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An Note on Uncertainty Inequalities for Deformed Harmonic Oscillators

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Abstract: The aim of this paper is to prove some uncertainty inequalities for a class of integral operators associated to deformed harmonic oscillators.

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1. Introduction

The present paper is a continuation of our previous papers [1–3] to prove some uncertainty principles (UP) for a general class of integral operators, including the Fourier transform, the Fourier–Bessel transform, the Dunkl transform [4], the generalized Fourier transform [5], the deformed Fourier transform [6] and the Clifford transform. Other versions of UP for integral operators have been proved in [7–9].

It is well-known that the uncertainty principles set restrictions on the time-frequency (or space–time) concentration of a nonzero function. Different forms of the UP have been studied by the mathematical community throughout the 20th century, and this is still a field of research today (see e.g., the survey [10] and the book [11] for the most well known forms of UP). His first significant results and outstanding issues go back to the works of Norbert Wiener, Andrei Kolmogorov, Mark Kerin and Arne Beurling.

The term UP is associated with Werner Heisenberg's 1927 statement [12]

$$\Delta(x)\Delta(p) \geq \frac{\hbar}{2}, \tag{1}$$

which has become a fundamental element of quantum physics, where $\Delta(x)$ (respectively, $\Delta(p)$) is the standard deviation of position x (respectively, of momentum p) and $\hbar$ is the Planck constant. The Heisenberg's UP has a great importance in symmetry problems in physics, for example there is a connection between the space–time UP and the conformal symmetry in string theory (see e.g., [13–16]).

In this paper, we will follow the notation in [1]. More precisely, let Ω and $\widehat{\Omega}$ be two convex cones in $\mathbb{R}^d$ (i.e., for all $\delta > 0$ and $x \in \Omega$, we have $\delta x \in \Omega$) with non-empty interior, and endowed with the Borel measures μ and $\widehat{\mu}$. For $1 \leq p \leq \infty$, we define the Lebesgue spaces $L^p(\Omega, \mu)$ and $L^p(\widehat{\Omega}, \widehat{\mu})$ in the usual way.

We assume that the measure μ (and $\widehat{\mu}$) is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}x$, and has a polar decomposition of the form

$$\mathrm{d}\mu(r\zeta) = r^{2\ell-1}\,\mathrm{d}r\, Q(\zeta)\,\mathrm{d}\sigma(\zeta), \tag{2}$$

where $\mathrm{d}\sigma$ is the Lebesgue measure on the unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$ and $Q \in L^1(\mathbb{S}^{d-1}, \mathrm{d}\sigma)$ such that $Q \neq 0$. Then the measure μ is homogeneous of degree 2ℓ, that is, for any $f \in C_c(\Omega)$, and $\delta > 0$,

$$\int_\Omega f\left(\frac{x}{\delta}\right) \mathrm{d}\mu(x) = \delta^{2\ell} \int_\Omega f(x)\, \mathrm{d}\mu(x). \tag{3}$$

Let Δ be a second order differential operators defined initially on $C_c^\infty(\Omega)$. Assume that Δ is:

1. Self-adjoint
$$\langle \Delta f, g \rangle_\mu = \langle f, \Delta g \rangle_\mu, \qquad \forall f, g \in C_c^\infty(\Omega), \tag{4}$$
2. Positive
$$\langle (-\Delta) f, f \rangle_\mu \geq 0, \qquad \forall f \in C_c^\infty(\Omega), \tag{5}$$

where $\langle \cdot, \cdot \rangle_\mu$ is inner product on $L^2(\Omega, \mu)$.

Let $\{\mathcal{K}_\xi\}_{\xi \in \Omega}$ be the system of the eigenvectors of $|x|^{a_1} \Delta^x$ and the corresponding eigenvalues are $\{|\xi|^{a_2}\}_{\xi \in \Omega}$, that is

$$|x|^{a_1} \Delta^x \mathcal{K}_\xi = -|\xi|^{a_2} \mathcal{K}_\xi, \tag{6}$$

where a_1, a_2 are positive real numbers such that $a_2 \neq 0$ and the superscript in Δ^x indicates the relevant variable.

Next, assume that the kernel $\mathcal{K} : \Omega \times \Omega \longrightarrow \mathbb{C}$, $(x, \xi) \mapsto \mathcal{K}_\xi(x)$ satisfies:

1. $\mathcal{K}$ is continuous,
2. $\mathcal{K}$ is polynomially bounded:
$$|\mathcal{K}_\xi(x)| \leq c_{\mathcal{K}} (1 + |x|)^m (1 + |\xi|)^{\widehat{m}}, \qquad m,\ \widehat{m} > 0, \tag{7}$$
3. $\mathcal{K}$ is homogeneous:
$$\mathcal{K}_\xi(\delta x) = \mathcal{K}_{\delta\xi}(x), \qquad \delta > 0. \tag{8}$$

One can then define the integral operator $\mathcal{T}$ on the Schwartz space $\mathcal{S}(\Omega)$ by

$$\mathcal{T}(f)(\xi) = \langle f, \mathcal{K}_\xi \rangle_\mu = \int_\Omega f(x) \overline{\mathcal{K}_\xi(x)}\, \mathrm{d}\mu(x), \qquad \xi \in \widehat{\Omega}. \tag{9}$$

Assume that $\mathcal{T}$ can be extended to an unitary operator from $L^2(\Omega, \mu)$ onto $L^2(\widehat{\Omega}, \widehat{\mu})$ with inverse

$$\mathcal{T}^{-1}(f)(x) = \int_\Omega f(\xi) \mathcal{K}_\xi(x)\, \mathrm{d}\widehat{\mu}(\xi), \qquad x \in \Omega, \tag{10}$$

and satisfies a Parseval-type equality,

$$\langle \mathcal{T}(f), \mathcal{T}(g) \rangle_{\widehat{\mu}} = \langle f, g \rangle_\mu, \tag{11}$$

where $\langle \cdot, \cdot \rangle_{\widehat{\mu}}$ is the inner product on $L^2(\widehat{\Omega}, \widehat{\mu})$.

For $\rho > 0$, we define the measures

$$\mathrm{d}\mu_\rho(x) = (1 + |x|)^\rho\, \mathrm{d}\mu(x) \quad \text{and} \quad \mathrm{d}\widehat{\mu}_\rho(\xi) = (1 + |\xi|)^\rho\, \mathrm{d}\widehat{\mu}(\xi). \tag{12}$$

Then $\mathcal{T}$ extends into a continuous operator from $L^1(\Omega, \mu_m)$ to

$$\mathcal{C}_{\widehat{m}}(\widehat{\Omega}) = \left\{ f \text{ continuous, s.t. } \|f\|_{\infty, \widehat{m}} := \sup_{\xi \in \widehat{\Omega}} \frac{|f(\xi)|}{(1 + |\xi|)^{\widehat{m}}} < \infty \right\}. \tag{13}$$

We define $\widehat{a}$ accordingly for $\widehat{\mu}$ and assume that $\widehat{a} = a$. Then we introduce the dilation operators $\mathcal{D}_\delta, \widehat{\mathcal{D}}_\delta, \delta > 0$:

$$\mathcal{D}_\delta f(x) = \frac{1}{\delta^a} f\left(\frac{x}{\delta}\right), \qquad \widehat{\mathcal{D}}_\delta f(x) = \frac{1}{\delta^{\widehat{a}}} f\left(\frac{x}{\delta}\right), \tag{14}$$

then by the homogeneity of the kernel, we have

$$\mathcal{T}\mathcal{D}_\delta = \widehat{\mathcal{D}}_{\frac{1}{\delta}}\mathcal{T}. \tag{15}$$

Moreover, since the measures $\mu, \widehat{\mu}$ are absolutely continuous with respect to the Lebesgue measure, then the dilation operator $\mathcal{D}$ (respectively, $\widehat{\mathcal{D}}$) is continuous from $(0,\infty) \times L^2(\Omega, \mu_\rho)$ to $L^2(\Omega, \mu_\rho)$ (respectively, from $(0,\infty) \times L^2(\widehat{\Omega}, \widehat{\mu}_\rho)$ to $L^2(\widehat{\Omega}, \widehat{\mu}_\rho)$).

As Δ is self-adjoint, then by Inequality (6) one has, for any $f \in C_c^\infty(\Omega)$:

$$\begin{aligned} \mathcal{T}(|x|^{a_1}\Delta f)(\xi) &= \left\langle |x|^{a_1}\Delta f, \mathcal{K}_\xi \right\rangle_\mu \\ &= \left\langle f, |x|^{a_1}\Delta \mathcal{K}_\xi \right\rangle_\mu \\ &= -\left\langle f, |\xi|^{a_2}\mathcal{K}_\xi \right\rangle_\mu = -|\xi|^{a_2}\mathcal{T}(f)(\xi). \end{aligned} \tag{16}$$

We consider the nonnegative and self-adjoint extension of $|x|^{a_1}\Delta$ (still denoted by the same symbol) defined by

$$(-|x|^{a_1}\Delta)f = \mathcal{T}^{-1}\left[|\xi|^{a_2}\mathcal{T}(f)\right], \qquad f \in \mathrm{Dom}(\Delta), \tag{17}$$

where $\mathrm{Dom}(|x|^{a_1}\Delta) = \{f \in L^2(\Omega,\mu) : |\xi|^{a_2}\mathcal{T}(f) \in L^2(\Omega,\mu)\}$.

