frac{a'(r)}{2}[\rho + p_r] + \frac{2}{r}[p_t - p_r] + F_e + F_{\text{mod}} = 0, \quad (38)$$

where

$$F_h = -\frac{dp_r}{dr}, \quad (\text{Hydrostatic Force}) \quad (39)$$

$$F_g = -\frac{a'}{2}(\rho + p_r), \quad (\text{Gravitational Force}) \quad (40)$$

$$F_a = \frac{2}{r}(p_t - p_r), \quad (\text{Anisotropic Force}) \quad (41)$$

$$F_e = \frac{1}{8\pi r^4} \frac{d}{dr}(q^2) = \frac{v^2 r}{2\pi}, \quad (\text{Electromagnetic Force}) \quad (42)$$

$$F_{\text{mod}} = \frac{1}{f_R} \left[(p_r - \frac{\rho}{3})f'_R + \frac{\epsilon}{3}(f_X X)' - \frac{\epsilon}{3}f_X X a' \right], \quad (\text{Modification Force}) \quad (43)$$

As shown in Figure 7, F_h and F_a are positive, whereas F_g is negative. Similarly, Figure 8 shows that F_e is positive and increasing, while the modified force is negative. The modification force F_{mod} originates from the non-minimal coupling between curvature and the scalar field. Near the throat, it is negative, indicating an attractive contribution that helps balance the repulsive hydrostatic and electromagnetic forces. As r increases, F_{mod} decays to zero, restoring the usual GR behavior in the asymptotic region. Thus, the modified TOV equation reveals how $f(R, \phi, X)$ gravity provides an additional stabilizing mechanism for traversable wormholes. It can also be noticed from the last panel of Figure 8 that the sum of all forces approaches zero, which confirms that the equilibrium condition is satisfied and demonstrates the stability of the wormhole geometry.

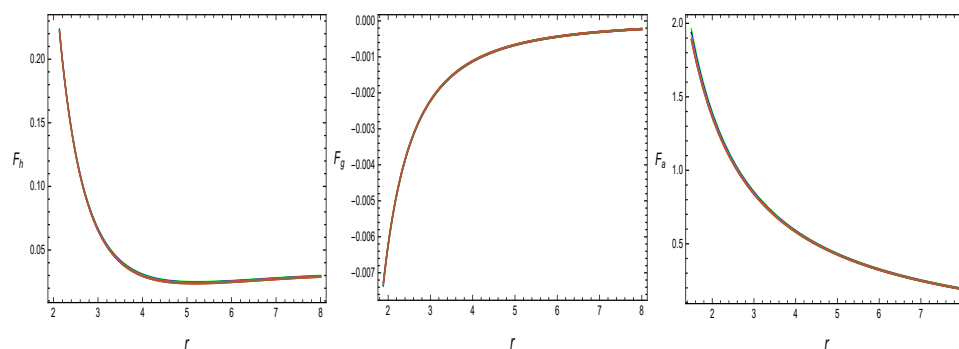


Figure 7. Graphical analysis of F_h , F_g and F_a .

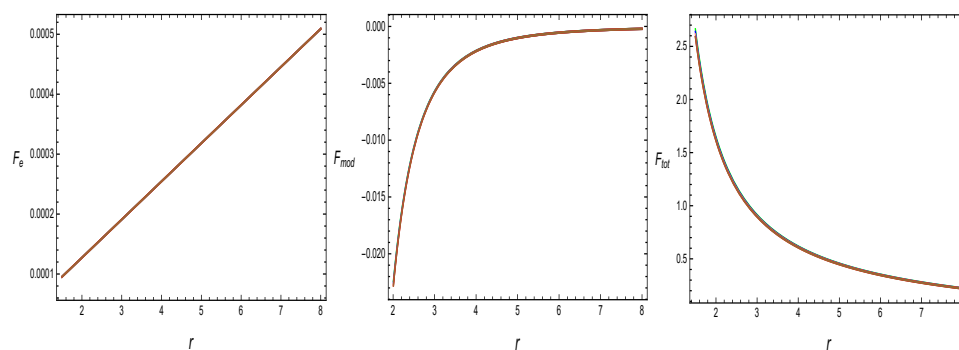


Figure 8. Graphical analysis of F_e , F_{mod} and equilibrium condition.

