A Novel Semi-Active Control Approach for Flexible Structures: Vibration Control through Boundary Conditioning Using Magnetorheological Elastomers

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Abstract: This research study explores an alternative method of vibration control of flexible beam type structures via boundary conditioning using magnetorheological elastomer at the support location. The Rayleigh–Ritz method has been used to formulate dynamic equations of motions of the beam with MRE support and to extract its natural frequencies and mode shapes. The MRE-based adaptive continuous beam is then converted into an equivalent single-degree-of-freedom system for the purpose of control implementation, assuming that the system’s response is dominated by its fundamental mode. Two different types of control strategies are formulated including proportional–integral–derivative and on–off control. The performance of controllers is evaluated for three different loading conditions including shock, harmonic, and random vibration excitations.

Keywords: magnetorheological elastomers; vibration control; boundary condition; Rayleigh–Ritz

1. Introduction

Magnetorheological (MR) materials are smart functional materials that are capable of changing their mechanical properties in the presence of an applied magnetic field. This phenomenon is called the MR effect which is attributed to the micron-sized ferromagnetic particles dispersed in the carrier substance of MR materials. Magnetorheological elastomers (MREs) contain ferromagnetic particles embedded in a nonmagnetic viscoelastic polymeric material. The use of a viscoelastic material empowers MREs to change both their stiffness and damping simultaneously under the application of an external magnetic field. The viscoelastic properties of MREs change in the presence of a magnetic field due to the alignment of ferromagnetic particles toward the direction of the applied magnetic field [1]. Because of the controllability of the stiffness and damping in MREs, they can effectively be used for the development of MRE-based adaptive vibration absorbers and isolators capable of altering their natural frequency.

MREs have been effectively used in the control of flexible structures such as continuous beams. Recent studies consider implementing MRE into the beam structure called an MRE sandwich beam, where an MRE is placed between two separate aluminum layers. Generally, the material of the outer layer sandwiching the MRE must not interfere with the magnetic field such that the MR effect is not disrupted. Selvaraj and Ramamoorthy [2] studied the effects of using carbon nanotubes (CNT) to reinforce the MRE layer and concluded that the use of CNT increased the storage modulus and loss factor. Ramesh et al. [3] conducted a vibration analysis, using laminated composite beams as the elastic layer. Wei et al. [4] performed a study showing that the MRE sandwich beam is capable of increasing its fundamental frequency and lower its amplitude of vibration in the presence of a homogeneous magnetic field. Furthermore, Hu et al. [5] demonstrated the behavior of the MRE sandwich beam under non-homogeneous magnetic field to activate portions of the MRE layer. The research performed by Hu et al. [5] becomes of interest since
the MRE layer in the sandwich structure can be activated in sections compared with full activation of the entire MRE. Long et al. [6] modelled the transverse motion of the MRE-based beam under non-homogeneous magnetic fields using the fine Mead–Markus model. Poojary [7] addressed the relationship between the change in frequency with a changing non-homogeneous magnetic field and concluded that the natural frequency decreases by increasing magnetic field strength and magnetic field placement towards the free end of a cantilever beam. A multidisciplinary design optimization study was conducted by Khanouki [8] to increase the frequency shift of an MRE-based sandwich beam structure using electromagnets on each end. Different electromagnet geometries were considered in order to maximize the shift in fundamental frequency while minimizing the weight. A U-shaped electromagnet was able to provide a shift of 42%. Szmidt et al. [9] explored the vibration control of a double-beam structure by embedding MRE within the double-beam but at the free end only. Ni, Ying, and Chen [10] extended the research to different composite structures using MRE as the core, such as composite floor and composite wall, to suppress micro-vibrations. Rasooli [11] applied the semi-active tunable vibration absorber to a five-layered beam with alternating layers of MRE and thin elastic plates. U-shaped electromagnets were mounted on the five-layered beam to activate the MRE layers, resulting in a 10% frequency shift. Soleymani [12] studied the aeroelastic stability of a piezo-MRE, which uses piezoelectric face layers, and showed that the use of MRE can help with resolving flutter effects. A study was also conducted by Bornassi [13] in using MRE as the core layer in turbomachine blades in reducing bending and torsional modes of flutter.

Alternative to MR materials, piezoelectric actuators can be used for vibration control. Unlike MR materials, the application of an electric field causes the piezoelectric material to strain. This type of actuation falls under the category of active control. In the reverse, piezoelectric materials can experience changes in the electric field when undergoing deformation. Piezoelectric materials are generally used in feedforward active noise control (ANC) or active vibration control (AVC). Feedforward control requires that the piezoelectric material vibrates out-of-phase with the primary vibration source. Magliacano [14] implemented piezoelectric materials as both a sensor and actuator for feedforward vibration control by using multiple piezoelectric components as error sensors.

Rajamohan et al. [15] performed a vibration control study of a magnetorheological fluid (MRF) sandwich beam using observer-based and flexural mode shape (FMS)-based optimal control. These control laws were tested for fully and partially treated MR beams and resulted in substantial reductions in settling time and beam tip deflection for full-state observer-based LQR control. The conclusion was also made that such systems would require a larger-sized electromagnet due to the size of the overall system. This can be a major drawback to using MR-based sandwich beams since the electromagnet would need to be mounted on the beam in order to activate the MR layer. Mounting an electromagnet to the beam would add additional weight, which will affect the dynamics of the beam.

In this study, a novel approach has been proposed to control the vibration of continuous, flexible beam structures through boundary conditioning using MRE-based systems at support locations. This proof-of-concept will be demonstrated for a cantilevered beam in which MRE is positioned at the cantilever end. The design requires that the MRE be activated and controlled through a grounded electromagnet using different control laws, such as proportion–integral–derivative (PID) and on–off control. The beam is to be analyzed under different loading conditions to validate the capability of MRE to control flexible structures through the boundary condition. Compared with previous vibration control studies of flexible beams using MRE, the novelty of this work is the electromagnet being used to control the rigidity of a boundary condition rather than the electromagnet being mounted on the beam to control the beam’s mechanical properties. Such an application does not put a constraint on the electromagnet size thus allowing for larger shifts on the fundamental frequency.
Section 2 outlines the mathematical modelling of the MRE relating the mechanical properties to the magnetic flux density. The electromagnet is then designed to activate the MRE, which will be used to convert a rigid beam support to one that is flexible. Both PID and on–off controller designs will also be presented. Section 3 contains the results and discussions of applying the different controllers under different loading conditions such as impulse, harmonic, and random loads.