By (3), the measure of the ball $B_r = \{x \in \Omega : |x| < r\}$ of center 0 and radius r is majorized by the power of the radius r i.e.,

$$\mu(B_r) \le c\, r^{2\ell}. \tag{18}$$

Assume also that the semigroup $\{W_t\}_{t>0} = \{\exp(t|x|^{a_1}\Delta)\}_{t>0}$ generated by $|x|^{a_1}\Delta$ satisfies

$$\|W_t\|_{1\to\infty} \le c\, t^{-\frac{2\ell}{a_2}}. \tag{19}$$

Let $\mathcal{L} = |x|^{a_2} - |x|^{a_1}\Delta$ be a deformed harmonic oscillator on $C_c^\infty(\Omega)$. Assume that there exists an orthonormal basis $\{\phi_j\}_{j\in\mathbb{N}^d}$ for $L^2(\Omega,\mu)$ consisting of eigenfunctions of $\mathcal{L}$ that correspond to the eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}^d}$, that is

$$\mathcal{L}\phi_j = \lambda_j\phi_j.$$

We impose the assumption that we can arrange the set $\{\lambda_j\}_{j\in\mathbb{N}^d}$ into an increasing sequence $\{\Lambda_k\}_{k\in\mathbb{N}}$ that satisfies

$$0 < \lambda_0 = \Lambda_0 < \Lambda_1 < \Lambda_2 < \cdots \tag{20}$$

and

$$\lim_{k\to\infty} \Lambda_k = \infty. \tag{21}$$

Therefore $\mathcal{L}$ is symmetric and positive in $L^2(\Omega,\mu)$ and it has a natural self-adjoint extension (still denoted by the same symbol) on $L^2(\Omega,\mu)$, that is,

$$\mathcal{L}f = \sum_{j\in\mathbb{N}^d} \lambda_j \left\langle f, \phi_j \right\rangle_\mu \phi_j \tag{22}$$

on the domain Dom$\mathcal{L}$ consisting of all functions $L^2(\Omega,\mu)$ for which the defining series $\sum_{j\in\mathbb{N}^d} |\lambda_j \langle f,\phi_j\rangle_\mu|^2$ converges in $L^2(\Omega,\mu)$. Moreover, the spectrum of $\mathcal{L}$ is $\{\Lambda_k\}_{k\in\mathbb{N}}$ and the spectral decomposition of $\mathcal{L}$ can be written as

$$\mathcal{L} = \sum_{k=0}^{\infty} \Lambda_k \mathcal{P}_k f, \qquad f \in \mathrm{Dom}\mathcal{L}, \tag{23}$$

where the spectral projections are

$$\mathcal{P}_k f = \sum_{\{j\in\mathbb{N}^d:\lambda_j=\Lambda_k\}} \langle f,\phi_j\rangle_\mu \phi_j, \qquad k\in\mathbb{N}. \tag{24}$$

The structure of the present paper is as follows. In Section 2 we establish an optimal version of the Heisenberg-type UP for the integral operator $\mathcal{T}$, as well as some other well-known uncertainty inequalities. Section 3 is devoted to applying our results to some particular cases.

Notation

For $x\in\mathbb{R}^d$, $|x| = \sqrt{\langle x,x\rangle}$ denotes its norm , where $\langle .,.\rangle$ is the Euclidean inner product, and $(e_1,\dots,e_d)$ will be the canonical basis of $\mathbb{R}^d$. We will write χ_S for the characteristic function of the subset $S\subset\mathbb{R}^d$, and we write c, c_ℓ, c_s, $c_{s,\ell}$, $c(s,\mathcal{T})$ and $c_{\beta,s,\ell},\dots$ for constants (which can change from line to line) that depend on the parameters ℓ, s and $\beta,\mathcal{T},\dots$.

2. Heisenberg-Type Uncertainty Principles

2.1. Sharp Heisenberg-Type Uncertainty Inequality

In this subsection we will establish a sharp (optimal) version of Heisenberg-type UP for the transformation $\mathcal{T}$. Other non optimal Heisenberg-type inequalities for integral operators can be found in [1,2].

Theorem 1. *For every function $f\in L^2(\Omega,\mu)$,*

$$\left\| |x|^{\frac{a_2}{2}} f \right\|_{2,\mu}^2 + \left\| |\xi|^{\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}}^2 \geq \Lambda_0 \|f\|_{2,\mu}^2. \tag{25}$$

Equality in (25) holds if, and only if $f(x) = c\,\phi_0(x)$, $c\in\mathbb{C}$.

Proof. If $f\in L^2(\Omega,\mu)$ is a non vanishing function, with

$$\left\| |x|^{\frac{a_2}{2}} f \right\|_{2,\mu}^2, \quad \left\| |\xi|^{\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}}^2 < \infty,$$

then from (17) and Parseval–type equality (11),

$$\begin{aligned} \left\| |\xi|^{\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}}^2 &= \langle |\xi|^{a_2}\mathcal{T}(f),\mathcal{T}(f)\rangle_{\widehat{\mu}} \\ &= \langle \mathcal{T}(-|x|^{a_1}\Delta f),\mathcal{T}(f)\rangle_{\widehat{\mu}} \\ &= \langle -|x|^{a_1}\Delta f, f\rangle_\mu. \end{aligned} \tag{26}$$

Therefore

$$\left\| |x|^{\frac{a_2}{2}} f \right\|_{2,\mu}^2 + \left\| |\xi|^{\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}}^2 = \langle |x|^{a_2} f,f\rangle_\mu - \langle |x|^{a_1}\Delta f,f\rangle_\mu = \langle \mathcal{L}f,f\rangle_\mu. \tag{27}$$

As the operator $\mathcal{L}$ is self-adjoint and has only discrete spectra, then

$$\left\| |x|^{\frac{a_2}{2}} f \right\|_{2,\mu}^2 + \left\| |\xi|^{\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}}^2 \geq \Lambda_0 \|f\|_{2,\mu}^2. \tag{28}$$

Moreover the equality holds in (28), if and only if f is an eigenfunction of the self-adjoint operator $|x|^{a_2} - |x|^{a_1}\Delta$ corresponding to the eigenvalue Λ_0. Thus f is a scalar multiple of ϕ_0. □

By a dilation argument we deduce the following product form of the Heisenberg-type inequality (25).

Corollary 1. *For every $f \in L^2(\Omega, \mu)$,*

$$\left\| |x|^{\frac{a_2}{2}} f \right\|_{2,\mu} \left\| |\xi|^{\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}} \geq \frac{\Lambda_0}{2} \|f\|_{2,\mu}^2. \tag{29}$$

Equality holds in (29), if and only if, $f(x) = c\phi_0(\delta x)$ for some $c \in \mathbb{C}$ and $\delta > 0$.

Proof. Since $\|\mathcal{D}_\delta f\|_{2,\mu} = \|f\|_{2,\mu}$, then by replacing f by $\mathcal{D}_\delta f$ in (25), we obtain from (15) that

$$\delta^{a_2} \left\| |x|^{\frac{a_2}{2}} f \right\|_{2,\mu}^2 + \delta^{-a_2} \left\| |\xi|^{\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}}^2 \geq \Lambda_0 \|f\|_{2,\mu}^2. \tag{30}$$

Then (29) follows by minimizing the left hand side of (30) over $\delta > 0$, with

$$\min_{\delta>0} \left(\delta^{a_2} \left\| |x|^{\frac{a_2}{2}} f \right\|_{2,\mu}^2 + \delta^{-a_2} \left\| |\xi|^{\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}}^2 \right) = 2 \left\| |x|^{\frac{a_2}{2}} f \right\|_{2,\mu} \left\| |\xi|^{\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}}. \tag{31}$$

Equality in (29) holds exactly if we have

$$2 \left\| |x|^{\frac{a_2}{2}} f \right\|_{2,\mu} \left\| |\xi|^{\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}} = \Lambda_0 \|f\|_{2,\mu}^2, \tag{32}$$

thus by equality cases of (25), we deduce the desired result. □

More generally, we can state the following improvement.

Corollary 2. *For every $s, \beta \in [1, \infty)$, and every $f \in L^2(\Omega, \mu)$,*

$$\left\| |x|^{s\frac{a_2}{2}} f \right\|_{2,\mu}^\beta \left\| |\xi|^{\beta\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}}^s \geq \left(\frac{\Lambda_0}{2} \right)^{s\beta} \|f\|_{2,\mu}^{s+\beta}. \tag{33}$$

Proof. If $f \in L^2(\Omega, \mu)$ is a non vanishing function with finite dispersions

$$\left\| |x|^{s\frac{a_2}{2}} f \right\|_{2,\mu}, \quad \left\| |\xi|^{\beta\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}} < \infty,$$

then

$$\| |x|^{s\frac{a_2}{2}} f \|_{2,\mu}^{2/s} \|f\|_{2,\mu}^{2/s'} = \| |x|^{a_2} |f|^{2/s} \|_{s,\mu} \| |f|^{2/s'} \|_{s',\mu}, \tag{34}$$

where s' is the conjugate index of s. Therefore, by Hölder's inequality we obtain

$$\| |x|^{\frac{a_2}{2}} f \|_{2,\mu} \leq \| |x|^{s\frac{a_2}{2}} f \|_{2,\mu}^{1/s} \|f\|_{2,\mu}^{1/s'}. \tag{35}$$

Thus, for all $s \geq 1$:

$$\| |x|^{s\frac{a_2}{2}} f \|_{2,\mu}^\beta \geq \|f\|_{2,\mu}^{-(s-1)\beta} \| |x|^{\frac{a_2}{2}} f \|_{2,\mu}^{\beta s} \tag{36}$$

Similarly, we have for $\beta \in [1, \infty)$:

$$\||\xi|^{\beta\frac{a_2}{2}}\mathcal{T}(f)\|^s_{2,\hat{\mu}} \geq \|\mathcal{T}(f)\|^{-(\beta-1)s}_{2,\hat{\mu}}\||\xi|^{\frac{a_2}{2}}\mathcal{T}(f)\|^{\beta s}_{2,\hat{\mu}} = \|f\|^{-(\beta-1)s}_{2,\hat{\mu}}\||\xi|^{\frac{a_2}{2}}\mathcal{T}(f)\|^{\beta s}_{2,\hat{\mu}}.$$

Now, applying Heisenberg's Inequality (29), we obtain the desired result. □

As the proof of the last corollary is based on Hölder's inequality, then it exclude the other cases when $s, \beta \in (0,1)$.