4.6. Adiabatic Index Stability Analysis

To further assess the dynamical stability of the wormhole configuration, we examine the radial adiabatic index Γ . For an anisotropic fluid, the adiabatic index is defined as [49–51]:

$$\Gamma = \frac{\rho + p_r}{p_r} \frac{dp_r}{d\rho}. \quad (44)$$

According to the well-established stability criterion, a relativistic fluid configuration is considered stable against radial adiabatic perturbations if $\Gamma > 4/3$ throughout the interior [30,39]. Conversely, $\Gamma \leq 4/3$ signals the onset of instability.

Figure 9 presents the radial profile of Γ for our wormhole solution. It is evident that Γ remains consistently greater than the critical threshold $4/3$ across the entire domain. This behavior confirms that the wormhole geometry is stable against radial perturbations, complementing the equilibrium analysis from the modified TOV equation. The satisfaction of the adiabatic index criterion further supports the physical viability of our traversable wormhole model within $f(R, \phi, X)$ gravity.

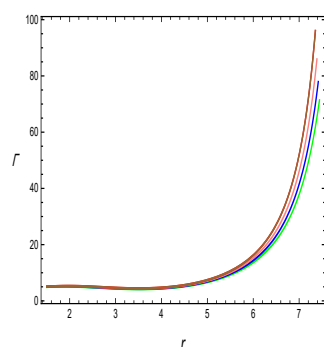


Figure 9. Graphical analysis of adiabatic index.

4.7. Comparative Discussion with Other Modified Gravity Wormhole Solutions

To highlight the advantages of the specific $f(R, \phi, X)$ model, we compare its key features with representative wormhole solutions in GR, $f(R)$ gravity, and $f(R, T)$ gravity. In classical GR traversable wormholes (Morris–Thorne), the energy density is negative near the throat, indicating a large amount of exotic matter [7,8]. In many $f(R)$ wormhole solutions, energy density can be positive only with fine-tuning [10]. In our model, the effective energy density is positive throughout the spacetime (Figure 3), which is a clear improvement.

Regarding energy condition violations, GR wormholes require violation of NEC [9]. In some $f(R, T)$ models, NEC can be avoided, but often at the cost of introducing unusual matter equations of state [11]. In our solution, the NEC for the matter-plus-charge sector is satisfied (Figure 4); only the strong energy condition (SEC) is violated. The flare-out condition is nevertheless fulfilled because the effective stress–energy tensor contains additional curvature–scalar–kinetic terms together with the charge contribution. These terms act as an effective source that supports the throat, while the physical matter-plus-charge sector remains non-exotic with respect to the NEC. This is less restrictive and reduces the required exoticity.

Stability analysis is another area where our work goes beyond typical studies. Most modified gravity wormhole papers only check energy conditions and the flare-out condition; stability is rarely addressed. We have performed two independent stability tests: (i) a modified TOV force balance demonstrating that the sum of all forces tends to zero (Figure 8); and (ii) an adiabatic index analysis confirming $\Gamma > 4/3$ (Figure 9). This dual verification is a novel contribution.

Asymptotic behavior is also well controlled. Our model restores asymptotic flatness and isotropy naturally: all pressures, EoS parameters, anisotropy, and the modification force decay to zero as r increases (extended range up to $r = 8$). This ensures that the wormhole geometry smoothly merges into a flat spacetime, a feature not always verified in other modified gravity works.

Finally, the presence of the kinetic term X and the Ricci-square correction αR^2 in our action provides extra degrees of freedom that allow the modification force F_{mod} to

be tuned. Near the throat, F_{mod} is negative (attractive), helping to balance the repulsive hydrostatic and electromagnetic forces. This flexibility is not available in simpler $f(R)$ or scalar-tensor models. In summary, our $f(R, \phi, X)$ model offers a compelling combination of positive energy density, absence of matter-plus-charge NEC violation, effective geometric support for the flare-out condition, rigorous stability verification, and asymptotic consistency—advantages that are not simultaneously present in many existing modified gravity wormhole solutions.