2. Materials and Methods

2.1. MRE Characterization and Modelling

The MRE used in this research study consists of a silicon rubber matrix integrated with 25% volume fraction of carbonyl iron particles. The experimental results describing the variation of storage and loss moduli of the MRE versus excitation frequency and amplitude as well as magnetic flux density have been previously reported by Liu [15]. Experimental results show that the storage and loss moduli experienced negligible changes for frequencies under 10 Hz. Furthermore, the storage and loss moduli decreased substantially as the strain amplitude increased, particularly under high magnetic flux densities. Reduction of storage modulus by increasing strain amplitude is mainly due to the strain softening effect as also observed in filled rubber due to the Payne effect. In this study, it was assumed that MREs operate in linear viscoelastic region for strain amplitudes up to 15%. Thus, in the design of MRE support for the beam, attempts were made to limit strain amplitude in MRE to under 15%.

The storage and loss moduli were further investigated at a frequency of 2 Hz and a shear strain amplitude of 15% in order to obtain their behavior under varying magnetic flux density [15]. The experimental results were curve-fitted with a 3rd-order polynomial for the magnetic flux density ranging from 0 to 1 T to derive explicit relations for the storage and loss moduli with respect to magnetic flux density as [15]:

\[
G' = -234.3 B^3 + 396.7 B^2 + 10.94 B + 63.04 \text{ kPa}
\]

\[
G'' = -103.2 B^3 + 151.1 B^2 + 7.79 B + 13.27 \text{ kPa}
\] (1)

Figure 1 shows the variation in storage and loss moduli with respect to magnetic flux density. Results show that both moduli approach an asymptote at 1 T due to magnetic saturation, where any further increase in magnetic flux density will cause negligible changes in both moduli [15]. Results clearly show that storage modulus increases from nearly 60 kPa to almost 240 kPa (300% increase) by increasing the magnetic flux density from 0 to 1 T.

![Figure 1. Comparison of Experimental Data and Curve Fitting Representation of Storage and Loss Moduli vs. Magnetic Flux Densities at 2 Hz and 15% Shear Strain [16].](image-url)
2.2. Design of Electromagnet to Activate MRE

The proposed electromagnet has a C-shape core made of 1008 steel with a density of 7861 kg/m$^3$. Low-carbon steels, such as 1008 steel, are preferable for the electromagnet core as they can be easily magnetized and demagnetized. The $B$-$H$ curve for the 1008 steel is used to obtain the relation between $H$ and $B$ using a 5th-order polynomial. The polynomial coefficients are identified using the least square minimization technique and are provided in Table 1 [17].

$$H_{steel} = R_{s0}B^5 + R_{s1}B^4 + R_{s2}B^3 + R_{s3}B^2 + R_{s4}B + R_{s5}$$  (2)

Table 1. Curve-Fitting Coefficients for $B$-$H$ Relationship of 1008 Steel [17].

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>$B \leq 1.5$ T</th>
<th>$B &gt; 1.5$ T</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{s0}$</td>
<td>0.00</td>
<td>-1419.52</td>
</tr>
<tr>
<td>$R_{s1}$</td>
<td>1.82</td>
<td>13,551.37</td>
</tr>
<tr>
<td>$R_{s2}$</td>
<td>-3.63</td>
<td>-50744.31</td>
</tr>
<tr>
<td>$R_{s3}$</td>
<td>1.782</td>
<td>93,520.50</td>
</tr>
<tr>
<td>$R_{s4}$</td>
<td>0.387</td>
<td>-5032.46</td>
</tr>
<tr>
<td>$R_{s5}$</td>
<td>0</td>
<td>30,566.42</td>
</tr>
</tbody>
</table>

The electromagnet core is coiled using 17 AWG wire, which has a diameter of $D = 1.15$ mm. To avoid overheating of the wires, the ampacity of the 17 AWG wire is about 2.9 A to operate within safe limits [18]. For a density of 8886 kg/m$^3$, the 17 AWG copper wire with diameter of 1.15 mm will have a mass of nearly 9.23 kg for the length of 1 km [19].

Two pads of MRE, having a density of 3500 kg/m$^3$ [8], are contained between the two permanent magnets. The MRE pads sandwich a layer of 1008 steel, which will act as the support of the beam. The magnetic properties of MRF are used since the magnetic properties of MRE are not readily available [8]. The MRE considered has a density of approximately 3500 kg/m$^3$. The $B$-$H$ relationship for the MRF follows a second-order polynomial of the form [8,20]:

$$H_{MRE} = 289B^2 + 34B$$  (3)

Permanent magnets are also included into the electromagnet design to be able to both increase and decrease the stiffness of the MRE with respect to the initial configuration with an input current of 0 A. N52 neodymium permanent magnets, that have a density of 7500 kg/m$^3$ [21], are included in the electromagnet design to help increase the magnetic flux density when the current direction to the electromagnet is positive and decrease when the current is negative. The N52 permanent magnet has a magnetic remanence of 1.464 T and a coercivity of 956 kA/m [22]. The initial magnetization $B$-$H$ curve for the N52 magnet is provided by Finite Element Magnetic Methods (FEMM) solver application; however, the provided curve must be offset by the remanence to obtain the demagnetization curve, which is shown in Figure 2.