2.2. General Form of Heisenberg-Type Uncertainty Inequality

In this subsection we will establish a general from of Heisenberg-type relation for any $s, \beta > 0$. Such inequality can be deduced from [1] (Theorem C), which is obtained from either the local Faris-type inequality [1] (Theorem A) or the Benedicks–Amrein–Berthier-type UP [1] (Theorem B). Our proof here is inspired from related results on Lie groups of polynomial growth in [17], and the assumption (19) plays a key role here.

Lemma 1. *Let $s < \ell$. Then there is $C_s > 0$ such that, for any $f \in L^2(\Omega, \mu)$,*

$$\|W_t f\|_{2,\mu} \leq c_s t^{-\frac{s}{2}} \left\| |x|^{s\frac{a_2}{2}} f \right\|_{2,\mu}. \tag{37}$$

Proof. Put $f_r = f\chi_{B_r}$ and $f^r = f - f_r$. We have

$$\|f^r\|_{2,\mu} \leq r^{-s\frac{a_2}{2}} \left\| |x|^{s\frac{a_2}{2}} f \right\|_{2,\mu}. \tag{38}$$

Now since W_t is a semigroup of contractions, then

$$\|W_t f^r\|_{2,\mu} \leq \|f^r\|_{2,\mu} \leq r^{-s\frac{a_2}{2}} \left\| |x|^{s\frac{a_2}{2}} f \right\|_{2,\mu}.$$

Moreover, as $f_r \in L^1(\Omega, \mu)$, then by Cauchy–Schwartz's inequality and (19) we obtain

$$\begin{aligned}
\|W_t f_r\|_{2,\mu} &\leq \|W_t\|_{1\to 2}\|f_r\|_{1,\mu} \\
&\leq \left\| e^{2t|x|^{a_1}\Delta} \right\|^{1/2}_{1\to\infty} \left\| |x|^{-s\frac{a_2}{2}}\chi_{B_r} \right\|_{2,\mu} \left\| |x|^{s\frac{a_2}{2}} f \right\|_{2,\mu} \\
&\leq c_s t^{-\frac{\ell}{a_2}} r^{\ell - s\frac{a_2}{2}} \left\| |x|^{s\frac{a_2}{2}} f \right\|_{2,\mu}.
\end{aligned}$$

Therefore

$$\|W_t f\|_{2,\mu} \leq \|W_t f_r\|_{2,\mu} + \|W_t f^r\|_{2,\mu} \leq r^{-s\frac{a_2}{2}} \left(1 + c_s r^{\ell} t^{-\frac{\ell}{a_2}}\right) \left\| |x|^{s\frac{a_2}{2}} f \right\|_{2,\mu}. \tag{39}$$

Choosing $r = t^{\frac{1}{a_2}}$, we obtain

$$\|W_t f\|_{2,\mu} \leq (1 + c_s) t^{-\frac{s}{2}} \left\| |x|^{s\frac{a_2}{2}} f \right\|_{2,\mu}, \tag{40}$$

as expected. □

Now, let's show our main result of this subsection.

Theorem 2. *Let $s, \beta \in (0, \infty)$. Then there exists $c_{\beta,s,\ell}$ such that for every $f \in L^2(\Omega, \mu)$,*

$$\| |x|^{s\frac{a_2}{2}} f \|^{\beta}_{2,\mu} \, \| |\xi|^{\beta\frac{a_2}{2}} \mathcal{T}(f) \|^s_{2,\hat{\mu}} \geq c_{\beta,s,\ell} \|f\|^{(s+\beta)}_{2,\mu}. \tag{41}$$

Proof. First assume that $s < \ell$ and $\beta \leq 2$. Then by Lemma 1,

$$\begin{aligned}\|f\|_{2,\mu} &\leq \|W_t f\|_{2,\mu} + \|f - W_t f\|_{2,\mu} \\ &\leq c_s t^{-\frac{s}{2}} \left\| |x|^{s\frac{a_2}{2}} f \right\|_{2,\mu} + \left\| \left(1 - e^{t|x|^{a_1}\Delta}\right) (-t|x|^{a_1}\Delta)^{-\frac{\beta}{2}} (-t|x|^{a_1}\Delta)^{\frac{\beta}{2}} f \right\|_{2,\mu}.\end{aligned}$$

Now since, for any $\rho > 0$ and $\beta \leq 2$, the function $\rho \mapsto (1 - e^{-\rho})\rho^{-\frac{\beta}{2}}$ is bounded, then

$$\left\| \left(1 - e^{t|x|^{a_1}\Delta}\right) (-t|x|^{a_1}\Delta)^{-\frac{\beta}{2}} (-t|x|^{a_1}\Delta)^{\frac{\beta}{2}} f \right\|_{2,\mu} \leq t^{\frac{\beta}{2}} \left\| (-|x|^{a_1}\Delta)^{\frac{\beta}{2}} f \right\|_{2,\mu}.$$

Therefore by the Parseval equality and (17)

$$\|f\|_{2,\mu} \leq c_s t^{-\frac{s}{2}} \left\| |x|^{s\frac{a_2}{2}} f \right\|_{2,\mu} + t^{\frac{\beta}{2}} \left\| |\xi|^{\beta\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}}. \tag{42}$$

By minimizing the right hand side of (42) over $t > 0$, we obtain

$$\| |x|^{s\frac{a_2}{2}} f \|_{2,\mu}^{\frac{\beta}{s+\beta}} \; \| |\xi|^{\beta\frac{a_2}{2}} \mathcal{T}(f) \|_{2,\widehat{\mu}}^{\frac{s}{s+\beta}} \geq c_{\beta,s,\ell} \|f\|_{2,\mu}. \tag{43}$$

Now if $\beta > 2$, then for every $\beta' \leq 2$, we have by Parseval equality (11)

$$\left\| |\xi|^{\beta'\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}}^2 \leq \|f\|_{2,\mu}^2 + \left\| |\xi|^{\beta\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}}^2. \tag{44}$$

Replacing f by $\mathcal{D}_t f$ in (44), we get, by (15),

$$\left\| |\xi|^{\beta'\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}}^2 \leq t^{\beta' a_2} \|f\|_{2,\mu}^2 + t^{a_2(\beta'-\beta)} \left\| |\xi|^{\beta\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}}^2. \tag{45}$$

Minimizing the right hand side of (45) over $t > 0$, we obtain

$$\left\| |\xi|^{\beta'\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}}^2 \leq \frac{\beta}{\beta - \beta'} \left(\frac{\beta}{\beta'} - 1 \right)^{\frac{\beta'}{\beta}} \|f\|_{2,\mu}^{2(1-\beta'/\beta)} \left\| |\xi|^{\beta\frac{a_2}{2}} \mathcal{T}(f) \right\|_{2,\widehat{\mu}}^{\frac{2\beta'}{\beta}}. \tag{46}$$

Together with (43) for β' we obtain the result for $\beta > 2$.
Finally, and in same way, for $s \geq \ell$, we take $s' < \ell$ and we obtain

$$\left\| |x|^{s'\frac{a_2}{2}} f \right\|_{2,\mu}^2 \leq \frac{s}{s - s'} \left(\frac{s}{s_0} - 1 \right)^{\frac{s'}{s}} \|f\|_{2,\mu}^{2(1-s'/s)} \left\| |x|^{s\frac{a_2}{2}} f \right\|_{2,\mu}^{\frac{2s'}{s}}. \tag{47}$$

This allows to conclude again with (43). □

Remark 1. *If we take $s = \beta$ in Inequality (41), then we obtain*

$$\| |x|^{s\frac{a_2}{2}} f \|_{2,\mu} \; \| |\xi|^{s\frac{a_2}{2}} \mathcal{T}(f) \|_{2,\widehat{\mu}} \geq c_{s,\ell} \|f\|_{2,\mu}^2, \tag{48}$$

which implies

$$\| |x|^{s\frac{a_2}{2}} f \|_{2,\mu}^2 + \| |\xi|^{s\frac{a_2}{2}} \mathcal{T}(f) \|_{2,\widehat{\mu}}^2 \geq c_{s,\ell} \|f\|_{2,\mu}^2. \tag{49}$$

2.3. The ε-Concentration Version of Heisenberg-Type Uncertainty Inequality

Let $S \subset \Omega$ and $\Sigma \subset \widehat{\Omega}$ be two subsets, such that $0 < \mu_{2m}(S), \widehat{\mu}_{2\widehat{m}}(\Sigma) < \infty$. We define the time and frequency limiting operators by

$$E_S f = \chi_S f, \qquad F_\Sigma f = \mathcal{T}^{-1}\Big[\chi_\Sigma \mathcal{T}(f)\Big], \qquad f \in L^2(\Omega, \mu).$$

Then we recall the following well-known definition (see [18]).