4.8. Comparison with Previous Work

Very recently, Li et al. [41] studied charged wormhole solutions in $f(R, \phi, X)$ gravity, analyzing energy conditions for three distinct models. However, their work did not examine the equation of state (EoS) parameters, the anisotropy factor, nor the detailed force-balance verification that confirms equilibrium. In contrast, the present manuscript goes beyond that study in several key aspects: (i) we derive and analyse the radial and tangential EoS parameters, as well as the anisotropy, which are essential for characterizing the exotic fluid and the flare-out condition; (ii) we present each force component from the modified TOV equation in separate panels and, crucially, explicitly demonstrate that the sum of all forces approaches zero, providing a direct quantitative proof of hydrostatic equilibrium—a check not performed in Li et al.; (iii) we extend all graphical analyses to larger radial distances (r up to 8) to confirm asymptotic behavior; (iv) we adopt a different $f(R, \phi, X)$ model, namely $R + \alpha R^2 + \phi(r) + h(X)$ with a linear kinetic term, which has not been used for wormholes before; and (v) we include an adiabatic index stability analysis, further supporting the viability of the wormhole configurations. These novel contributions collectively offer a more complete and physically insightful description of wormhole geometries in $f(R, \phi, X)$ gravity.

5. Conclusions

In our work, we have investigated the traversable wormhole solutions in $f(R, \phi, X)$ modified gravity theory. For this work, we have adopted the charge and Karmarkar condition approach for the wormhole geometry's investigation. The detailed outcome of our work is as follows:

- The graphical behavior of the shape function is shown in Figures 1 and 2, where it is well defined and satisfies all the required properties.
- The graphical analysis in Figure 3 shows that the energy density and radial pressure are positive, whereas the tangential pressure is negative.
- The graphical analysis of the NEC, WEC, SEC, and DEC is presented in Figures 4 and 5. From these figures, it can be observed that the NEC for the matter-plus-charge sector is satisfied while SEC is violated. Thus, the existence of the wormhole is not attributed to NEC-violating ordinary matter alone; rather, the required flare-out support comes from the effective geometric contribution of the modified $f(R, \phi, X)$ sector.
- The radial EoS parameter (left panel of Figure 6) is positive. By contrast, the tangential EoS parameter (middle panel) and anisotropy (right panel) are negative, as shown in Figure 6.
- In summary, the combination of a decreasing positive w_r , an increasing negative w_t , and a negative Δ that decays to zero describes an exotic fluid that is strongly anisotropic near the throat, providing the required tangential support and radial tension, while smoothly transitioning to an isotropic, pressureless state at infinity. This behavior is physically reasonable and supports the viability of the traversable wormhole solution within $f(R, \phi, X)$ gravity.

- The graphical behavior of the modified TOV equation, including all force components (F_h , F_g , F_a , F_e and F_{mod}), is shown in Figures 7 and 8. The forces F_h , F_e , and F_a are positive, whereas F_g and F_{mod} are negative. The sum of all forces approaches zero, confirming the equilibrium condition and supporting the stability of the wormhole configuration.
- Finally, the graphical analysis of the adiabatic index Γ is shown in Figure 9. It is evident that Γ remains consistently greater than the critical threshold $4/3$ across the entire domain $r \geq r_0$, which further supports the physical viability of our traversable wormhole model within $f(R, \phi, X)$ gravity.

In summary, the wormhole configuration examined in this study within $f(R, \phi, X)$ gravity satisfies the necessary conditions for a traversable wormhole solution. These results contribute to the ongoing study of exotic spacetime geometries and their relevance in modified gravity frameworks. They also open new directions for investigating the nature of traversable wormholes and their possible astrophysical implications. Future research may explore specific functional forms of $f(R, \phi, X)$ that further enhance wormhole stability and observable signatures.