Figure 3 shows the layout of the proposed electromagnet design including the relevant materials and geometrical dimensions along with a section view of the solenoid. It is noted that extending beyond the permanent magnet will result in the copper windings interfering with the MRE’s deflection. The number of windings is approximately determined by dividing the area covered by the wires by the cross-sectional area of the wire and can be obtained as follows:

$$N_w = \frac{b(c + k)}{D^2}$$  (4)
The total length of the wire is approximated by measuring the perimeter formed by a line centered in the coiled region, which is represented by the red dashed line in section A-A of Figure 3. Therefore, the length can be approximated by the following:

\[
L_w = N_w [2w + 2t + 2(c + k)]
\]  

\[ (5) \]

The proposed electromagnet is developed in FEMM, which is an open-source finite element solver, to conduct magnetostatic analysis numerically. FEMM is a two-dimensional solver, which assumes that no variations in the magnetic field exist across the depth. The geometrical parameters of the electromagnet shown in Figure 3 have been found through trial and error to generate a maximum flux density of nearly 1 T in the MRE active regions at an applied current of 3 A. The electromagnet dimensions are summarized in Table 2. For the identified dimensions, the weight of the electromagnet is found to be approximately 5.5 kg, where the bulk of the weight is from the windings. A summary of the component weights is found in Table 3.
Table 2. Electromagnet Dimensions.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thickness, ( t )</td>
<td>10 mm</td>
</tr>
<tr>
<td>1008 Steel Uncoiled Length, ( c )</td>
<td>20 mm</td>
</tr>
<tr>
<td>1008 Steel Coiled Length, ( b )</td>
<td>50 mm</td>
</tr>
<tr>
<td>Electromagnet Width, ( w )</td>
<td>50 mm</td>
</tr>
<tr>
<td>N52 Magnet Thickness, ( k )</td>
<td>3 mm</td>
</tr>
<tr>
<td>MRE Thickness, ( r )</td>
<td>8 mm</td>
</tr>
<tr>
<td>1008 Steel between MRE, ( g )</td>
<td>5 mm</td>
</tr>
<tr>
<td>Wire Diameter, ( D )</td>
<td>1.15 mm</td>
</tr>
<tr>
<td>Number of Turns, ( N_N )</td>
<td>870 turns</td>
</tr>
<tr>
<td>Wire Length, ( L_{aw} )</td>
<td>144.4 m</td>
</tr>
</tbody>
</table>

Table 3. Summary of Mass of the Electromagnet Components.

<table>
<thead>
<tr>
<th>Component</th>
<th>Density (kg/m(^3))</th>
<th>Volume (m(^3))</th>
<th>Mass (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1008 Steel</td>
<td>7861</td>
<td>1.26 \times 10^{-4}</td>
<td>0.990</td>
</tr>
<tr>
<td>N52</td>
<td>7500</td>
<td>3.00 \times 10^{-6}</td>
<td>0.022</td>
</tr>
<tr>
<td>MRE</td>
<td>3500</td>
<td>8.00 \times 10^{-6}</td>
<td>0.028</td>
</tr>
<tr>
<td>17 AWG Wire</td>
<td>8886</td>
<td>5.83 \times 10^{-4}</td>
<td>5.180</td>
</tr>
</tbody>
</table>

Figure 4 shows the distribution of magnetic flux density dependent on the input current of \(-3\) and \(3\) A. Results show that the designed electromagnet is capable of generating magnetic flux density of around 1 T in MRE active region at a maximum current of \(3\) A and 0 T at a current of \(-3\) A. Table 4 shows the results obtained using the 2-D magnetostatic FE model at different input currents.

A functional relationship between the input current and the magnetic flux density is required for controlling the mechanical properties of the MRE. The relationship is to be obtained using the results from FEMM due to its numerical accuracy. The functional relationship will be determined through least-squares optimization where the curve fitting function takes on the form of a polynomial. The resulting 4th-order curve-fitting polynomial takes the following form:

\[
B(I) = -0.00079 I^4 + 0.0015 I^3 - 0.012 I^2 + 0.15 I + 0.67
\] (6)
Table 4. Summary of Magnetic Flux Density using FEMM and Analytical Magnetic Circuit Model.

<table>
<thead>
<tr>
<th>Input Current (A)</th>
<th>Magnetic Flux Density (T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>−3</td>
<td>0.012</td>
</tr>
<tr>
<td>−2</td>
<td>0.300</td>
</tr>
<tr>
<td>−1</td>
<td>0.501</td>
</tr>
<tr>
<td>0</td>
<td>0.665</td>
</tr>
<tr>
<td>1</td>
<td>0.806</td>
</tr>
<tr>
<td>2</td>
<td>0.913</td>
</tr>
<tr>
<td>3</td>
<td>0.983</td>
</tr>
</tbody>
</table>

Figure 5 shows the curve-fitting results with the data obtained from FEMM.

**2.3. Mathematical Modelling of the Beam with MRE Support**

Control at the boundary condition is only possible if the rigidity of the support is variable, which is possible by implementation of an MRE-based system. Beams can be considered statically similar if they have the same geometric and natural boundary conditions. The geometric boundary conditions of a cantilever beam must have zero deflection and zero slope at the supported end. The natural boundary conditions are such that the shear and bending moments are non-zero at the fixed end. Dynamic similarity is verified by comparing the natural frequency of the alternative beam model to that of the cantilever beam. The comparison is conducted by considering the ratio of fundamental frequencies between the alternative model, \( \omega_{\text{equiv}} \), and the cantilever model, \( \omega_{\text{rigid}} \) as follows:

\[
r = \frac{\omega_{\text{equiv}}}{\omega_{\text{rigid}}} \tag{7}
\]

Equation (7) explains how far the dynamics of the alternative model deviates from that of the cantilever model. In order for the dynamics to be as similar as possible, then the ratio of fundamental frequencies should approach unity.

A beam with an intermediate linear spring support at \( x = x_s \) is used as the alternative beam model. This model is otherwise known as an overhanging beam, as shown in Figure 6a, and may represent the cantilever beam if the linear spring is infinitely rigid. The model having the intermediate spring is selected since the MRE can be placed in such a
way that it is in direct shear as shown in Figure 6b. Two MRE pads placed in a parallel configuration are used as per the electromagnet design and to increase the stiffness at the support. The equivalent total complex stiffness due to the MRE in shear can be defined as follows:

\[ k_s = \frac{2G^*A_{MRE}}{t_{MRE}} \]  

(8)

where \( t_{MRE} \) is the thickness of each MRE pad and \( G^* = G' + jG'' \) is the total complex shear modulus having both storage \((G')\) and loss \((G'')\) moduli, and \( A_{MRE} \) is the shear area on the MRE.