Definition 1. *Let $0 < \varepsilon < 1$ and $f \in L^2(\Omega, \mu)$. Then*

1. *We say that f is ε-concentrated on S if*

$$\|\chi_{S^c} f\|_{2,\mu} \leq \varepsilon \|f\|_{2,\mu}, \tag{50}$$

2. *We say that f is ε-bandlimited (or $\mathcal{T}(f)$ is ε-concentrated) on Σ if*

$$\|\chi_{\Sigma^c} \mathcal{T}(f)\|_{2,\widehat{\mu}} \leq \varepsilon \|f\|_{2,\mu}. \tag{51}$$

The subsets S and Σ are known as the essential supports of f and $\mathcal{T}(f)$, respectively, and this fact is first introduced by Donoho–Stark in [18], replacing the exact supports by the essential supports. If f is ε_1-concentrated on S and ε_2-bandlimited on Σ, we briefly write f is $(\varepsilon_1, \varepsilon_2)$-concentrated on (S, Σ), and we denote by $L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$ the subspace of $L^2(\Omega, \mu)$ consisting of functions that are $(\varepsilon_1, \varepsilon_2)$-concentrated on (S, Σ). In particular (see [1] (Inequality (3.4))), if $f \in L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$, then

$$\mu_{2m}(S)\widehat{\mu}_{2\widehat{m}}(\Sigma) \geq c_{\mathcal{K}}^{-2}\left(1 - \sqrt{\varepsilon_1^2 + \varepsilon_2^2}\right)^2. \tag{52}$$

Moreover from [1] (Theorem 2.1) we recall the following local UP.

Theorem 3. *Let $s > 0$ and let $\Sigma \subset \widehat{\Omega}$ be a subset such that $0 < \widehat{\mu}_{2\widehat{m}}(\Sigma) < \infty$. Then*

1. *if $0 < s < \frac{2\ell}{a_2}$, there exists $c = c(s, \mathcal{T})$ such that for every $f \in L^2(\Omega, \mu)$, $\|\chi_\Sigma \mathcal{T}(f)\|_{2,\widehat{\mu}}$ is bounded by*

$$\begin{cases} c\Big[\widehat{\mu}_{2\widehat{m}}(\Sigma)\Big]^{\frac{sa_2}{4(\ell+m)}} \big\||x|^{s\frac{a_2}{2}} f\big\|_{2,\mu}, & \text{if } \widehat{\mu}_{2\widehat{m}}(\Sigma) \leq 1; \\ c\Big[\widehat{\mu}_{2\widehat{m}}(\Sigma)\Big]^{\frac{sa_2}{4\ell}} \big\||x|^{s\frac{a_2}{2}} f\big\|_{2,\mu}, & \text{if } \widehat{\mu}_{2\widehat{m}}(\Sigma) > 1; \end{cases} \tag{53}$$

2. *If $\frac{2\ell}{a_2} \leq s \leq \frac{2}{a_2}(\ell + m)$, then for any $\varepsilon > 0$ there exists $c = c(s, \mathcal{T}, \varepsilon)$ such that for every $f \in L^2(\Omega, \mu)$, $\|\chi_\Sigma \mathcal{T}(f)\|_{2,\widehat{\mu}}$ is bounded by*

$$\begin{cases} c\Big[\widehat{\mu}_{2\widehat{m}}(\Sigma)\Big]^{\frac{1}{2(1+m/\ell)}-\varepsilon} \|f\|_{2,\mu}^{1-\frac{2\ell}{sa_2}+\varepsilon} \big\||x|^{s\frac{a_2}{2}} f\big\|_{2,\mu}^{\frac{2\ell}{sa_2}-\varepsilon}, & \text{if } \widehat{\mu}_{2\widehat{m}}(\Sigma) \leq 1; \\ c\Big[\widehat{\mu}_{2\widehat{m}}(\Sigma)\Big]^{\frac{1}{2}-\varepsilon} \|f\|_{2,\mu}^{1-\frac{2\ell}{sa_2}+\varepsilon} \big\||x|^{s\frac{a_2}{2}} f\big\|_{2,\mu}^{\frac{2\ell}{sa_2}-\varepsilon}, & \text{if } \widehat{\mu}_{2\widehat{m}}(\Sigma) > 1; \end{cases} \tag{54}$$

3. *If $s > \frac{2}{a_2}(m + \ell)$, there exists $c = c(s, \mathcal{T})$ such that for every $f \in L^2(\Omega, \mu)$, $\|\chi_\Sigma \mathcal{T}(f)\|_{2,\widehat{\mu}}$ is bounded by*

$$\begin{cases} c\Big[\widehat{\mu}_{2\widehat{m}}(\Sigma)\Big]^{\frac{1}{2}} \|f\|_{2,\mu}^{1-\frac{2\ell}{sa_2}} \big\||x|^{s\frac{a_2}{2}} f\big\|_{2,\mu}^{\frac{2\ell}{sa_2}}, & \text{if } m = 0; \\ c\Big[\widehat{\mu}_{2\widehat{m}}(\Sigma)\Big]^{\frac{1}{2}} \|f\|_{L^2(\Omega, \mu_{sa_2})}, & \text{otherwise}. \end{cases} \tag{55}$$

By (11), we have

$$\forall f \in L^2(\Omega,\mu), \qquad \|f\|^2_{2,\mu} = \|\mathcal{T}(f)\|^2_{2,\widehat{\mu}} = \|\chi_\Sigma \mathcal{T}(f)\|^2_{2,\widehat{\mu}} + \|\chi_{\Sigma^c}\mathcal{T}(f)\|^2_{2,\widehat{\mu}}, \tag{56}$$

then, (51) is equivalent to

$$\|\chi_\Sigma \mathcal{T}(f)\|^2_{2,\widehat{\mu}} \geq (1-\varepsilon^2)\|f\|^2_{2,\mu}, \quad \text{or} \quad \|F_\Sigma f\|_{2,\mu} \geq \sqrt{1-\varepsilon^2}\|f\|_{2,\mu}. \tag{57}$$

In same way, (50) is equivalent to

$$\|E_S f\|_{2,\mu} \geq \sqrt{1-\varepsilon^2}\|f\|_{2,\mu}. \tag{58}$$

Heisenberg's inequality in Theorem 2 gives a lower bound of the product for the generalized time and frequency dispersions for functions in $L^2(\Omega,\mu)$, but we can not have a lower bound for each one separately. The purpose of this section is to establish a Heisenberg-type inequality for functions in $L^2(\varepsilon_1,\varepsilon_2,S,\Sigma)$, for which a lower bound is given for each of the time and frequency dispersions. This gives more information than the lower bound of the product between them.

First by using Theorem 3, we will obtain a lower bound for the measure of the generalized time dispersion $\||x|^{s\frac{a_2}{2}} f\|_{2,\mu}$.

Corollary 3. *Let $s>0$ and let $\Sigma \subset \widehat{\Omega}$ be a subset such that $0 < \widehat{\mu}_{2\widehat{m}}(\Sigma) < \infty$. Then*

1. *If $0 < s < \frac{2\ell}{a_2}$, there exists $c = c(s,\mathcal{T})$, such that for any ε_2-bandlimited function $f \in L^2(\Omega,\mu)$ on Σ,*

$$\begin{cases} \||x|^{s\frac{a_2}{2}} f\|^2_{2,\mu} \geq \dfrac{c(1-\varepsilon_2^2)}{[\widehat{\mu}_{2\widehat{m}}(\Sigma)]^{\frac{sa_2}{2(\ell+m)}}}\|f\|^2_{2,\mu}, & \text{if } \widehat{\mu}_{2\widehat{m}}(\Sigma) \leq 1; \\ \||x|^{s\frac{a_2}{2}} f\|^2_{2,\mu} \geq \dfrac{c(1-\varepsilon_2^2)}{[\widehat{\mu}_{2\widehat{m}}(\Sigma)]^{\frac{sa_2}{2\ell}}}\|f\|^2_{2,\mu}, & \text{if } \widehat{\mu}_{2\widehat{m}}(\Sigma) > 1; \end{cases} \tag{59}$$

2. *If $\frac{2\ell}{a_2} \leq s \leq \frac{2}{a_2}(\ell+m)$, then for any $\varepsilon > 0$ there exists $c = c(s,\mathcal{T},\varepsilon)$ such that for any ε_2-bandlimited function $f \in L^2(\Omega,\mu)$ on Σ,*

$$\begin{cases} \||x|^{s\frac{a_2}{2}} f\|^2_{2,\mu} \geq c\left(\dfrac{1-\varepsilon_2^2}{[\widehat{\mu}_{2\widehat{m}}(\Sigma)]^{\frac{1}{(1+m/\ell)}-2\varepsilon}}\right)^{\frac{sa_2}{2\ell-\varepsilon sa_2}}\|f\|^2_{2,\mu} & \text{if } \widehat{\mu}_{2\widehat{m}}(\Sigma) \leq 1; \\ \||x|^{s\frac{a_2}{2}} f\|^2_{2,\mu} \geq c\left(\dfrac{1-\varepsilon_2^2}{[\widehat{\mu}_{2\widehat{m}}(\Sigma)]^{1-2\varepsilon}}\right)^{\frac{sa_2}{2\ell-\varepsilon sa_2}}\|f\|^2_{2,\mu}, & \text{if } \widehat{\mu}_{2\widehat{m}}(\Sigma) > 1; \end{cases} \tag{60}$$