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Appendix A

In the following expressions, the compact notations f_0 , f_1 , and f_2 denote f_R , df_R/dr , and d^2f_R/dr^2 , respectively. These abbreviations are introduced only to keep the lengthy formulae readable.

$$\rho = \frac{1}{8} \left[\frac{r^2 \gamma^2}{\pi} + 8f_2 \left(-1 + \frac{\epsilon(r)}{r} \right) - \frac{4f_1(r - \epsilon(r))(4 + rb'(r))}{r^2} + \frac{2f_0(r - \epsilon(r))(a'(r)(4 + ra'(r) - rb'(r)) + 2ra''(r))}{r^2} \right. \\ \left. + 2 \left(-2k_2 + \frac{4}{r^2} - 2r^\beta - \frac{4(r - \epsilon(r))}{r^3} + k_1 r^{-3+2\beta} \beta^2 \eta(r - \epsilon(r)) + \frac{4(-r + \epsilon(r))a'(r)}{r^2} \right) \right. \\ \left. + \frac{(-r + \epsilon(r))[a'(r)]^2}{r} + \frac{4(r - \epsilon(r))b'(r)}{r^2} + \frac{(r - \epsilon(r))a'(r)b'(r)}{r} + \frac{2(-r + \epsilon(r))a''(r)}{r} \right. \\ \left. - \frac{\alpha(r^2(-4 + ra'(r))(a'(r) - b'(r)) - 2ra''(r)) + \epsilon(r)(4 + r((4 + ra'(r))(a'(r) - b'(r)) + 2ra''(r)))^2}{2r^6} \right], \quad (\text{A1})$$

$$p_r = \frac{1}{8} \left[-\frac{r^2 \gamma^2}{\pi} + \frac{4f_1(r - \epsilon(r))(4 + ra'(r) + 2rb'(r))}{r^2} - \frac{2f_0(r - \epsilon(r))([a'(r)]^2 - 2b'(r) - a'(r)b'(r) + 2a''(r))}{r} \right. \\ \left. + 2(-2k_2 + \frac{4}{r^2} - 2r^\beta - \frac{4(r - \epsilon(r))}{r^3} + k_1 r^{-3+2\beta} \beta^2 \eta(r - \epsilon(r)) + \frac{4(-r + \epsilon(r))a'(r)}{r^2} \right. \\ \left. + \frac{(-r + \epsilon(r))[a'(r)]^2}{r} + \frac{4(r - \epsilon(r))b'(r)}{r^2} + \frac{(r - \epsilon(r))a'(r)b'(r)}{r} + \frac{2(-r + \epsilon(r))a''(r)}{r} \right. \\ \left. - \frac{\alpha(r^2(-(4 + ra'(r))(a'(r) - b'(r)) - 2ra''(r)) + \epsilon(r)(4 + r((4 + ra'(r))(a'(r) - b'(r)) + 2ra''(r))))^2}{2r^6} \right], \quad (A2)$$

$$p_t = \frac{1}{8} \left[\frac{r^2 \gamma^2}{\pi} + \frac{8f_2(r - \epsilon(r))}{r} - \frac{4f_0(r - \epsilon(r))(2\epsilon(r) + r^2(a'(r) - b'(r)))}{r^2} + \frac{4f_1(2 + r(a'(r) + b'(r)))}{r - \epsilon(r)} \right. \\ \left. + 4(k_2 - \frac{2}{r^2} + r^\beta + \frac{2(r - \epsilon(r))}{r^3} + \frac{1}{2}k_1 r^{-3+2\beta} \beta^2 \eta(-r + \epsilon(r)) + \frac{2(r - \epsilon(r))a'(r)}{r^2} \right. \\ \left. + \frac{(r - \epsilon(r))[a'(r)]^2}{2r} + \frac{2(-r + \epsilon(r))b'(r)}{r^2} + \frac{(-r + \epsilon(r))a'(r)b'(r)}{2r} + \frac{(r - \epsilon(r))a''(r)}{r} \right. \\ \left. + \frac{\alpha(r^2(-(4 + ra'(r))(a'(r) - b'(r)) - 2ra''(r)) + \epsilon(r)(4 + r((4 + ra'(r))(a'(r) - b'(r)) + 2ra''(r))))^2}{4r^6} \right]. \quad (A3)$$