From Figure 7, it can be realized that the shear and bending moment diagrams of the overhanging beam past the intermediate support are similar to that of a cantilever beam. Therefore, the overhang section of the beam can be considered cantilevered. This approach satisfies the natural boundary conditions starting from the overhang section; however, it does not fully satisfy the geometric boundary conditions. If the intermediate support has infinite stiffness, then there would be zero deflection at the overhang support, but the slope would be non-zero at the same location. Although this is true, similitude in boundary conditions is not totally enforced since the boundary itself is now subject to control.

\[ X(x) = \sum_{i=1}^{n} c_i \phi_i(x) \]  

(9)

**2.4. Modelling the Continuous Beam Using Rayleigh–Ritz**

The Rayleigh–Ritz method, which is an approximate method, can be effectively used to evaluate the natural frequencies of continuous systems, especially the first few lower modes [23]. The Rayleigh–Ritz method assumes a mode shape, \( X(x) \), which is a linear combination of admissible basis functions, \( \phi_i(x) \). The mode shape must satisfy the geometric boundary conditions and can be generalized as follows:

Figure 6. (a) Overhang Beam with Spring at Overhang Support; (b) Equivalent Beam Model with MRE in Direct shear.

Figure 7. Shear and Bending Moment Shown in Red of (a) Cantilever Beam and (b) Overhanging Beam.
where \( c_i \) are the coefficients associated to each admissible basis function. The natural frequencies of the continuous system are determined by the Rayleigh’s quotient, which is dependent on the maximum strain energy, \( \pi_{\text{max}} \), and kinetic energy, \( T_{\text{max}}^* \). Rayleigh’s quotient also happens to be related to the harmonic frequency of the system, and is formulated as [23]

\[
R = \frac{\pi_{\text{max}}}{T_{\text{max}}^*} = \omega^2
\] (10)

The unknown coefficients, \( c_i \), of the mode shape function are determined such that the Rayleigh’s quotient is minimized. The minimization is applied to Rayleigh’s quotient for each coefficient, and is stated as follows:

\[
\frac{\partial R}{\partial c_i} = 0
\] (11)

The accuracy of the Rayleigh–Ritz method in determining the natural frequencies depends on the number of admissible functions used. The estimated frequency will always overestimate the actual frequency but as the number of admissible functions increases, the estimated frequency will approach the exact solution [23].

The set of orthonormal polynomials are generated by using the Gram–Schmidt recursive relation [24,25]. Let us consider that the non-dimensional distance along the beam length can be defined as

\[
\xi = \frac{x}{L}
\] (12)

The Rayleigh–Ritz method is initiated by first selecting appropriate admissible functions; however, selection of the admissible functions must be less constrained such that the springs provide the displacement resistance to satisfy the geometric boundary conditions [26]. Therefore, the Gram–Schmidt recursive algorithm is initialized with a non-normalized polynomial of the form \( p_1 = \xi \).

Once the beam’s mode shape function is defined by the admissible functions, it can then be used to derive the associated beam energies. The potential energy must include both the stiffness of the beam and the stiffness of the spring support. The beam potential energy is defined by its flexural strength and the potential energy due to the overhang spring. The kinetic and potential energies are defined as follows:

\[
T_{\text{max}}^* = \frac{1}{2} \vec{c}^T \left[ m \right] \vec{c}
\] (13)

\[
\pi_{\text{beam}} = \frac{1}{2} \vec{c}^T \left[ k \right] \vec{c}
\] (14)

where the elements in the mass, \( [m] \), and stiffness, \( [k] \), matrices are defined as follows:

\[
m_{ij} = \int_0^L \rho A \phi_i \phi_j dx
\] (15)

\[
k_{ij} = \int_0^L E I \frac{d^2 \phi_i}{dx^2} \frac{d^2 \phi_j}{dx^2} dx + k_s \phi_i(x_s) \phi_j(x_s)
\] (16)

Finally, the fundamental frequencies are determined by inserting Equations (13)–(16) into Equation (10), which leads to an eigenvalue problem that can be stated as follows:

\[
[k] = \omega^2 [m]
\] (17)

Solution to the above eigenvalue problem results in the natural frequencies and mode shapes, or eigenvalues and eigenvectors.
Considering that the mode shape can be arbitrary, then it would be useful to normalize the mode shape. It turns out that the coefficients in Equation (9) are the coefficients of the associated mode shape. Let us consider the normalized eigenvector, \( \tilde{C} \), obtained from the general eigenvector, \( C \), through some normalization factor \( n_c \):

\[
\tilde{C} = n_c C
\]  

(18)

where the general eigenvector and normalized eigenvector are of the form:

\[
C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_n \end{bmatrix}
\]  

(19)

The eigenvector is normalized such that the associated mode shape in Equation (9) satisfies \( X(x = L) = 1 \). The normalization factor is determined by inserting the coefficients of Equation (18) into Equation (9) with respect to the previously mentioned condition, resulting in

\[
n_c = \frac{1}{\sum_{i=1}^{n} c_i \phi_i(x = L)}
\]  

(20)

The beam dynamics are dependent on the geometry, which is determined by nonlinear mathematical programming optimization technique as outlined in Appendix A. The optimized beam geometry has a length of 777 mm, width of 10 mm, and thickness of 2 mm. The support location is placed 6.5% the length of the beam from the pinned support. The optimized geometry provides the highest frequency shift of 44%.