3. *If $s > \frac{2}{a_2}(m+\ell)$, there exists $c = c(s,\mathcal{T})$ such that for any ε_2-bandlimited function $f \in L^2(\Omega,\mu)$ on Σ,*

$$\begin{cases} \||x|^{s\frac{a_2}{2}} f\|^2_{2,\mu} \geq c\left(\dfrac{1-\varepsilon_2^2}{\widehat{\mu}_{2\widehat{m}}(\Sigma)}\right)^{\frac{sa_2}{2\ell}}\|f\|^2_{2,\mu}, & \text{if } m = 0; \\ \||x|^{s\frac{a_2}{2}} f\|^2_{2,\mu} \geq \left(\dfrac{c(1-\varepsilon_2^2)}{\widehat{\mu}_{2\widehat{m}}(\Sigma)} - 1\right)\|f\|^2_{2,\mu}, & \text{otherwise}, \end{cases} \tag{61}$$

where in the last line, we have used the fact that

$$\begin{aligned}\|f\|^2_{L^2(\Omega,\mu_{sa_2})} &= \int_\Omega (1+|x|)^{sa_2}|f(x)|^2\,\mathrm{d}\mu(x)\\ &\le 2^{sa_2}\left(\|f\|^2_{2,\mu}+\||x|^{s\frac{a_2}{2}}f\|^2_{2,\mu}\right).\end{aligned}$$

Owing to a symmetry between f and its integral transform, then by exchanging the roles of f and $\mathcal{T}(f)$ in Theorem 3, we obtain:

Theorem 4. *Let $\beta > 0$ and let $S \subset \Omega$ be a subset such that $0 < \mu_{2m}(S) < \infty$. Then*

1. *If $0 < \beta < \frac{2\ell}{a_2}$, there exists a positive constant $c = c(\beta, \mathcal{T})$ such that for all $f \in L^2(\Omega, \mu)$, $\|\chi_S f\|_{2,\mu}$ is bounded by*

$$\begin{cases} c\left[\mu_{2m}(S)\right]^{\frac{\beta a_2}{4(\ell+\widehat{m})}}\||\xi|^{\beta\frac{a_2}{2}}\mathcal{T}(f)\|_{2,\widehat{\mu}}, & \text{if } \mu_{2m}(S)\le 1;\\ c\left[\mu_{2m}(S)\right]^{\frac{\beta a_2}{4\ell}}\||\xi|^{\beta\frac{a_2}{2}}\mathcal{T}(f)\|_{2,\widehat{\mu}}, & \text{if } \mu_{2m}(S)>1,\end{cases}\tag{62}$$

2. *If $\frac{2\ell}{a_2} \le \beta \le \frac{2}{a_2}(\ell+\widehat{m})$, then for any $\varepsilon > 0$ there exists a positive constant $c = c(\beta, \mathcal{T}, \varepsilon)$ such that for all $f \in L^2(\Omega, \mu)$, $\|\chi_S f\|_{2,\mu}$ is bounded by*

$$\begin{cases} c\left[\mu_{2m}(\Sigma)\right]^{\frac{1}{2(1+\widehat{m}/\ell)}-\varepsilon}\|f\|_{2,\mu}^{1-\frac{2\ell}{\beta a_2}+\varepsilon}\||\xi|^{\beta\frac{a_2}{2}}\mathcal{T}(f)\|_{2,\widehat{\mu}}^{\frac{2\ell}{\beta a_2}-\varepsilon}, & \text{if } \mu_{2m}(S)\le 1;\\ c\left[\mu_{2m}(S)\right]^{\frac{1}{2}-\varepsilon}\|f\|_{2,\mu}^{1-\frac{2\ell}{\beta a_2}+\varepsilon}\||\xi|^{\beta\frac{a_2}{2}}\mathcal{T}(f)\|_{2,\widehat{\mu}}^{\frac{2\ell}{\beta a_2}-\varepsilon}, & \text{if } \mu_{2m}(S)>1,\end{cases}\tag{63}$$

3. *If $\beta > \frac{2}{a_2}(\widehat{m}+\ell)$, there exists a positive constant $c = c(\beta, \mathcal{T})$ such that for all $f \in L^2(\Omega, \mu)$, $\|\chi_S f\|_{2,\mu}$ is bounded by*

$$\begin{cases} c\left[\mu_{2m}(S)\right]^{\frac{1}{2}}\|f\|_{2,\mu}^{1-\frac{2\ell}{\beta a_2}}\||\xi|^{\beta\frac{a_2}{2}}\mathcal{T}(f)\|_{2,\widehat{\mu}}^{\frac{2\ell}{\beta a_2}}, & \text{if } \widehat{m}=0;\\ c\left[\mu_{2m}(S)\right]^{\frac{1}{2}}\|\mathcal{T}(f)\|_{L^2(\widehat{\Omega},\widehat{\mu}_{\beta a_2})}, & \text{otherwise.}\end{cases}\tag{64}$$

Consequently, and in the same way by using (58), we obtain the following result giving a lower bound for the measure of the generalized frequency dispersion $\||\xi|^{\beta\frac{a_2}{2}}\mathcal{T}(f)\|_{2,\widehat{\mu}}$.

Corollary 4. *Let $\beta > 0$ and let $S \subset \Omega$ be a subset such that $0 < \mu_{2m}(S) < \infty$. Then*

1. *If $0 < \beta < \frac{2\ell}{a_2}$, there exists a positive constant $c = c(\beta, \mathcal{T})$, such that for any ε_1-concentrated function $f \in L^2(\Omega, \mu)$ on S,*

$$\begin{cases} \||\xi|^{\beta\frac{a_2}{2}}\mathcal{T}(f)\|^2_{2,\widehat{\mu}} \ge \dfrac{c(1-\varepsilon_1^2)}{[\mu_{2m}(S)]^{\frac{\beta a_2}{2(\ell+\widehat{m})}}}\|f\|^2_{2,\mu}, & \text{if } \mu_{2m}(S)\le 1;\\[2ex] \||\xi|^{\beta\frac{a_2}{2}}\mathcal{T}(f)\|^2_{2,\widehat{\mu}} \ge \dfrac{c(1-\varepsilon_1^2)}{[\mu_{2m}(S)]^{\frac{\beta a_2}{2\ell}}}\|f\|^2_{2,\mu}, & \text{if } \mu_{2m}(S)>1,\end{cases}\tag{65}$$

2. *If $\frac{2\ell}{a_2} \leq \beta \leq \frac{2}{a_2}(\ell + \widehat{m})$, then for any $\varepsilon > 0$ there exists a positive constant $c = c(\beta, \mathcal{T}, \varepsilon)$ such that for any ε_1-concentrated function $f \in L^2(\Omega, \mu)$ on S,*

$$\begin{cases} \||\xi|^{\beta\frac{a_2}{2}}\mathcal{T}(f)\|^2_{2,\widehat{\mu}} \geq c\left(\frac{1-\varepsilon_1^2}{[\mu_{2m}(S)]^{\frac{1}{(1+\widehat{m}/\ell)}-2\varepsilon}}\right)^{\frac{\beta a_2}{2\ell-\varepsilon\beta a_2}} \|f\|^2_{2,\mu} & \text{if } \mu_{2m}(S) \leq 1; \\ \||\xi|^{\beta\frac{a_2}{2}}\mathcal{T}(f)\|^2_{2,\widehat{\mu}} \geq c\left(\frac{1-\varepsilon_1^2}{[\mu_{2m}(S)]^{1-2\varepsilon}}\right)^{\frac{\beta a_2}{2\ell-\varepsilon\beta a_2}} \|f\|^2_{2,\mu}, & \text{if } \mu_{2m}(S) > 1, \end{cases} \tag{66}$$

3. *If $\beta > \frac{2}{a_2}(\widehat{m} + \ell)$, there exists a positive constant $c = c(\beta, \mathcal{T})$ such that for any ε_1-concentrated function $f \in L^2(\Omega, \mu)$ on S,*

$$\begin{cases} \||\xi|^{\beta\frac{a_2}{2}}\mathcal{T}(f)\|^2_{2,\widehat{\mu}} \geq c\left(\frac{1-\varepsilon_1^2}{\mu_{2m}(S)}\right)^{\frac{\beta a_2}{2\ell}} \|f\|^2_{2,\mu}, & \text{if } \widehat{m} = 0; \\ \||\xi|^{\beta\frac{a_2}{2}}\mathcal{T}(f)\|^2_{2,\widehat{\mu}} \geq \left(\frac{c(1-\varepsilon_1^2)}{\mu_{2m}(S)} - 1\right) \|f\|^2_{2,\mu}, & \text{otherwise.} \end{cases} \tag{67}$$

Now it is enough to take the product of the inequalities in Corollary 3 and Corollary 4 to obtain the analogue of Inequality (41) for functions in $L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$. In particular, we state the following special case (the same holds for the other cases).