$$w_r = \frac{1}{8\rho} \left(-\frac{r^2 \gamma^2}{\pi} + \frac{4f_1(r - \epsilon[r])(4 + ra'(r) + 2rb'(r))}{r^2} - \frac{2f_0(r - \epsilon[r])([a'(r)]^2 - 2b'(r) - a'(r)b'(r) + 2a''(r))}{r} \right. \\ \left. + 2(-2k_2 + \frac{4}{r^2} - 2r^\beta - \frac{4(r - \epsilon[r])}{r^3} + k_1 r^{-3+2\beta} \beta^2 \eta(r - \epsilon[r]) + \frac{4(-r + \epsilon[r])a'(r)}{r^2} + \frac{(-r + \epsilon[r])[a'(r)]^2}{r} \right. \\ \left. + \frac{4(r - \epsilon[r])b'(r)}{r^2} + \frac{(r - \epsilon[r])a'(r)b'(r)}{r} + \frac{2(-r + \epsilon[r])a''(r)}{r} - \frac{\alpha}{2r^6} (r^2(-(4 + ra'(r))(a'(r) - b'(r)) \right. \\ \left. - 2ra''(r)) + \epsilon[r](4 + r((4 + ra'(r))(a'(r) - b'(r)) + 2ra''(r))))^2 \right) \quad (A4)$$

$$w_t = \frac{1}{8\rho} \left(\frac{r^2 \gamma^2}{\pi} + \frac{8f_2(r - \epsilon[r])}{r} - \frac{4f_0(r - \epsilon[r])(2\epsilon[r] + r^2(a'(r) - b'(r)))}{r^2} + \frac{4f_1(2 + r(a'(r) + b'(r)))}{r - \epsilon[r]} + 4(k_2 \right. \\ \left. - \frac{2}{r^2} + r^\beta + \frac{2(r - \epsilon[r])}{r^3} + \frac{1}{2}k_1 r^{-3+2\beta} \beta^2 \eta(-r + \epsilon[r]) + \frac{2(r - \epsilon[r])a'(r)}{r^2} + \frac{(r - \epsilon[r])[a'(r)]^2}{2r} + \frac{2(-r + \epsilon[r])b'(r)}{r^2} \right. \\ \left. + \frac{(-r + \epsilon[r])a'(r)b'(r)}{2r} + \frac{(r - \epsilon[r])a''(r)}{r} + \frac{\alpha}{4r^6} (r^2(-(4 + ra'(r))(a'(r) - b'(r)) - 2ra''(r)) + \epsilon[r](4 + r((4 \right. \\ \left. + ra'(r))(a'(r) - b'(r)) + 2ra''(r))))^2 \right) \quad (A5)$$

$$\Delta = \frac{1}{4} \left(-4k_2 + \frac{8}{r^2} - 4r^\beta - \frac{r^2 \gamma^2}{\pi} - \frac{8(r - \epsilon[r])}{r^3} + 2k_1 r^{-3+2\beta} \beta^2 \eta(r - \epsilon[r]) + 4f_2(-1 + \frac{\epsilon[r]}{r}) \right. \\ \left. + \frac{8(-r + \epsilon[r])a'(r)}{r^2} + \frac{2(-r + \epsilon[r])[a'(r)]^2}{r} + \frac{2f_0(r - \epsilon[r])(2\epsilon[r] + r^2(a'(r) - b'(r)))}{r^2} + \frac{8(r - \epsilon[r])b'(r)}{r^2} \right. \\ \left. + \frac{2(r - \epsilon[r])a'(r)b'(r)}{r} + \frac{2f_1(r - \epsilon[r])(4 + ra'(r) + 2rb'(r))}{r^2} - \frac{2f_1(2 + r(a'(r) + b'(r)))}{r - \epsilon[r]} \right. \\ \left. + \frac{4(-r + \epsilon[r])a''(r)}{r} - \frac{f_0(r - \epsilon[r])([a'(r)]^2 - 2b'(r) - a'(r)b'(r) + 2a''(r))}{r} - \frac{\alpha}{r^6} (r^2(-(4 + ra'(r))(a'(r) \right. \\ \left. - b'(r)) - 2ra''(r)) + \epsilon[r](4 + r((4 + ra'(r))(a'(r) - b'(r)) + 2ra''(r))))^2 \right) \quad (A6)$$

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