2.5. Equivalent Single-Degree-of-Freedom (SDOF) System

The newly designed continuous model with an intermediate linear MRE spring is subsequently converted to a single-degree-of-freedom (SDOF) model, assuming that the dynamics of the continuous beam is dominated by its fundamental mode. One general method of finding equivalent systems is by performing an energy comparison to determine an equivalent mass and complex stiffness. The equivalent SDOF model, as shown in Figure 8, is obtained by equating the kinetic and potential energies of the equivalent SDOF system to those of the continuous beam \[27,28\]. By considering the displacement of the beam tip, \( q(t) \), the dynamic beam displacement, \( w(x, t) \), can be expressed in separate time and space domains as

\[
w(x, t) = X(x)q(t)
\]  

(21)

where \( X(x) \) is the mode shape and \( q(t) \) is the time-dependent generalized motion of the beam tip. The kinetic and potential energy of the SDOF and continuous system are equated and stated as follows:

\[
\frac{1}{2} k_{eq}q^2 = \frac{1}{2} \int_0^L E_1 \left( \frac{dw}{dx} \right)^2 dx + \frac{1}{2} k_s [w(x_s)]^2
\]  

(22)

\[
\frac{1}{2} m_{eq}q^2 = \frac{1}{2} \int_0^L \rho A w^2 dx
\]  

(23)
Substituting \( k_f \) from Equation (8) into Equation (22), yields a complex equivalent stiffness in the form of

\[
k_{eq} = k' + jk''
\]

(24)

Now having defined the equivalent mass and stiffness, the governing equation of the SDOF system shown in Figure 8 can be formulated as follows:

\[
m_{eq} \ddot{x} + k'x = f(t)
\]

(25)

Considering that the equivalent stiffness is in the complex form due to the MRE viscoelastic nature, then Equation (25) can be rewritten as

\[
m_{eq} \ddot{x} + (k' + jk'')x = f(t)
\]

(26)

Noting that the loss modulus describes the dissipation of energy in the complex stiffness, then the loss modulus can be replaced by an equivalent viscous damping. Therefore, the governing differential equation for the equivalent SDOF system can be described as follows:

\[
m_{eq} \ddot{x} + c_{eq} \dot{x} + k'x = f(t)
\]

(27)

For structures having complex stiffness, the equivalent damping depends on both the loss component of stiffness and the operating frequency. For a harmonic input, the equivalent damping depends on the input frequency. For any other input, the beam should naturally vibrate at its fundamental frequency.

Equations (22) and (23) show that the equivalent mass and equivalent stiffness are dependent on the mode shape, which is also dependent on the stiffness of the MRE. The equivalent damping under harmonic loading is simply the loss component of stiffness scaled down by the operating frequency. Therefore, the equivalent mass, spring, and damper are dependent on the input current as this drives the stiffness of the MRE. Curve fitting using the least-squares minimization method is used to obtain the functional relationship between each equivalent system parameter to the input current. Figure 9 visually shows the goodness of fit for each obtained polynomial by least-squares minimization to the data points obtained. The field-dependent functional relations for storage and loss components of stiffness, equivalent mass, and equivalent damping with respect to applied current are found to be

\[
k'[I(t)] = -0.0047[I(t)]^5 + 0.0070[I(t)]^4 + 0.0632[I(t)]^3 - 0.2725[I(t)]^2 + 0.8334[I(t)] + 11.8511
\]

(28)

\[
k''[I(t)] = 0.0141[I(t)]^3 - 0.0489[I(t)]^2 - 0.0446[I(t)] + 1.7226
\]

(29)

\[
m_{eq}[I(t)] = 0.0066
\]

(30)

\[
c_{eq}[I(t)] = \begin{cases} \frac{k''[I(t)]}{\omega} & \text{Forced Harmonic Input} \\ 0.0003[I(t)]^3 - 0.0011[I(t)]^2 - 0.0010[I(t)] + 0.0381 & \text{Otherwise} \end{cases}
\]

(31)
fitting using the least-squares minimization method is used to obtain the functional relationship between each equivalent system parameter to the input current. Figure 9 visually shows the goodness of fit for each obtained polynomial by least-squares minimization to the data points obtained. The field-dependent functional relations for storage and loss components of stiffness, equivalent mass, and equivalent damping with respect to applied current are found to be

$$k''[I(t)] = -0.0047[I(t)]^2 + 0.0070[I(t)]^3 + 0.0632[I(t)]^4 - 0.2725[I(t)]^5 + 0.8334[I(t)] + 11.8511$$ (28)

$$k'[I(t)] = 0.0141[I(t)]^2 - 0.0489[I(t)]^3 - 0.0446[I(t)]^4 + 1.7226$$ (29)

$$m_{eq}[I(t)] = 0.0066$$ (30)

$$c_{eq}[I(t)] = c_{eq}[0] + \Delta c_{eq}[I(t)]$$ (31)

Figure 9. Curve Fit for Equivalent Stiffness Storage and Loss Components, Equivalent Mass, and Equivalent Damping.

2.6. Modelling the PID Control

The PID controller will control the electromagnet current to modify the stiffness and damping of the system [29]. Furthermore, the electromagnet current must be clipped between $-3$ A and 3 A as per the electromagnet design. The ordinary differential equation corresponding to the semi-active system can be transformed into a form resembling that of the active system, by accounting for the alterations in damping and spring forces as the actuation. The MRE stiffness and damping properties of the MRE under applied current, $I(t)$, to the electromagnet can be described as follows:

$$k'[I(t)] = k'[0] + \Delta k'[I(t)]$$ (32)

$$c_{eq}[I(t)] = c_{eq}[0] + \Delta c_{eq}[I(t)]$$ (33)

in which $k'[0]$ and $c_{eq}[0]$ are the stiffness and damping of the MRE under zero current and $\Delta k'[I(t)]$ and $\Delta c_{eq}[I(t)]$ are the changes in the stiffness and damping due to the applied current. Equations (32) and (33) show that the stiffness and damping can be expressed as a nominal term at a current of 0 A and an incremental term dependent on the input current. Replacing the field-dependent equivalent damping and stiffness in Equations (32) and (33) into Equation (27) for the case of free vibration ($f(t) = 0$) yields

$$m_{eq}\ddot{x} + c_{eq}[0]\dot{x} + k'[0]x = u(t)$$ (34)
where the input actuation is defined as
\[
u(t) = -\left(\Delta c_{eq}[I(t)]\dot{x} + \Delta k'[I(t)]x\right)
\] (35)

The actuation is now dependent on the generated loads due to the changes in both stiffness and damping as a function of the input current.