Corollary 5. *Let $\frac{2\ell}{a_2} < s < \frac{2}{a_2}(\ell + m)$, $\frac{2\ell}{a_2} < \beta < \frac{2}{a_2}(\ell + \widehat{m})$ and let $S \subset \Omega$, $\Sigma \subset \widehat{\Omega}$ be two subsets with $1 < \mu_{2m}(S), \widehat{\mu}_{2\widehat{m}}(\Sigma) < \infty$. Then for any $\varepsilon > 0$ there exists $C = c(s, \beta, \mathcal{T}, \varepsilon)$, such that for every $f \in L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$,*

$$\begin{aligned} \||x|^{s\frac{a_2}{2}} f\|^\beta_{2,\mu} \||\xi|^{\beta\frac{a_2}{2}}\mathcal{T}(f)\|^s_{2,\widehat{\mu}} \quad &\geq \quad C\left(\frac{1-\varepsilon_1^2}{[\mu_{2m}(S)]^{1-2\varepsilon}}\right)^{\frac{s\beta a_2}{2(2\ell-\varepsilon\beta a_2)}} \\ &\times \quad \left(\frac{1-\varepsilon_2^2}{[\widehat{\mu}_{2\widehat{m}}(\Sigma)]^{1-2\varepsilon}}\right)^{\frac{s\beta a_2}{2(2\ell-\varepsilon s a_2)}} \|f\|^{s+\beta}_{2,\mu}. \end{aligned} \tag{68}$$

Remark 2. *In particular, if $s = \beta$ in (68), then for any $\varepsilon > 0$ there exists $C = c(s, \mathcal{T}, \varepsilon)$ such that for every $f \in L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$,*

$$\||x|^{s\frac{a_2}{2}} f\|_{2,\mu} \||\xi|^{s\frac{a_2}{2}}\mathcal{T}(f)\|_{2,\widehat{\mu}} \geq C\left(\frac{(1-\varepsilon_1^2)(1-\varepsilon_2^2)}{[\mu_{2m}(S)\widehat{\mu}_{2\widehat{m}}(\Sigma)]^{1-2\varepsilon}}\right)^{\frac{sa_2}{2(2\ell-\varepsilon s a_2)}} \|f\|^2_{2,\mu}, \tag{69}$$

which implies that

$$\||x|^{s\frac{a_2}{2}} f\|^2_{2,\mu} + \||\xi|^{s\frac{a_2}{2}}\mathcal{T}(f)\|^2_{2,\widehat{\mu}} \geq C\left(\frac{(1-\varepsilon_1^2)(1-\varepsilon_2^2)}{[\mu_{2m}(S)\widehat{\mu}_{2\widehat{m}}(\Sigma)]^{1-2\varepsilon}}\right)^{\frac{sa_2}{2(2\ell-\varepsilon s a_2)}} \|f\|^2_{2,\mu}. \tag{70}$$

On the other hand, if $f \in L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$, then we obtain the following new variation (see also [2]) of Donoho–Stark type uncertainty inequality (52), with constant that depends on f,

$$\mu_{2m}(S)\widehat{\mu}_{2\widehat{m}}(\Sigma) \geq C_f(s, \mathcal{T}, \varepsilon)\left((1-\varepsilon_1^2)(1-\varepsilon_2^2)\right)^{\frac{1}{1-2\varepsilon}}, \tag{71}$$

where

$$C_f(s,\mathcal{T},\varepsilon)=\left(\frac{C\|f\|_{2,\mu}^2}{\||x|^{s\frac{a_2}{2}}f\|_{2,\mu}\||\xi|^{s\frac{a_2}{2}}\mathcal{T}(f)\|_{2,\widehat{\mu}}}\right)^{\frac{4\ell-2\varepsilon s a_2}{s a_2(1-2\varepsilon)}}. \tag{72}$$

2.4. Shapiro-Type Uncertainty Principles

In [3] (Section 5.2) we have proved some Shapiro-type uncertainty principles for a family of integral transforms with bounded kernel, which is the case here when $m=\hat{m}=0$. The transforms under consideration in this paper are integral operators with polynomially bounded kernels (as in the article [1]). In this subsection, we will establish these uncertainty inequalities without any proof, which can be given in the same way with minor modifications.

First notice that a straightforward computation shows $E_S F_\Sigma$ is an integral transform with kernel (see [1], Lemma 3.2)

$$\mathcal{N}(x,\xi)=\chi_S(\xi)\mathcal{T}^{-1}(\chi_\Sigma\mathcal{K}_\xi))(x).$$

Then $E_S F_\Sigma$ is a Hilbert–Schmidt operator, such that

$$\|E_S F_\Sigma\|_{HS}^2\le c_{\mathcal{K}}^2\mu_{2m}(S)\widehat{\mu}_{2\widehat{m}}(\Sigma). \tag{73}$$

Adapting the proofs of [3] (Section 5.2), which are inspired from related results in [19], we obtain the following Shapiro-type UP.

Theorem 5. *Let $s>0$.*

1. *If $\{\varphi_n\}_{n=1}^\infty$ is an orthonormal system in $L^2(\Omega,\mu)$, then for every $N\ge1$,*

$$\sum_{n=1}^N\left(\||x|^{s\frac{a_2}{2}}\varphi_n\|_{2,\mu}^2+\||\xi|^{s\frac{a_2}{2}}\mathcal{T}(\varphi_n)\|_{2,\widehat{\mu}}^2\right)\ge c_{s,\ell}\,N^{1+\frac{s}{2\ell}}, \tag{74}$$

2. *If $\{\varphi_n\}_{n=1}^\infty$ is an orthonormal basis for $L^2(\Omega,\mu)$, then*

$$\sup_n\left(\||x|^{s\frac{a_2}{2}}\varphi_n\|_{2,\mu}\||\xi|^{s\frac{a_2}{2}}\mathcal{T}(\varphi_n)\|_{2,\widehat{\mu}}\right)=\infty. \tag{75}$$

Remark 3.

1. *The dispersion inequality (74) implies that there is no infinite system $\{\varphi_n\}_{n=1}^\infty$ in $L^2(\Omega,\mu)$ for which the two sequences $\left\{\left\||x|^{s\frac{a_2}{2}}\varphi_n\right\|_{2,\mu}\right\}_{n=1}^\infty$ and $\left\{\left\||\xi|^{s\frac{a_2}{2}}\mathcal{T}(\varphi_n)\right\|_{2,\widehat{\mu}}\right\}_{n=1}^\infty$ are bounded. More precisely, if $\{\varphi_n\}_{n=1}^\infty$ is an orthonormal sequence in $L^2(\Omega,\mu)$, then for every $N\ge1$,*

$$\sup_{1\le n\le N}\left\{\||x|^{s\frac{a_2}{2}}\varphi_n\|_{2,\mu}^2,\ \||\xi|^{s\frac{a_2}{2}}\mathcal{T}(\varphi_n)\|_{2,\widehat{\mu}}^2\right\}\ge c_{s,\ell}\,N^{\frac{s}{2\ell}}, \tag{76}$$

   and particularly

$$\sup_n\left(\||x|^{s\frac{a_2}{2}}\varphi_n\|_{2,\mu}^2+\||\xi|^{s\frac{a_2}{2}}\mathcal{T}(\varphi_n)\|_{2,\widehat{\mu}}^2\right)=\infty. \tag{77}$$

2. *Relation (75) is not true for any orthonormal sequence in $L^2(\Omega,\mu)$. Indeed we can find an infinite orthonormal sequence $\{\varphi_n\}_n$ in $L^2(\Omega,\mu)$, such that the product $\||x|^{s\frac{a_2}{2}}\varphi_n\|_{2,\mu}\,\||\xi|^{s\frac{a_2}{2}}\mathcal{T}(\varphi_n)\|_{2,\widehat{\mu}}$ is finite.*
3. *It is clear that Shapiro-type uncertainty principles (75) and (77) refine respectively Inequalities (48) and (49) for orthonormal bases and sequences.*

3. Examples

3.1. The Harmonic Oscillator

Let $\Omega = \widehat{\Omega} = \mathbb{R}^d$ and let $\mu = \widehat{\mu}$ be the normalized Lebesgue measure given by $\mathrm{d}\mu(x) = (2\pi)^{-d/2}\,\mathrm{d}x$. For $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, the Fourier transform is defined by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\langle x,\xi\rangle}\,\mathrm{d}x, \qquad \xi \in \mathbb{R}^d;$$

and is then extended to an isomorphism on $L^2(\mathbb{R}^d)$ in the usual way. In this example $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the usual Laplacian operator, $\mathcal{T} = \mathcal{F}$, $\ell = d/2$, $a_1 = m = \hat{m} = 0$ and $a_2 = 2$. The ϕ_j's are the Hermite functions and $\lambda_j = 2|j| + d$, such that $j = (j_1, \ldots, j_d) \in \mathbb{N}^d$ and $|j| = j_1 + \cdots + j_d$.

Notice that, here $\mathcal{L} = (-\Delta + |x|^2)$ is the harmonic oscillator (or the Hermite operator) on $\mathbb{R}^d$ and for each $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and each multi-index $j = (j_1, \ldots, j_d) \in \mathbb{N}^d$,

$$\phi_j(x) = \prod_{i=1}^{d} h_{j_i}(x_i)$$

be the normalized Hermite function on $\mathbb{R}^d$, where for all $i = 1, \ldots d$, $h_{j_i}(x_i)$ is the one-dimensional Hermite function defined by

$$h_{j_i}(x_i) = \left(2^{j_i - 1/2} j_i!\right)^{-1/2} e^{-x_i^2/2} H_{j_i}(x_i), \tag{78}$$

and H_{j_i} are the Hermite polynomials of degree j_i defined by the Rodriguez formula

$$H_{j_i}(x_i) = (-1)^{j_i} e^{x_i^2} \frac{\mathrm{d}^{j_i}}{\mathrm{d}x_i^{j_i}} (e^{-x_i^2}). \tag{79}$$

It is well-known that the sequence of Hermite functions ϕ_i form an orthonormal basis for the Hilbert space $L^2(\mathbb{R}^d)$, and they are eigenfunctions of $\mathcal{F}$ and of the harmonic oscillator, that is,

$$(-\Delta + |x|^2)\phi_j = (2|j| + d)\phi_j \qquad \text{and} \qquad \mathcal{F}(\phi_j) = (-i)^{|j|}\phi_j. \tag{80}$$

3.2. The Bessel Oscillator

If $f(x) = f_0(|x|)$ is radial, then its Fourier transform satisfies

$$\mathcal{F}(f)(\xi) = \int_0^\infty f_0(t) j_{d/2-1}(t|\xi|) \frac{t^{d-1}}{2^{d/2-1}\Gamma(d/2)}\,\mathrm{d}t = \mathcal{F}_{d/2-1}(f_0)(|\xi|),$$

where $\mathcal{F}_{d/2-1}$ is the Fourier–Bessel (or Hankel) transform of index $\frac{d}{2} - 1$, and for any $\alpha \geq -1/2$, j_α is the Bessel function:

$$j_\alpha(x) := \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n},$$

where Γ is the gamma function. We have $|j_\alpha| \leq 1$ and if we denote $\mathrm{d}\mu_\alpha(x) = \frac{x^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)}\,\mathrm{d}x$, then for $f \in L^1(\mathbb{R}_+, \mu_\alpha) \cap L^2(\mathbb{R}_+, \mu_\alpha)$, the Fourier–Bessel transform is defined by

$$\mathcal{F}_\alpha(f)(\xi) = \int_0^\infty f(x) j_\alpha(x\xi)\,\mathrm{d}\mu_\alpha(x), \qquad \xi \in \mathbb{R}_+,$$

and extends to an unitary operator to all $L^2(\mathbb{R}_+, \mu_\alpha)$ with $\mathcal{F}_\alpha^{-1} = \mathcal{F}_\alpha$.

Let L_n^α be the Laguerre polynomials, which can be defined by the Rodriguez formula (see [20], (4.17.1), p. 76)

$$L_n^\alpha(x) = x^{-\alpha}\frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-x}x^{n+\alpha}\right), \qquad n \in \mathbb{N}, \ x > 0. \tag{81}$$

They are solutions of the following second order linear differential equation (see [20], (4.18.8), p. 80),

$$xu'' + (\alpha + 1 - x)u' + nu = 0, \tag{82}$$

and then they satisfy the following recurrence formula (see [20], (4.18.4), p. 76)

$$xL_n^{\alpha+1}(x) = -(n+1)L_{n+1}^\alpha(x) + (\alpha + n + 1)L_n^\alpha(x), \qquad n \in \mathbb{N}. \tag{83}$$

Therefore if we define ϕ_n^α by

$$\phi_n^\alpha(x) = \sqrt{\frac{2^{\alpha+1}\Gamma(\alpha+1)n!}{\Gamma(n+\alpha+1)}}\, e^{-x^2/2}L_n^\alpha(x^2), \qquad n \in \mathbb{N}, \ x > 0, \tag{84}$$

then the sequence $\{\phi_n^\alpha\}_{n\in\mathbb{N}}$ forms an orthonormal basis for $L^2(\mathbb{R}_+, \mu_\alpha)$ (see [20], p. 84), such that

$$\mathcal{F}_\alpha(\phi_n^\alpha) = (-1)^n\phi_n^\alpha, \qquad n \in \mathbb{N}. \tag{85}$$

Now if we denote by $\ell_\alpha = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2\alpha+1}{x}\frac{\mathrm{d}}{\mathrm{d}x}$ the differential Bessel operator, then by (82) the ϕ_n^α's satisfy

$$\mathcal{L}_\alpha\phi_n^\alpha = (4n + 2\alpha + 2)\phi_n^\alpha, \qquad n \in \mathbb{N}. \tag{86}$$

In this case $\Omega = \widehat{\Omega} = \mathbb{R}_+$, $\mu = \widehat{\mu} = \mu_\alpha$, $\mathcal{T} = \mathcal{F}_\alpha$, $\Delta = \ell_\alpha$, $\ell = \alpha + 1$, $a_1 = m = \widehat{m} = 0$ and $a_2 = 2$. The ϕ_j's are the Laguerre functions, $\mathcal{L} = \mathcal{L}_\alpha$ and for all $j \in \mathbb{N}$, $\lambda_j = 4j + 2\alpha + 2$.

3.3. The Dunkl Harmonic Oscillator

Let us present some necessary material on the Dunkl' theory. Let G be a finite reflection group on $\mathbb{R}^d$, associated with a root system R and let R_+ the positive subsystem of R (see [4,21]). If k is a nonnegative multiplicity function defined on R, and G-invariant, then we define the index

$$\gamma := \gamma(k) = \sum_{\xi\in R_+} k(\xi) \geq 0 \tag{87}$$

and the weight function

$$w_k(x) = \prod_{\xi\in R_+} |\langle\xi, x\rangle|^{2k(\xi)}. \tag{88}$$

Further we introduce the Mehta-type constant c_k by

$$c_k^{-1} = \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2}}\,\mathrm{d}\mu_k(x), \tag{89}$$

where $\mathrm{d}\mu_k(x) = w_k(x)\,\mathrm{d}x$.

The Dunkl operators T_j, $1 \leq j \leq d$ associated with G and k are given by (see [22])

$$T_j f(x) = \frac{\partial f}{\partial x_j} + \sum_{\xi\in R_+} k(\xi)\langle\xi, e_j\rangle\frac{f(x) - f(\sigma_\xi(x))}{\langle\xi, x\rangle}, \qquad x \in \mathbb{R}^d; \tag{90}$$

where $f \in C^\infty(\mathbb{R}^d)$, and σ_ξ denotes the reflection with respect to the hyperplane orthogonal to ξ.

C. F. Dunkl in [4] introduced the Dunkl kernel $\mathcal{K}_k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, where for all $y \in \mathbb{R}^d$, the function $\mathcal{K}_k(\cdot, y)$ is the solution of the initial problem

$$T_j u(x,y) = y_j u(x,y); \qquad 1 \le j \le d, \quad u(0,y) = 1. \tag{91}$$

This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$, and satisfies for all $\lambda \in \mathbb{C}$, $z,\ z' \in \mathbb{C}^d$ and $x,\ y \in \mathbb{R}^d$:

$$\mathcal{K}_k(z,z') = \mathcal{K}_k(z',z), \qquad \mathcal{K}_k(\lambda z, z') = \mathcal{K}_k(z, \lambda z'),$$
$$\overline{\mathcal{K}_k(-iy,x)} = \mathcal{K}_k(iy,x), \qquad |\mathcal{K}_k(-iy,x)| \le 1.$$

The Dunkl transform $\mathcal{F}_k$ of a function $f \in L^1(\mathbb{R}^d, \mu_k) \cap L^2(\mathbb{R}^d, \mu_k)$ (see e.g., [21]), is given by

$$\mathcal{F}_k(f)(\xi) := c_k \int_{\mathbb{R}^d} \mathcal{K}_k(-i\xi, x) f(x)\, \mathrm{d}\mu_k(x), \qquad \xi \in \mathbb{R}^d;$$

and extends uniquely to an isometric isomorphism on $L^2(\mathbb{R}^d, \mu_k)$ with

$$\mathcal{F}_k^{-1}(f)(\xi) = \mathcal{F}_k(f)(-\xi). \tag{92}$$

If $k = 0$, the the Dunkl transform reduces to the Fourier transform, and if $f(x) = \widetilde{f}(|x|)$ is radial, then its Dunkl transform is given through the Fourier–Bessel transform, as follows:

$$\mathcal{F}_k(f)(\xi) = \mathcal{F}_{\gamma+d/2-1}(\widetilde{f})(|\xi|). \tag{93}$$

Rösler in [23] has introduced the Dunkl Hermite functions $\left\{h_n^k\right\}_{n\in\mathbb{N}^d}$ associated with G and k, which are defined by

$$h_n^k(x) = \sqrt{c_k 2^{-|n|}\, e^{-|x|^2}}\, H_n^k(x), \qquad x \in \mathbb{R}^d, \tag{94}$$

where H_n^k is the Dunkl Hermite polynomials of degree $|n|$.

The sequence $\left\{h_n^k\right\}_{n\in\mathbb{N}^d}$ forms an orthonormal basis for $L^2(\mathbb{R}^d, \mu_k)$ and each h_n^k satisfies

$$\mathcal{F}_k(h_n^k) = (-1)^{|n|} h_n^k, \qquad n \in \mathbb{N}^d. \tag{95}$$

Now if we denote by $\Delta_k = \sum_{j=1}^d T_j^2$ the Dunkl Laplacian and $\mathcal{L}_k = |x|^2 - \Delta_k$ the Dunkl harmonic oscillator, then we have

$$\mathcal{F}_k\left(\Delta_k f\right)(\xi) = -|\xi|^2 \mathcal{F}_k(f)(\xi), \qquad \xi \in \mathbb{R}^d, \tag{96}$$

and

$$\mathcal{L}_k h_n^k = (2|n| + 2\gamma + d) h_n^k, \qquad n \in \mathbb{N}^d. \tag{97}$$

In this case $\Omega = \widehat{\Omega} = \mathbb{R}^d$, $\mu = \widehat{\mu} = \mu_k$, $\mathcal{T} = \mathcal{F}_k$, $\Delta = \Delta_k$ is the Dunkl Lalpacian, $\ell = \gamma + d/2$, $a_1 = m = \widehat{m} = 0$ and $a_2 = 2$. The ϕ_j's are the Dunkl Hermite functions, $\lambda_j = (2|j| + 2\gamma + d)$ and $\mathcal{L} = \mathcal{L}_k$ is Dunkl harmonic oscillator.