Generally, the tuning parameters for the PID controller are determined by trial-and-error to meet a certain performance. Rather than performing trial-and-error, the tuning parameters are determined by nonlinear-mathematical optimization programming technique in Appendix B. Three sets of tuning parameters are determined by individually minimizing different objectives in the cost function. The PID\_Ts minimizes the settling time, the PID\_X minimizes the state energy, and the PID\_P minimizes the percent overshoot. Table 5 summarizes the results of the optimization problem.

<table>
<thead>
<tr>
<th>Controller</th>
<th>K_P</th>
<th>K_I</th>
<th>K_D</th>
</tr>
</thead>
<tbody>
<tr>
<td>PID_P</td>
<td>26,472</td>
<td>16,178</td>
<td>-93,056</td>
</tr>
<tr>
<td>PID_X</td>
<td>1169</td>
<td>-92,175</td>
<td>-150</td>
</tr>
<tr>
<td>PID_Ts</td>
<td>35,751</td>
<td>-89,030</td>
<td>-1186</td>
</tr>
</tbody>
</table>

2.7. Modelling the On–Off Control

The time-optimal control law known as the on–off controller has been formulated for the equivalent SDOF system featuring MRE. To understand the on–off controller, the input actuation force defined in Equation (34) must be defined with some logic function, \(g(t)\), which represents the on and off conditions of the current. The input function can then be formulated as
\[
u(t) = -\left(\Delta c_{eq}[g(t)]\dot{x} + \Delta k'[g(t)]x\right)
\] (36)

Gu et al. [30] proposed an on–off controller that activates the MRE during deviations from the equilibrium point. The control law for \(g(t)\) in this study follows a similar logic to that of the skyhook controller, where the MRE is activated as the system moves away from its equilibrium position. The electromagnet design in this study, however, considers the off state as having a current of \(-3\) A and the on state as having a current of \(3\) A. Therefore, the control law for \(g(t)\) in this study is defined as
\[
g(t) = \begin{cases} 
3 & x\dot{x} > 0 \\
-3 & x\dot{x} \leq 0
\end{cases}
\] (37)

This control law demonstrates that the system’s stiffness and damping reach their peak when the system deviates from equilibrium, resulting in heightened resistance to movement. Conversely, as the system approaches equilibrium, stiffness and damping are minimized, expediting the system’s convergence to the equilibrium point by reducing resistance to motion.

3. Results and Discussion

In this section, the effectiveness of the developed control strategy using boundary conditioning is examined under transient and force excitations as described in the following sections.

3.1. Shock (Free Vibration) Response

A shock, or impulse, response can be represented as a free vibration problem with zero initial displacement and non-zero initial velocity. Under an initial velocity of \(1\) m/s and zero initial displacement, each controller is tested to examine their capabilities in decreasing both settling time and peak value. The PID\_Ts controller is considered due to its capability of minimizing settling time. The summary of the results for the settling time
and peak response of the MRE-based equivalent SDOF system under shock response for different control methods are provided in Table 6. Figure 10 compares the time response and input current time history of the passive system with those of semi-active systems using $PID_T$, and on–off control strategies. Each controller type is capable of lowering the settling time when comparing with the passive system. However, the peak value for the $PID_T$-controlled system is higher than the passive case, which is because the controller has an input value of 3 A during the first peak, resulting in a higher percent overshoot due to a decrease in damping ratio. The on–off, overall, is the better performer as it is capable of approximately halving the settling time while slightly decreasing the peak value.

Table 6. Settling Time and Peak Comparison.

<table>
<thead>
<tr>
<th>Method</th>
<th>Settling Time (s)</th>
<th>Peak (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passive</td>
<td>1.38</td>
<td>21.3</td>
</tr>
<tr>
<td>$PID_T$</td>
<td>1.34 (−3.3%)</td>
<td>23.1 (+8.4%)</td>
</tr>
<tr>
<td>On–Off</td>
<td>0.65 (−52.7%)</td>
<td>20.7 (−3.1%)</td>
</tr>
</tbody>
</table>

Figure 10. Comparison of Transient Response and Input Current between Passive and Semi-active Systems due to different Control Strategies.

3.2. Harmonic Response

The harmonic response of the MRE-based SDOF system is obtained under a harmonic load having an amplitude of 0.01 N to assure that the MRE operates under small shear strain. To ensure proper steady-state results, a simulation time of 10 s is used in order to ensure that all transient effects due to initial conditions have decayed and that the response will be purely due to the harmonic load. The system will be tested for operating frequencies of 5.14 Hz and 7.06 Hz, which represent the natural frequencies when using an input current of $−3$ and 3 A, respectively. The $PID_X$ controller is used since it is designed to minimize state energy, which in this case causes a lowering of steady-state amplitude.
In the case of harmonic loads, vibration attenuation is achieved by lowering the steady-state amplitude. Results for the steady-state harmonic response and control current under harmonic force with excitation frequencies set at 5.14 Hz and 7.06 Hz are shown in Figure 11. The steady-state amplitudes are summarized in Table 7. Overall, the on–off controller provides safe vibrations for all input frequencies except when operating at lowest frequency of 5.14 Hz. Amplification at 5.14 Hz occurs since the on–off controller has a fundamental frequency of 5.14 Hz when at $-3$ A. The PID$_X$ controller can provide good attenuation at all frequencies except at 7.06 Hz because of the tendency of having an input current of 3 A, which causes the input frequency to match the fundamental frequency.

![Figure 11. Steady-State Time Response and Control Current for Different Controllers under Harmonic Input at (a) 5.14 Hz and (b) at 7.06 Hz.](image)
Table 7. Amplitude Comparison under Harmonic Input.

<table>
<thead>
<tr>
<th>Controller</th>
<th>5.14 Hz (mm)</th>
<th>7.06 Hz (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passive</td>
<td>1.9</td>
<td>4.8</td>
</tr>
<tr>
<td>PID&lt;sub&gt;x&lt;/sub&gt;</td>
<td>1.6 (−17.0%)</td>
<td>6.5 (+35.3%)</td>
</tr>
<tr>
<td>On–Off</td>
<td>2.4 (+25.0%)</td>
<td>2.2 (−54.4%)</td>
</tr>
</tbody>
</table>

3.3. Random Vibration Response

The random response is generated with respect to a white noise power input of 1 mW, so that the MRE deflects with small shear strain. A summary of the random vibration response can be found in Table 8. The PID<sub>T<sub>s</sub></sub> controller is selected over the other PID-type controllers because it has a lower RMS displacement. Figure 12 show the time response over a time interval of 50 s. Visualizing the input current over a smaller interval helps in understanding the behavior of the different controllers. A comparison of the different controllers shows that the on–off controller is better at attenuating the RMS displacement by 23.5%. The PID<sub>T<sub>s</sub></sub> controller has almost no change in RMS displacement compared with the passive case. Although the on–off controller is capable of good vibration suppression, the control input for the on–off controller shown in Figure 12 is subject to chatter [31]. In context, this would mean that there would be a high amount of switching between the on and off state over a short period of time.

Table 8. Mean and RMS Response for Random Input.

<table>
<thead>
<tr>
<th>Controller</th>
<th>Mean Displacement (mm)</th>
<th>RMS Displacement (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passive</td>
<td>0.0</td>
<td>33.2</td>
</tr>
<tr>
<td>PID&lt;sub&gt;P&lt;/sub&gt;</td>
<td>−0.7</td>
<td>45.3 (36.4%)</td>
</tr>
<tr>
<td>PID&lt;sub&gt;X&lt;/sub&gt;</td>
<td>0.0</td>
<td>34.0 (2.4%)</td>
</tr>
<tr>
<td>PID&lt;sub&gt;T&lt;sub&gt;s&lt;/sub&gt;&lt;/sub&gt;</td>
<td>0.2</td>
<td>33.2 (0%)</td>
</tr>
<tr>
<td>On–Off</td>
<td>0.1</td>
<td>25.4 (−23.5%)</td>
</tr>
</tbody>
</table>
4. Conclusions

This study demonstrates the use of MRE as a means of controlling flexible structures through boundary conditioning. The proof-of-concept was conducted for a cantilever beam using an overhanging beam as an alternative such that the overhang support can be made

Figure 12. Time Response and Controller Input for Random Input over (a) 50 Seconds and (b) 5 Seconds.
flexible using MRE. PID and on–off control methods were implemented on the proposed model to adaptively tune the MRE for vibration attenuation.

Control via boundary conditioning demonstrated that the electromagnet is not limited by its weight. In general, a larger-sized electromagnet is required to generate higher magnetic flux densities. Previous control strategies required the electromagnet be mounted on the beam; thus, size and weight were a constraint. Without this limitation, the electromagnet can be designed to reach a magnetic flux density which would saturate the mechanical properties of the MRE. The larger-sized electromagnet allows the MRE-based support to have a wide range of stiffness and damping, thus leading to a larger frequency shift under changing magnetic flux densities.

The present research study has provided essential guidance on the application of MRE technology to semi-actively attenuate vibration through boundary conditioning. Several limitations and assumptions have been made to facilitate the development of the model and design optimization method. For instance, the MRE model was assumed to be independent of frequency and strain amplitude in the studied frequency range, thus the effects of strain softening and frequency stiffening typically observed in MREs were ignored. Also, the proposed overhang beam with MRE support does not accurately represent the cantilever beam model. The electromagnet to activate the MRE has been designed through trial and error as the weight of the electromagnet was not a primary constraint. Moreover, while the on–off control performed well, they are inherently subject to chattering. Therefore, the modelling of the MRE can be improved by considering the dependency on both frequency and amplitude. Moreover, other control laws can be considered, which may perform better than the on–off controller.

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**Data Availability Statement:** The data presented in this study are available on request from the corresponding authors as the data are part of an ongoing research study.

**Conflicts of Interest:** The authors declare no conflicts of interest.

**Appendix A Beam Geometry Optimization**

The beam dimensions are selected through a multi-objective constrained optimization problem. The cost function consists of a combination of two objective functions using weighting factors. The first objective function is to make the fundamental frequency of the equivalent model close to that of the cantilever model at the maximum magnetic flux density of 1 T. This can be done by ensuring that the ratio at 1 T, \( r_{1T} \), defined by Equation (7) be maximized. The physical meaning behind this condition is that the beam can be considered closely cantilevered when the MRE is at its maximum stiffness. The second objective function is to have a maximum shift in fundamental frequency during a change in magnetic flux density from 0 to 1 T. By using Equation (7), the non-dimensional frequency shift can be defined as

\[
\Delta r = r_{1T} - r_{0T}
\]  

This ensures the vibration control capability using MRE for a wide range of frequencies.

It is assumed that MREs operate in the linear viscoelastic regime to prevent a strain softening effect due to large amplitude deformation. The MRE considered in this research study approximately experiences linear viscoelastic behavior under a shear strain below 15%. This 15% shear strain is then accounted for between the static and dynamic behavior
of the beam. Generally, any system under vibratory motion oscillates about a static equilibrium point, which is generally dependent on the weight of the system. In this case, the static equilibrium point is enforced assuming a maximum shear strain of 5%, as shown in Figure A1. The remaining 10% shear strain is considered to be due to oscillatory motion about the static equilibrium point.

![Figure A1. Static and Dynamic Shear Strain Representations on Deflected MRE.](image)

Figure A2 illustrates the free-body diagram under static loading, which provides the relationship between the weight of the beam and the static deflection of the MRE. This functional relationship will provide a constraint in the optimization problem. The total strain is to be considered in the later chapters when the beam undergoes dynamic loading.