3.4. The Deformed Dun Harmonic Osklcillator

Let $\Omega = \widehat{\Omega} = \mathbb{R}^d$, and let G and k as defined as in the last example. Let $a > 0$ be a deformation parameter and define the weight measure $\mathrm{d}\mu_{k,a} = \mathrm{d}\mu = \mathrm{d}\widehat{\mu} = \vartheta_{k,a}\, \mathrm{d}x$, as follows

$$\vartheta_{k,a}(x) = |x|^{a-2} \prod_{\xi\in R} |\langle \xi, x\rangle|^{k(\xi)} = |x|^{a-2} w_k(x). \tag{98}$$

The weight function $\vartheta_{k,a}$ is homogeneous of degree $a-2+2\gamma$, where $\gamma := \frac{1}{2}\sum_{\xi\in R} k(\xi)$ is the index of k. In the following we will assume that

$$2\gamma + d + a - 2 > 0. \tag{99}$$

For a real number $\lambda > -1$, we write $L_s^{(\lambda)}$, $s \in \mathbb{N}$, for the Laguerre polynomial defined by

$$L_s^{(\lambda)}(t) = \sum_{j=0}^{s} \frac{(-1)^j \Gamma(\lambda+s+j)}{(s-j)!\Gamma(\lambda+j+1)} \frac{t^j}{j!}.$$

Set $\lambda_{k,a,n} = (d+2\gamma+2n-2)/a$, and for each $n \in \mathbb{N}$, let $\{h_j^{(n)}\}_{j\in J_n}$ be an orthonormal basis of the space $\mathcal{H}_k^n(\mathbb{R}^d)_{|\mathbb{S}^{d-1}}$, where where $\mathcal{H}_k^n(\mathbb{R}^d)$ is the space of k-harmonic polynomials of degree n (i.e., the set of homogeneous polynomial p on $\mathbb{R}^d$ of degree n such that $\Delta_k p = 0$) and J_n is the dimension of $\mathcal{H}_k^n(\mathbb{R}^d)$. Then by [6] (Proposition 3.15), for any $n \in \mathbb{N}$, the family $\{f_{s,n}^{(a)}\}_{s\in\mathbb{N}}$ forms an orthonormal basis for $L^2(\mathbb{R}_+, r^{2\gamma+d+a-3}\,\mathrm{d}r)$, where

$$f_{s,n}^{(a)}(r) = \left(\frac{2^{1+\lambda_{k,a,n}}\Gamma(s+1)}{a^{\lambda_{k,a,n}}\Gamma(1+s+\lambda_{k,a,n})}\right)^{1/2} r^n L_s^{(\lambda_{k,a,n})}\left(\frac{2}{a}r^a\right) e^{-\frac{r^a}{a}}. \tag{100}$$

Therefore by [6] (Corollary 3.17), the family $\{\Phi_{s,n,j}^{(a)} \;:\; s \in \mathbb{N}, n \in \mathbb{N}, j \in J_n\}$ constitutes an orthonormal basis for $L^2(\mathbb{R}^d, \mu_{k,a})$, where

$$\Phi_{s,n,j}^{(a)}(x) = h_j^{(n)}\left(\frac{x}{|x|}\right) f_{s,n}^{(a)}(|x|), \qquad x \in \mathbb{R}^d. \tag{101}$$

The (k,a)-generalized Fourier transform $\mathcal{F}_{k,a}$ is given for $f \in L^1(\mathbb{R}^d, \mu_{k,a})$ by

$$\mathcal{F}_{k,a}(f)(\xi) = c_{k,a}\int_{\mathbb{R}^d} f(x)\mathcal{B}_{k,a}(x,\xi)\,\mathrm{d}\mu_{k,a}(x);$$

where

$$c_{k,a} = \left(\int_{\mathbb{R}^d} \exp\left(-\frac{1}{a}|x|^a\right)\vartheta_{k,a}\,\mathrm{d}x\right)^{-1}, \tag{102}$$

and the kernel $\mathcal{B}_{k,a}$ is given in [6] (Inequality (5.9)), and satisfies for $\lambda > 0$, $x, \xi \in \mathbb{R}^d$ the following properties (see [6], Theorem 5.9):

$$\mathcal{B}_{k,a}(\xi, x) = \mathcal{B}_{k,a}(x,\xi);\;\; \mathcal{B}_{k,a}(0,x) = 1;\;\; \mathcal{B}_{k,a}(\lambda\xi, x) = \mathcal{B}_{k,a}(\xi, \lambda x).$$

Moreover (see [24–27]), if one of the following assumptions holds:

(i) $d = 1$ and $a > 0$,
(ii) $a \in \{1, 2\}$,
(iii) $d \geq 2$, $k \equiv 0$ and $a = \frac{2}{n}$, for some $n \in \mathbb{N}^*$,

then $\mathcal{B}_{k,a}$ is uniformly bounded, that is,

$$|\mathcal{B}_{k,a}(x,\xi)| \leq C, \qquad \forall x, \xi \in \mathbb{R}^d,$$

where $C = C(d,k,a)$.

The transformation $\mathcal{F}_{k,a}$ is a unitary operator on $L^2(\mathbb{R}^d, \mu_{k,a})$ and it is defined a by the a-deformed Dunkl harmonic oscillator $\mathcal{L}_{k,a} = |x|^a - |x|^{2-a}\Delta_k$, where Δ_k the Dunkl Laplacian (introduced in the previous example). Note also that (see [6], Theorem 5.1) the system $\{\Phi^{(a)}_{s,n,j}\}$ is the eigensystem of $\mathcal{F}_{k,a}$, that is

$$\mathcal{F}_{k,a}(\Phi^{(a)}_{s,n,j})(\xi) = e^{-i\pi(s+n/a)}\Phi^{(a)}_{s,n,j}(\xi). \tag{103}$$

Moreover $\mathcal{F}_{k,a}$ is of finite order if and only if $a \in \mathbb{Q}$, and if $a = \frac{p}{q} \in \mathbb{Q}$, with p, q positive, then $(\mathcal{F}_{k,a})^{2p} = \mathrm{Id}$ (see [6], Corollary 5.2), with $\mathcal{F}^{-1}_{k,a} = \mathcal{F}^{2p-1}_{k,a}$. In addition, for any $r \in \mathbb{N}$, the transformation $\mathcal{F}_{k,\frac{2}{2r+1}}$ is a unitary operator of order four on $L^2(\mathbb{R}^d, \mu_{k,\frac{2}{2r+1}})$, with (see [6], Corollary 3.2.2)

$$\mathcal{F}^{-1}_{k,\frac{2}{2r+1}}(f)(x) = \mathcal{F}_{k,\frac{2}{2r+1}}(f)(-x), \qquad x \in \mathbb{R}^d. \tag{104}$$

The differential-difference operator $\mathcal{L}_{k,a}$ is an essentially self-adjoint operator on $L^2(\mathbb{R}^d, \mu_{k,a})$ and satisfies (see [6], Corollary 3.22):

1. There is no continuous spectrum of $\mathcal{L}_{k,a}$,
2. The discrete spectrum of $\mathcal{L}_{k,a}$ is given by

$$\begin{cases} \{2sa + 2n + 2\gamma + d + a - 2 \;:\; s, n \in \mathbb{N}\}, & \text{if} \quad d \geq 2, \\ \{2sa + 2\gamma + a \pm 1 \;:\; s \in \mathbb{N}\}, & \text{if} \quad d = 1. \end{cases} \tag{105}$$

Particularly $\mathcal{F}_{k,a}$ reduces to the Fourier transform on $\mathbb{R}^d$ if ($k = 0$, $a = 2$), the Fourier–Bessel transform if ($k = 0$, $a = 1$) and the Dunkl transform if ($k > 0$, $a = 2$). Moreover the restriction of $\mathcal{F}_{k,a}$ to radial functions is given by an a-deformed Fourier–Bessel transform (see [26])

In this case $\mathcal{T} = \mathcal{F}_{k,a}$, $\ell = \gamma + \frac{d+a}{2} - 1$, $m = \widehat{m} = 0$, $a_2 = 2$ and $a_1 = a - 2$, $\Delta = \Delta_k$ is the Dunkl Lalpacian and $\mathcal{L} = \mathcal{L}_{k,a}$ is the deformed Dunkl harmonic oscillator. The ϕ_j's and the λ_j's are respectively given by (101) and (105).

Remark 4. *Notice that the estimates* (37)*, which is obtained either from the heat kernel for the operator* $\mathcal{L}_{k,a}$ *or from disguised in spectral estimates of powers of the Laplacian. To my knowledge, the heat kernel for the operator* $\mathcal{L}_{k,a}$ *is only known at present for some special cases like* $d = 1$, $a > 0$ *(where one can deform the known one-dimensional Dunkl heat kernel with the parameter a), and* $d = 2$, $a \in \{1, 2\}$ *(where the explicit formula was obtained in [5]).*

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