![Figure A2. Free-Body Diagram of Proposed Beam Model under Static Loading.](image)

The design variables are the beam dimensions including the beam length, \( L \), with a rectangular cross-section having width, \( b \), and height, \( h \). The placement of the MRE support is considered as a design variable and is defined in the form of a non-dimensional location along the beam, \( p \), as

\[
p = \frac{x_s}{L} \tag{A2}
\]

At static equilibrium, a moment balance about the support yields

\[
(pgbhL)\frac{L}{2} = x_s k_s \delta \tag{A3}
\]

The static shear strain, \( \gamma \), and the deflection at the support, \( \delta \), depend on the thickness of the MRE through the following expression:

\[
\gamma = \frac{\delta}{t_{MRE}} \tag{A4}
\]

Since the system is under static condition, then the stiffness due to the MRE is only dependent on the storage modulus. Furthermore, to ensure that the 5% static strain is
respected, then equilibrium should be maintained when the MRE is under a magnetic flux density of 0 T since this represents the minimum stiffness. Hence,

\[ k_s = 2 \frac{C_0' T A_{MRE}}{L_{MRE}} \]  \hspace{1cm} (A5)

Substituting Equations (A2), (A4) and (A5) into Equation (A3) yields

\[ \rho gbhL = 4pg_0' T A_{MRE} \gamma \]  \hspace{1cm} (A6)

Now to assure that the shear strain in the MRE under static loading is less than the allowable static strain, \( \gamma_{\text{static}} \), of 5\%, then the following condition should be met:

\[ \rho gbhL \leq 4pg_0' T A_{MRE} \gamma_{\text{static}} \]  \hspace{1cm} (A7)

Additional side constraints are applied to the geometry to prevent unrealistic dimensions and numerical inability during optimization iteration. From the dynamic conditions on the MRE, the fundamental frequency under a magnetic flux density of 1 T should remain under 10 Hz. Finally, the constrained optimization problem can be formally formulated as

Maximize \[ J = Cr_1 T + (1 - C)(r_1 T - r_0 T) \]

Subject to \[ \rho gbhL \leq 4pg_0' T A_{MRE} \gamma_{\text{static}} \]

\[ b \geq h \]

1 mm \( \leq L \leq 500 \) mm

10 mm \( \leq b \leq 500 \) mm

2 mm \( \leq h \leq 500 \) mm

\( f_1 T \leq 10 \) Hz

The MATLAB R2022A optimization toolbox was used to solve the optimization problem using the combination of the genetic algorithm (GA) and sequential programming algorithm (SQP). GA is initially used to find a near-region of the global optimum solution; this is then followed by sequential quadratic programming (SQP) to find the accurate global optimum in the vicinity defined by GA. The beam material is chosen to be aluminum to prevent interference with the applied magnetic field due to its very low magnetic permeability of \( 1.26 \times 10^{-6} \) H/m compared with steel [32]. An additional reason for the selection of aluminum is that it is mainly used for lightweight applications, where the density of aluminum is 2700 kg/m\(^3\). The geometry of the MRE has a width of 10 mm, length of 50 mm, and thickness of 8 mm. The storage modulus at 0 T, as per Equation (1), is 63.040 kPa.

The optimization was tested for different weights on the cost function. The area of the beam appeared to be unaffected by the weighting factor, but the beam length and non-dimensional support location varied according to the applied weighting factor. The results of the optimization problem under different weighting factors are provided in Table A1.

**Table A1. Summary of Geometry Results for Different Cost Weights.**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Cost Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.2</td>
</tr>
<tr>
<td>Beam Length, ( l ) (mm)</td>
<td>437</td>
</tr>
<tr>
<td>Beam Width, ( b ) (mm)</td>
<td>10</td>
</tr>
<tr>
<td>Beam Thickness, ( h ) (mm)</td>
<td>2</td>
</tr>
<tr>
<td>Support Location, ( p ) (%L)</td>
<td>3.7</td>
</tr>
<tr>
<td>Cantilever Fund. Freq. (Hz)</td>
<td>9.29</td>
</tr>
<tr>
<td>Fund. Freq. at 0 T (Hz)</td>
<td>4.99</td>
</tr>
<tr>
<td>Fund. Freq. at 1 T (Hz)</td>
<td>7.19</td>
</tr>
</tbody>
</table>
Appendix B  PID Tuning Optimization

The primary drawback of PID control lies in the process of selecting the tuning parameters, which typically involves a trial-and-error approach. To address this limitation, in this study, the tuning parameters for the PID controller are established through optimization, employing a multi-objective cost function. The multi-objective function consists of the settling time, peak value, and state energy, each multiplied by a weighting factor. For enhanced numerical analysis, the settling time and peak value of the PID-controlled system are normalized using the settling time and peak value of the passive system. The optimization problem is constrained to follow the dynamics defined by Equations (34) and (35). Theoretically, the tuning parameters can be any real number but to have more realistic values, the tuning parameters are also constrained such that their magnitudes do not exceed $1 \times 10^5$. For active systems, the tuning parameters are generally positive to prevent the creation of an unstable pole; however, for semi-active systems, negative tuning parameters are possible since they will not cause the system to be unstable. The optimization problem to find the proportional, integral, and differential tuning parameters ($k_p$, $k_i$, $k_d$) of the PID controller can thus be formally formulated as

$$\min_{k_p,k_i,k_d} J = C_1 \frac{T_{r,\text{passive}}}{T_{r,\text{passive}}} + C_2 \frac{x_{\text{peak,passive}}}{x_{\text{peak,passive}}} + C_3 \int_0^T x^2 dt$$

Subject to $m_{eq} \ddot{x} + c_{eq}[0] \dot{x} + k_{eq}[0]x = - (\Delta c_{eq}[I(t)] \dot{x} + \Delta k_{eq}[I(t)]x)$

$$x(0) = x_0, \quad \dot{x}(0) = v_0$$

$$-1E5 \leq k_p \leq 1E5$$

$$-1E5 \leq k_i \leq 1E5$$

$$-1E5 \leq k_d \leq 1E5$$

$$-3 \leq I(t) \leq 3$$

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